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STRENGTH CALCULATION OF LATERALY LOADED TWO DIMENSIONAL PLANE STRUCTURAL ELEMENT

BOGDAN S. BOGDANOVIĆ
SERBIA, bogdanovic00@gmail.com

TONKO A. MIHOVILOVIĆ
UTVA AI, SERBIA, tonkojetonko@gmail.com

Abstract: *Laterally loaded plane structural elements have very wide use in aircraft structures. Dimensions of these elements should be defined very carefully, because of demands for smaller mass and reliable exploatation. These, plane elements behave like plates, membranes and most often like mixture of them. It means that corresponded strength calculation is very complex. For practical use it was very useful to make strength calculation much simpler. It could be done by introducing some assumptions, which do not change the esence of element behavior. In this paper some simlificated procedure, based on assumptions like these, for plane element strength calculation is shown. Also a numerical example is given.*

Keywords: *plane structural element, membrane, plate, strenght calculation.*

1. INTRODUCTION

Two dimensional plane structural elements are most frequently used as locally loaded parts of secondary structures. Term lateral load means that load acts in direction normal to the element`s plane. Usually, elements are walls of:

- passenger cabin under pressure,
- Some fluid compartment, gasoline, or oil tanks.

Main role of elements of such type is to transfer different types of lateral load, concentrated or uniformly distributed, to the primary structure elements. Elements could be reinforced, or rarely without reinforcements; it does depend on:

- Element size,
- Element load, or
- Both.

2. BEHAVIOR OF LATERALLY LOADED PLANE ELEMENT

Term “element behavior” describes the way how element:

- accepts lateral load, and
- transfers it to the surrounding (primary) structure.

All this is defined by corresponding stresses and deformation of element. There are two different ways of behavior for considered element type under lateral load such as:

- Plate, and/or
- Membrane

It should remark that both ways of behavior are more or less always present. Dominance of one or other way depends on:

- Element flexural rigidity, and
- Load intensity.

Usually, elements of greater flexural rigidity behaves more like plate, smaller flexural rigidity elements have membrane like behavior. On the other side lower load intensity produces plate, and higher load intensity membrane behavior. Generally, best criterion for elements behavior estimation is the relative magnitude of transverse (lateral) deflection under load compared with total element thickness. Transverse deflection of plates is approximately one or two tenths of thickness, for membrane it is about ten times elements thickness, or even more.

2.1. Element behaves like plate

Some basic characteristics of elements plate behavior are:

- Loads are resisted by bending and shear of elements cross sections,
- Lateral deflections and slopes are relatively small, so that $\frac{\delta^2 w}{\delta x^2}$ and $\frac{\delta^2 w}{\delta y^2}$ are good approximation to the curvature
- Elements supports on the surrounding primary structure are loaded mostly (only) in vertical plane (normal to the element)

Relations, connecting bending, and twisting moments, with internal shear forces and lateral loads are:

$$Q_X = \frac{\partial M_x}{\partial_x} + \frac{\partial M_{yx}}{\partial_y} \tag{1}$$

$$Q_y = \frac{\partial M_y}{\partial_y} + \frac{\partial M_{yx}}{\partial_x} \quad (2)$$

$$q = \frac{\partial Q_y}{\partial X} + \frac{\partial Q_y}{\partial_x} \quad (3)$$

Here are:

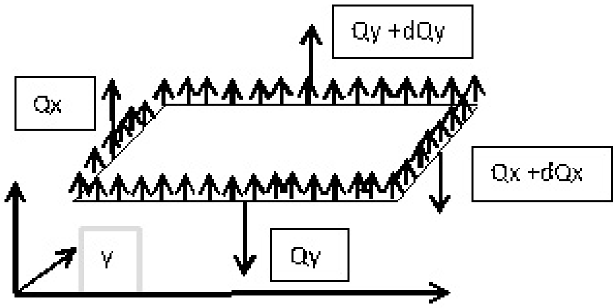

Figure 1. Load and internal shear forces on plate element

Equations (1), (2).. (3) are derived under assumption that:

- No stress acts in the middle surface of element, and
- It means that deformed surface is developable.

Basic relation between lateral loads (q), deflection (w) and element rigidity (D) is given:

$$\frac{\delta^4 w}{\delta x^4} + 2\frac{\delta^4 w}{\delta x^2 \delta y^2} + \frac{\delta^4 w}{\delta y^4} = \frac{q}{D} \quad (4)$$

Usually, most important results for practical use are:

- maximum stress and deformations values

These values should be obtained by performing analytic computation, but also using lot of tables and expressions [1]:

$$W_{\max} = W_{MAX} = \propto \frac{qa^4}{Et^3} \quad (5)$$

$$S_{MAX} = \beta \frac{qa^2}{t^2} \quad (6)$$

Marks in (5), (6) are:

W_{MAX} [m] - maximum deflection,

S_{MAX} [bar] – maximum stress values

α, β [ ] – deflection, stress coefficient

a [m], t [m] – plate short side, thicknes

E [bar] – plate material modulus of elasticity

∂ *Mx*, ∂ *My*, ∂ *Mxy*, [Nm] – bending, twisting moments

Qx, *Qy*, [N] – internal shear forces,

Coefficient values α, β are given in corresponding

q [N/m^2] – lateral load tables in literature [1].

2.2. Element behaves like membrane

Membrane length (a):

Some basic characteristics of elements membrane behavior are:

- Loads are resisted by tension and stretching of middle surface,
- Lateral deflections and slopes are relatively great,
- Elements supports on the surrounding primary structure are loaded in both vertical and horizontal planes

$$\delta = \int_0^a dl - a \quad (9)$$

After using equation (8) and integration it is obtained:

$$\delta = \frac{q^2}{24} \frac{a^2}{s^2} \frac{}{t^2} \quad (10)$$

On the other side is:

One dimensional problem retains the general problem feature. It will be treated first because of simplicity Relation connecting tension stress (force) with deformation and load, is:

$$\frac{d^2 w}{dx} = -\frac{q}{st} \quad (7)$$

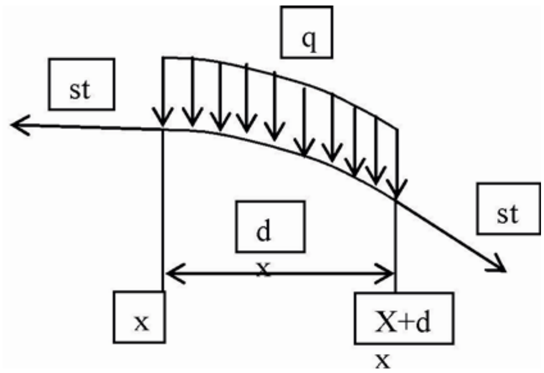

Figure 2. Load and internal tension forces on one dimensional membrane element

Equation (7) is differential equation of a parabola and its solution is:

$$w = \frac{qx}{2st}(a-x) \quad (8)$$

In equation (8) there are two unknowns:

s – stress and

w - deformation

Firstly, stress value (s) will be determined by computing the membrane stretch as a difference between curved arc length (dl) and original

$$\delta = \frac{s \cdot a}{E} \quad (11)$$

Combining (10) and (11) the final expression for stress is:

$$s = 0.347\left[E\left(\frac{q\cdot a}{t}\right)^{2}\right]^{\frac{1}{3}} \tag{12}$$

Expression (12) shows the nonlinearity of problem because stress varies as fractional exponent of load.

Solution for complete two dimensional problem is performed. Obtained expressions for maximum stress and deformation have similar shape as these for one dimensional membrane and for plate:

$$W_{MAX} = n_1 a\left(\frac{q\cdot a}{E\cdot t}\right)^{\frac{1}{3}} \tag{13}$$

$$S_{MAX} = n_2\left(E\frac{q\cdot a}{t}\right)^{\frac{1}{2}} \tag{14}$$

It is worth to note that maximum stress value occurs at the middle of the long panel side.

Marks in (13), (14) are:

w_{MAX}[m] - maximum deflection,

s_{MAX}[bar] – maximum stress values

n_1,n_2, [ ] – deflection, stress coefficient

a [m], t [m] – membrane long side, thickness

Coefficient values n_1, n_2 are given in corresponding tables in literature [1].

2.3. Element behaves like plate membrane combination

In most often cases laterally loaded two dimensional element behaves like some plate – membrane combination. It means that one part of lateral load is reacted by plate, and the other one by membrane action. A relatively good approximation for maximum values deformation calculation could be obtained simply by adding together parts of load carried through plate and membrane actions:

$$q = q_{PLATE} + q_{MEM} \tag{15}$$

Values q_{PLATE}, q_{MEM} could be obtained by solving equations (5) and (13) for corresponding loads part:

$$q_{PLATE} = \frac{w_{MAX}\cdot E\cdot t^3}{a\cdot a^4} \tag{16}$$

$$q_{MEM} = \frac{w_{MAX}^3\cdot E\cdot t}{n_1^3 a^4} \tag{17}$$

Putting (16) and (17) in (15) and rearranging it is obtained:

$$q = \frac{1}{a}\frac{E\cdot t^3}{a^4\cdot\left(\frac{b}{a}\right)^4}\cdot W_{MAX} + \frac{1}{n_1^3}\frac{E\cdot t}{a^4}\cdot W_{MAX}^3 \tag{18}$$

In (18) there is not any interaction between plate and membrane behavior, and since the system is NONLINEAR obtained result will be approximative

Table 1. Plate coefficient

b/a	α	β
1.0	0.0443	0.2874
1.2	0.0616	0.3756
1.4	0.0770	0.4518
1.6	0.0906	0.5172
1.8	0.1017	0.5688
2.0	0.1106	0.6102
3.0	0.1336	0.7134
Al edges are simply supported		

Table 2. Membrane coefficient

a/b	1.0	1.5	2.0	2.5
n_1	0.318	0.228	0.16	0.125
n_2	0.356	0.37	0.336	0.304

3. PLATE AND MEMBRANE DEFORMATION AND STRESS CALCULATION

There are several steps for plate and membrane deformation and stress calculation:

- Maximum values for deflection W_{MAX} will be obtained by solving cubic equation (18) using corresponding procedure
- Values for q_{PLATE} and q_{MEM} will be obtained using W_{MAX} and expressions (16) and (17)
- Corresponding stress values for plate s_{PLATE} and membrane s_{MEM} ,will be calculated using q_{PLATE} and q_{MEM} and expression (6) and (14)

4. NUMERICAL EXAMPLE

As a numerical example maximum deformation, maximum plate, membrane and total stresses for a uniformly laterally loaded plane two dimensional simply supported element will be calculated. Load data are shown in Table 3:

Table 3. Load data

Load		
$F_{total\ 1}$	A	q_1
[daN	[cm^2]	[daN/cm^2]
216	296.36	0.728843

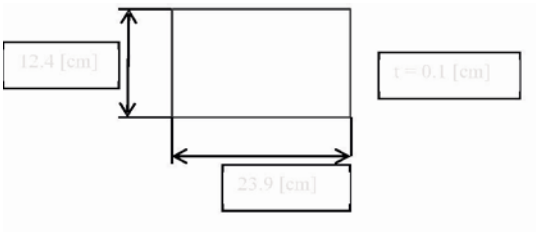

Figure 3. Two dimensional plane element

Calculation is made of several steps:

1. Definition of plate deflection and stress coefficients,
2. Definition of membrane deflection and stress coefficients,
3. Cubic equation solution – w $_{MAX}$ calculation
4. Calculation of load part reacted by element like a plate
5. Calculation of load part reacted by element like a membrane
6. Calculation of plate bending stresses
7. Calculation of membrane tension stress

Step 1

Plate deflection and stress coefficients are obtained by interpolation using data from Table 1; obtained results are shown in Tables 4 and 5

Table 4. Plate deflection coefficient α

a	b	b/a	b/a_1	b/a_2	$\Delta\ b/a_1$
[cm]	[cm]	[ ]	[ ]	[ ]	[ ]
12.4	23.9	1.9274	1.8	2	0.2
α (1.8)	α (2)	$\Delta\alpha 1$	$\Delta\ b/a_2$	$\Delta\ \alpha_2$	α
[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
0.1017	0.1106	0.0089	0.1274	0.0057	0.1074

Table 5. Plate stress coefficient β

a	b	b/a	b/a_1	b/a_2	$\Delta\ b/a_1$
[cm]	[cm]	[ ]	[ ]	[ ]	[ ]
12.4	23.9	1.9274	1.8	2	0.2
β (1.8)	β (2)	$\Delta\ \beta_1$	$\Delta\ b/a_2$	$\Delta\ \beta_2$	β
[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
0.5688	0.6102	0.0414	0.1274	0.0264	0.5952

Step 2

Membrane deflection and stress coefficients are obtained by interpolation using data from Table 2; obtained results are shown in Tables 6 and 7.

Table 6. Membrane deflection coefficient n_1

a	b	a/b	a/b_1	a/b_2	$\Delta\ a/b_1$
[cm]	[cm]	[ ]	[ ]	[ ]	[ ]
12.4	23.9	1.9274	1.5	2	0.5
n_1(1.5)	$n1$ (2)	$\Delta\ n_{11}$	$\Delta\ a/b_2$	$\Delta\ n_2$	n_1
[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
0.228	0.16	-0.068	0.4274	-0.0581	0.1699

Table 7. Membrane stress coefficient n_1

a	b	a/b	a/b_1	a/b_2	$\Delta\ a/b_1$
[cm]	[cm]	[ ]	[ ]	[ ]	[ ]
12.4	23.9	1.9274	1.5	2	0.5
n_2(1.5)	n_2(2)	$\Delta\ n_2 1$	$\Delta\ b/a_2$	$\Delta\ n_2$	n_2
[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
0.37	0.336	-0.034	0.4274	-0.0291	0.3410

Step 3

Equation (18) could be rearranged. It is incomplete (or reduced) cubic equation; without second degree member:

$$\frac{1}{n_1^3}\frac{E\cdot t}{a^4}\cdot W_{MAX}^3+\frac{1}{a}\frac{E\cdot t}{a^4\left(\frac{b}{a}\right)^4}\cdot W_{MAX}-q=0 \quad (19.1)$$

or

$$K_{31}w^3+K_{11}w+q_1=0 \quad (19.2)$$

Starting values for coefficients K_{11} and K_{11} are calculated using data from tables 4 and 6, obtained results are shown in Table 8.

Table 8. Cubic equation coefficient – starting values

E	t	t^3	α
[bar]	[cm]	[cm^3]	[ ]
735000	0.1	0.001	0.10737
b	b^4	$A=Et^3$	$B=\alpha b^4$
[cm]	[cm^4]	[daNcm]	[cm^4]
23.9	326280.9	735	35032.83
$K_{11}=A/B$	a	a^4	n_1
[ ]	[cm]	[cm^4]	[ ]
0.02098	23.9	326280.9	0.169871
n_1^3	$C=Et$	$D=n_1^3a^4$	$K_{31}=C/D$
[ ]	[ ]	[cm^4]	[ ]
0.004902	73500	1599.371	45.95558

Using values from Table 7 and 3, equation (19.2) is now:

$$45.95558w^3 + 0.02098w - 0.728843 = 0 \quad (20)$$

Before solving, cubic equation (20) should be rearranged to:

$$K_{32}w^3+K_{12}w+q_2=0 \quad (21)$$

Values for coefficients K_{32}, K_{12} and q_2 are given in Table 9.

Table 9. Cubic equation coefficient – calculation values

K_{32}	K_{12}	q_2
1	0.000457	- 0.01586

Cubic equation - shape for solution is:

$$w^3+0.000457w-0.01586=0 \quad (22)$$

First step in solution of eq. (22) is to calculate its discriminant; calculation and obtained results are shown in Table 10.

Table 10. Cubic equation - discriminant

K_{12}	q_2	q_2^2	$q_2^2/4$
0.000457	-0.01586	0.000252	6.29E-05
K_{12}^3	$K_{12}^3/27$	D	
9.52E-11	3.52E-12	6.29E-05	

Obtained value for discriminant is:

$$D = 6.29E\text{ - }05 > 0$$

In case like this Cardano´s formula for cubic equation roots solution will be applied. Such type of cubic equation has one real root value x_1, and the other two are conjugate complex.

$$x_1 = \sqrt{-\frac{q_2}{2} + \sqrt{\frac{q_2^2}{4} + \frac{K_{12}^3}{27}}} + \sqrt[3]{-\frac{q_2}{2} - \sqrt{\frac{q_2^2}{4} + \frac{K_{12}^3}{27}}} \qquad (23)$$

Complete real root calculation and obtained results are given in Table 11.

Table 11. Real root calculation and obtained value

q_2	K_{12}	$q_{2/2}$	$A=(q_{2/2})^2$
-0.01586	0.000457	-0.00793	6.29E-05
$K_{12/3}$	$B=(K_{12/3})^3$	$(A + B)$	
0.000152	3.52E-12	6.29E-05	
$C=(A+B)^{1/2}$	$E=-(q_{2/2})+C$	$F=-(q_{2/2})-C$	$G=E^{1/3}$
0.00793	0.01586	-2.2E-10	0.25125
$H=F^{1/3}$	$x_1=G + H$	w [cm]	
-0.00061	0.25064	0.25064	

Load distribution; calculation and results, corresponding stress values for plate and membrane are shown in Tables 12 13, 14, 15.

Table 12. Part of load accepted by membrane action

w	w^3	E	t	n_1	n_1^3
[cm]	[cm^3]	[bar]	[cm]	[ ]	[ ]
0.2506	0.0157	735000	0.1	0.17	0.0049
a	a^4	$A=w^3Et$	$B=n_1^3a^4$	$q_{мем}$	
[cm]	[cm^4]	[daNcm^2]	[cm^4]	[bar]	
23.9	326281	1157.28	1599.4	0.724	

Table 13. Part of load accepted by plate action

w	E	t	t^3	α
[cm]	[bar]	[cm]	[cm]	[ ]
0.2506	735000	0.1	0.001	0.1073
a	a^4	$A=wEt^3$	$B=\alpha a_4$	$q_{plate.}$
[cm]	[cm^4]	[daNcm^2]	[cm]	[bar]
23.9	326281	184.220	35033.	0.0053

Table 14. Membrane stress in element - tension

q_{mem}	a	t	$A = qa$	$B=A/t$	$C=B^2$
[bar]	[cm]	[cm]	[daN/cm]	[bar]	[bar^2]
0.7236	23.9	0.1	17.2937	172.97	29907
E	$D = EC$	$F=D^{(1/3)}$	n_2	$\sigma_{tens.}$	
[bar]	[bar^3]	[bar]	[ ]	[bar]	
735000	2.2E+10	2801.3	0.34094	955.1	

Table 15. Plate stress in element – tension/compression

q_{plate}	a	a^2	t	t^2
[bar]	[cm]	[cm^2]	[cm]	[cm^2]
0.005259	12.4	153.76	0.1	0.01
β	$A = a^2q$	$B = A/t^2$	σ comp.	σ tens.
[ ]	[daN]	[bar]	[bar]	[bar]
0.595176	0.808548	80.85482	48.12283	-48.1228

Table 16. Total stresses in element

$\sigma_{mem.\ tens.}$	$\sigma_{plate\ tens.}$	$\sigma_{plate\ comp.}$	$\sigma_{tot.\ 1}$	$\sigma_{tot.\ 2}$
[bar]	[bar]	[bar]	[bar]	[bar]
955.050	48.123	-48.123	1003.17	906.93

5. CONCLUSION

The simplified procedure for stresses and strain values definition in a laterally loaded two dimensional structural elements is shown. Numerical example is presented. Procedure should be the basis for detailed calculation of much complicated load cases, particularly on the elements of the secondary aircraft structures, which must include the stiffness of surrounding structural elements.

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