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Finite Time and Practical Stability of Linear Continuous Time Delay Systems - Classical and Modern Approach: An Overview

Dragutin Debeljković [1)]
Sreten Stojanović [2)]
Tamara Nestorović [3)]
Nebojša Dimitrijević [4)]

This paper gives a detailed overview of the work and the results of the authors of this paper in the area of Non-Lyapunov (finite time stability, technical stability, practical stability, final stability) for a particular class of linear continuous time delays systems.
This survey covers the period since 2000 and has a strong intention to present the main concepts and contributions that have been derived during the mentioned period, published in the international journals or presented at respectable workshops or prestigious conferences.

Key words: **continuous system, time delay system, finite time interval system, system stability, non-Lyapunov stability, practical stability.**

Introduction

THE problem of investigation of time delay systems has been exploited over many years. Delay is very often encountered in different technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time delay, regardless if it is present in the control or/and state, may cause an undesirable system transient response, or generally, even an instability. Consequently, the problem of the stability analysis of this class of systems has been one of the main interests of many researchers. In general, the introduction of time lag factors makes the analysis much more complicated. In the existing stability criteria, mainly two ways of approach have been adopted. Namely, one direction is to contrive the stability condition which does not include information on the delay, and the other is the method which takes it into account..

The former case is often called the delay-independent criteria and generally provides nice algebraic conditions.

Numerous reports have been published on this matter, with a particular emphasis on the application of Lyapunov's second method, or on using the idea of matrix measure.

In practice, there is not only an interest in system stability (e.g. in the sense of Lyapunov), but also in the bounds of system trajectories. A system could be stable but still completely useless because it possesses undesirable transient performances.

Thus, it may be useful to consider the stability of such systems with respect to certain subsets of state-space which are defined *a priori* in a given problem.

Besides that, it is of particular significance to consider the behavior of dynamical systems only over a finite time interval.

These boundedness properties of system responses, i.e. the solution of system models, are very important from the engineering point of view. Realizing this fact, numerous definitions of the so-called technical and practical stability were introduced. Roughly speaking, these definitions are essentially based on the predefined boundaries for the perturbation of initial conditions and allowable perturbation of a system response. In the engineering applications of control systems, this fact becomes very important and sometimes crucial, for the purpose of characterizing in advance, in a quantitative manner, possible deviations of a system response.

Thus, the analysis of these particular boundedness properties of solutions is an important step which precedes the design of control signals, when finite time or practical stability control is concerned.

Chronological preview of the previous results

In the short overview that follows, we will be familiarized only with the results achieved for *linear continuous time delay systems* in the area of Non - Lyapunov stability.

Motivated by a "brief discussion" on practical stability in the monograph *La Salle*, *Lefchet*, (1961), *Weiss* and *Infante* (1965, 1967) have introduced various notations of stability over a finite time interval for continuous-time systems and

[1)] University of Belgrade, Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Belgrade, SERBIA
[2)] University of Niš, Faculty of Technological Engineering, Bulevar oslobođenja 124, 16000 Leskovac, SERBIA
[3)] Ruhr University of Bochum, Bochum, GERMANY
[4)] High School of Applied Professional Studies, 17500 Vranje, SERBIA

constant set trajectory bounds.

Further development of these results was accredited to many other authors.

In the context of finite or practical stability for a particular class of *nonlinear singularly perturbed multiple time delay systems,* various results were, *for the first time*, obtained in *Feng*, *Hunsarg* (1996). It seems that their definitions are very similar to those in *Weiss*, *Infante* (1965, 1967), clearly addopted to time delay systems.

In the paper of *Debeljkovic et al.* (1997) and *Nenadic et al.* (1975, 1997), some basic results from the area of finite time and practical stability were extended to the particular class of linear continuous time delay systems.

Stability sufficient conditions dependent on delay, expressed in terms of a time delay fundamental system matrix, have been derived. In addition, in the circumstances when it is possible to establish a suitable connection between the fundamental matrices of linear time delay and non-delay systems, the presented results enable an efficient procedure for testing practical as well the finite time stability of time delay systems.

The matrix measure approach has been, for the first time, applied in *Debeljkovic et al.* (1997) for the analysis of practical and finite time stability of linear time delayed systems.

With the Coppel's inequality and the introduction of the matrix measure approach, one provides very simple delay-dependent sufficient conditions of practical and finite time stability with no need for time delay fundamental matrix calculations.

Another approach, based on a very well-known Bellman – Gronwall Lemma, was applied in *Debeljkovic et al.* (2000), to provide new, more efficient sufficient delay-dependent conditions for checking the finite and practical stability of continuous systems with state delay.

An overview of all previous results and contributions was presented in the paper of *Debeljkovic et al.* (1999) with overall comments and a slightly modified Bellman – Gronwall approach.

Finally, a modified Bellman – Gronwall principle has been extended to the particular class of continuous *non-autonomous* time delayed systems operating over the finite time interval, *Debeljkovic et al.* (2000.a, 2000.b, 2000.c).

The concept of Lyapunov asymptotic stability is largely known to the control community.

However, Lyapunov asymptotic stability is often not enough for practical applications, because there are some cases where large values of the state are not acceptable.

For these purposes, the concept of finite time stability can be used.

A system is said to be finite time stability if, once a time interval is fixed, its state does not exceed some bounds during this time interval.

It is important to point out that finite time stability and Lyapunov stability are independent concepts: for instance, a system which is finite-time stable could be Lyapunov unstable, while a Lyapunov stable system could be finite-time unstable if its state exceeds the prescribed bounds during the transient period.

Recently, the concept of finite time stability has been revisited in the light of the linear matrix inequality theory, which has allowed to find less conservative conditions guaranteeing finite time stability and finite-time stabilization of continuous time systems.

Many valuable results have been obtained for this type of stability; see, for instance, *Amato et al.* (2001), *Moulay*, *Perruquetti* (2006), *Moulay et al.* (2008), *Amato et al.* (1998, 2003, 2006), *Garcia* (2009).

Similar to systems without delay, we also need to investigate finite time stability and finite time stabilization for a class of time-delay systems.

There are few results concerning finite time stability and finite time stability stabilization of time-delay systems.

Some early results on finite time stability of time-delay systems can be found in *Debeljkovic et al* (2000), as mentioned before.

The results of these investigations are conservative because they use the boundedness proprieties of the system response, i.e. of the solution of system models.

Recently, based on the linear matrix inequality theory, some results have been obtained for finite time stability and finite-time boundedness for some particular classes of time-delay systems, *Shen et al.* (2007).

The papers *Wang et al.* (2010) consider the problem of finite-time boundedness of the delayed neural networks.

In *Lin et al.* (2011), finite-time boundedness of switched linear systems with time-varying delay and exogenous disturbances are studied. Based on the average dwell-time technique, sufficient conditions which can ensure finite-time boundedness, finite-time weighted L_2-gain *Lin et al.* (2011) and H_∞ finite-time boundedness *Liu*, *Shen* (2011) are given. Finite time stability and the time stability stabilization of retarded-type functional differential equations are developed in *Moulay et al.* (2008).

The papers *Gao et ai.* (2011) and *Shang et al.* (2011) investigate the time stability stabilization problem for networked control systems with time-varying delay.

In *Shang et al.* (2011) a particular linear transformation is introduced to convert the original time-delay system into a delay-free form.

According to the author's knowledge, there is no result available yet on robust finite time stability and finite time stability stabilization of linear uncertain time-delay systems using linear matrix inequality.

Here we present the problem of sufficient conditions that enable system trajectories to stay within the *a priori* given sets for a particular class of time-delay systems.

Notations and preliminaries

A linear, multivariable time-delay system can be represented by a differential equation:

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B_0\mathbf{u}(t) + B_1\mathbf{u}(t-\tau), \quad (1)$$

and with the associated function of the initial state:

$$\mathbf{x}(t) = \boldsymbol{\varphi}_x(t), \quad \mathbf{u}(t) = \boldsymbol{\varphi}_u(t), \quad -\tau \le t \le 0. \quad (2)$$

Equation (1) is referred to as *nonhomogenous* or forced state equation, $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ the control vector, A_0, A_1, B_0 and B_1 are the constant system matrices of appropriate dimensions, and τ is pure time delay, $\tau = const.$ ($\tau > 0$).

Dynamical behavior system (1) with initial functions (2) is defined over the time interval $\Im = \{t_0,\ t_0 + T\}$, where the quantity T may be either a positive real number or the symbol $+\infty$, so finite time stability and practical stability can be treated simultaneously.

It is obvious that $\Im \in \mathbb{R}$.

Time invariant sets, used as the bounds of system trajectories, are assumed to be open, connected and bounded.

Let the index β stands for the set of all allowable states of the system and the index α for the set of all initial states of the system, such that the set $S_\alpha \subseteq S_\beta$.

In general, one may write:

$$S_\rho = \left\{ \mathbf{x}(t):\ \|\mathbf{x}(t)\|_Q^2 < \rho \right\}\ , \tag{3}$$

where Q will be assumed to be a symmetric, positive-definite, real matrix.

S_ε denotes the set of all allowable control actions.

Let $\|\mathbf{x}(\cdot)\|_{(\cdot)}$ be any vector norm (e.g., $\cdot$ = 1, 2, ∞) and ||(·)|| the matrix norm induced by this vector.

Here, we use $\|\mathbf{x}(t)\|_2 \overset{\Delta}{=} \left(\mathbf{x}^T(t)\mathbf{x}(t)\right)^{1/2}$ and $\|\ \|_2 = \lambda_{\max}^{1/2}\left(A^* A\right)$.

Upper indices * and T denote transpose conjugate and transpose, respectively.

The matrix measure has been widely used in the literature when dealing with stability of time delay systems.

The matrix measure $\mu(\cdot)$ for any matrix $A \in \mathbb{C}^{n\times n}$ is defined as follows

$$\mu(A) \overset{\Delta}{=} \lim_{\varepsilon\to 0} \frac{\|1+\wp A\|-1}{\wp}\ . \tag{4}$$

The matrix measure defined in (4) can be subdefined in three different ways, depending on the norm used in its definitions, *Coppel* (1965), *Desoer*, *Vidysagar* (1975):

$$\mu_1(A) = \max_k \left(\operatorname{Re}(a_{kk}) + \sum_{\substack{i=1\\ i\neq k}}^{n} |a_{ik}| \right), \tag{5.a}$$

$$\mu_2(A) = \frac{1}{2}\max_i \lambda_i\left(A^* + A\right), \tag{5.b}$$

and

$$\mu_\infty(A) = \max_i \left(\operatorname{Re}(a_{ii}) + \sum_{\substack{k=1\\ k\neq i}}^{n} |a_{ki}| \right), \tag{5.c}$$

From *Mori* (1988), the following inequality holds:

$$-\|F\|_2 \le -\mu(-F) \le \mu(F) \le \|F\|_2\ . \tag{5.d}$$

Basic notations

$\mathbb{R}$	-	Real vector space
$\mathbb{C}$	-	Complex vector space
$F = (f_{ij}) \in \mathbb{R}^{n\times n}$,	-	real matrix
F^T	-	Transpose of the matrix F
$F > 0$	-	Positive definite matrix
$F \ge 0$	-	Positive semi definite matrix
$\lambda(F)$	-	Eigenvalue of the matrix F
$\sigma_{(\ )}(F)$	-	Singular values of the matrix F
$\sigma\{F\}$	-	Spectrum of the matrix F
Euclidean matrix norm $\|F\| = \sqrt{\lambda_{\max}\left(A^T A\right)}$		
$\Rightarrow$	-	Follows
$\mapsto$	-	Such that

Time invariant time delay systems finite time stability

Stability definitions

In the context of finite or practical stability for a particular class of *nonlinear singularly perturbed multiple time delay systems* various results were, *for the first time*, obtained in *Hsiao*, *Hwang* (1996).

It seems that their definitions are very similar to those in *Weiss*, *Infante* (1965, 1967), clearly addopted to time delay systems.

It should be noticed that those definitions are significantly different from the definition presented by the autor of this paper.

Definition 1. A system is *stable* with respect to the set $\{\alpha,\ \beta,\ -\tau,\ T,\ \|\mathbf{x}\|\}$, $\alpha \le \beta$ if for any trajectory $\mathbf{x}(t)$ condition $\|\mathbf{x}_0\| < \alpha$ implies $\|\mathbf{x}(t)\| < \beta$ $\forall t \in [-\Delta,\ T]$, $\Delta = \tau_{\max}$, *Hsiao*, *Hwang* (1996).

Definition 2. An autonomous system is *contractively stable* with respect to the set $\{\alpha,\ \beta,\ -\tau,\ T,\ \|\mathbf{x}\|\}$, $\gamma < \alpha < \beta$, if for any trajectory $\mathbf{x}(t)$ condition $\|\mathbf{x}_0\| < \alpha$ implies:

(i) Stability w.r.t. $\{\alpha,\ \beta,\ -\tau,\ T,\ \|\mathbf{x}\|\}$,

(ii) There exists $t^* \in \left]0,\ T\right[$ such that $\|\mathbf{x}(t)\| < \gamma$ for all $\forall t \in \left]t^*,\ T\right[$ *Hsiao*, *Hwang* (1996).

Definition 3. Autonomous system (1) satisfying initial condition (2) is finite time stable w.r.t. $\{\zeta(t),\ \beta,\ \Im\}$ if and only if $\|\boldsymbol{\varphi}_x(t)\|^2 < \zeta(t)$, implies: $\|\mathbf{x}(t)\|^2 < \beta,\ \ t \in \Im$, $\zeta(t)$ being a scalar function with the property $0 < \zeta(t) \le \alpha,\ \ -\tau \le t \le 0$, where α is a real positive number and $\beta \in \mathbb{R}$ and $\beta > \alpha$, *Debeljkovic et al.* (2001), *Nenadic et al.* (1997).

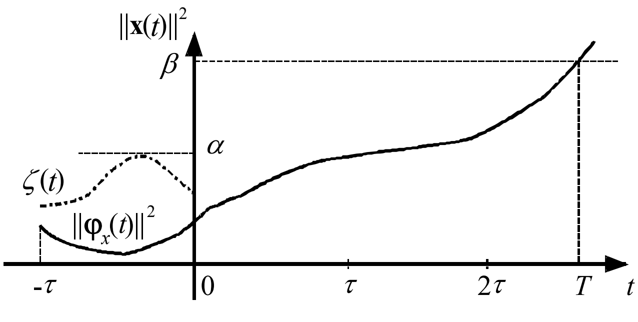

Figure 1. Illustration of the preceding definition

Definition 4. System (1), with $\mathbf{u}(t-\tau) \equiv 0,\ \forall t$, satisfying initial condition (2) is finite time stable w.r.t. $\{\zeta(t),\ \beta,\ \varepsilon,\ \tau,\ \Im,\ \mu(A_0 \neq 0)\}$ if and only if

$$\boldsymbol{\varphi}_x(t) \in S_\alpha,\quad \forall t \in [-\tau,\ 0],$$

and

$$\mathbf{u}(t) \in S_{\varepsilon}, \quad \forall t \in \Im,$$

imply

$$\mathbf{x}(t_0, t, \mathbf{x}_0) \in S_{\beta}, \quad \forall t \in \Im,$$

Debeljkovic et al. (2001)

Definition 5. System (1) satisfying initial condition (2) is finite time stable w.r.t. $\{\alpha, \beta, \varepsilon_{\varphi}, \varepsilon, \tau, \Im, \mu_2(A_0) \neq 0\}$ if and only if

$$\boldsymbol{\varphi}_x(t) \in S_{\alpha}, \quad \boldsymbol{\varphi}_u(t) \in S_{\varepsilon_{\boldsymbol{\varphi}}}, \quad \forall t \in [-\tau, \ 0], \tag{13}$$

$$\mathbf{u}(t) \in S_{\varepsilon}, \quad \forall t \in \Im,$$

imply:

$$\mathbf{x}(t, t_0, \mathbf{x}_0, \mathbf{u}(t)) \in S_{\beta}, \quad \forall t \in \Im,$$

Debeljkovic et al. (2001).

Definition 6. System (6.a) satisfying given initial condition (6.b) is finite time stable with respect to $\{\alpha, \beta, \Im\}$, where $0 \leq \alpha < \beta$, if $\sup_{t \in [-\tau, 0]} \boldsymbol{\varphi}_x^T(t)\boldsymbol{\varphi}_x^T(t) \leq \alpha$ implies $\|\mathbf{x}(t)\|^2 < \beta, \quad t \in \Im$, *Stojanovic et al* (2012).

Dependent delay stability conditions

Stability theorems

Theorem 1. Autonomous system (**1**) with initial function (2) is finite time stable with respect to $\{\alpha, \beta, \tau, J\}$ if the following condition is satisfied

$$\|\Phi(t)\| < \frac{\sqrt{\beta / \alpha}}{1 + \tau \|A_1\|_2}, \quad \forall t \in \Im, \tag{6}$$

where $\|(\cdot)\|$ is the Euclidean norm and $\Phi(t)$ is the fundamental matrix of system (1), *Nenadic et al.* (1997), *Debeljkovic et al.* (2001).

Theorem 2: Autonomous system (**1**) with initial function (2) is finite time stable w.r.t. $\{\alpha, \beta, \tau, T\}$ if the following condition is satisfied

$$e^{\mu_2(A_0)t} < \frac{\sqrt{\beta / \alpha}}{1 + \tau \|A_1\|_2}, \quad \forall t \in \Im, \tag{7}$$

where $\|(\cdot)\|$ denotes the Euclidean norm, *Debeljkovic et al.* (2001).

Theorem 3. Autonomous system (1) with initial function (2) is finite time stable with respect to $\{\alpha, \beta, \tau, T, \mu_2(A_0) \neq 0\}$ if the following condition is satisfied

$$e^{\mu_2(A_0)t} < \frac{\beta / \alpha}{1 + \mu_2^{-1}(A_0) \cdot \| A_1 \|_2 \cdot \left(1 - e^{-\mu_2(A_0)\tau}\right)} \tag{8}$$
$$\forall t \in \Im[0, \ T]$$

Debeljkovic et al. (2001).

Theorem 4. System (1), with initial function (2) is finite time stable w.r.t. $\{\zeta(t), \beta, \varepsilon, \tau, \Im, \mu_2(A_0) \neq 0, B_1 = 0\}$ if the following condition is satisfied

$$e^{\mu_2(A_0)t} < \frac{\beta / \alpha}{\phi}, \tag{9}$$

$$\begin{aligned} \phi = \mu^{-1}(A_0)\left(\mu(A_0) + \| A_1 \|_2 \left(1 - e^{-\mu_2(A_0)\tau}\right)\right) \\ + \mu^{-1}(A_0)\gamma \| B_0 \|_2 \left(1 - e^{-\mu_2(A_0)t}\right), \quad \forall t \in J. \end{aligned} \tag{10}$$

where,

$$\gamma = \frac{\varepsilon}{\alpha}, \quad \mu_2(A_0) = \frac{1}{2}\lambda_{\max}\left(A_0 + A_0^T\right), \tag{11}$$

Debeljkovic et al. (2001).

Theorem 5. System (1), with $\mathbf{u}(t - \tau) \equiv \mathbf{0}$, $\forall t$, satisfying initial condition (2) is finite time stable w.r.t. $\{\zeta(t), \beta, \varepsilon, \tau, T, \mu_2(A_0) \neq 0, B_1 = 0\}$, if the following condition is satisfied

$$\left(1 + \tau \| A_1 \|_2\right) + \gamma \| B_0 \|_2 \, t < \frac{\beta}{\alpha}, \quad \forall t \in J, \tag{12}$$

where γ is given with (33), *Debeljkovic et al.* (2001).

Theorem 6. Autonomous system (1) with initial function (2) is finite time stable with respect to $\{\sqrt{\alpha}, \sqrt{\beta}, \tau, T, \mu(A_0) = 0\}$ if the following condition is satisfied

$$1 + \tau \| A_1 \|_2 < \sqrt{\beta / \alpha}, \ \forall t \in [0, \ T], \tag{13}$$

Debeljkovic et al. (2001):

Theorem 7. System given by (1), with initial function (2) is finite time stable w.r.t. $\{\alpha, \beta, \varepsilon_{\boldsymbol{\varphi}}, \varepsilon, \tau, \Im, \mu_2(A_0) \neq 0\}$ if the following condition is satisfied

$$\frac{e^{\mu_2(A_0)t}}{\mu_2(A_0)} < \beta / \alpha \cdot \delta, \quad \forall t \in \Im, \tag{14}$$

where

$$\begin{aligned} \delta = \mu_2(A_0) + a_1\left(\pi_1\left(1 - e^{-\mu_2(A_0)\tau}\right) + \right. \\ \left. + \pi_2\left(1 - e^{-\mu_2(A_0)t}\right)\right) \end{aligned} \tag{15.a}$$

$$\pi_1 = 1 + b_1(\gamma + \gamma_{\boldsymbol{\varphi}}), \qquad \pi_2 = \gamma(b_0 + b_1), \tag{15.b}$$

$$a_1 = \| A_1 \|, \ b_1 = \frac{\| B_0 \|}{a_1}, \ b_0 = \frac{\| B_0 \|}{a_1}, \tag{15.c}$$

$$\gamma = \frac{\varepsilon}{\alpha}, \quad \gamma_{\boldsymbol{\varphi}} = \frac{\varepsilon_{\boldsymbol{\varphi}}}{\alpha} \tag{15.d}$$

Debeljkovic et al. (2001).

Theorem 8 Time delayed system (1) is *finite time stable* with respect to $\{t_0, \Im, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if the following condition is satisfied:

$$e^{\Lambda_{\max}(\Pi)(t - t_0)} < \frac{\beta}{\alpha}, \quad \forall t \in \Im, \tag{16}$$

where:

$$\Lambda_{\max}(\Pi) = \lambda_{1\max}(\ ) + \tau \cdot \lambda_{2\max}(\ ). \tag{17}$$

and:

$$\lambda_{1\max}(\ ) = \lambda_{1\max}\left(\left(A_0^T + A_0\right) + \left(A_1^T + A_1\right)\right) \tag{18}$$

$$\lambda_{2\max}(\ ) = \lambda_{2\max}\left(\wp\left(A_1 A_0 A_0^T A_1^T + A_1 A_1 A_1^T A_1^T\right) + +2\frac{q^2}{\wp} I\right) \tag{19}$$

with: $\wp > 0$ and $q > 1$, *Debeljkovic et al* (2011.a).

Proof. Let us consider the continuous linear time delay system described by differential delay equation (1).

Let $\mathbf{x}(t)$, $t \geq 0$ be the solution of (1) if the initial moment and state are $\mathbf{0}$ and $\boldsymbol{\varphi}_{\mathbf{x}}(t)$.

It is very well known that if the $\mathbf{x}(t)$ is continuously differentiable for $t \geq 0$, one can write:

$$\mathbf{x}(t-\tau) = \mathbf{x}(t) - \int_{-\tau}^{0} \left(A_0 \mathbf{x}(t+\vartheta) + A_1 \mathbf{x}(t-\tau+\vartheta)\right) d\vartheta, \tag{21}$$

for $t \geq \tau$, *Hale* (1977), so the basic system dynamics in (**1**) can be rewritten as:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \left(A_0 + A_1\right)\mathbf{x}(t) - \\ &-A_1 \int_{-\tau}^{0} \left(A_0 \mathbf{x}(t+\vartheta) + A_1 \mathbf{x}(t-\tau+\vartheta)\right) d\vartheta\end{aligned} \tag{20}$$

for an arbitrary continuous initial function $\boldsymbol{\varphi}(t)$ on the time interval $t \in [-2\vartheta,\ 0]$.

It is stated in *Hale* (1977) that the asymptotic stability of (20) can assure the asymptotic stability of the original system (1), since the basic system (1) is only a spacial case of a system whose dynamics is described by (20).

This important fact will be directly used in the second part of this investigation, namely when the *attractive practical stability* is considerd.

In this section, for the sake of simplicity, we will use (20) to obtain a sufficient condition of finite time stability of (**1**) since the conditions to be fulfilled are less severe than to achive asymptotic stability,

Let us define the tentative aggregation function as:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{x}(t). \tag{21}$$

The total time derivative $\dot{V}(\mathbf{x}(t))$ along the trajectories of the system yields:

$$\begin{aligned}\dot{V}(\mathbf{x}(t)) &= \mathbf{x}^T(t)\left(\left(A_0^T + A_0\right) + \left(A_1^T + A_1\right)\right)\mathbf{x}(t) \\ &-2\int_{-\tau}^{0} \mathbf{x}^T(t) A_1 \left(A_0 \mathbf{x}(t+\vartheta) + A_1 \mathbf{x}(t-\tau+\vartheta)\right) d\vartheta\end{aligned} \tag{22}$$

Now, we follow *Su* (1994).

Following the Ruzumikhin-type theorem, *Hale* (1977), we assume that for some real constant $q > 1$ the following inequality holds:

$$V(\mathbf{x}(\xi)) < q^2 V(\mathbf{x}(t)), \quad t - 2\vartheta \leq \xi \leq t. \tag{23}$$

Furthermore, by using the following inequality, for any real constant $\wp > 0$ and any symmetric, positive definite matrix Ξ, $\Xi = \Xi^T > 0$:

$$-2\mathbf{u}^T(t)\mathbf{v}(t) \leq \wp \mathbf{u}^T(t)\Xi^{-1}\mathbf{u}(t) + \frac{1}{\wp}\mathbf{v}^T(t)\Xi\mathbf{v}(t), \tag{24}$$

we obtain:

$$\begin{aligned}&-2\int_{-\tau}^{0} \mathbf{x}^T(t) A_1 A_0 \mathbf{x}(t+\vartheta) d\vartheta \leq \\ &\leq \int_{-\tau}^{0} \left(\wp \mathbf{x}^T(t) A_1 A_0 A_0^T A_1^T \mathbf{x}(t) + \frac{1}{\wp}\mathbf{x}^T(t+\vartheta)\mathbf{x}(t+\vartheta)\right) d\vartheta \\ &< \tau\, \mathbf{x}^T(t)\left(\wp A_1 A_0 A_0^T A_1^T + \frac{q^2}{\wp} I\right)\mathbf{x}(t) \\ &< \tau\, \lambda_{\max}\left(\wp A_1 A_0 A_0^T A_1^T + \frac{q^2}{\wp} I\right)\mathbf{x}^T(t)\mathbf{x}(t)\end{aligned}, \tag{25}$$

and in the same way:

$$\begin{aligned}&-2\int_{-\tau}^{0} \mathbf{x}^T(t) A_1 A_1 \mathbf{x}(t-\tau+\vartheta) d\vartheta < \\ &< \tau\, \lambda_{\max}\left(\wp A_1 A_1 A_1^T A_1^T + \frac{q^2}{\wp} I\right)\mathbf{x}^T(t)\mathbf{x}^T(t)\end{aligned}. \tag{26}$$

Using (25) and (26), one may have:

$$\begin{aligned}&\dot{V}(\mathbf{x}(t)) < \\ &< \lambda_{1\max}\left(\left(A_0^T + A_0\right) + \left(A_1^T + A_1\right)\right)\mathbf{x}^T(t)\mathbf{x}^T(t) \\ &+\tau\, \lambda_{2\max}\left(\wp\left(A_1 A_0 A_0^T A_1^T + A_1 A_1 A_1^T A_1^T\right) + 2\frac{q^2}{\wp} I\right)\mathbf{x}^T(t)\mathbf{x}^T(t) \\ &< \Lambda_{\max}(\Pi)\mathbf{x}^T(t)\mathbf{x}^T(t)\end{aligned} \tag{27}$$

$$\Lambda_{\max}(\Pi) = \Lambda_{\max}\left(A_0, A_1, \tau\right) = \lambda_{1\max}(\ ) + \tau\, \lambda_{2\max}(\ ) \tag{28}$$

From (27) one can get:

$$\frac{d\left(\mathbf{x}^T(t)\mathbf{x}(t)\right)}{\mathbf{x}^T(t)\mathbf{x}(t)} < \lambda_{\max}(\Pi)\, dt \quad ,(29)$$

or:

$$\int_{t_0}^{t} \frac{d\left(\mathbf{x}^T(t)\mathbf{x}(t)\right)}{\mathbf{x}^T(t)\mathbf{x}(t)} < \int_{t_0}^{t} \lambda_{\max}(\Pi)\, dt, \tag{30}$$

and:

$$\mathbf{x}^T(t)\mathbf{x}(t) < \mathbf{x}^T(t_0)\mathbf{x}(t_0) e^{\lambda_{\max}(\Pi)(t-t_0)}. \tag{31}$$

Finally, if one uses the first condition of *Definition* 3, then:

$$\mathbf{x}^T(t)\mathbf{x}(t) < \alpha \cdot e^{\lambda_{\max}(\Pi)(t-t_0)}, \tag{34}$$

and finally by (16), yields to:

$$\mathbf{x}^T(t)\mathbf{x}(t) < \alpha \cdot \frac{\beta}{\alpha} < \beta, \quad \forall t \in \Im, \tag{35}$$

which has to be proved. **Q.E.D.**

Theorem 9. System (1) is finite time stable with respect to (α,β,T), $\alpha < \beta$, if there are nonnegative scalar $\wp$ and positive define symmetric matrices P and Q such that the following conditions hold

$$\Omega = \begin{pmatrix} A_0^T P + PA_0 + Q - \wp P & PA_1 \\ A_1^T P & -Q \end{pmatrix} < 0, \tag{36}$$

$$\frac{1}{\lambda_{\min}(P)}\left(\lambda_{\max}(P) + \tau \cdot \lambda_{\max}(Q)\right)e^{\wp T} < \frac{\beta}{\alpha}, \tag{37}$$

Stojanovic, Debeljkovic, Antic (2012)

Proof. Let us consider the following Lyapunov-like function

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)P\mathbf{x}(t) + \int_{t-\tau}^{t} \mathbf{x}^T(\vartheta)Q\mathbf{x}(\vartheta)d\vartheta \tag{38}$$

Then, the time derivative of $V(\mathbf{x}(t))$ along the solution of (1) gives

$$\begin{aligned}\dot V(\mathbf{x}(t)) &= \mathbf{x}^T(t)\left(A_0^T P + PA_0 + Q\right)\mathbf{x}(t) \\ &\quad +2\mathbf{x}^T(t)PA_1\mathbf{x}(t-\tau) - \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau) \\ &= \xi(t)^T \Gamma\, \xi(t)\end{aligned} \tag{39}$$

where

$$\begin{aligned}\xi(t) &= \left[\mathbf{x}^T(t) \;\; \mathbf{x}^T(t-\tau)\right]^T, \\ \Gamma &= \begin{pmatrix} A_0^T P + PA_0 + Q & PA_1 \\ A_1^T P & -Q \end{pmatrix}\end{aligned} \tag{40}$$

Hence

$$\begin{aligned}\dot V(\mathbf{x}(t)) &= \xi^T(t)\Gamma\xi(t) = \xi^T(t)\left(\Omega - \begin{pmatrix} -\wp P & 0 \\ 0 & 0\end{pmatrix}\right)\xi(t) \\ &< \xi^T(t)\Omega\xi(t) + \xi^T(t)\begin{pmatrix} \wp P & 0 \\ 0 & 0\end{pmatrix}\xi(t) \\ &< \xi^T(t)\begin{pmatrix} \wp P & 0 \\ 0 & 0\end{pmatrix}\xi(t) = \wp\,\mathbf{x}^T(t)P\mathbf{x}(t) \\ &< \wp\left(\mathbf{x}^T(t)P\mathbf{x}(t) + \int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)Q\mathbf{x}(\vartheta)d\vartheta\right) \\ &< \wp V(\mathbf{x}(t))\end{aligned} \tag{41}$$

Multiplying (41) by $e^{-\wp \cdot t}$, we can obtain

$$\frac{d}{dt}\left(e^{-\wp\cdot t}V(\mathbf{x}(t))\right) < 0 \tag{42}$$

Integrating (42) from 0 to t, with $t \in [0,T]$, we have

$$V(\mathbf{x}(t)) < e^{\wp\cdot t}V(\mathbf{x}(0)) \tag{43}$$

Then

$$\begin{aligned}V(\mathbf{x}(0)) &= \mathbf{x}^T(0)P\mathbf{x}(0) + \int_{-\tau}^{0}\boldsymbol{\varphi}^T(\vartheta)Q\boldsymbol{\varphi}(\vartheta)d\vartheta \\ &\le \lambda_{\max}(P)\alpha + \lambda_{\max}(Q)\tau\cdot\alpha \\ &= \alpha\left(\lambda_{\max}(P)\alpha + \lambda_{\max}(Q)\tau\right)\end{aligned} \tag{44}$$

On the other hand,

$$V(\mathbf{x}(t)) > \mathbf{x}^T(t)P\mathbf{x}(t) \ge \lambda_{\min}(P)\mathbf{x}^T(t)\mathbf{x}(t) \tag{45}$$

Combining (43), (44) and (45), we get

$$\begin{aligned}\mathbf{x}^T(t)\mathbf{x}(t) &< \frac{\alpha}{\lambda_{\min}(P)}\left(\lambda_{\max}(P) + \tau\cdot\lambda_{\max}(Q)\right)e^{\wp\cdot t} \\ &< \frac{\alpha}{\lambda_{\min}(P)}\left(\lambda_{\max}(P) + \tau\cdot\lambda_{\max}(Q)\right)e^{\wp\cdot t}, \ \forall t \in [0,T]\end{aligned} \tag{46}$$

If

$$\frac{\alpha}{\lambda_{\min}(P)}\left(\lambda_{\max}(P) + \tau\cdot\lambda_{\max}(Q)\right)e^{\wp\cdot t} < \beta \tag{47}$$

then

$$\mathbf{x}^T(t)\mathbf{x}(t) < \beta, \ \forall t \in [0,T]. \tag{48}$$

Q.E.D.

Remark 1 It should be pointed out that the condition in Theorem 9 is not standard LMIs condition with respective to $\wp$ and P. However, once we fix $\wp$, it can be turned into an LMIs based feasibility problem which can be solved via existing software.

Theorem 10. Suppose that the certain matrix $\left(A_0^T + A_0 + A_1^T A_1 + I\right)$ is positive definite.

Then the autonomous system (6.a) with initial function (6.b) is finite time stable with respect to $\{\alpha, \beta, \tau, T\}$, if $\alpha < \beta$, such that the following conditions hold:

$$(1+\tau)e^{\lambda_{\max}(\Pi)\cdot t} < \frac{\beta}{\alpha}, \tag{49}$$

Proof: Let us consider the following Lyapunov-like aggregation function:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t)\mathbf{x}(t) + \int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)\mathbf{x}(\vartheta)d\vartheta, \tag{50}$$

Denote by $\dot V(\mathbf{x}(t))$ the time derivative of $V(\mathbf{x}(t))$ along the trajectory of system (1), so one can obtain:

$$\begin{aligned}\dot V(\mathbf{x}(t)) &= \dot{\mathbf{x}}^T(t)\mathbf{x}(t) + \mathbf{x}^T(t)\dot{\mathbf{x}}(t) + \frac{d}{dt}\int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)\mathbf{x}(\vartheta)d\vartheta \\ &= \mathbf{x}^T(t)\left(A_0^T + A_0\right)\mathbf{x}(t) + 2\mathbf{x}^T(t)A_1\mathbf{x}(t-\tau) \\ &\quad +\mathbf{x}^T(t)\mathbf{x}(t) - \mathbf{x}^T(t-\tau)\mathbf{x}(t-\tau)\end{aligned} \tag{51}$$

Based on the well-known inequality[1] and with the particular choice:

$$\mathbf{x}^T(t)\Gamma\mathbf{x}(t) = \mathbf{x}^T(t)I\mathbf{x}(t) > 0, \ \forall \mathbf{x}(t) \in \mathcal{S}_\beta, \tag{52}$$

[1] $2\mathbf{u}^T(t)\mathbf{v}(t-\tau) \le \mathbf{u}^T(t)\Gamma^{-1}\mathbf{u}(t) + \mathbf{v}^T(t-\tau)\Gamma\mathbf{v}(t-\tau), \ \Gamma > 0$

so that:

$$\begin{aligned}\dot V(\mathbf{x}(t)) &\leq \mathbf{x}^T(t)\left(A_0^T + A_0 + A_1A_1^T + I\right)\mathbf{x}(t)\\ &\leq \mathbf{x}^T(t)\Pi\,\mathbf{x}(t)\\ &\leq \lambda_{\max}(\Pi)\mathbf{x}^T\mathbf{x}(t)\end{aligned}, \quad (53)$$

under the assumption given by *Theorem* 10.

Moreover, it is easy to see that:

$$\begin{aligned}\dot V(\mathbf{x}(t)) &< \lambda_{\max}(\Pi)\mathbf{x}^T\mathbf{x}(t) + \lambda_{\max}(\Pi)\int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)\mathbf{x}(\vartheta)d\vartheta\\ &< \lambda_{\max}(\Pi)\left(\mathbf{x}^T\mathbf{x}(t) + \int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)\mathbf{x}(\vartheta)d\vartheta\right)\\ &< \lambda_{\max}(\Pi)V(\mathbf{x}(t))\end{aligned}, \quad (54)$$

since $\int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)\mathbf{x}(\vartheta)d\vartheta > 0$ and $\lambda_{\max}(\Pi) > 0$.

Multiplying (54) by $e^{-\lambda_{\max}(\Pi)\cdot t}$, we can obtain:

$$\frac{d}{dt}\left(e^{-\lambda_{\max}(\Pi)\cdot t}V(\mathbf{x}(t))\right) < 0. \quad (55)$$

Integrating (22) from 0 to *t,* with $t \in \Im$, we get:

$$V(\mathbf{x}(t)) < e^{\lambda_{\max}(\Pi)\cdot t}\cdot V(0). \quad (56)$$

From (50) it can be seen:

$$\begin{aligned}V(0) &= \mathbf{x}^T(0)\mathbf{x}(0) + \int_{-\tau}^{0}\boldsymbol{\varphi}^T(0)\boldsymbol{\varphi}(0)d\vartheta\\ &\leq \mathbf{x}^T(0)\mathbf{x}(0) + \boldsymbol{\varphi}^T(0)\boldsymbol{\varphi}(0)\int_{-\tau}^{0}d\vartheta\\ &\leq \alpha + \alpha\cdot\tau = \alpha(1+\tau)\end{aligned}, \quad (57)$$

in the light of *Definition* 6**.**

Combining (56) and (57) leads to:

$$V(\mathbf{x}(t)) < \alpha(1+\tau)\cdot e^{\lambda_{\max}(\Pi)\cdot t} \quad (58)$$

On the other hand:

$$\begin{aligned}&\mathbf{x}^T(t)\mathbf{x}(t) < \mathbf{x}^T(t)\mathbf{x}(t) + \int_{t-\tau}^{t}\mathbf{x}^T(\vartheta)\mathbf{x}(\vartheta)d\vartheta =\\ &= V(\mathbf{x}(t)) < \alpha(1+\tau)\cdot e^{\lambda_{\max}(\Pi)\cdot t}\end{aligned}, \quad (59)$$

Condition (50) and the above inequality imply:

$$\mathbf{x}^T(t)\mathbf{x}(t) < \alpha(1+\tau)\cdot e^{\lambda_{\max}(\Pi)\cdot t} < \beta, \ \forall t \in \Im, \quad (60)$$

which has to be proved.

In the case of a non-delay system, e.g., when $\tau = 0$ or $A_1 = 0$, the result given in (49) is reduced to *Angelo* (1974).

Independent delay stability conditions

The results that will be presented in the sequel enable the checking of finite time stability of the autonomous system to be considered, namely the system given by (1) and (2), without finding the fundamental matrix or corresponding matrix measure.

Eq. (2) can be rewritten in its general form as:

$$\mathbf{x}(t_0+\vartheta) = \boldsymbol{\varphi}_x(\vartheta),\ \boldsymbol{\varphi}_x(\vartheta) \in \mathcal{C}[-\tau,\ 0], \quad -\tau \leq \vartheta \leq 0 \quad (61)$$

where t_0 is the initial time of observation of system (1) and $\mathcal{C}[-\tau,\ 0]$ is the *Banach space* of continuous functions over a time interval of length τ, mapping the interval $[(t-\tau),\ t]$ into $\mathbb{R}^n$ with the norm defined in the following manner:

$$\|\boldsymbol{\varphi}\|_{\mathcal{C}} = \max_{-\tau \leq \vartheta \leq 0}\|\boldsymbol{\varphi}(\vartheta)\|. \quad (62)$$

It is assumed that the usual smoothness conditions are present so that there is no difficulty with questions of existence, uniqueness, and continuity of solutions with respect to the initial data.

Moreover, one can write:

$$\mathbf{x}(t_0+\vartheta) = \boldsymbol{\varphi}_x(\vartheta), \quad (63)$$

as well as:

$$\dot{\mathbf{x}}(t_0) = \mathbf{f}(t_0, \boldsymbol{\varphi}_x(\vartheta)). \quad (64)$$

Theorem 11 The autonomous system given by (1) with initial function (2) is finite time stable w.r.t. $\{t_0, \Im, \alpha, \beta\}$ if the following condition is satisfied

$$\left(1+(t-t_0)\sigma_{\max}\right)^2 e^{2(t-t_0)\sigma_{\max}} < \frac{\beta}{\alpha}, \quad \forall t \in \Im, \quad (65)$$

$\sigma_{\max}(\cdot)$ being the largest singular value of the matrix $(\cdot)$, namely

$$\sigma_{\max} = \sigma_{\max}(A_0) + \sigma_{\max}(A_1). \quad (66)$$

Debeljkovic et al. (2001).

Theorem 12. The autonomous system given by **(**1**)** with initial function (2) is finite time stable w.r.t. $\{t_0, \Im, \alpha, \beta\}$ if the following condition is satisfied

$$e^{2(t-t_0)\cdot\sigma_{\max}} < \frac{\beta}{\alpha}, \quad \forall t \in \Im, \quad (67)$$

where $\sigma_{\max}(\cdot)$ is defined in (66), *Debeljkovic et al.* (2001).

Remark 2. In the case when in *Theorem* 11 $A_1 = 0$, e.g. A_1 is null matrix, we have the result similar to that presented in *Angelo* (1974).

Practical stability

Delay dependent stability conditions

Stability definitions

Definition 7. System (6.a) with initial function (6.b), is *attractive practically stable* w.r.t. $\{t_0, \Im, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$, if:

$$\|\mathbf{x}(t_0)\|^2 = \|\mathbf{x}_0\|^2 < \alpha,$$

implies:

$$\left\|\mathbf{x}(t)\right\|^2 < \beta, \quad \forall t \in \Im,$$

with a property that:

$$\lim_{k\to\infty} \left\|\mathbf{x}(t)\right\|^2 \to 0 .$$

Stability theorems

Theorem 13. System (6.a) with initial function (6.b), is *attractive practically stable* with respect to $\left\{t_0, \Im, \alpha, \beta, \left\|(\cdot)\right\|^2\right\}$, $\alpha < \beta$, if there exists the matrix $P = P^T > 0$ being a solution of:

$$A_0^T P + PA_0 = -Q, \tag{68}$$

with the matrice $Q = Q^T > 0$, and if the following conditions are satisfied:

$$\left\|A_1\right\| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}P\right), \tag{69}$$

and:

$$e^{\Lambda_{\max}(\Sigma)(t-t_0)} < \frac{\beta}{\alpha}, \quad \forall t \in \Im, \tag{70}$$

where:

$$\Lambda_{\max}(\Xi) = \lambda_{1\max}(\ ) + \tau\,\lambda_{2\max}(\ ). \tag{71}$$

with $\lambda_{1\max}(\ )$ and $\lambda_{2\max}(\ )$ given by (18) and (19) with $\wp > 0$ and $q > 1$, *Debeljkovic et al* (2011.b)

Remark 3. The asymptotic stability of **(1)** is quaranted by (68) and (69).

Theorem 14. System (1) with initial function (2), is *attractive practically stable* with respect to $\left\{t_0, \Im, \alpha, \beta, \left\|(\cdot)\right\|^2\right\}$, $\alpha < \beta$, if there exists the matrix $P = P^T > 0$ being a solution of:

$$A_0^T P + PA_0 = -2I, \tag{72}$$

and if the following conditions are satisfied:

$$\left\|A_1\right\| < \frac{1}{\lambda_{\max}(P)}, \tag{73}$$

and:

$$e^{\Lambda_{\max}(\Sigma)(t-t_0)} < \frac{\beta}{\alpha}, \quad \forall t \in \Im, \tag{74}$$

where:

$$\Lambda_{\max}(\Sigma) = \lambda_{1\max}(\ ) + \tau\,\lambda_{2\max}(\ ). \tag{75}$$

with $\lambda_{1\max}(\ )$ and $\lambda_{2\max}(\ )$ given by (18) and (19) and $\wp > 0$ and $1 < q < \frac{1}{\left\|A_1\right\|}\lambda_{\max}^{-1}(P)$, *Debeljkovic et al* (2011.a).

Proof. Define the tentative aggregation function as:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t) P\, \mathbf{x}(t). \tag{76}$$

The proof of the asymptotic properties of the system under consideration is identical to that presented in *Xu*, *Liu* (1994) for the particular case when:

$$\mathbf{f}\left(t, \mathbf{x}\left(t-\tau(t)\right)\right) = A_1\mathbf{x}(t-\tau). \tag{77}$$

It is clear that the asymptotic stability of (6) is quaranted by (72) and (73), based on the results presented in *Xu*, *Liu* (1994) and the additional corrections in *Mao* (1997). **Q.E.D**

Theorem 15. System (**1**) with initial function (2), is *attractive practically stable* with respect to $\left\{t_0, \Im, \alpha, \beta, \left\|(\cdot)\right\|^2\right\}$, $\alpha < \beta$, if there exists the matrix $P = P^T > 0$ being the solution of:

$$A_0^T P + PA_0 = -Q, \tag{78}$$

with the matrix $Q = Q^T > 0$, and if the following conditions are satisfied:

$$\left\|A_1\right\| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}P\right), \tag{79}$$

and:

$$e^{\mu_2(A_0)t} < \frac{\sqrt{\beta/\alpha}}{1+\tau\left\|A_1\right\|_2}, \quad \forall t \in \Im, \tag{80}$$

$\mu_2(A_0)$ being any matrix measure, *Debeljkovic et al* (2011.a).

Proof. The asymptotic stability of (1) is guaranteed by (78) and (79), based on the ideas presented in *Tissir*, *Hmamed* (1996).

To prove *finite time stability,* one should start with the solution of (1) with the given initial function (2) and the approach given in *Debeljkovic et al* (2011.a).

A detailed overview of different contributions in the area of finite time stability for particular classes of time delay and singular – descriptor time delay systems can be found in papers *Debbbeljkovic et al.* (2010.a,2010b, 2011.c, 2011.d, 2012.a, 2012.b)

APPENDIX A

Some additional results

Lemma 1. Let $Q(t)$ be an $n \times n$ characteristic matrix for autonomous system (1) with initial function (2), also continuous and differentiable in $[0,\ \tau]$ and zero elsewhere.

Define the following vector:

$$\mathbf{y}(t) = \mathbf{x}(t) + \int_0^{\tau} Q(t)\mathbf{x}(t-\theta)\,d\theta, \tag{A.1}$$

where the matrix $Q(t)$ satisfy the following matrix equation:

$$\dot{Q}(\theta) = \left(A_0 + Q(0)\right)\cdot Q(\theta)Q, \quad \theta \in [0,\ \tau], \tag{A.2}$$

with the boundary value:

$$Q(\tau) = A_1, \tag{A.3}$$

Lee, *Diant* (1981).

If

$$V(\mathbf{y}(t)) = \mathbf{y}^T(t)\mathbf{y}(t), \tag{A.4}$$

is the aggregation function for system (1), then

$$\dot{V}\left(\mathbf{y}(t)\right)=\mathbf{y}^{T}(t)(-R)\mathbf{y}(t), \qquad (A.5)$$

where:

$$-R=\left(A_0+Q(0)\right)^{T}+\left(A_0+Q(0)\right), \qquad (A.6)$$

The proof is omitted, for the sake of brevity and can be found in *Lee* and *Diant* (1981).

Theorem A.1 If λ_M is the maximum eigenvalue of the matrix $(-R)$ being defined by (A.6), then

$$\int_0^{\tau}\| Q(\theta)\mathbf{x}(t-\theta)\| d\theta \le \| Q(0)\| \int_0^{\tau} e^{\frac{\lambda_M}{2}\theta}\|\mathbf{x}(t-\theta)\| d\theta \quad (A.7)$$

Debeljkovic et al. (2001).

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Praktična stabilnost i stabilnost na konačnom vremenskom intervalu linearnih kontinualnih sistema sa čistim vremenskim kašnjenjem - klasičan i moderan pristup: Pregled rezultata

Ovaj rad daje detaljan pregled radova i rezultata autora ovog članka na polju neljapunovske stabilnosti (stabilnost na konačnom vremenskom intervalu, tehnička stabilnost, praktična stabilnost, krajnja stabilnost) posebne klase linearnih kontinualnih sistema sa čistim vremenskim kašnjenjem u stanju.

Ovaj pregled obuhvata period posle 2000.godine, pa sve do današnjih dana i ima snažnu nameru da predstavi glavene koncepte i doprinose koji su stvoreni u pomenutom periodu a koji su publikovani u međunarodnim časopisima ili prezentovani repektabilnim workshop-ovima na prestižnim međunarodnim konferencijama.

Ključne reči: **kontinualni sistem, sistem sa kašnjenjem, sistem na konačnom vremenskom intervalu, stabilnost sistema, neljapunovska stabilnost, praktična stabilnost.**

Стабильность в практике и стабильность на конечном интервале времени линейных непрерывных систем с запаздывающим аргументом – классический и современный подход: Обзор результатов

Эта статья содержит подробный обзор работ и результатов автора этой статьи на поле неляпуновской устойчивости (стабильность на конечном интервале времени, техническая стабильность, практическая стабильность, высокая стабильность) специального класса линейных непрерывных систем с запаздывающим аргументом в состоянии. Этот обзор охватывает период после 2000 года и по сей день и имеет сильное намерение представить основные понятия и взносы, которые созданы в этот период, а которые были опубликованы в авторитетных международных журналах или представлены на престижных международных конференциях, а некоторые из них и на престижных мировых семинарах.

Ключевые слова: **непрерывная система, система с запаздыванием, система на конечном интервале времени, устойчивость системы, неляпуновская стабильность, практическая стабильность.**

Stabilité pratique et stabilité sur l'intervalle temporelle finie des systèmes linéaires continus à délai temporel pur – approche classique et approche moderne: Tableau des résultats

Ce papier donne un tableau détaillé des travaux et des résultats de l'auteur de cet article réalisés dans le domaine de la stabilité de non Lyapunov (stabilité sur l'intervalle temporelle finie, stabilité technique, stabilité pratique, stabilité finale) de classe particulière des systèmes linéaires continus à délai temporel pur. Ce tableau comprend la période après l'an 2000 jusqu'à nos jours et son intension est de présenter les concepts principaux et les contributions réalisés pendant la période citée et publiés dans des revues internationales ou bien exposés aux ateliers de renom, lors des conférences internationales de prestige.

Mots clés: **système continu, système à délai, système sur l'intervalle temporelle finie, stabilité de système, stabilité de non Lyapunov, stabilité pratique.**