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Iterative Learning Control of Integer and Noninteger Order: an Overview

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This paper provides an overview of the recently presented and published results relating to the use of iterative learning control (ILC) based on and integer and fractional order. ILC is one of the recent topics in control theories and it is a powerful control concept that iteratively improves the behavior of processes that are repetitive in nature. ILC is suitable for controlling a wider class of mechatronic systems - it is especially suitable for motion control of robotic systems that attract and hold an important position in biomechatronical, technical systems involving the application, military industry, etc. The first part of the paper presents the results relating to the application of higher integer order PD type ILC with numerical simulation. Also, another integer order ILC scheme is proposed for a given robotic system with three degrees of freedom for task-space trajectory tracking where the effectiveness of the suggested control is demonstrated through a simulation procedure. In the second part, the results related to the application of the fractional order of ILC are presented where PD^{α} type of ILC is proposed firstly, for a fractional order linear time invariant system. It is shown that under some sufficient conditions which include the learning operators, convergence of the learning system can be guaranteed. Also, $PI^{\beta}D^{\alpha}$ type of ILC is suggested for a fractional order linear time delay system. Finally, sufficient conditions for the convergence in the time domain of the proposed ILC were given by the corresponding theorem together with its proof.

Key words: **theory of control, iterative learning control, learning control, integer order, fractional order, robotic system, overview.**

Introduction

IN the classical control theory, state feedback and output feedback are two important techniques in system control. In recent years, there has been a great deal of study to overcome limitations of conventional controllers against uncertainty due to inaccurate modeling and/or parameter variations. Particularly, in many motion control tasks, it is usually required to follow a trajectory repeatedly where conventional control algorithms do not take advantage of the repetitiveness. Iterative learning control (ILC) is one of the recent topics in control theories and it is a powerful control concept that iteratively improves the behavior of processes that are repetitive in nature[1, 2]. ILC is an effective technique that attempts to achieve a perfect output tracking of systems with repetitive nature over a finite time interval. ILC is a simple and effective control technique and can progressively reduce tracking errors and improve system performance from iteration to iteration. The ILC approach is more or less an imitation of the learning process of every intelligent being. Intelligent beings tend to learn by performing a trial (i.e. selecting a control input) and observing what was the end result of this control input selection. After that, they try to change their behavior in order to get an improved performance during the next trial. Emulating human learning, ILC uses knowledge obtained from the previous trial to adjust the control input for the current trial so that a better performance can be achieved. Namely, ILC is a trajectory tracking improvement technique for control systems, which can perform the same task repetitively in a finite time interval to improve the transient response of a system using the previous motion. ILC incorporates past control information such as tracking errors and their corresponding control input signals into constructing the present control action. Therefore, ILC requires less *a priori* knowledge about the controlled system in the controller design phase and also less computational effort than many other kinds of control. Owing to its simplicity and effectiveness, ILC has been found to be a good alternative in many areas and applications, e.g., see recent surveys [3, 4] for detailed results. For an example, mechatronic systems, such as production machines or industrial robots, often perform the same task repeatedly. In many applications the task is represented by a reference signal y_d that needs to be tracked by the system's output y_i. Traditional controllers provide the same performance each time the motion is repeated, even if that performance is suboptimal, for example due to model plant mismatch or repeating disturbances. Moreover, many factors can reduce the performance over time, such as slowly changing operating conditions or dynamics. During the past decades, ILC has been shown to be one of the most effective control strategies in dealing with repeated tracking control or periodic disturbance rejection for nonlinear dynamic systems. ILC is a trajectory-tracking improvement technique for systems performing a prescribed task. The ILC system improves its control performance by a self-tuning process without using an accurate system model. The advantages of the ILC algorithm are shown in its applications to the nonlinear systems and the systems with uncertainty or unknown structure information.

In 1978, the concept of ILC was originally proposed by Uchiyama when he presented the initial explicit formulation of ILC in Japanese [5]. In 1984, Arimoto et al. first introduced this method in English [6] where they proposed ILC for

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accurate tracking of robot trajectories. These contributions are widely considered to be the origins of ILC. One motivation for the development of ILC is the industrial robot, which repeats the same task from trial to trial. To overcome this problem, Arimoto, one of the inventors of ILC, [5-7] suggested that both the information from the previous tasks or "trials" and the current task should be used to improve the control action during the current trial. In other words, the controller should learn iteratively the correct control actions in order to minimize the difference between the output of the system and the given reference signal. He called this method "*betterment process*" [6]. Regarding the past of ILC, it is clear that the pioneering work of Arimoto and his colleagues stimulated a new approach to controlling certain types of repetitive systems. The concept of iterative learning is quite natural but had not been expressed in the algorithmic form of ILC until the early 1980's.

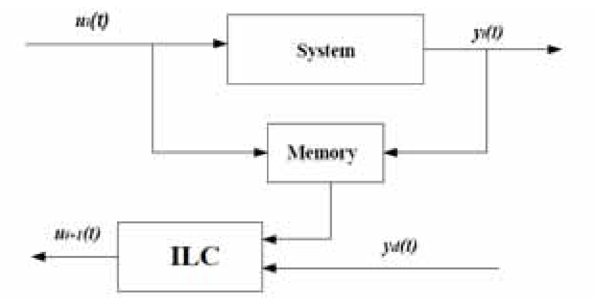


Figure 1. The basic scheme of iterative learning control

The basic idea of ILC is illustrated in Fig.1, where $u_i(t)$ and $y_i(t)$ are, respectively, the system input and output in the i_{th} iteration, $u_{i+1}(t)$ is the system input of the $(i+1)_{th}$ trial, and $y_d(t)$ is the given desired trajectory. The goal of ILC is that $\lim_{i\to\infty} y_i(t) = y_d(t)$ for all $t \in [0,T]$, where T is a fixed constant. All in the sense, $u_i(t)$ is a function that is adaptively tuned. That is, we seek a sequence $u_{i+1}(t) = f(u_i(t), y_i(t), y_d(t), t) \quad \forall t \in [0,T]$. Namely, the intuitive notion of "improving performance progressively" can be refined to a convergence condition on the error, i.e., (in some norm topology) $\lim_{i\to\infty} \|e_i(.)\| = 0$, $e_i(t) = y_d(t) - y_i(t)$.

The ILC system operates in two dimensions, one is in the time domain and the other is in the iteration domain.A typical ILC, in the time domain, is under a simple open-loop (feedforward) control (*off-line* ILC) and/or closed-loop (feedback) (*on-line* ILC). The conventional ILC is an *open-loop* strategy, which refines the current iteration control input by only employing information from the previous iterations and, hence, cannot improve the tracking performance along the time axis within a single cycle. To overcome such drawbacks, an ILC scheme is usually performed together with a feedback controller for compensation. For example, the ILC design and analysis has been addressed from the viewpoints of both high-gain feedback [8] and fixed-gain feedback [9], whereas in most cases, the feedback controller is directly incorporated into the ILC schemes design, resulting in the so-called feedback ILC. With respect to the learning controller (learning update law), ILC can be categorized as to the type (P, I, D, PD, PI, PID) [1, 10]. In a similar way, a closed-loop ILC can be classified applying a corresponding combination of previous types. Algorithms that only use information of the past trial are called *first order algorithms,* and can be distinguished from *higher order algorithms* that use multiple past trials or current trial algorithms, which incorporate a feedback loop. Since theories and learning algorithms on ILC were firstly proposed, ILC has attracted considerable interests [2, 10] due to its simplicity and effectiveness of the learning algorithm and its ability to deal with problems with nonlinear, time-delay, uncertainties and recently singular systems.

Recently, increasing attentions have been paid to fractional calculus (FC) and its application in various science and engineering fields, [11-13]. As an important application of FC, fractional-order control systems [13] have attracted more and more interests in the last several years. Particularly, the application of ILC to the fractional-order system has become a new topic, where authors [14] were the first to propose the fractional order *D*-type iterative learning control algorithm and the convergence was proved in the frequency domain. Then, the time domain analyses of fractional-order ILC are obtained and presented in the papers, [15-17], as well as in [18-20].

Preliminaries and basics of fractional calculus

The λ -norm, maximum norm, induced norm

For later use in proving the convergence of the proposed learning control, the following norms are introduced [4] for n-dimensional Euclidean space R^n: the sup-norm $\|x\|_\infty = \sup_{1\le k\le n} |x_k|$, $x=[x_1, x_2, \ldots x_n]^T$, $|x_k|$-absolute value; maximum norm $\|x\|_s = \max_{0\le t\le T} |x(t)|$, $x(t)=[x_1(t), x_2(t), \ldots, x_n(t)]^T$; the matrix norm as $\|A\|_\infty = \max_{1\le k\le m}\left(\sum_{j=1}^{n} |g_k|\right)$, $A=[a_{kj}]_{mxn}$ and the λ -norm for a real function:

$$h(t), \ (t\in[0,T]), \ h:[0,T]\to \Re^n \\ \|h(t)\|_\lambda = \sup_{t\in[0,T]} e^{-\lambda t}\|h(t)\|, \ \lambda>0 \cdot \tag{1}$$

A useful property associated with the λ-norm the following inequality.

Property 1: λ norm has the next property

$$\sup_{t\in[0,T]} e^{-\lambda t}\int_0^t \|f(.)\|e^{a(t-\tau)}d\tau = \\ \sup_{t\in[0,T]} \int_0^t e^{-\lambda t}\|f(.)\|e^{a(t-\tau)(a-\lambda)}d\tau \le \frac{1-e^{(a-\lambda)T}}{\lambda-a}\|f(.)\|_\lambda \tag{2}$$

The induced norm of a matrix A is defined as:

$$\|A\| = \sup\left\{\frac{\|Ax\|}{x} : x\in X \ \text{ with } \|x\|\ne 0\right\} \text{ with,} \tag{3}$$

where $\|(.)\|$ denotes an arbitrary vector norm. In case $\|(.)\|_\infty$ it follows that

$$\|Ax\|_\infty \le \|A\|_\infty \|x\|_\infty , \tag{4}$$

where $\|A\|_\infty$ denotes the maximum value of the matrix A. For the previous norms, note that

$$\|h(t)\|_\lambda \le \|h(t)\|_\infty \le e^{\lambda T}\|h(t)\|_\lambda . \tag{5}$$

The λ - norm is thus equivalent to the ∞ - norm. For simplicity, in applying the norm $\|(.)\|_\infty$ the index ∞ will be omitted. Before giving the main results, we first give the following Lemma 1, [21].

Lemma 1. Suppose a real positive series $\{a_n\}_1^\infty$ satisfies

$$\begin{aligned} &a_n \le \rho_1 a_{n-1} + \rho_2 a_{n-2} + \ldots + \rho_N a_{n-N} + \varepsilon \\ &(n = N+1,\ N+2,\ \ldots,\ ) \end{aligned} \tag{6}$$

where $\rho_1 \ge 0$ $(i = 1,2,\ldots,N)$ $\varepsilon = 0$ and $\rho = \sum_{i=1}^{N}\rho_i < 1$. Then the following holds:

$$\lim_{n\to\infty} a_n \le \varepsilon/(1-\rho) . \tag{7}$$

Basics of fractional calculus

Fractional calculus (FC) is a generalization of classical calculus concerned with operations of integration and differentiation of non-integer (fractional) order. The concept of fractional operators has been introduced almost simultaneously with the development of the classical ones. This question consequently attracted the interest of many well- known mathematicians, including Euler, Liouville, Laplace, Riemann, Grünwald, Letnikov and many others. Since the 19th century, the theory of fractional calculus developed rapidly, mostly as a foundation for a number of applied disciplines, including fractional geometry, fractional differential equations (FDE) and fractional dynamics. The applications of FC are very wide nowadays,[11-13], [22-25]. It is safe to say that almost no discipline of modern engineering and science in general, remains untouched by the tools and techniques of fractional calculus. For example, wide and fruitful applications can be found in rheology, viscoelasticity, acoustics, optics, chemical and statistical physics, robotics, control theory, electrical and mechanical engineering, bioengineering, etc. The main reason for the success of FC applications is that these new fractional-order models are often more accurate than integer-order ones, i.e. there are more degrees of freedom in the fractional order model than in the corresponding classical one. All fractional operators consider the entire history of the process being considered, thus being able to model the non-local and distributed effects often encountered in natural and technical phenomena.

There exist today many different forms of fractional integral operators, ranging from divided-difference types to infinite-sum types, Riemann-Liouville fractional derivative, Grunwald–Letnikov fractional derivative, Caputo's, Weyl's and Erdely-Kober left and right fractional derivatives, etc. [12]. The three most frequently used definitions for the general fractional differintegral are: the Grunwald-Letnikov (GL) definition, the Riemann-Liouville (RL) and the Caputo definitions, [11, 12]. Also, fractional order dynamic systems and controllers have been increasing in interest in many areas of science and engineering in the last few years. In most cases, our objective of using fractional calculus is to apply the fractional order controller to enhance the system control performance, [13, 25].

Here, we review some basic properties of fractional integrals and derivatives, which we will need later in the obtaining ILC algorithms schemes. Sets of natural, real, integer real and complex numbers are denoted, respectively, by $\mathbb{N},\mathbb{R},\mathbb{Z},\mathbb{C}$. Also $L^p((a,b)) = L^p([a,b])$, $p \ge 1$, is the space of the measurable functions for which $\left(\int_a^b |f(x)|^p dx\right)^{1/p} < \infty$.

Definition 1. The left Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$ is given by

$$\begin{aligned} &{}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \\ &t \in [a,b],\ \operatorname{Re}\alpha > 0. \end{aligned} \tag{8}$$

In the special case of a positive real α $(\alpha \in \mathbb{R}_+)$ and $f \in L^1(a,b)$, the integral ${}_aI_t^\alpha f$ exists for almost all $t \in [a,b]$ as well as ${}_aI_t^\alpha f \in L^1(a,b)$, [26].

Definition 2. The right Riemann-Liouville fractional integral of order $\alpha \in \mathbb{C}$ is given by

$$\begin{aligned} &{}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\tau-t)^{\alpha-1} f(\tau)d\tau, \\ &t \in [a,b],\ \operatorname{Re}\alpha > 0. \end{aligned} \tag{9}$$

The existence is the same as in the case of the left RL fractional integral given above.

Defintion 3. The left and right RL fractional derivatives ${}_aD_t^\alpha f$, and ${}_tD_b^\alpha f$ of the order $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha \ge 0$, $n-1 \le \operatorname{Re}\alpha < n$, $n \in \mathbb{N}$, are defined as

$$\begin{aligned} {}_a^{RL}D_t^\alpha f(t) &= \frac{d^n}{dt^n}\left({}_aI_t^{n-\alpha} f(t)\right) = \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau,\ t \in (a,b), \end{aligned} \tag{10}$$

$$\begin{aligned} {}_t^{RL}D_b^\alpha f(t) &= (-1)^n \frac{d^n}{dt^n}\left({}_tI_b^{n-\alpha} f(t)\right) = \\ &= (-1)^n \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_t^b \frac{f(\tau)}{(\tau-t)^{\alpha-n+1}}d\tau,\ t \in (a,b), \end{aligned} \tag{11}$$

Also, for the special case $0 \le \alpha < 1$ where $t > a$ and $t < b$, we have

$${}_a^{RL}D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t \frac{f(\tau)}{(t-\tau)^\alpha}d\tau, \tag{12a}$$

and

$${}_t^{RL}D_b^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^b \frac{f(\tau)}{(\tau-t)^\alpha}d\tau, \tag{12b}$$

Definition 4. The left Caputo fractional derivative of a function of order α, denoted by ${}_a^CD_t^\alpha f$, is given, [27]

$${}_a^CD_t^\alpha f(t) = {}_aI_t^{n-\alpha}\left(\frac{d^n}{dt^n} f(t)\right) \tag{13}$$

where ${}_aI_t^{n-\alpha}$ is the left RL fractional integral (8) or in the explicit form as follows:

$$ {}_a^CD_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & n-1 \le \alpha < n, \\ \frac{d^n}{dt^n} f(t), & \alpha = n, \end{cases} \quad t \in [a,b] \tag{14} $$

Definition 5. The right Caputo fractional derivative of a function of order α, denoted by ${}_t^CD_b^\alpha f$, is defined as

$$ {}_t^CD_b^\alpha f(t) = (-1)^n \, {}_tI_b^{n-\alpha} \left(\frac{d^n}{dt^n} f(t) \right) \tag{15} $$

where ${}_tI_b^{n-\alpha}$ is the right RL fractional integral (9), or in the explicit form as follows:

$$ {}_t^CD_b^\alpha f(t) = \begin{cases} (-1)^n \frac{1}{\Gamma(n-\alpha)} \int_t^b \frac{f^{(n)}(\tau)}{(\tau-t)^{\alpha+1-n}} d\tau, & n-1 \le \alpha < n, \\ (-1)^n \frac{d^n}{dt^n} f(t), & \alpha = n, \end{cases} \quad t \in [a,b] \tag{16} $$

Also, for the special case $0 \le \alpha < 1$ where $t > a$ and $t < b$, we have

$$ {}_a^CD_t^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f^{(1)}(\tau)}{(t-\tau)^\alpha} d\tau, & 0 \le \alpha < 1, \\ \frac{d}{dt} f(t), & \alpha = 1, \end{cases} \tag{17} $$

$$ {}_t^CD_b^\alpha f(t) = \begin{cases} -\frac{1}{\Gamma(1-\alpha)} \int_t^b \frac{f^{(1)}(\tau)}{(\tau-t)^\alpha} d\tau, & 0 \le \alpha < 1, \\ -\frac{d}{dt} f(t), & \alpha = 1, \end{cases} \tag{18} $$

The Riemann-Liouville fractional derivatives and the Caputo fractional derivatives are connected with each other by the following relations:

$$ {}_a^{RL}D_t^\alpha f(t) = {}_a^CD_t^\alpha f(t) + \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}, \tag{19} $$

$$ {}_t^{RL}D_b^\alpha f(t) = {}_t^CD_b^\alpha f(t) + \sum_{k=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-t)^{k-\alpha}, \tag{20} $$

The Caputo and Riemann-Liouville formulations coincide when the initial conditions are zero, [11, 12].

Lemma 2. If the function $f(t,x)$ is continuous, then the initial value problem

$$ \begin{cases} {}_{t_0}^CD_t^\alpha x(t) = f(t, x(t)), & 0 < \alpha < 1 \\ x(t_0) = x(0) \end{cases} \tag{21} $$

is the equivalent to the following nonlinear Volterra integral equation:

$$ x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s, x(s)) ds \tag{22} $$

and its solutions are continuous, [28].

Higher integer order PD -type iterative learning control for LTI systems

ILC is a relatively recent but well-established area of study in control theory. ILC, which can be categorized as an intelligent control methodology, is an approach for improving the transient performance of systems that operate repetitively over a fixed time interval. Although control theory provides numerous design tools for improving the response of a dynamic system, it is not always possible to achieve desired performance requirements, due to the presence of unmodeled dynamics or parametric uncertainties that are exhibited during actual system operation or to the lack of suitable design techniques [29, 30]. Thus, it is not easy to achieve perfect tracking using traditional control theories. ILC is a design tool that can be used to overcome the shortcomings of traditional controller design, especially for obtaining a desired transient response, for the special case when the system of interest operates repetitively. For such systems, ILC can often be used to achieve perfect tracking, even when the model is uncertain or unknown and we have no information about the system structure and nonlinearity. Various definitions of ILC have been given in the literature and common emphasis of these definitions is the idea of “repetition”. For an example, it is well known that robot manipulators are generally used in repetitive tasks (e.g., automotive manufacturing industries). Therefore, it is interesting to take advantage of the fact that the reference trajectory is repeated over a given operation time. In this context, ILC techniques can be applied in order to enhance the tracking performance from operation to operation. In [6], the input update utilizes the derivative signals of the previous error signal and the learning law is termed D-type ILC,

$$ u_{i+1}(t) = u_i(t) + \Gamma \dot{e}_i(t) \tag{23} $$

where Γ is the learning gain designed based on partial knowledge of the system under investigation; $u_{i+1}(t)$ is the control input at the iteration $i+1$ while $e_i(t) = y_d(t) - y_i(t)$ is the tracking error between the actual output $y_i(t)$ and the desired trajectory $y_d(t)$ at the iteration i. Time $t \in [0,T]$ where T is finite and fixed. D-type ILC is a simple but effective law. Besides, high-order ILC schemes can be used to improve the transient learning behavior along the learning iteration number direction, [4, 29]. By using the difference $\dot{e}_i(t) - \dot{e}_{i-1}(t)$ as the derivative approximation along the i-direction, the PID controller in the i-direction will result in the following form of the ILC updating law, [4]

$$ u_{i+1}(t) = u_i(t) + \Gamma \dot{e}_i(t) + \Gamma_1 \dot{e}_{i-1}(t) + \Gamma_2 \dot{e}_{i-2}(t) \tag{24} $$

In this section, we present basic as well as higher integer order ILC algorithms, continuous time, and their convergence properties. The linear system described in the form of state space and output equation is considered here.

$$ \dot{x}_i(t) = Ax_i(t) + Bu_i(t) \tag{25} $$

$$y_i(t) = Cx_i(t) \tag{26}$$

In these equations $t \in [0,T]$, $t \in \mathbb{R}_+$, where T presents terminal time which is known; $x_i \in R^n, u_i \in R^m, y_i \in R^r$, and i denotes the *i*-th repetitive operation of the system. A, B and C are matrices with appropriate dimensions and it is assumed that CB is a full rank matrix. Let $x_d(t)$ be the desired state trajectory and $y_d(t)$ be the corresponding output trajectory. The control task is to servo the output $y_i(t)$ to track the desired output $y_d(t)$ on a fixed interval $t \in [0,T]$, as the iteration i increases. In classical ILC, the following basic postulates are required, although in recent ILC research some of these postulations have been relaxed:

a) every trial (pass, cycle, batch, iteration, repetition) ends in a fixed time of duration;
b) repetition of the initial setting is satisfied, i.e the initial state $x_i(0)$ of the objective system can be set to the same point at the beginning of each iteration;
c) invariance of the system dynamics is ensured throughout the repetition;
d) $y_d(t), x_d(t)$, (t) are continuously differentiable on $[0,T]$.

Also tracking error and their first and second derivative are defined as:

$$e_i(t) = y_d(t) - y_i(t),\ \dot{e}_i(t) = de_i(t)/dt = \dot{y}_d(t) - \dot{y}_i(t),$$
$$\ddot{e}_i(t) = \ddot{y}_d(t) - \ddot{y}_i(t), \tag{27}$$

The original ILC scheme proposed in [6] is a D-type, i.e., (23) where Γ is a diagonal learning gain matrix, ensures that $\lim_{i\to\infty} y_i(t) \to y_d(t)$ for all $t \in [0,T]$, if $\|I - CB\Gamma\|_i < 1$ where $\|(..)\|_i$ is an operator norm and $i \in \{1,2,...,\infty\}$. One can notice that the basic formula for selecting the learning gain does not require information about the system matrix A, which implies that ILC can be effective for model-uncertain systems, which is a key characteristic of ILC. Also, for instance, a "*PID-like*" update law can be given as [30]

$$u_{i+1}(t) = u_i(t) + \Pi e_i(t) + \Gamma \dot{e}_i(t) + \mathrm{H}\int e_i(t)dt \tag{28}$$

A higher order ILC (HOILC) meaning information from more than one previous trial is used in the ILC algorithm PID-like update rule [31] can be formulated as

$$u_{i+1}(t) = \sum_{k=1}^{N}(I-\Lambda)P_k u_{i-k+1}(t) + \Lambda u_0(t) + \sum_{k=1}^{N}\left(\Pi_k e_{i-k+1}(t) + \Gamma_k \dot{e}_{i-k+1}(t) + \mathrm{H}_k \int e_{i-k+1}(t)dt\right) \tag{29}$$

where $N \geq 1$ is the order of the ILC algorithm; Λ is a weighting parameter to restrain the large fluctuation of the control input at the beginning of ILC iterations. Moreover, paper [32] proposed a new PD- type ILC (HOILC) which contains a higher order derivative of $e_i(t)$ and $\dot{u}_i(t)$ such as:

$$u_{i+1}(t) = u_i(t) + \Gamma CB\dot{u}_i(t) + \Gamma\left(\ddot{e}_i(t) + Q\dot{e}_i(t) + Re_i(t)\right) \tag{30}$$

where Γ and Q, R are gain learning matrices. The P - component in a continuous-time D - type ILC scheme, with properly chosen learning gains, enables robustness to non-zero initialization errors. Also, the following assumption is imposed: initial state error at each iteration is an a neighbourhood of x_0 such that $\|x_i(0) - x_0\| \leq h_0$. Introducing $\dot{u}_i$, $\ddot{e}_i$ in (30) and consequently a matrix A in the convergence condition, one can obtain more flexible ILC updating law with three gain learning matrices.

Convergence analysis

In ILC, a fundamental problem is to guarantee the ILC convergence property, i.e. to guarantee the system is output trajectory converging to the desired one within a prescribed desired accuracy as the number of ILC iterations increases. The sufficient condition will be presented which guarantees the convergence of the proposed algorithm.

Theorem 1: Suppose that the update law (30), is applied to the system (25,26) and the initial state at each iteration satisfies. If matrices Γ, Q, exist such that

$$\left\|\left[I - \Gamma(CA + QC)B\right]\right\| \leq \rho < 1, \tag{31}$$

then, when $i \to \infty$ the bounds of the tracking errors $\|x_d(t) - x_i(t)\|$, $\|y_d(t) - y_i(t)\|$, $\|u_d(t) - u_i(t)\|$ converge asymptotically to a residual ball cantered at the origin.

***Proof*:** Let

$$\delta x_i = x_d(t) - x_i(t),\ \delta\dot{x}_i = \dot{x}_d(t) - \dot{x}_i(t) \tag{32}$$

$$\delta u_i = u_d(t) - u_i(t),\ \delta\dot{u}_i = \dot{u}_d(t) - \dot{u}_i(t)$$

Also, the tracking error can be presented as:

$$\dot{e}_i = C\delta\dot{x}_i = CA\delta x_i + CB\delta u_i \tag{33}$$

$$\ddot{e}_i = d(\dot{e}_i)/dt = C\delta\ddot{x}_i = CA^2\delta x_i + CAB\delta u_i + CB\delta\dot{u}_i \tag{34}$$

Taking the proposed control law and (30, 32-34),gives:

$$\delta u_{i+1} = \delta u_i - \Gamma CB\dot{u}_i(t) - \Gamma\big(CA^2\delta x_i + CAB\delta u_i + CB\delta\dot{u}_i + QCA\delta x_i + QCB\delta u_i + RC\delta x_i\big) \tag{35}$$

or

$$\delta u_{1+1} = \left[I - \Gamma(CA+QC)B\right]\delta u_i - \left[\Gamma\left((CA+QC)A + RC\right)\right]\delta x_i - \Gamma CB\dot{u}_d \tag{36}$$

Estimating the norms of (36) with $\|(.)\|$ and using the condition of Theorem 1 gives

$$\|\delta u_{i+1}\| \leq \rho\|\delta u_i\| + \beta\|\delta x_i\| + \gamma \tag{37}$$

where

$$\beta = \left\|\left[\Gamma\left((CA+QC)A + RC\right)\right]\right\| \tag{38a}$$

and

$$\gamma = \sup_{t\in[0,T]}\|\dot{u}_d(t)\| \cdot \|\Gamma CB\| \tag{38b}$$

Also,

$$\delta x_i = \delta x_i(0) + \int_0^t \left(A\delta x_i(\tau) + B\delta u_i(\tau) \right) d\tau \qquad (39)$$

For any function $x_i(t) \in R^n$, $t \in [0,T]$ the λ-norm form $\int_0^t \|x_i(s)\| ds$ is, (Eq.4)

$$\sup e^{-\lambda t} \int_0^t \|x_i(s)\| ds \le \|\delta x_i(t)\|_\lambda \, O(\lambda^{-1}), \qquad (40a)$$

where

$$O(\lambda^{-1}) = \left(1 - e^{-\lambda T}\right)/\lambda \le 1/\lambda \qquad (40b)$$

New performing the λ - norm operation for (39) and using (40) one obtains

$$\|\delta x_i\|_\lambda \le \eta + O_\eta(\lambda^{-1}) \|\delta u_i\|_\lambda \qquad (41)$$

where

$$\eta = h_0 / \left(1 - h_A O(\lambda^{-1})\right), \qquad (42a)$$

$$O_\eta(\lambda^{-1}) = h_B O(\lambda^{-1}) / \left(1 - h_A O(\lambda^{-1})\right) \qquad (42b)$$

If a sufficiently large λ is used, one can obtain that $h_A O(\lambda^{-1}) < 1$. Taking the λ-norm of (37) with the substitution of (41) simply yields

$$\|\delta u_{i+1}\|_\lambda \le \rho' \|\delta u_i\|_\lambda + \varepsilon_0 \qquad (43)$$

where also one can make by using a sufficiently large λ

$$\rho' = \rho + \beta O_\eta(\lambda^{-1}) < 1, \; \varepsilon_0 = \beta\eta + \gamma. \qquad (44)$$

According to Lemma 1 [30], it can be concluded that

$$\lim_{i\to\infty} \|\delta u_{i+1}\|_\lambda \le \varepsilon_0 / (1 - \rho') \qquad (45a)$$

and

$$\lim_{i\to\infty} \|\delta e_i\|_\lambda \le h_c \left(\eta + \varepsilon_0 O_\eta(\lambda^{-1}) / (1 - \rho')\right), \quad h_C = \|C\|_\infty \qquad (45b)$$

This completes the proof of Theorem 1. To demonstrate the benefits from using the proposed PD type ILC algorithm, the following example is used for the simulation study. The dynamics of the system is described by:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \; y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \qquad (46)$$

The desired output trajectory and the initial state are given as follows.

$$y_d(t) = 0.03t(20 - t), \; 0 \le t \le 20, \; x_i(0) = x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \qquad (47)$$

According to Theorem 1 and condition that

$$y = \sup_{t\in[0,T]} \|\dot{u}_d(t)\| \cdot \|\Gamma CB\| = \sup_{t\in[0,T]} \|\dot{u}_d(t)\| \cdot \|\Gamma\| \to 0 \qquad (48)$$

the Γ can be chosen as 0.04 and $Q = 27$, $R = 1$. Fig.2 shows the output trajectories at the 50th iteration and Fig.3 presents the iteration histories of control and error trajectory.

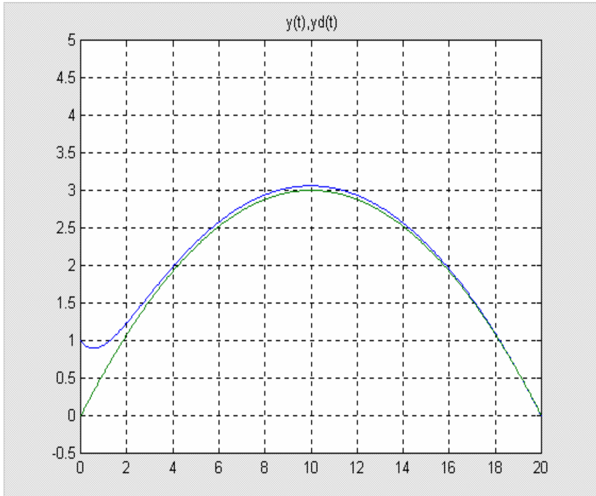


Figure 2. The output and desired output

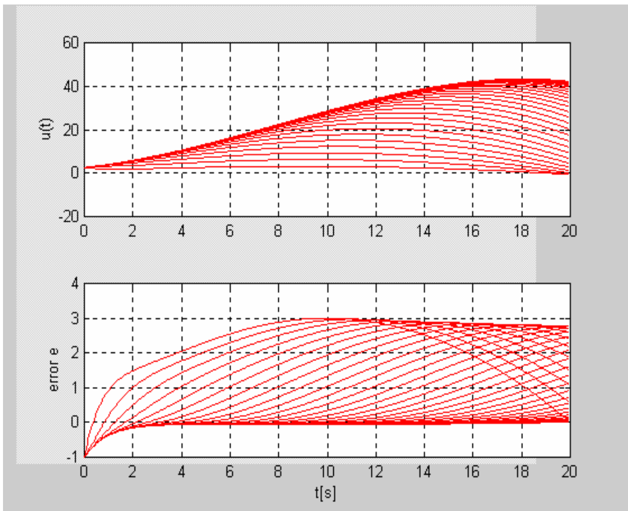


Figure 3. The trend of $e_i(t)$, $u_i(t)$

Feedforward-feedback PD type ILC for a robotic system with three degrees of freedom

Here, we are interested in a PD type ILC control of robotic system with $n = 3$ DOFs. The robotic system is considered as an open linkage consisting of $n+1$ rigid bodies $[V_i]$ interconnected by n one-degree-of–freedom joints formed kinematical pairs of the fifth class, where the robotic system possesses n degrees of freedom. Specially, the Rodriguez` method, [33, 34] is proposed for modelling the kinematics and dynamics of the robotic system. The configuration of the mechanical model of robot can be defined by the vector of joint (internal) generalized coordinates q of the dimension n, $(q) = (q^1, q^2, ..., q^n)^T$ with relative angles of rotation (in the case of revolute joints) and relative displacements (in the case of prismatic joints). The equations of motion of the robotic system can be expressed in a covariant form of Langrange`s equation of the second kind as follows [34]

$$\sum_{\alpha=1}^{n} a_{\gamma\alpha}(q)\ddot{q}_{\alpha} + \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \Gamma_{\alpha\beta,\gamma}(q)\dot{q}_{\alpha}\dot{q}_{\beta} = Q_{\gamma}$$
$$\gamma = 1,2,...n \tag{49}$$

where the coefficients $a_{\alpha\beta}$ are the covariant coordinates of the basic metric tensor $[a_{\gamma\alpha}] \in R^{n\times n}$ and $\Gamma_{\alpha\beta,\gamma}$ $\alpha,\beta,\gamma = 1,2,...,n$ presents Christoffel symbols of the first kind and Q_{γ}, $\gamma = 1,2,...,n$ generalized forces.The generalized forces Q_{γ} can be presented in the following expression (50) where $Q_{\gamma}^{g}, Q_{\gamma}^{u}$ denote the generalized gravitational and control forces, respectively.

$$Q_{\gamma} = Q_{\gamma}^{g} + Q_{\gamma}^{u},\ \ \gamma = 1,2,...n \tag{50}$$

where are vector of generalized gravitational forces Q_{γ}^{g}, $\gamma = 1,2,...n$ are bounded $|Q_{\gamma}^{g}| \le g_{\gamma}$, $\gamma = 1,2,...n$ as well as the vector of generalized control forces, Q_{γ}^{u}, $\gamma = 1,2,...n$ as follows $|Q_{\gamma}^{u}| \le h_{\gamma}$, $\gamma = 1,2,...n$. In a condensed form, eq. (49) can be presented as

$$a(q)\ddot{q} + C(q,\dot{q}) = Q^{u} \tag{51}$$

The following assumptions on system (51) are imposed.

A1. The desired trajectories $q_d(t) = (q_d^1, q_d^2, ..., q_d^n)^T$ are continuously differentiable on $[0,T]$ which are obtained on higher tactical level of planning trajectories (e.g. solving the inverse kinematics task).

A2. For the given desired trajectory $q_d(t)$, there exists a control input $u_d(t)$ such that

$$a(q_d)\ddot{q}_d + C(q_d,\dot{q}_d) = Q_d{}^{u} = u_d \tag{52}$$

A3. Robotic system is completely controllable i.e. they satisfy the following inequalities, [35]

$$g_{\gamma} = \sup|Q_{\gamma}^{g}| < h_{\gamma},\ \gamma = 1,2,...n \tag{53}$$

A4. The initial resetting conditionsholds for all iterations, i.e. $q_i(0) = q_d(0),\ \dot{q}_i(0) = \dot{q}_d(0)\ i = 0,1,2...$

A5. It is possible to determine the vector $Q^{g}(q_d(0))$ in the initial moment when the robot is in the referent configuration.

Given a desired trajectory $q_d(t)$, $t \in [0,T]$, our objective is to know whether the error $q_d(t) - q(t)$, can converge to zero for all $t \in [0,T]$ as $i \to \infty$, i.e. to determine the control of the vector $Q^{u}(t)$ so that the robotic system follows the predefined trajectory $q_d(t)$. Here, the feedforward-feedback fractional order PD learning algorithm is suggested which comprises two types of control laws: a feed-forward PD control law and a PD feedback law, see Fig.4.

$$u_i(t) = u_{fi}(t) + u_{fbi}(t) \tag{54}$$

In the feedback control loop, it is proposed that a PD - type ILC updating law for the given system is:

$$u_{fbi}(t) = Q(q_{d0}) - Kq_{di}(t) - L\dot{q}_i(t) \tag{55}$$

taking into account assumption A5. In the feed-forward control loop, a PD - type ILC updating law is given

$$u_{fi}(t) = u_{fi-1}(t) + \Gamma\left(K(q_d - q_i) + L(\dot{q}_d - \dot{q}_i)\right) \tag{56}$$

In the first iteration $i = 1$ we can choose

$$u_{f1}(t) = Kq_d(t) + L\dot{q}_d(t) \tag{57}$$

The value of learning gain Γ needs to satisfy the condition $0 < \Gamma \le 1$ which is obtained from the condition of convergence of the ILC, [36, 37].Also, the analysis in the $i-$th iteration of the linearized time-varying systems along $q_d(t)$ can show that the choice of the elements of the diagonal positive definite matrices K, L are large enough to ensure the stability of the desired trajectory. Therefore, as it was previously pointed out, precise knowledge of the vector $Q^{g}(q)$ is not necessary and it seems an interesting and important application of this type of ILC control.

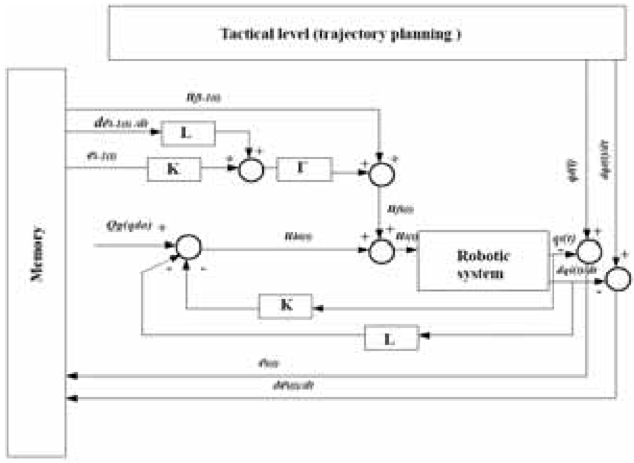


Figure 4. Block diagram of the feedforward-feedback PD^{α} type of ILC for a robotic system

Simulation results

Using a suitable example of a robotic manipulator with three DOF, Fig.5, a simulation is performed in proving the efficiency of the proposed algorithm of ILC. In Table 1 there are listed salient features of the robotic system.

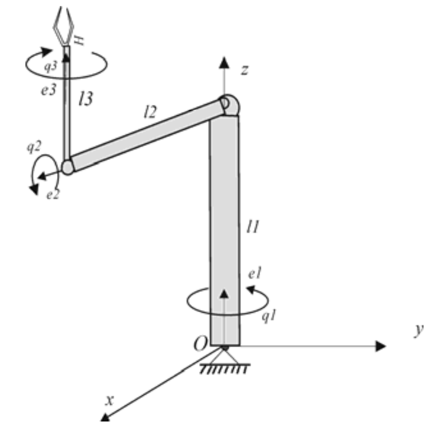


Figure 5. The robotic system with 3 DOF in the reference configuration

Table 1.

segment	1	2	3
m[kg]	2	5	3
L [m]	0.8	0.6	0.6
$I_C\left[kgm^2\right]$	$\begin{bmatrix} 0.279 & 0 & 0 \\ 0 & 0.279 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}$	$\begin{bmatrix} 0.241 & 0 & 0 \\ 0 & 0.239 & 0 \\ 0 & 0 & 0.15 \end{bmatrix}$	$\begin{bmatrix} 0.241 & 0 & 0 \\ 0 & 0.239 & 0 \\ 0 & 0 & 0.15 \end{bmatrix}$

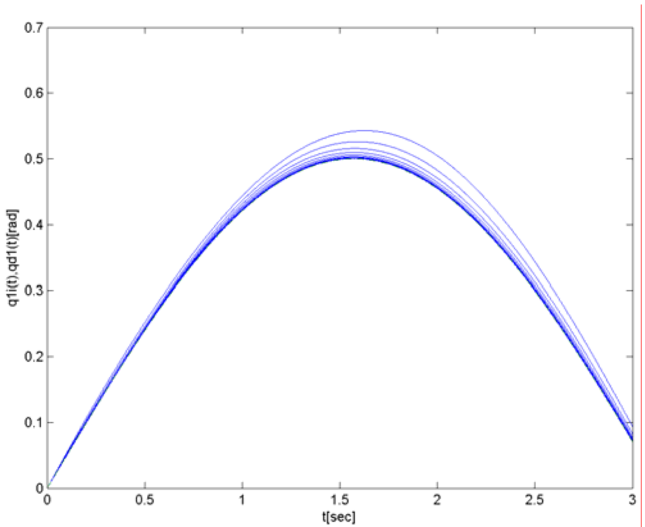
The equations of motion of the robotic system are determined by the basic metric tensor $a(q)$, $\Gamma_{\alpha\beta,\gamma}$ $\alpha,\beta,\gamma = 1,2,...,n$ Christoffel symbols of the first kind as well as the vector of generalized gravitational forces $Q^g(q)$, and one can find in [38]. Particularly, for later discussion, we can determine $Q_1^g = 0,\ Q_3^g = 0,\ Q_2^g = 21.582\sin q_2$. The desired trajectories are given as

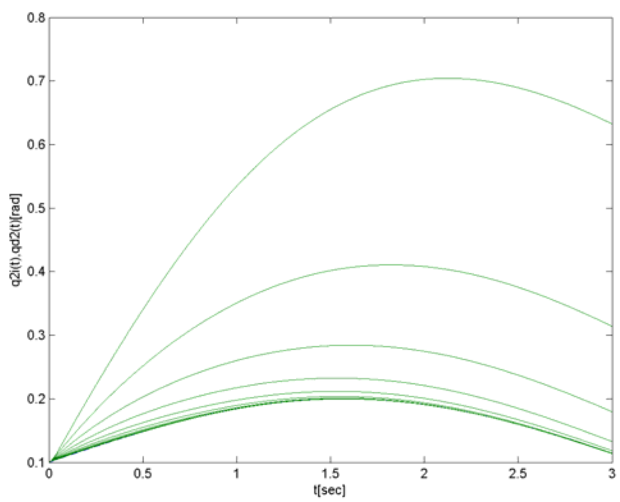
$$q_{d1}(t) = 0.5\sin t,\ \ q_{d2}(t) = 0.1\sin t + 0.1, \\ q_{d3}(t) = 1 - e^{-t},\ \ \forall t \in [0,3] \tag{58}$$

and for the elements of learning matrices *K, L*, the following values are adopted:

$$K = diag\{30,30,30\} \quad L = diag\{10,10,10\} \tag{59}$$

and the learning gain Γ is $\Gamma = 0.6$.





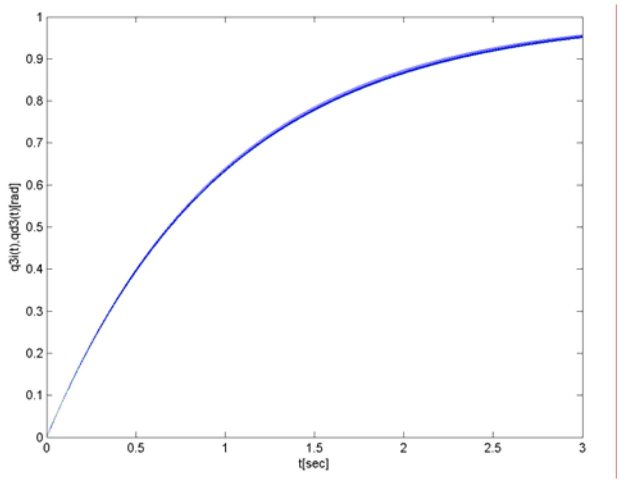


Figure 6. The tracking performance of the system trajectory ($q(t)$, $k=1,2,3$: *solid line*, $q_d(t)$, $k=1,2,3$: *bold line*)

It is shown that it is possible to achieve control of a robotic system with three degrees of freedom using the proposed ILC. Reaching within the prescribed accuracy, the desired trajectory is achieved after 8 iterations. In doing so, the maximum deviation is observed for $q_2 \to q_{2d}$ in the first iteration because $Q_2^g \neq 0$ ($Q_2^g = 21.582\sin q_2$) contrary to $Q_1^g = 0,\ Q_3^g = 0$.

PD^α -Type iterative learning control for a fractional LTI system

In recent years, the application of ILC to the fractional-order system has become a new topic. The development of fractional-order ILC algorithms, which belong to a branch of fractional-order control [7–9], is urgently needed for advanced control systems. In this section, a PD^α type of ILC is proposed for fractional linear time invariant systems. It is shown that under some sufficient conditions which include the learning operators, convergence of the learning system can be guaranteed. The proportional component in the ILC updating law do not affect the ILC stability but the P learning operator can be used as a design factor to make better ILC performance, [16]. The case of a D^α type of ILC is proposed for the fractional order LTI system obtained and presented in [15].

System description: fractional–order pseudo state space

Applying FC [11, 13] to dynamic systems control is a recent topic of interest. The real objects are generally fractional, but for many of them the fractionality is very low. Here, we considered the non-integer (fractional) linear system [39, 40] described in the form of pseudo state space and output equations. This description is convenient for simple models of systems with only one fractional-order derivation.

$$\mathbf{x}_i^{(\alpha)}(t) = A\mathbf{x}_i(t) + B\mathbf{u}_i(t),\ \ x_i(0) = x_d(0),\quad 0 \le \alpha < 1 \\ \mathbf{y}_i(t) = C\mathbf{x}_i(t), \tag{60}$$

where is $0 \le \alpha < 1$ fractional order derivative, A, B and C are matrices with appropriate dimensions. Also, it is assumed that CB is a full rank matrix. Here, Riemann-Liouville definition for the fractional derivative is used, (12) where in short, it can be written ${}_a^{RL,C}D_t^\alpha f(t) = f^{(\alpha)}$.

PD^{α} - type ILC updating law

Now, we introduce the PD^{α} - type ILC updating law for given system (60) such as:

$$\begin{aligned} \boldsymbol{u}_{i+1}(t) &= \boldsymbol{u}_i(t) + \Gamma \boldsymbol{e}_i(t) + \Pi \frac{d^{(\alpha)}\boldsymbol{e}_i(t)}{dt^{(\alpha)}} = \\ &= \boldsymbol{u}_i(t) + \Gamma \boldsymbol{e}_i(t) + \Pi \boldsymbol{e}_i^{(\alpha)}(t) \end{aligned} \quad (61)$$

where Γ, Π are the gain matrices appropriate dimensions. A sufficient condition for convergence of a proposed ILC is given by Theorem 2 and proved as follows.

Theorem 2: Suppose that the update law (61), is applied to the system (60) and the initial state at each iteration satisfies $x_i(0) = x_d(0)$. If matrix Π exists such that

$$\left\| [I - \Pi CB] \right\| \le \rho < 1, \quad (62)$$

then, when $i \to \infty$ the bounds of the tracking errors $\|x_d(t) - x_i(t)\|$, $\|y_d(t) - y_i(t)\|$, $\|u_d(t) - u_i(t)\|$, converge asymptotically to zero.

Proof: Let

$$\begin{aligned} \delta h_i &= h_d(t) - h_i(t), \\ \delta h_i^{(\alpha)} &= h_d^{(\alpha)}(t) - h_i^{(\alpha)}(t), \\ h &= x, x_d, u, u_d \end{aligned} \quad (63)$$

Tracking error can be obtained as follows:

$$\begin{aligned} e_i^{(\alpha)} &= \frac{d^{(\alpha)}(e_i)}{dt^{(\alpha)}} = \frac{d^{(\alpha)}}{dt^{(\alpha)}}(y_d - y_i) = \\ &= C\delta x_i^{(\alpha)} = CA\delta x_i + CB\delta u_i \end{aligned} \quad (64)$$

Taking the proposed control law gives:

$$\begin{aligned} \delta u_{i+1} &= u_d - u_{i+1} = u_d - u_i - \Pi e_i^{(\alpha)} - \Gamma e_i = \\ &= \delta u_i - \Pi e_i^{(\alpha)} - \Gamma e_i \end{aligned} \quad (65)$$

or, taking (64) it yields:

$$\delta u_{i+1} = [I - \Pi CB]\delta u_i - [\Pi CA + \Gamma C]\delta x_i \quad (66)$$

Estimating the norms of (66) with $\|(.)\|$ and using the condition of Theorem 2 implies

$$\|\delta u_{i+1}\| \le \rho \|\delta u_i\| + \gamma \|\delta x_i\|, \qquad \gamma = \|\Gamma C + \Pi CA\| \quad (67)$$

Also, one can write the solutions of (60) in the form of the equivalent Volterra integral equations, [11]:

$$\begin{aligned} x_i(t) &= x_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(Ax_i(s) + Bu_i(s) \right) ds \\ x_d(t) &= x_d(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(Ax_d(s) + Bu_d(s) \right) ds \end{aligned} \quad (68)$$

or

$$\delta x_i(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(A\delta x_i(s) + B\delta u_i(s) \right) ds \quad (69)$$

Applying the norms and by taking into account uniqueness of solution [11], it yields:

$$\begin{aligned} \|\delta x_i(t)\| &\le \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} A\delta x_i(s) ds \right\| + \\ &+ \left\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} B\delta u_i(s) ds \right\| \\ &\le \frac{1}{\Gamma(\alpha)} k_A \int_0^t \left|(t-s)^{\alpha-1}\right| \|\delta x_i(s)\| ds + \\ &+ \frac{1}{\Gamma(\alpha)} k_B \int_0^t \left|(t-s)^{\alpha-1}\right| \|\delta u_i(s)\| ds \end{aligned} \quad (70)$$

Moreover, applying the λ norm to both sides of previous (70), it follows

$$\begin{aligned} \|\delta x_i(t)\|_\lambda \le \sup_{0\le t\le T} e^{-\lambda t} &\left\{ \frac{k_A}{\Gamma(\alpha)} \int_0^t \left|(t-s)^{\alpha-1}\right| \|\delta x_i(s)\| ds + \right. \\ &\left. + \frac{k_B}{\Gamma(\alpha)} \int_0^t \left|(t-s)^{\alpha-1}\right| \|\delta u_i(s)\| ds \right\}_\lambda \end{aligned} , \quad (71)$$

$$\begin{aligned} &\le \sup_{0\le t\le T} \int_0^t e^{-\lambda(t-s)} \frac{|(t-s)|^{\alpha-1}}{\Gamma(\alpha)} \sup_{0\le t\le T} e^{-\lambda s} \cdot \\ &\left[k_A \|\delta x_i(s)\| + k_B \|\delta u_i(s)\| \right] ds \\ &\le \left(k_A \|\delta x_i(t)\|_\lambda + k_B \|\delta u_i(t)\|_\lambda \right) \cdot \\ &\cdot \sup_{0\le t\le T} \int_0^t e^{-\lambda(t-s)} ds \sup_{0\le t\le T} \int_0^t \frac{|(t-s)|^{\alpha-1}}{\Gamma(\alpha)} ds \end{aligned} \quad (72)$$

or,

$$\|\delta x_i(t)\|_\lambda \le \left(k_A \|\delta x_i(t)\|_\lambda + k_B \|\delta u_i(t)\|_\lambda \right) \cdot \frac{\left(1 - e^{-\lambda T}\right)}{\lambda} \frac{T^\alpha}{\Gamma(\alpha+1)} \quad (73)$$

Introducing $O\left(\lambda^{-1}\right)$, as

$$O\left(\lambda^{-1}\right) = \frac{\left(1 - e^{-\lambda T}\right)}{\lambda} \frac{T^\alpha}{\Gamma(\alpha+1)}, \quad (74)$$

where (73) simplifies to

$$\|\delta x_i(t)\|_\lambda \le \left(k_A \|\delta x_i(t)\|_\lambda + k_B \|\delta u_i(t)\|_\lambda \right) O\left(\lambda^{-1}\right) \quad (75)$$

or, one may conclude

$$\begin{aligned} \|\delta x_i(t)\|_\lambda &\le \frac{k_B O\left(\lambda^{-1}\right)}{\left(1 - k_A O\left(\lambda^{-1}\right)\right)} \|\delta u_i(t)\|_\lambda \le \\ &\le O_\eta\left(\lambda^{-1}\right) \|\delta u_i(t)\|_\lambda \end{aligned} , \quad (76)$$

then, if it is used a sufficiently large λ, one can obtain that:

$$T^\alpha k_A O_o(\lambda^{-1}) / \Gamma(\alpha+1) < 1 \quad (77)$$

where is $O_0\left(\lambda^{-1}\right) = (1 - e^{-\lambda T}) / \lambda$. Taking the λ-norm of (67) with the substitution of (76) simply yields

$$\|\delta u_{i+1}\|_\lambda \le \rho' \|\delta u_i\|_\lambda \quad (78)$$

where, also one can make by using a sufficiently large λ

$$\rho' = \rho + \gamma O_\eta(\lambda^{-1}) < 1, \qquad (79)$$

According to Lemma1 [30] it can be concluded that

$$\lim_{i\to\infty}\|\delta u_{i+1}\|_\lambda \to 0 \text{ and } \lim_{i\to\infty}\|\delta e_i\|_\lambda \to 0, \qquad (80)$$

This completes the proof of Theorem 2.

ILC for fractional order linear time delay system: $PI^\beta D^\alpha$ type

Further, a $PI^\beta D^\alpha$ type of iterative learning control is proposed for a fractional order linear time delay system, [41]. Some examples of fractional order time delay systems are presented in [42-45]. For example, in their paper [43], the authors considered finite-dimensional fractional time delay systems. Fractional order (non)linear time delay systems also can be presented in the form of the pseudo state space and output equation. This description is convenient for simple linear models of systems with only one fractional-order derivation:

$$\begin{aligned} &\boldsymbol{x}_i^{(\alpha)}(t) = A_0\boldsymbol{x}_i(t) + A_1\boldsymbol{x}_i(t-\tau) + B\boldsymbol{u}_i(t), \\ &\boldsymbol{y}_i(t) = C\boldsymbol{x}_i(t), \qquad x_i(0) = x_d(0) = 0,\ \ 0<\alpha<1 \end{aligned}, \qquad (81)$$

where is $0 \le \alpha < 1$ fractional order derivative, A_0, A_1, B and C are the matrices with appropriate dimensions and $\tau \le T$ denotes the known pure time-delay. The following assumptions on system (81) are imposed.

A1. The desired trajectories $y_d(t)$, $x_d(t)$ are continuously differentiable on $[0,T]$.

A2. The system (81) is causal and when $t<0$ it is assumed $x_i(t)=0$.

A3. The input-output coupling matrix CB is of full column rank.

$PI^\beta D^\alpha$ - type ILC updating law

Now, we introduce the $PI^\beta D^\alpha$ -type ILC updating law for given system (81) such as:

$$u_{i+1}(t) = u_i(t) + \Gamma e_i(t) + \Pi\, {}_0D_t^\alpha e_i(t) + \mathrm{H}\, {}_0D_t^{-\beta} e_i(t), \qquad (82)$$

where Γ, Π, H are the gain matrices appropriate dimensions. A sufficient condition for convergence of a proposed ILC is given by Theorem 3 and proved as follows.

Theorem 3: Let system (81) satisfy assumptions (A1-A3) as well as the initial state at each iteration satisfies $x_i(0) = x_d(0) = 0$. If updating law (82) is applied with the learning gain matrix Π being designed such that

$$\|[I - \Pi CB]\| \le \rho < 1, \qquad (83)$$

then, when $i\to\infty$ the bounds of the tracking errors $\|x_d(t)-x_i(t)\|$, $\|y_d(t)-y_i(t)\|$, $\|u_d(t)-u_i(t)\|$, converge asymptotically to zero.

Proof. Let

$$\delta h_i = h_d(t) - h_i(t),\quad h_i^{(\alpha)} = h_d^{(\alpha)}(t) - h_i^{(\alpha)}(t),\quad h = x, x_d, u, u_d$$

Tracking error can be obtained as follows:

$$\begin{aligned} e_i^{(\alpha)}(t) &= \frac{d^{(\alpha)}}{dt^{(\alpha)}}\left(y_d(t) - y_i(t)\right) = C\delta x_i^{(\alpha)}(t) = \\ &= CA_0\delta x_i(t) + CA_1\delta x_i(t-\tau) + CB\delta u_i(t) \end{aligned}, \qquad (84)$$

Taking the proposed control law and (84) gives:

$$\begin{aligned} \delta u_{i+1}(t) &= [I - \Pi CB]\delta u_i(t) - [\Gamma C + \Pi CA_0]\delta x_i(t) - \\ &-[\Pi CA_1]\delta x_i(t-\tau) - HC\left(\delta x_i(t)\right)^{(-\beta)} \end{aligned} \qquad (85),$$

Estimating the norms of (85) with $\|(.)\|$ and using the condition of Theorem 3 implies:

$$\begin{aligned} &\|\delta u_{i+1}\| \le \rho\|\delta u_i\| + \beta_0\|\delta x_i\| + \beta_1\|\delta x_i(t-\tau)\| + \\ &+\beta_2\left\|(\delta x_i)^{(-\beta)}\right\|, \\ &\beta_0 = \|\Gamma C + \Pi CA_0\|, \\ &\beta_1 = \|\Pi CA_1\|, \\ &\beta_2 = \|HC\| \end{aligned} \qquad (86)$$

where, taking into account (8) one can get:

$$\begin{aligned} \left\|{}_0D_t^{-\beta}\delta x_i\right\| &= \frac{1}{\Gamma(\beta)}\int_0^t |t-s|^{\beta-1}\|\delta x_i\|ds \le \\ &\le \frac{T^\beta}{\Gamma(\beta+1)}\int_0^t \|\delta x_i(s)\|ds \end{aligned} \qquad (87)$$

Also, one can obtain the solution of (81) in the form of the equivalent Volterra integral equations, [11] using assumption A2 and conditions of Theorem 3, as:

$$\begin{aligned} \delta x_i(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(A_0\delta x_i(s) + A_1\delta x_i(s-\tau) + \right. \\ \left. + B\delta u_i(s)\right)ds \end{aligned}, \qquad (88)$$

By applying the norms $\|(.)\|$ on equation (88), and by taking into account the uniqueness of solution of (81), it yields

$$\begin{aligned} \|\delta x_i(t)\| &\le \frac{1}{\Gamma(\alpha)}\|A_0\|\int_0^t \left\|(t-s)^{\alpha-1}\right\|\|\delta x_i(s)\|ds + \\ &+\frac{1}{\Gamma(\alpha)}\|A_1\|\int_0^t \left\|(t-s)^{\alpha-1}\right\|\|\delta x_i(s-\tau)\|ds \\ &+\frac{1}{\Gamma(\alpha)}\|B_0\|\int_0^t \left\|(t-s)^{\alpha-1}\right\|\|\delta u_i(s)\|ds \le \\ &\le \frac{T^\alpha}{\Gamma(\alpha+1)}a_0\int_0^t \|\delta x_i(s)\|ds + \\ &+\frac{T^\alpha}{\Gamma(\alpha+1)}a_1\int_0^t \|\delta x_i(s-\tau)\|ds + \frac{T^\alpha}{\Gamma(\alpha+1)}b_0\int_0^t \|\delta u_i(s)\|ds \end{aligned} \qquad (89)$$

For any function $x_i(t)\in R^n$, $t\in[0,T]$ the λ - norm for $\int_0^t \|x_i(\tau)\|d\tau$ is:

$$\sup_{t\in[0,T]} e^{-\lambda t}\int_0^t \|x_i(s)\| e^{-\lambda(s-s)}ds \le \|\delta x_i(t)\|_\lambda \cdot$$
$$\cdot \sup_{t\in[0,T]} e^{-\lambda t}\int_0^t e^{\lambda s}ds \le \|\delta x_i(t)\|_\lambda O(\lambda^{-1}) \quad (90)$$

where is $O(\lambda^{-1}) = (1-e^{-\lambda T})/\lambda \le 1/\lambda$. Due to the fact that $\|x(t-\tau)\|_\lambda \le \|x(t)\|_\lambda$ by referring (90), one can find that:

$$\sup_{t\in[0,T]} e^{-\lambda t}\int_0^t \|x(s-\tau)\| ds \le \|x(t)\|_\lambda O(\lambda^{-1})e^{-\lambda\tau}, \quad (91)$$

New performing the λ-norm operation for (89) one obtains:

$$\|\delta x_i(t)\|_\lambda \le \frac{T^\alpha a_0}{\Gamma(\alpha+1)}\|\delta x_i(t)\|_\lambda O(\lambda^{-1}) + \frac{a_1 T^\alpha}{\Gamma(\alpha+1)}\|\delta x_i(t)\|_\lambda O(\lambda^{-1})e^{-\lambda\tau} + \frac{b_0 T^\alpha}{\Gamma(\alpha+1)}\|\delta u_i(t)\|_\lambda O(\lambda^{-1}) \quad (92)$$

or

$$\|\delta x_i(t)\|_\lambda \le O_\eta(\lambda^{-1})\|\delta u_i(t)\|_\lambda \quad (93)$$

$$O_\eta(\lambda^{-1}) = b_0 O(\lambda^{-1}) / \left(\Gamma(\alpha+1)/T^\alpha - a_0 O(\lambda^{-1}) - a_1 e^{-\lambda\tau}O(\lambda^{-1})\right), \quad (94)$$

If a sufficiently large λ is used, one can obtain that:

$$\Gamma(\alpha+1)/T^\alpha - a_0 O(\lambda^{-1}) - a_1 e^{-\lambda\tau}O(\lambda^{-1}) > 0 \quad (95)$$

Taking the λ-norm of (86) with the substitution of (93) simply yields:

$$\|\delta u_{i+1}\|_\lambda \le [\ \rho + \beta_0 O_\eta(\lambda^{-1}) + \beta_1 O_\eta(\lambda^{-1}) + \beta_2 T^\beta O_\eta(\lambda^{-1}) O(\lambda^{-1})/\Gamma(\beta+1)]\|\delta u_i\|_\lambda \quad (96)$$

or:

$$\|\delta u_{i+1}\|_\lambda \le \rho' \|\delta u_i\|_\lambda, \quad (97)$$

where, also one can make by using a sufficiently large λ:

$$\rho' = \left[\rho + \beta_0 O_\eta(\lambda^{-1}) + \beta_1 O_\eta(\lambda^{-1}) + \beta_2 T^\beta O_\eta(\lambda^{-1}) O(\lambda^{-1})/\Gamma(\beta+1)\right] < 1, \quad (98)$$

According to Lemma 1 [30] it can be concluded that:

$$\lim_{i\to\infty}\|\delta u_{i+1}\|_\lambda \to 0, \text{ and } \lim_{i\to\infty}\|\delta e_i\|_\lambda \to 0 \quad (99)$$

This completes the proof of Theorem 3.

Conclusions

An overview of the recently presented and published results relating to the use of ILC based on and integer and non-integer (fractional) order are presented. ILC, which can be categorized as an intelligent control methodology, is an approach for improving the transient performance of systems that operate repetitively over a fixed time interval such as motion control of robotic systems, etc. First, the results relating to the application of higher integer order PD type ILC with suitable numerical simulation are considered and presented. Also, another integer order ILC scheme is presented for a given robotic system with three degrees of freedom for task-space trajectory tracking where the effectiveness of the suggested control is demonstrated through a simulation procedure. In the second part of this paper the applications of fractional order of ILC to the fractional order system are exposed. The PD^α - type of ILC is proposed for a given class of fractional order linear systems. It is shown that under some sufficient conditions which include the learning operators, convergence of the learning system can be guaranteed. The proportional component in the ILC updating law do not affect the ILC stability but the P learning operator can be used as a design factor to make better ILC performance. Also, a $PI^\beta D^\alpha$ ILC scheme for a class of fractional order time delay systems is suggested and proved. In the same manner, sufficient conditions for the convergence in time domain of the proposed ILC are given by the corresponding theorem together with its proof.

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Iterativno upravljanje putem učenja celog i necelog reda: pregledni prikaz

Apstrakt: Ovaj rad daje pregledni prikaz nedavno prezentiranih i objavljenih rezultata autora koji se odnose na primenu iterativnog upravljanja putem učenja (ILC) i to celog reda kao i necelog reda. ILC predstavlja jedno od važnih oblasti u teoriji upravljanja i ono je moćan koncept upravljanja koji na iterativan način poboljšava ponašanje procesa koji su po prirodi ponovljivi. ILC je pogodno za upravljanje šire klase mehatroničkih sistema i posebno su pogodni za upravljanje kretanja robotskih sistema koji imaju važnu ulogu u biomehatroničkim, tehničkim sistemima koji uključuju primenu i vojnoj industriju itd. U prvom delu rada predstavljeni su rezultati koji se odnose na primenu višeg celobrojnog reda PD tipa sa pratećom numeričkom simulacijom. Takođe, još jedna druga ILC šema celobrojnog reda je predložena za dati robotski sistem sa tri stepena slobode u rešavanju zadatka praćenja što je i verifikovano kroz simulacioni primer. U drugom delu, predstavljeni su rezultati koji se odnose na primenu ILC frakcionog reda gde je prvo PD^{α} tip predložen za linearni sistem frakcionog reda. Pokazano je da se pod odredjenim dovoljnim uslovima koji uključuju operatore učenja konvergencija datog sistema može biti garantovana. Takodje, $PI^{\beta}D^{\alpha}$ tip ILC upravljanja je predložen za linearni sistem frakcionog reda sa kašnjenjem. Konačno, dovoljni uslovi za konvergenciju u vremenskom domenu predloženog ILC upravljanja su dati odgovarajućom teoremom sa pratećim dokazom.

Ključne reči: **teorija upravljanja, iterativno upravljanje, upravljanje učenjem, celobrojni red, necelobrojni red, robotski sistem, pregledni prikaz.**

Итерационный контроль обучения через целый и дробный порядок: обзор

Эта статья представляет собой обзор недавно представленных и опубликованных результатов по виду использования итерационного контроля обучения (ИКО) и то целого и дробного порядка. ИКО является одним из важных направлений в теории управления, и это является мощной концепцией управления на постоянной основе улучшающей поведение процессов, которые по своей сути повторяются. ИКО подходит для более широкого управления класса мехатронных систем и особенно подходят для управления движением робототехнических систем, которые играют важную роль в биомехатронических, технических системах, включающий и применение в военной промышленности и.т.д. В первой части статьи представлены результаты, связанные с использованием высшего целого порядка PD-типа с соответствующим численным моделированием. Кроме того, ещё одна ИКО схема целого порядка предлагается для данной роботизированной системы с тремя степенями свободы в решении задачи мониторинга, которая проверена через пример моделирования. Во второй части представлены результаты, касающиеся использования дробного порядка ИКО, где первый тип PD^{α} предлагается для линейной системы дробного порядка. Показано, что при определённых достаточных условиях, связанных с операторамы обучения, сходимость данной системы может быть гарантирована. Кроме того, $PI^{\beta}D^{\alpha}$ тип ИКО управления предлагается для линейных систем дробного порядка с запаздыванием. Наконец, достаточные условия сходимости в временной области предлагаемого ИКО управления приведены соответствующей теоремой с подтверждающими доказательствами.

Ключевые слова: **теория управления, итеративное управление, обучение управления, целый порядок, дробный порядок, роботизированная система, обзор.**

Contrôle itératif par la théorie de l’ordre entier et de l’ordre fractionnel: une vue d’ensemble

Ce papier présente une vue d’ensemble des résultats présentés et publiés récemment qui se rapportent à l’application du contrôle itératif par la théorie basée sur l’ordre entier et sur l’ordre fractionnel. Cette théorie représente un domaine important dans la théorie de contrôle et c’est un concept puissant de contrôle qui de façon itérative améliore le comportement des processus itératifs par la nature. La théorie citée est commode pour le contrôle de nombreuses classes des systèmes mécatroniques en particulier pour le contrôle des mouvements chez les systèmes robotiques qui jouent un rôle signifiant dans les systèmes bio mécatroniques et techniques y compris l’emploi dans l’industrie militaire etc. Dans la première partie de ce travail on a présenté les résultats concernant l’application de l’ordre entier supérieur du type PD accompagné par la simulation numérique. On a proposé également un autre schéma de l’ordre entier pour le système robotique donné à trois degrés de liberté pour résoudre les taches de suivi ce qui a été vérifié par l’exemple de simulation. Dans la seconde partie on a donné les résultats liés a l’application de l’ordre fractionnel où on propose le type PDα pour le système linéaire de cet ordre. On a démontré que sous les conditions suffisantes déterminées y compris les opérateurs de la théorie la convergence du système donné peut être garantie. Pour le système linéaire de l’ordre fractionnel on a proposé Piβ Dα du contrôle à retard. Enfin les conditions nécessaires pour la convergence dans le domaine temporel du contrôle ont été exposées avec le théorème correspondant accompagné par la preuve.

Mots clés: **théorie de contrôle, contrôle itératif, contrôle par l’étude, ordre entier, ordre fractionnel, système robotisé, vue d’ensemble**

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