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Free Transverse Vibration Analysis of a Rayleigh Double-Beam System With a Keer Middle Layer Subjected to Compressive Axial Load

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Free transverse vibration of a Rayleigh double-beam system with the effect of compressive axial load with a Keer layer in-between is studied in this paper. It is assumed that the two beams of the system are continuously joined by the Keer layer. The equations of the motion for this system are described by a set of three homogenous differential equations. The classic Bernoulli-Fourier method was used for solving this system of differential equations. In the research found in literature on the subject of the Keer model, the most common methods used are the theories of Timoshenko and Euler-Bernoulli. It is for that exact reason that the research goal of this paper is determining analytical and numerical characteristics (natural frequency, associated amplitude ratio) of the considered model by applying the Rayleigh theory. What is also researched is the effect of elastic in-between layer on system's frequency and amplitudes. The numerical analysis of the system was performed using software tools. The numerical results obtained are shown in the form of a plot diagram. Presented numerical results in this paper confirm those obtained in the literature.

Key words: **natural frequency, Rayleigh double-beam, compressive axial load, Keer layer.**

Introductin

A large number of mechanical systems is of a complex structure composed of two or more basic mechanical systems, whose dynamic behavior is conditioned by their mutual interaction. Systems connected by an elastic layer compose a group of such mechanical structures, and they have a wide appliance in mechanical and civil industry. Oscillations and stability of such systems are the subject of scientific and practical work spanning several decades.

Oniszczuk [1, 2] analyzed the problem of free and forced oscillations of two elastically bound Euler-Bernoulli carriers. The paper determines analytical solutions of eigen frequencies for amplitude functions and oscillation forms. In the paper by Zhang et al. [3], on the basis of the Bernoulli–Euler beam theory, the properties of free transverse vibration and buckling of a double beam system under compressive axial loading are investigated. It is found that the effects of compressive axial loading on the natural frequencies of the system and associated amplitude ratios are more significant with the increase of axial compression. Also, the effects of compressive axial loading on the higher natural frequency and the amplitude ratios are significantly dependent on the axial compression ratio whereas that on the lower natural frequency is almost independent of it. It is concluded that the critical buckling load gets smaller with the increase of the ratio of the axial load F_2 to F_1 and the diminishment of the stiffness modulus K of the Winkler elastic layer.

In the paper by Stojanovic et al. [4], free transverse vibration and buckling of a double-beam continuously joined by a Winkler elastic layer under compressive axial loading with the influence of rotary inertia and shear are considered. The motion of the system is described by a homogeneous set of two partial differential equations, which is solved by using the classical Bernoulli–Fourier method. A structural model of a layered-beam system composed of two parallel Euler beams of uniform properties axially loaded with a flexible Winkler elastic layer in-between was used to study all the desired effects. It is determined that the influence of rotary inertia and shear on natural frequencies is manifested by the reduction of their values. In addition, it is found that the rotary inertia does not influence the critical buckling load model of a layered-beam system composed of two parallel Rayleigh beams, yet when the model of a layered-beam system composed of two parallel Timoshenko beams is considered, the influence of transverse shear causes a decrease in the critical buckling load.

The paper by Kozic et al. [5], an analytical theory to define the dynamic characteristics of the elastically connected parallel-beams under compressive axial loading. It is assumed that the two parallel-beams of the system are simply supported and continuously joined by a Kerr-type three parameter model. The motion of the system is described by a set of three homogeneous partial differential equations, which are solved by using the classical Bernoulli–Fourier method. The natural frequencies, associated amplitude ratio and the critical buckling load for complex system are determined. The

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model is tested numerically, and the results were compared with other numerical models.

In the paper by Mohammadi and Nasirshoaibi [6], the forced transverse vibrations of an elastically connected simply supported double-beam system with a Pasternak middle layer subjected to compressive axial load are investigated using the Rayleigh beam theory. The properties of the forced transverse vibrations of the system are found to be significantly dependent on the compressive axial load and shear foundation modulus of the Pasternak layer. The axial compression and shear foundation modulus of the Pasternak layer affects the magnitudes of the steady-state vibration amplitudes of the beam. Also, the ratios (φ_1) and (φ_2) decrease with increasing of the shear foundation modulus of the Pasternak layer G_o. This research can be used in an optimal design of a dynamic rotation absorber. The paper by Younesian et al. [7] represents a comprehensive review on different theoretical elastic and viscoelastic foundation models in oscillatory systems, different models of structure on foundation, most common solution methods, as well as practical implementations.

This article is organized as follows. In Section 2 a mathematical model of a double beam system with the Keer layer in between was formed using the Rayleigh beam theory [8, 9]. In Section 3 we solved formed differential equations in previous section and obtained analytical expressions of natural frequency and associated amplitude ratio. In Section 4, the critical axial buckling load of two elastically connected beams will be determined for the case where the physical properties and cross-sections of the two beams are identical. In Section 5, the numerical analysis of the system was performed using Matlab R2019a. Finally, in section 6, the conclusions are drawn, briefly.

Mathematical model

Fig.1 shows a double-beam system with the Keer layer [10] in-between with the length of l subjected to axial compressions F_1 and F_2. The model assumes that the axial forces F_1 and F_2 are not changed with time, the two beams have the same effective material constants, the rotary inertia and shear deformation are negligible, the behavior of the beam material is linear elastic and the cross-section is rigid and constant throughout the length of the beam and has one plane of symmetry.

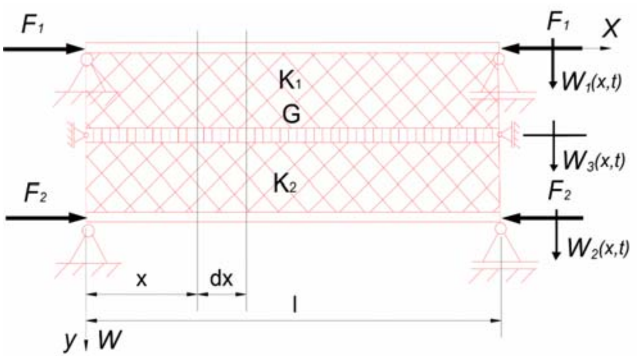

Figure 1. Double-beam dynamic system with a Kerr middle layer

The equations of free transverse vibration of a Rayleigh double-beam system with the effect of compressive axial load with a Kerr layer in-between have the following form:

$$\begin{aligned}&\rho A_1\frac{\partial^4 w_1}{\partial t^2}-\rho I_1\frac{\partial^4 w_1}{\partial x^2\partial t^2}+EI_1\frac{\partial^4 w_1}{\partial x^4}+\\&+F_1\frac{\partial^2 w_1}{\partial x^2}+K_1\left(w_1-w_3\right)=0\end{aligned} \tag{1}$$

$$G\frac{\partial^2 w_3}{\partial x^2}-\left(K_1+K_2\right)w_3+K_1w_1+K_2w_2=0 \tag{2}$$

$$\begin{aligned}&\rho A_2\frac{\partial^4 w_2}{\partial t^2}-\rho I_2\frac{\partial^4 w_2}{\partial x^2\partial t^2}+EI_2\frac{\partial^4 w_2}{\partial x^4}+\\&+F_2\frac{\partial^2 w_2}{\partial x^2}+K_2\left(w_3-w_2\right)=0\end{aligned} \tag{3}$$

where ρ is the mass density, A is the cross-sectional area of the beam, I_x is the moment of inertia of the beam cross-section and E Young's modules.

Eliminating w_3 from equations (1) and (3), one can obtain two sixth order coupled governing differential equations:

$$\begin{aligned}&\frac{GEI_1}{K_1}\frac{\partial^6 w_1}{\partial x^6}-\frac{G\rho I_1}{K_1}\frac{\partial^6 w_1}{\partial x^4\partial t^2}+\left[\frac{GF_1}{K_1}-EI_1\left(1+\frac{K_2}{K_1}\right)\right]\frac{\partial^4 w_1}{\partial x^4}\\&+\frac{G\rho A_1}{K_1}\frac{\partial^4 w_1}{\partial x^2\partial t^2}+\left[G-F_1\left(1+\frac{K_2}{K_1}\right)\right]\frac{\partial^2 w_1}{\partial x^2}-\\&\rho A_1\left(1+\frac{K_2}{K_1}\right)\frac{\partial^2 w_1}{\partial t^2}-K_2w_1+K_2w_2=0\end{aligned} \tag{4}$$

$$\begin{aligned}&\frac{GEI_2}{K_2}\frac{\partial^6 w_2}{\partial x^6}-\frac{G\rho I_2}{K_2}\frac{\partial^6 w_2}{\partial x^4\partial t^2}+\left[\frac{GF_2}{K_2}-EI_2\left(1+\frac{K_2}{K_1}\right)\right]\frac{\partial^4 w_2}{\partial x^4}\\&+\frac{G\rho A_2}{K_2}\frac{\partial^4 w_2}{\partial x^2\partial t^2}+\left[G-F_2\left(1+\frac{K_2}{K_1}\right)\right]\frac{\partial^2 w_2}{\partial x^2}\\&-\rho A_2\left(1+\frac{K_2}{K_1}\right)\frac{\partial^2 w_2}{\partial t^2}-K_1w_2+K_1w_1=0\end{aligned} \tag{5}$$

The initial conditions in general form and boundary conditions for simply supported beams of the same length are assumed as follows:

$$w_i\left(x,0\right)=w_{i0}\left(x\right),\ \dot{w}_i\left(x,0\right)=v_{i0}\left(x\right), \tag{6}$$

$$w_i''\left(0,t\right)=w_i\left(0,t\right)=w_i\left(l,t\right)=w_i''\left(l,t\right),\ \ i=1,2. \tag{7}$$

Solution of the problem

Assuming time harmonic motion and using separation of variables, the solutions to Eqs. (3) and (4) with the governing boundary conditions can be written in the form:

$$w_i\left(x,t\right)=\sum_{n=1}^{\infty}X_n\left(x\right)T_{ni}\left(t\right), \tag{8}$$

where $T_{in}(t)$ denotes the unknown functions, and $X_n(x)$ is the known mode shape function which is defined as:

$$X_n\left(x\right)=\sin\left(k_nx\right),\ \ k_n=\frac{n\pi}{l},\ \ n=1,2,3,.... \tag{9}$$

Substitution of Eq. (8) into Eqs. (3), (4) yields ordinary differential equations for the Rayleigh double-beam system. Therefore:

$$\sum_{n=1}^{\infty}\left\{-a_2\frac{\partial^2 T_{2n}}{\partial t^2}-b_2T_{2n}+H_2T_{1n}\right\}X_n=0 \tag{10}$$

$$\sum_{n=1}^{\infty}\left\{-a_1\frac{\partial^2 T_{1n}}{\partial t^2}-b_1T_{1n}+H_1T_{2n}\right\}X_n=0 \tag{11}$$

where

$$H_1=\frac{K_2}{\rho A_1}\quad H_2=\frac{K_1}{\rho A_2}$$

$$a_1=\frac{GI_1k_n^4}{A_1K_1}+\frac{Gk_n^2}{K_1}+\left(1+\frac{K_2}{K_1}\right)$$

$$a_2=\frac{GI_2k_n^4}{A_2K_2}+\frac{Gk_n^2}{K_2}+\left(1+\frac{K_2}{K_1}\right)$$

$$b_1=\frac{GEI_1k_n^6}{\rho A_1K_1}-\frac{GF_1k_n^4}{\rho A_1K_1}+\frac{EI_1k_n^4}{\rho A_1}\left(1+\frac{K_2}{K_1}\right)$$

$$+\frac{Gk_n^2}{\rho A_1}-\frac{F_1k_n^2}{\rho A_1}\left(1+\frac{K_2}{K_1}\right)+\frac{K_2}{\rho A_1}$$

$$b_2=\frac{GEI_2k_n^6}{\rho A_2K_2}-\frac{GF_2k_n^4}{\rho A_2K_2}+\frac{EI_2k_n^4}{\rho A_2}\left(1+\frac{K_2}{K_1}\right)$$

$$+\frac{Gk_n^2}{\rho A_2}-\frac{F_2k_n^2}{\rho A_2}\left(1+\frac{K_2}{K_1}\right)+\frac{K_1}{\rho A_2}$$

The solutions of equations (10) and (11) can be obtained by:

$$T_{n1}=C_ne^{i\omega_nt},\ T_{n2}=D_ne^{i\omega_nt}. \tag{12}$$

where ω_n denotes the natural frequency of the system. Substituting equation (12) into equations (10) and (11), we obtained:

$$\left(b_1-a_1\omega_n^2\right)D_n-H_1D_n=0 \tag{13}$$

$$\left(b_2-a_2\omega_n^2\right)D_n-H_2C_n=0 \tag{14}$$

When the determinant of the coefficients in Eqs. (13), (14) vanishes, non-trivial solutions for the constants C_n and D_n can be obtained, which yields the following frequency (characteristic) equation:

$$a_1a_2\omega_n^4-\left(b_1a_2+a_1b_2\right)\omega_n^2+b_1b_2-H_1H_2=0 \tag{15}$$

Then from the characteristic equation (15), we obtained

$$\omega_{n1}^2=\frac{\left(b_1a_2+a_1b_2\right)-\sqrt{\left(b_1a_2+a_1b_2\right)^2-4a_1a_2\left(b_1b_2-H_1H_2\right)}}{2a_1a_2} \tag{16}$$

$$\omega_{n2}^2=\frac{\left(b_1a_2+a_1b_2\right)+\sqrt{\left(b_1a_2+a_1b_2\right)^2-4a_1a_2\left(b_1b_2-H_1H_2\right)}}{2a_1a_2} \tag{17}$$

For each of the natural frequencies, the associated amplitude ratio of vibration modes of the two beams is given by:

$$\alpha_{ni}=\frac{C_n}{D_n}=\frac{H_1}{\left(b_1-a_1\omega_{ni}^2\right)}=\frac{\left(b_2-a_2\omega_{ni}^2\right)}{H_2} \tag{18}$$

From the above analysis we know that solutions (12) can be rewritten as:

$$T_{1n}(t)=C_{1n}e^{j\omega_{n1}t}+C_{2n}e^{-j\omega_{n1}t}+C_{3n}e^{j\omega_{n2}t}+C_{4n}e^{-j\omega_{n2}t}, \tag{19}$$

$$T_{2n}(t)=D_{1n}e^{j\omega_{n1}t}+D_{2n}e^{-j\omega_{n1}t}+D_{3n}e^{j\omega_{n2}t}+D_{4n}e^{-j\omega_{n2}t}, \tag{20}$$

or introducing the trigonometric functions we get:

$$T_{1n}(t)=\sum_{i=1}^{2}\left[A_{ni}\sin\left(\omega_{ni}t\right)+B_{ni}\cos\left(\omega_{ni}t\right)\right], \tag{21}$$

$$T_{2n}(t)=\sum_{i=1}^{2}\alpha_{ni}^{-1}\left[A_{ni}\sin\left(\omega_{ni}t\right)+B_{ni}\cos\left(\omega_{ni}t\right)\right], \tag{22}$$

where A_{ni} and B_{ni} $(i=1,2)$ are unknown constants which are determined from initial conditions [3, 5].

$$A_{n1}=\frac{2\alpha_{n1}}{\omega_{n1}\left(\alpha_{n2}-\alpha_{n1}\right)l}\int_0^l\left(\dot{v}_{20}\alpha_{n2}-\dot{v}_{10}\right)\sin\left(k_nx\right)dx, \tag{23}$$

$$A_{n2}=\frac{2\alpha_{n2}}{\omega_{n2}\left(\alpha_{n1}-\alpha_{n2}\right)l}\int_0^l\left(\dot{v}_{20}\alpha_{n1}-\dot{v}_{10}\right)\sin\left(k_nx\right)dx, \tag{24}$$

$$B_{n1}=\frac{2\alpha_{n1}}{\left(\alpha_{n2}-\alpha_{n1}\right)l}\int_0^l\left(v_{20}\alpha_{n2}-v_{10}\right)\sin\left(k_nx\right)dx, \tag{25}$$

$$B_{n2}=\frac{2\alpha_{n2}}{\left(\alpha_{n1}-\alpha_{n2}\right)l}\int_0^l\left(v_{20}\alpha_{n1}-v_{10}\right)\sin\left(k_nx\right)dx, \tag{26}$$

Finally, the free transverse vibrations of an elastically connected double beam system described by the following equations:

$$w_1(x,t)= \\ =\sum_{n=1}^{\infty}\sin\left(k_nz\right)\sum_{i=1}^{2}\left[A_{ni}\sin\left(\omega_{ni}t\right)+B_{ni}\cos\left(\omega_{ni}t\right)\right], \tag{27}$$

$$w_2(x,t)= \\ =\sum_{n=1}^{\infty}\sin\left(k_nz\right)\sum_{i=1}^{2}\alpha_{ni}^{-1}\left[A_{ni}\sin\left(\omega_{ni}t\right)+B_{ni}\cos\left(\omega_{ni}t\right)\right], \tag{28}$$

Numerical results

In the numerical experiment, a model of two beams of identical geometrical and physical properties was used [1, 3]: A=0.05 m^2, E=10^{10} Nm^{-2}, l=10m, I=0.0004 m^4, K_0=200000 Nm^{-2}, G=(0,0.5,1)K_0, ρ=2000 kgm^{-3}.

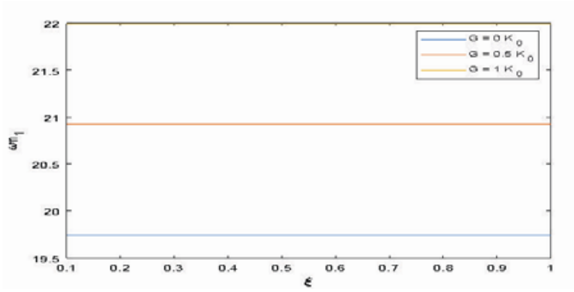

Figure 4. Effect of the axial load ratio ζ on the lower natural frequency ω_{n1} for K=K_0=K_1=K_2, (F_1=0)

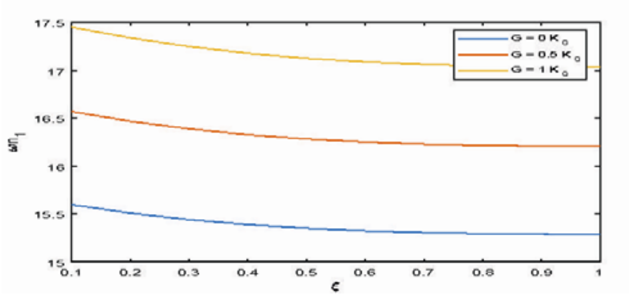

Figure 5. Effect of the axial load ratio ζ on the lower natural frequency ω_{n1} for $K=K_0=K_1=K_2$, (F_1=0.4 F_b^{cr})

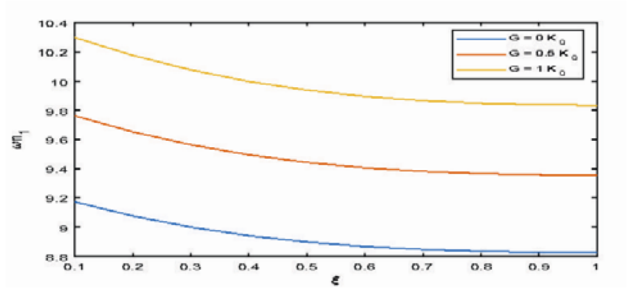

Figure 6. Effect of the axial load ratio ζ on the lower natural frequency ω_{n1} for $K=K_0=K_1=K_2$, (F_1=0.8 F_b^{cr})

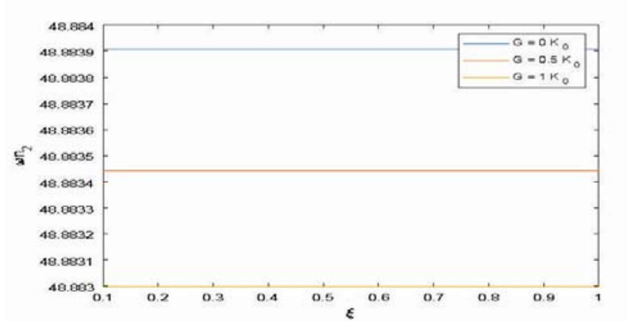

Figure 7. Effect of the axial load ratio ζ on the higher natural frequency ω_{n2} for $K=K_0=K_1=K_2$, (F_1=0)

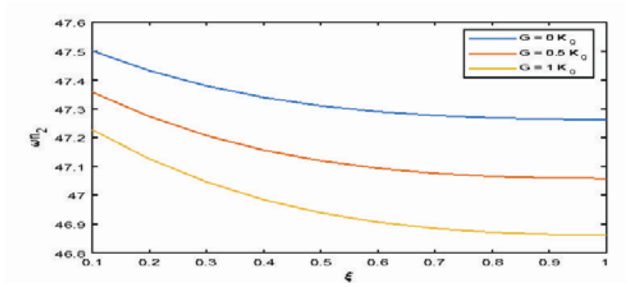

Figure 8. Effect of the axial load ratio ζ on the higher natural frequency ω_{n2} for $K=K_0=K_1=K_2$, (F_1=0.4 F_b^{cr})

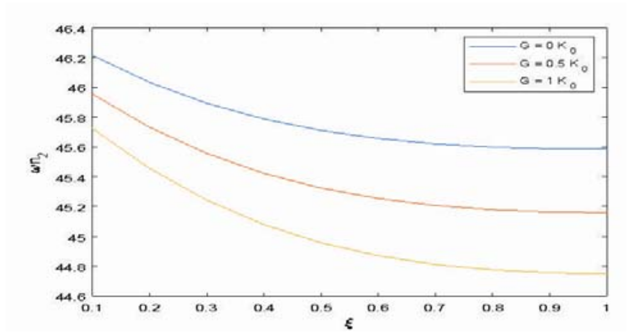

Figure 9. Effect of the axial load ratio ζ on the higher natural frequency ω_{n2} for $K=K_0=K_1=K_2$, (F_1=0.8 F_b^{cr})

From Figures 4-9, it is seen that the increase of compressive axial load seriatim F_1 causes the reduction of the natural frequency. We also notice that the increase of the shear layer G causes the reduction of the higher natural frequency ω_{n2} and increasing of the lower natural frequency ω_{n1}.

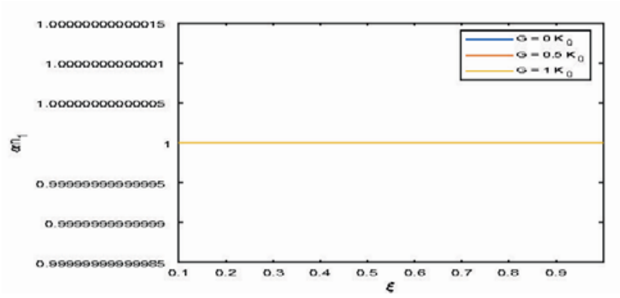

Figure 10. Effect of the axial load ratio ζ on the amplitude ratio ω_{n1} for $K=K_0=K_1=K_2$, (F_1=0)

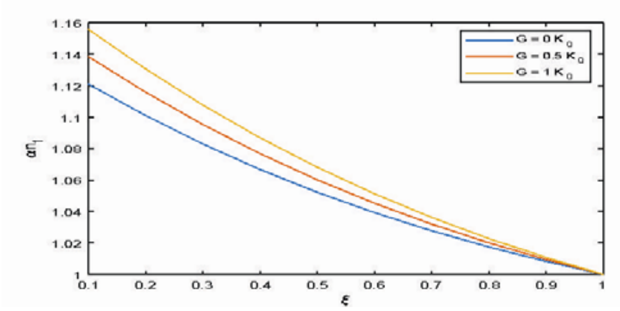

Figure 11. Effect of the axial load ratio ζ on the amplitude ratio ω_{n1} for $K=K_0=K_1=K_2$, (F_1=0.4 F_b^{cr})

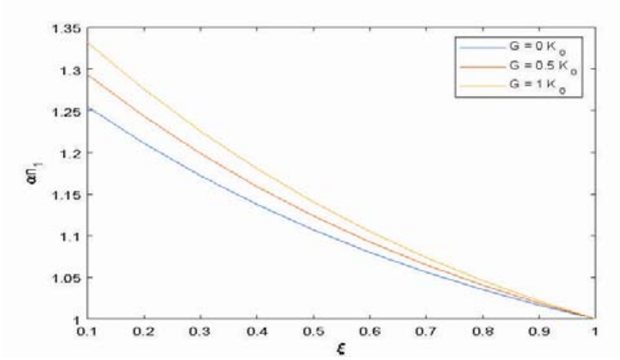

Figure 12. Effect of the axial load ratio ζ on the amplitude ratio ω_{n1} for $K=K_0=K_1=K_2$, (F_1=0.8 F_b^{cr})

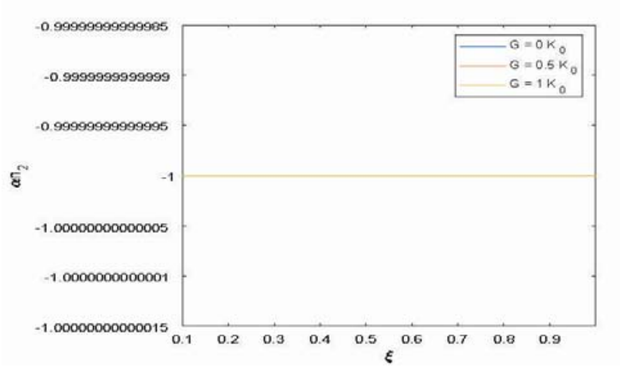

Figure 13. Effect of the axial load ratio ζ on the amplitude ratio ω_{n2} for $K=K_0=K_1=K_2$, (F_1=0)

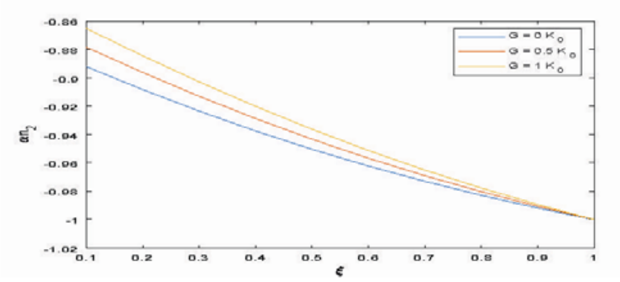

Figure 14. Effect of the axial load ratio ζ on the amplitude ratio ω_{n2} for K=K_0=K_1=K_2, (F_1=0.4 F_b^{cr})

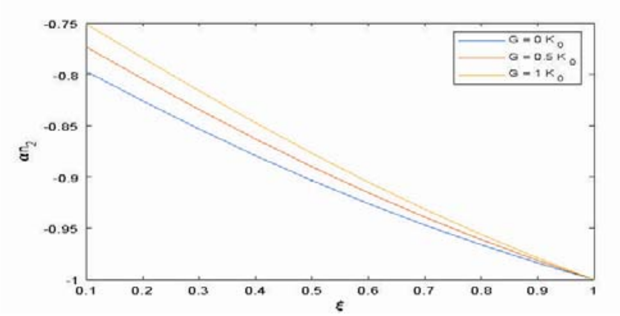

Figure 15. Effect of the axial load ratio ζ on the amplitude ratio ω_{n2} for K=K_0=K_1=K_2, (F_1=0.8 F_b^{cr})

From Figures 10-15 it is seen that the amplitude ratios α_{n1} and α_{n2} are influenced by the shear layer constant G and by the axial ratio ζ. Amplitude ratios α_{n1} and α_{n2} are becoming higher with the increase in the shear layer constant G, but smaller with the increase in compressive axial load ratio ζ.

Conclusion

Based on the Rayleigh beam theory, the free transverse vibration of an elastically connected simply supported Rayleigh double-beam, with the Kerr middle layer under compressive axial loading for one case of particular excitation loading are studied in this paper.

Using the classical Bernoulli-Fourier method, the solutions of differential equations of motion for double-beam system are formulated. The explicit expressions are presented for natural frequency, associated amplitude ratio of the two beams.

From Figures 4-9 it can be concluded that the lower natural frequency ω_{n1} is more sensitive than higher natural frequency ω_{n2} to the compressive axial loading. We also notice that the increase of the shear layer G causes the reduction of the higher natural frequency ω_{n2} and increasing of the lower natural frequency ω_{n1}.

From Figures 10-15 it can be concluded that the amplitude ratios α_{n1} and α_{n2} are influenced by the shear layer constant G and by the axial load ratio ζ. The amplitude ratios α_{n1} and α_{n2} are becoming smaller with the increase in the compressive axial load ratio ζ, but increase with the increase in shear layer constant G.

Future research could include the improvement of the considered model. A model of two connected parallel Rayleigh beams with the Kerr layer in between including inerter will be examined.

Acknowledgments

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References

[1] ONISZCZUK,Z.: *Forced transverse vibrations of an elastically connected complex simply supported double-beam system*, Journal of Sound and Vibration 264 (2003), pp. 273–86.

[2] ONISZCZUK,Z.: *Free transverse vibrations of elastically connected simply supported double-beam complex system*, Journal of Sound and Vibration 232 (2000), pp. 387–403.

[3] ZHANG,Y.Q., LU,Y., WANG,S.L., LIU,X.: *Vibration and buckling of a double-beam system under compressive axial loading*, Journal of Sound and Vibration 318 (2008), pp. 341–352.

[4] STOJANOVIC,V., KOZIC,P., PAVLOVIC,R., JANEVSKI,G.: *Effect of rotary inertia and shear on vibration and buckling of a double beam system under compressive axial loading*, Arch Appl Mech (2011) 81:1993–2005, DOI: 10.1007/s00419-011-0532-1

[5] KOZIC,P., PAVLOVIC,R., KARLICIC,D.: *The flexural vibration and buckling of the elastically connected parallel-beams with a Kerr-type layer in between*, Mechanics Research Communications 56 (2014), pp. 83–89.

[6] MOHAMMADI,N., NASIRSHOAIBI., *Forced transverse vibration analysis of a Rayleigh double-beam system with a Pasternak middle layer subjected to compressive axial load*, JVE International LTD. Journal of Vibroengineering, 17(2015), pp. 4545-4559.

[7] YOUNESIAN,D., HOSSEINKHANI,A., ASKARI,H., ESMAILZADEH,E.: *Elastic and viscoelastic foundations: a review on linear and nonlinear vibration modeling and applications*, Nonlinear Dyn (2019) 97:853–895, DOI: https://doi.org/10.1007/s11071-019-04977-9

[8] RAŠKOVIĆ,D.: *Theory of Oscillations*, (Scientific book, 1965, in Serbian).

[9] RAO,S.S.: *Mechanical Vibrations*, 4th Edition, Pearson-Prentice Hall, Upper Saddle River, 2004.

[10] KERR,D.A.: 1964. *Elastic and viscoelastic foundation models*, Journal of Applied Mechanics ASME 31, pp. 491-498.

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Analiza slobodnih transverzalnih oscilacija dva elastično povezana nosača slojem Keerovog tipa izloženog dejstvu aksijalnih pritisnih sila

U ovom radu biće prikazana analiza slobodnih transverzalnih oscilacija Rayleigh sistema od dve grede koje su izložene dejstvu aksijalnih pritisnih sila, pri čemu je uzeto da je Keer sloj u sredini. Pretpostavlja se da su dve grede koje sačinjavaju sistem kontinualno spojene Keerovim slojem. Jednačine kretanja ovog sistema su opisane trima homogenim diferencijalnim jednačinama. Uobičajeni metod Bernoulli-Fourier je korišćen za rešavanje ovog sistema diferencijalnih jednačina. U dosadašnjoj literature iz Kerrovog modela, najčešće korišćene metode su Timoshenko i Euler-Bernoulli. Iz tog razloga je cilj ovog rada određivanje analitičkih i numeričkih karakteristika (prirodna frekvencija, odnos amplituda) razmatranog modela, koristeći Rayleigh teoriju. Ono što je takođe razmatrano je uticaj elastičnog središnjeg sloja na amplitudu i frekvencije sistema. Numerička analiza sistema je izvršena korišćenjem softverskih alata. Numerički rezultati su prikazani pomoću grafika, i potvrđuju rezultate iz literature.

Ključne reči: **prirodne frekvencije, Rayleigh teorija, aksijalne pritisne sile, Keerov sloj.**