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Nonlinear Differential Equations in Current Research of System Nonlinear Dynamics

Marija Stamenković[1)]

This paper deals with basic nonlinear differential equations describing non-linear phenomena in the dynamics of systems with one or more degrees of freedom. It presents the most important results from the papers presented at the Non-linear Dynamics Mini Symposium held during the Congress of Serbian Society for Mechanics IConSSM 2011.

Key words: **nonlinear dynamics, nonlinear differential equations, nonlinear phenomena, nonlinear analysis, system dynamics.**

Introduction

LET us start with the list of the nonlinear differential equations describing non-linear oscillations presented in the classical books such as “Theory of Oscillations” by D.Rašković [12] and K.Hedrih (Stevanović), [1-2]. The asymptotic methods for obtaining approximations of the solutions of nonlinear differential equations are presented in monographs [4-11] written by Mitropolskiy. The series of characteristic nonlinear differential equations of non-linear dynamics in a mechanical engineering system abstracted into a theoretical model of nonlinear dynamics of systems with one degree of freedom is presented in the series of the papers [3,14] written by K.Hedrih (Stevanovic).

Out of the papers listed in References II [15-24] presented at the Non-linear Dynamics Mini Symposium Proceedings IConSSM 2011, Serbian Society of Mechanics, the author made a selection for a suitable presentation in this review paper. The equations serving as a basis for new equations are presented together with the newly obtained ones. The methods used for equation solving are given as well.

A list of selected characteristic nonlinear differential equations on the basis of non-linear oscillations

Differential equations for the case of non-linear free oscillations of a conservative system

One of the basic nonlinear differential equations describing the nonlinear dynamics of a conservative system with one degree of freedom is in the form [1-2, 4-11, 12]:

$$a\ddot{q} + F_e(q) = 0 \tag{2.1}$$

where: a is the coefficient of inertia, $F_e(q)$ is the restitution force which depends on the generalized coordinate q.

The equation can be written as:

$$\ddot{q} + k^2 f(q) = 0 \tag{2.2}$$

The previous ordinary nonlinear differential equation corresponds to the free nonlinear dynamics of a mechanical system with one degree of freedom in which the total mechanical energy is constant during the system dynamics.

Differential equations for the case of simple forced non-linear oscillations without a damping force

One of the basic nonlinear differential equations describing the nonlinear forced dynamics of a conservative system with one degree of freedom loaded by an external single frequency force is in the form [1-2, 4-11, 12]:

$$a\ddot{q} + F_e(q) = Q(t) = Q_0 \sin\Omega t \tag{2.3}$$

where $Q(t) = Q_0 \sin\Omega t$ external single frequency force with the amplitude Q_0 and the circular frequency Ω .

Differential equations for the case of simple forced non-linear oscillations with a nonlinear damping force

One of the basic nonlinear differential equations describing the nonlinear forced dynamics of a non conservative system with one degree of freedom loaded by an external single frequency force is in the form [1-2, 4-11, 12]:

$$a\ddot{q} + \Phi(\dot{q}) + cq = Q\sin\Omega t \tag{2.4}$$

where the damping force is a nonlinear function of the system velocity, and the problem is in many cases reduced to simple forced oscillations with a linear damping force or proportional to the square of the system velocity as it is determined by Jacobsen.

General form of the equations of free reo-linear (rheonomic) oscillations

One of the basic rheolinear differential equations describing the rheolinear free dynamics of a rheonomic system with one degree of freedom is in the form [1-2, 4-11, 12]:

$$\ddot{x} + P(t)\dot{x} + Q(t)x = 0 \text{ or } m\ddot{x} + b\dot{x} + cx = 0 \tag{2.5}$$

where: P, Q, m, b, c, are the continuous functions of the time t. This differential equation is with a coefficient as a function of time.

1) Mathematical Institute SANU, Kneza Mihaila 36/III, 11000 Belgrade, SERBIA

Mathieu's differential equation of the second order

A special case of the previous rheolinear differential equation is in the form of Mathieu's differential equation of the second order in the following form [1-2, 4-11, 12]:

$$\frac{d^2\phi}{d\tau^2}+(\lambda+\gamma\cos\tau)x=0 \tag{2.6}$$

or

$$\ddot{x}+(\lambda+\gamma\cos\tau)x=0 \tag{2.7}$$

where: $\tau=\Omega t$ is a variable, but $\lambda=g/\Omega^2=\omega^2/\Omega^2$ and $\gamma=a/l$ are its parameters.

Review of the characteristic equations of non-linear dynamics by the author's choice

Nonlinear differential equations of motion and the phase trajectory equation of the heavy mass particle dynamics along the rough parabolic, cycloid and circular line:

a) An ordinary nonlinear differential double equation of motion of the heavy mass particle nonlinear dynamics along the rough parabolic line is derived by K.Hedrih (Stvanovic) and presented in reference [3] in the following form:

$$\ddot{\phi}+(3tg\phi\pm\mu)\dot{\phi}^2+\frac{g\cos^3\phi}{p}(\sin\phi\pm\mu\cos\phi)=0 \tag{3.1}$$

In addition, the corresponding double equation of the phase trajectory is in the following form:

$$\dot{\phi}^2=\cos^6\phi\left(-\frac{g}{p\cos^2\phi}+Ce^{\mp2\mu\phi}\right) \tag{3.2}$$

and presents the first integral of the previous ordinary differential double equation of motion of the heavy mass particle nonlinear dynamics along the rough parabolic line.

b) Ordinary differential double equation of the heavy mass particle along the cycloid rough line is presented in reference [3] in the form:

$$\dot{\phi}^2=\cos^6\phi\left(-\frac{g}{p\cos^2\phi}+Ce^{\mp2\mu\phi}\right) \tag{3.3}$$

Also, the corresponding double equation of the phase trajectory is in the following form:

$$\begin{gathered}\dot{\phi}^2=-\frac{\left(\frac{g}{2R}\right)}{1+4\mu^2}\frac{1}{\cos^2\frac{\phi}{2}}\\ \left[(\pm3\mu)\sin\phi-\left(1-2\mu^2\right)\cos\phi+\frac{1+4\mu^2}{2}+Ce^{\mp2\mu\phi}\right]\end{gathered} \tag{3.4}$$

and presents the first integral of the previous ordinary differential double equation of motion of the heavy mass particle nonlinear dynamics along the rough cycloid line.

c) Ordinary differential double equation of the heavy mass particle along the rough circular line is presented in reference [3] in the form:

$$\ddot{\phi}\pm\dot{\phi}^2tg\alpha_0+\frac{g}{R\cos\alpha_0}\sin(\phi\pm\alpha_0)=0 \tag{3.5}$$

Also, the corresponding double equation of the phase trajectory is in the following form:

$$\begin{gathered}\dot{\phi}(\phi)^2=\frac{2g}{\left(1+4tg^2\alpha_0\right)R\cos\alpha_0}\\ \cdot\left[\cos(\phi\pm\alpha_0)-2tg\alpha_0\sin(\phi\pm\alpha_0)\right]+Ce^{\mp2\phi tg\alpha_0}\end{gathered} \tag{3.6}$$

and presents the first integral of the previous ordinary differential double equation of motion of the heavy mass particle nonlinear dynamics along the rough circular line.

The nonlinear differential equation of the heavy gyrorotor-disk self rotation in the case of coupled rotations around two orthogonal axes is in the following form (*see reference* [18])*:*

$$\ddot{\phi}_2+\Omega^2(\lambda-\cos\phi_2)\sin\phi_2+\Omega^2\psi\cos\phi_2=0 \tag{3.7}$$

and presents the equation of motion of the heavy gyrorotor disk with one degree of freedom and the coupled rotations when one component of rotation is programmed by a constant angular velocity.

The dynamics of Watt's regulator is described by this nonlinear differential equation.

The nonlinear differential equation of the relative motion of the heavy material particle along a circle which rotates around the skewly positioned axis with respect to the horizon, is in the following form (*see reference* [14]):

$$\begin{gathered}\ddot{\phi}+\Omega^2(\lambda-\cos\phi)\sin\phi-\Omega^2\varepsilon\cos\phi=\\ =\Omega^2\lambda ctg\alpha\cos\phi\cos\Omega t\end{gathered} \tag{3.8}$$

The nonlinear differential equation describes the equation of the motion of the heavy material particle along a circle rotating around the fixed axis. Based on a simple model, the nonlinear dynamics with one degree of freedom of oscillatory motion is obtained.

The Van der Pol non-linear differential equation is in the following form (*see reference* [20]):

d) Non-dimensional equation of the Van der Pol oscillator

$$\frac{d^2\xi}{d\tau^2}+\varepsilon f\left(\xi,\frac{d\xi}{d\tau}\right)+\mathrm{sgn}(\xi)|\xi|^\alpha=F\cos\Omega\tau \tag{3.9}$$

e) Differential equation which corresponds to the Van der Pol damping

$$f\left(\xi,\frac{d\xi}{d\tau}\right)=\left(\xi^2-1\right)\frac{d\xi}{d\tau} \tag{3.10}$$

where: ξ is the non-dimensional displacement; τ is the non-dimensional time; ε is a small constant, i.e. $\varepsilon << 1$; F and Ω are the magnitude and the frequency of the harmonic excitation.

The non-linear damping force defined by relation (3.10) corresponds to the Van der Pol damping which dissipates energy for large displacements and supplies energy to the system for small displacements. As such, it gives rise to limit cycle oscillations of free oscillators modeled by equations (3.9) and (3.10) with F=0.

The nonlinear equations of motion *presented in reference* [21] *in the form:*

$$M\ddot{u}+f_D(\dot{u})+f_S(u)=p(t) \tag{3.11}$$

This equation is obtained based on the dynamic equilibrium of the external, internal and inertia forces, where: $f_I(t) = M\ddot{u}$ is the inertia forces, $f_D(t) = f_D(\dot{u})$ the damping forces (nonlinear function), $f_S(t) = f_S(u)$ the elastic forces (nonlinear function) and *p(t)* are the external excitation forces; $u=u(t)$ is the displacement vector.

Equation (3.11) presents the basic equation of motion with which J.T.Katsikadelis's [21] new direct time integration method is presented for the solution of the equations of motion describing the dynamic response of structural linear and nonlinear multi-degree of freedom systems.

Analysis of characteristic equations in the basis of non-linear oscillations in reference [12]

As a primary equation for the case of free non-linear oscillations, equation (2.2) is given (see reference [12]). Based on it, the first integral is done as well as the law of motion in the form of:

$$\dot{q}(q) = v(q) = \mp k\sqrt{2\int_q^{q_0} f(\xi)d\xi} \tag{4.1}$$

$$t(q) = \frac{1}{k\sqrt{2}}\int_0^q \frac{d\eta}{\sqrt{J(q_0) - J(\eta)}} \tag{4.2}$$

This is the case when the function *f*(*q*) is given by an analytical expression.

For example, with the rectilinear harmonic oscillation *f*(*q*)=*q*=*x*, so *k*=*ω*, the law of motion is presented in the form:

$$t = \frac{1}{\omega}\int_0^x \frac{d\eta}{\sqrt{x_0^2 - \eta^2}} = \frac{1}{\omega}\arcsin\frac{x}{x_0} \tag{4.3}$$

The sine and parabolic characteristics are further considered.

The law of motion for sine characteristics (pendulum) according to (4.2) is presented in the form:

$$\begin{aligned} t &= \frac{1}{k\sqrt{2}}\int_0^{\phi} \frac{d\eta}{\sqrt{\cos\eta - \cos\phi_0}} = \\ &= \frac{1}{k}\int_0^{\theta} \frac{d\theta}{\sqrt{1-\varepsilon^2\sin^2\theta}} = \frac{1}{k}F(\varepsilon,\theta) \end{aligned} \tag{4.4}$$

where: φ is the generalized coordinate, ε is the module, θ is the amplitude.

The period of oscillating is also determined, depending on the module ε:

$$T = \frac{4}{k}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\varepsilon^2\sin^2\theta}} = \frac{4}{k}K(\varepsilon;\pi/2) \tag{4.5}$$

The equation of motion with parabolic characteristics where the movement is observed for a quarter of the period in reference [12]:

$$\ddot{q} + k^2 q^n = 0 \tag{4.6}$$

The law of motion and the period of oscillations for parabolic characteristics are:

$$t(q) = \frac{1}{k}\sqrt{\frac{n+1}{2q_0^{n-1}}}\int_0^{q/q_0} \frac{d\zeta}{\sqrt{1-\zeta^{n+1}}} \tag{4.7}$$

$$T = \frac{4}{k}\frac{1}{\sqrt{q_0^{n-1}}}\left[\sqrt{\frac{n+1}{2}}\int_0^1 \frac{d\zeta}{\sqrt{1-\zeta^{n+1}}}\right] = \frac{4}{k}\frac{1}{\sqrt{q_0^{n-1}}}\psi(n) \tag{4.8}$$

where: $\zeta = \eta/q_0$, and $\psi(n)$ are the introduced functions.

All this is illustrated in more details in D. Rašković [12].

Based on equation (2.3) (see [1-2, 4-11, 12]) which presents the differential equation of movement in the case of simple forced oscillations without the resistant force, the expression for the law of motion and the amplitude in the case of resonance is defined as:

$$q = C\sin\Omega t - \frac{1}{36}\left(\frac{k}{\Omega}\right)^2 \alpha_2 C^3 \sin 3\Omega t \tag{4.9}$$

$$C = \sqrt[3]{4h/3k^2\alpha_2} \tag{4.10}$$

While the amplitude for the case of forced oscillations with the resistant force is obtained based on (2.4) in the form:

$$C = N = \frac{h}{\sqrt{\left(k^2 - \Omega^2\right)^2 + 4\delta^2\psi^2 C^{2(n-1)}\Omega^{2n}}} \tag{4.11}$$

A general method for solving differential equation (2.5) is the integration using the chains of T. Pejovic [13].

When the quasi-elastic coefficient c from equation (2.5) changes linearly with time, then the differential equation of free oscillations without amortization is obtained in the form:

$$\ddot{x} + tx = 0 \tag{4.12}$$

and its solution is in the form of the Macloren's chain.

$$x = A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n + \cdots = \sum_{v=0}^{\infty} A_v t^v \tag{4.13}$$

A_v are constants that have to be determined in order to satisfy equation (4.12)

The general integral of equation (4.12) is presented in the form:

$$x = A_0 p_1(t) + A_1 p_2(t) \tag{4.14}$$

In the case of free amortized oscillations, the differential equation is in the form:

$$\ddot{x} + t\dot{x} + x = 0 \tag{4.15}$$

whose general solution is:

$$x = A_0 p_1(t) + A_1 p_2(t) \tag{4.16}$$

where A_0 and A_1 are the integration constants, and $p_1(t)$ and $p_2(t)$ are the particular integrals also presented in D. Raskovic [12].

The general solution of equation (2.6) is obtained in the form:

$$x = Ae^{\mu t} p_1(t) + Be^{-\mu t} p_2(t) \tag{4.17}$$

where A and B are the integration constants which have to satisfy the initial conditions, μ is the characteristic exponent, and $p_1(t)$ and $p_2(t)$ are the periodic functions of t, period 2π.

Based on Mathieu's equation (2.6) in D. Rašković [12], the differential equation of small oscillations is in the form:

$$l\ddot{\phi}+2l\dot{\phi}+g\phi=0 \tag{4.18}$$

as well as Mathieu's differential equation for the variable τ.

$$\ddot{T}+\left(\lambda+\gamma\cos\tau\right)T=0 \tag{4.19}$$

Analysis of the characteristic differential equations of the non-linear dynamics presented in References II

Based on the results of K.Hedrih (Stevanović) [3] *and the equations from* (3.1) *to* (3.6) *in the work by Jovic and Raičević in reference* [19]*, the analyses of the vibro-impact system dynamics of two heavy mass particles moving freely along non-ideal lines of rough curvilinear paths in the vertical plane in the shapes of parabola, cycloid and circle are performed. The oscillator is composed of one heavy mass particle with one degree of freedom of motion limited by one or two stabile elongation limiters.*

Using the previously listed nonlinear differential equations of motion, in the work by S.Jovic and V.Raičević in reference [19], the expressions for the phase trajectory equation in the phase plane $\left(\phi,\dot{\phi}\right)$ necessary for the energy analysis of the dynamics of vibro-impact systems together with the energy equation curves are illustrated. In addition, the methodology of the energy transfer investigation among the elements of the observed vibro-impact system is presented.

In this paper, only the expressions for the total mechanical energy $E_i(\varphi)$ are presented in the following forms:

$$\begin{aligned}E_i\left(\phi\right)&=E_{k_i}(\phi)+E_{p_i}(\phi)=\\&=\frac{1}{2}mp^2\Big(-\frac{g}{p\cos^2\phi}+C_ie^{\mp2\mu\phi}\Big)+\frac{1}{2}\frac{mgp}{\cos^2\phi}\end{aligned} \tag{5.1}$$

$$\begin{aligned}E_i\left(\phi\right)&=E_{k_i}(\phi)+E_{p_i}(\phi)=\\&=2mR^2\cos^2\frac{\phi}{2}\left(-\frac{\left(\dfrac{g}{2R}\right)}{1+4\mu^2}\frac{1}{\cos^2\dfrac{\phi}{2}}\left[(\pm3\mu)\sin\phi-\left(1-2\mu^2\right)\cos\phi+\frac{1+4\mu^2}{2}+C_ie^{\mp2\mu\phi}\right]\right)+mgR(1-\cos\phi)\end{aligned} \tag{5.2}$$

$$E_i\left(\phi\right)=E_{k_i}(\phi)+E_{p_i}(\phi)=\frac{1}{2}mR^2\left(\frac{2g}{\left(1+4tg^2\alpha_0\right)R\cos\alpha_0}\left[\cos\left(\phi\pm\alpha_0\right)-2tg\alpha_0\sin\left(\phi\pm\alpha_0\right)\right]+C_ie^{\mp2\phi tg\alpha_0}\right)+mgR(1-\cos\phi) \tag{5.3}$$

The graphs of visualization of the energy analysis of the observed vibro-impact system are also given. For every separate branch of the phase portrait, there is a graphic presentation of the alternation of F_N, P_μ, E_k, E_p and E from the initial moment of motion until the moment when the heavy mass particle returns into the equilibrium position.

Based on equation (3.7) *in the work by K.Hedrih (Stevanović) and Lj.Veljović* (*see reference* [18]), *the expression for the phase trajectory equation is obtained, and the nonlinear dynamics of the rotation of the heavy gyrorotor-disk around its shaft axis is possible to be presented with the phase portrait method.*

Nonlinear differential equation (3.7), based on the model shown in Fig.1, is obtained:

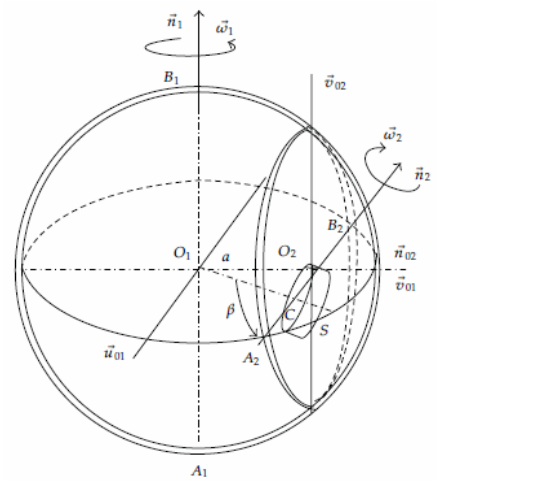

Figure 1. Model of a heavy gyrorotor with two component coupled rotations around the orthogonal axis without intersections

The forms of phase trajectories and their transformations by changes of the initial conditions, for different cases of the disk eccentricity and its skew angle as well as for different values of the orthogonal distance between the axes of component rotations, may present the character of nonlinear oscillations.

For that reason, it is necessary to find the first integral of differential equation (3.7) which is in the following form:

$$\begin{aligned}\dot{\phi}_2^2&=\dot{\phi}_{02}^2+2\Omega^2\left(\lambda\cos\phi_2-\frac{1}{2}\cos^2\phi_2+\psi\sin\phi_2\right)-\\&-2\Omega^2\left(\lambda\cos\phi_{02}-\frac{1}{2}\cos^2\phi_{02}+\psi\sin\phi_{02}\right)\end{aligned} \tag{5.4}$$

In the cited paper (see reference [18]), the trajectories of the vector rotators are presented with the series of three parameters transformations.

Nonlinear differential equation (3.8) *is presented in the paper by K.Hedrih (Stevanović)* (*see reference* [14]) *and in that paper a suitable linearized approximation and an optimal control of nonlinear dynamics is performed.*

Nonlinear differential equation (3.8), based on the model shown in Fig.2, is obtained:

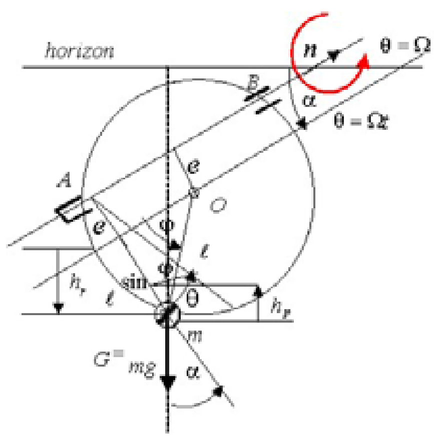

Figure 2. Motion of the heavy material particle along a circle rotating around the fixed axis. Simple model of the nonlinear dynamics.

As for linearized approximation around stationary points (which correspond to relative equilibrium positions), the paper considers special cases of the heavy material particle of free and forced dynamics in the case when the eccentricity e is equal to zero. Let us present these special cases:

1* For the case when $\lambda > 1$, small oscillations around the stable relative equilibrium position $\varphi = 0$, by using a corresponding linearization of the differential equation, are studied in the form (see reference [14]):

$$\ddot{\varphi} + \Omega^2(\lambda - 1)\varphi \approx \Omega^2 \lambda ctg\alpha \cos\Omega t \tag{5.5}$$

For that case, we can see that a phenomenon of a similar resonance is possible for the relation between the system parameters: $\lambda \approx 2$.

2* For the case when $\lambda < 1$, small oscillations around the stable relative equilibrium position $\varphi_s = \pm \arccos\lambda$ are studied by applying a corresponding linearization of the differential equation and a change of the generalized coordinate φ is made by following $\varphi_s + \varphi$. After linearization, the following linearized equation is obtained (see reference [14]):

$$\ddot{\varphi} + \Omega^2\left(1 - \lambda^2\right)\left[1 + \frac{\lambda ctg\alpha}{\sqrt{1-\lambda^2}}\cos\Omega t\right]\varphi \approx \Omega^2 \lambda ctg\alpha \cos\Omega t \tag{5.6}$$

From this linearized differential equation – differential equation with time dependent coefficients (Mathieu Hill type), we can see that the forced nonlinear dynamics around the equilibrium positions must be investigated taking into consideration the area of the stability or non-stability regimes.

Nonlinear differential equation (3.9) *presented in the paper by Z. Rakaric, and I. Kovacic* [20] *is given as the initial equations of the Van der Pol oscillator and it served for the following considerations:*

1. ***Motion of conservative oscillators,*** where the period of oscillations T and the frequency of the elliptic function are obtained, implies that there is a specific non-linear power-form relationship between the amplitude of free oscillations and the frequency of the elliptic function.

The period of oscillations:

$$T = 4\int_0^a \frac{d\xi}{\left|\frac{d\xi}{d\tau}\right|} = 4\sqrt{\frac{\alpha+1}{2}}\int_0^a \frac{d\xi}{\sqrt{|\overline{a}|^{\alpha+1} - |\xi|^{\alpha+1}}} = T_{ex}^{ND} a^{\frac{1-\alpha}{2}} \tag{5.7}$$

The frequency of the elliptic function:

$$\omega = a^{\frac{\alpha-1}{2}} \tag{5.8}$$

2. ***Motion of forced non-conservative oscillators*** where based on the equation of response a system of forced non-conservative oscillators motion is obtained:

$$\xi(\tau) = a(\tau) cn\left[\psi(\tau), m(\tau)\right] \tag{5.9}$$

During the detailed equation solution in the work, the final result is obtained:

$$\xi(\tau) = a(\tau) cn\left[\frac{2K}{\pi}(\Omega\tau + \varphi), m(\tau)\right] \tag{5.10}$$

which shows that the frequency of the free oscillation modeled by the Jacobi elliptic function is close to the excitation frequency ω=Ω.

3. ***Motion of forced oscillators with van der Pol damping*** where the expression for stability of the steady-state is obtained.

The expression for the steady–state value of φ is derived:

$$\tan\varphi = \frac{\pi P\varepsilon\left(d_1 - d_6 a^2\right)a^{\frac{\alpha-1}{2}}}{\bar{D}Ka_1\left[\left(\frac{\pi}{2K}\right)^2 a^{\alpha-1} - \Omega^2\right]} \tag{5.11}$$

The paper Z.Rakaric and I.Kovacic [20] represents oscillators with a non-negative real-power restoring force and considers the Van der Pol damping. The entrainment phenomenon has been investigated, when the frequency of the unforced limit cycle oscillations and the excitation frequency synchronize, so that the response occurs only at the excitation frequency. A new elliptic averaging method has been developed, which does not have any limitations regarding the value of the power of the non-linear restoring force, as this power can be any non-negative real number. The solution for motion is expressed in the form of the Jacobian elliptic function and has excellent accuracy with respect to the numerical solution.

Based on nonlinear differential equations (3.11) *from reference* [21]*, the problem consists of establishing the time history u=u(t) where* $t \in [0, T], T > 0$ *satisfying Eq.* (3.11) *with the initial conditions:*

$$u(0) = u_0,\ \dot{u}(0) = \dot{u}_0 \tag{5.12}$$

The forces $f_D(\dot{u})$ and $f_S(u)$ are in general non-linear functions of their arguments.

For linear problems they are given as $f_D(t) = C\dot{u}$ and $f_S(t) = Ku$ and equation (3.11) becomes

$$M\ddot{u} + C\dot{u} + Ku = p(t) \tag{5.13}$$

where M, C and K are the mass, damping and stiffness matrix of the structure, respectively.

In the paper by J.T.Katsikadelis [21], a system with one-degree-of-freedom and multi-degree-of-freedom systems with their stability of the numerical scheme and errors analysis and convergence are presented, using the analysis of accuracy in solving numerical examples.

A special highlight matrix for the one-degree-of-freedom systems and for multi-degree-of-freedom systems was investigated.

For the one-degree-of-freedom systems, it is in the form:

$$\begin{bmatrix} m & c & k \\ \beta c_1 & -h & 1 \\ -\beta c_2 & 1 & 0 \end{bmatrix}\begin{Bmatrix} q_n \\ \dot{u}_n \\ u_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -\alpha c_1 & 0 & 1 \\ \alpha c_2 & 1 & 0 \end{bmatrix}\begin{Bmatrix} q_{n-1} \\ \dot{u}_{n-1} \\ u_{n-1} \end{Bmatrix} + \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} p_n \tag{5.14}$$

For the multi-degree-of-freedom systems, it is in the following form:

$$\begin{bmatrix} M & C & K \\ \frac{c_1}{2}I & -hI & I \\ -\frac{c_1}{2}I & I & 0 \end{bmatrix}\begin{Bmatrix} q_n \\ \dot{u}_n \\ u_n \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{c_1}{2}I & 0 & I \\ \frac{c_1}{2}I & I & 0 \end{bmatrix}\begin{Bmatrix} q_{n-1} \\ \dot{u}_{n-1} \\ u_{n-1} \end{Bmatrix} + \begin{Bmatrix} I \\ 0 \\ 0 \end{Bmatrix} p_n \tag{5.15}$$

Matrixes (5.14) and (5.15) are obtained based on equations from (17) to (19) in the paper by J.T.Katsikadelis [21].

In this paper, the nonlinear initial value problem is given for multi-degree of freedom systems and is described as:

$$M\ddot{u} + F(\dot{u},u) = p(t) \tag{5.16}$$

$$u(0) = u_0 \,,\ \dot{u}(0) = \dot{u}_0 \tag{5.17}$$

From equation (5.16) for *t=0* it follows that the initial acceleration vector is:

$$q_0 = M^{-1}\left[p_0 - F(\dot{u}_0,u_0)\right],\ \ q_0 = \ddot{u} \tag{5.18}$$

Then it is applied for $t=t_n$ and the equation has been obtained in the following form:

$$Mq_n + F(\dot{u}_n,u_n) = p_n \tag{5.19}$$

For calculating $q_n, \dot{u}_n, u_n$, equation (5.19) and the equation that is presented in the following form are used (see reference [21]):

$$\begin{bmatrix} -hI & I \\ I & 0 \end{bmatrix}\begin{Bmatrix} \dot{u}_n \\ u_n \end{Bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{Bmatrix} \dot{u}_{n-1} \\ u_{n-1} \end{Bmatrix} + \\ + \begin{bmatrix} \dfrac{c_1}{2} I \\ -\dfrac{c_2}{2} I \end{bmatrix} q_n + \begin{bmatrix} -\dfrac{c_1}{2} I \\ \dfrac{c_2}{2} I \end{bmatrix} q_{n-1} \tag{5.20}$$

In this paper by J.T.Katsikadelis [21], a new direct time integration method is presented for the solution of the equations of motion describing the dynamic response of structural linear and nonlinear multi-degree of freedom systems. According to this principle, the system of the *N* coupled equations of motion, linear or non-linear, is replaced by a set of uncoupled linear single term quasi-static equations each of which includes only one unknown displacement and are subjected to appropriate unknown fictitious external loads. These fictitious loads are established numerically from the integral representation of the solution and the requirement that the equations of motion are satisfied at discrete times.

An analysis of a mechanical model of the human voice production systems is presented in the review paper written by L. Cvetićanin [15].

In this paper, the basic model of the vocal cords/vocal tracts and vocal folds is considered as a two mass non-linear oscillator system which is assumed to be the basic one for a mechanical description in voice production on the basis of listed references published in the world scientific literature.The corresponding mathematical model is a system of two coupled second order non-linear differential equations. Usually, this system of equations does not have an exact closed form solution and various analytical and numerical solving methods are applied. The solutions describe the self-excited vibrations of the mechanical elements of voice production. The influence of air flow in glottis is additionally modeled and included into the previously developed mechanical system.

Analyzing the corresponding mathematical models, it is evident that besides the self-excited oscillations of vocal cords some additional vibrations appear. The vibrations may be regular but also irregular like bifurcation and chaos. The numerical simulation gives the parameter values for proper and improper voice production. Based on the results given in the review, the objectives for future investigation into the matter are given.

In the work by L. Cveticanin [15], according to the obtained results, it is concluded that the mathematical models based on the physical model of human voice production give a very good qualitative description of the phenomenon; however, in spite of the fact that the clinically observed and measured parameters are used for modeling, the obtained results quantitatively differ from the real ones. This requires the improvement of the accuracy of the models.

Therefore, seven suggestions are given in the paper (see reference [15]).

In paper [17] *written by A.Hedrih, and K.Hedrih (Stevanović), a model of double DNA helix main chain forced vibrations is analytically investigated.*

The DNA transcription process is well described at the biochemical level. During transcription, double DNA interacts with transcription proteins; a part of double DNA is unzipped, and only one chain helix is used as a matrix for transcription. In order to understand better the DNA transcription process and its behavior from the biomechanical point of view, we consider double DNA (dDNA) as an oscillatory system that oscillates in forced regimes. In this paper, the analytical expressions of the forced oscillations of the dDNA helix chains are presented for both introduced models, an ideally elastic one as well as a fractional order model. On the basis of the previous results (DNA mathematical nonlinear models published by N. Kovaleva, L. Manevich in 2005 and 2007), and the multipendulum models where main chain subsystems of the double DNA helix are obtained, the analysis of the forced vibrations is done as a new result. There are different cases of the resonant state in one of the main chains, and there are no interactions between the main chains. The possibility of appearance of resonant regimes only in one of the two main chains is proved as well as the dynamical absorption under forced excitations of external frequency.

The geometrical aspects of nonholonomic mechanical systems are investigated by C. Frigioiu and published in paper [16].

In this paper, the author presents the nonholonomic mechanical systems studied from the geometric point of view using the Lagrange Geometry. The geometrization of nonholonomic mechanical systems is given using the geometry of tangent bundle and the Lagrange equations are obtained.

The dynamic systems admitting Morse functions that do not increase along trajectories with time are considered by R. M. Bulatović and M. Kažić and presented in paper [22] *entitled "On the degree of instability of mechanical systems".*

The main relations between the indices of inertia of these functions and the instability degrees of the equilibria are indicated, and these results are supplemented with several new statements. The results are applied to two classes of mechanical systems.

A. Obradović, S. Šalinić, O. Jeremić1, Z. Mitrović are the authors of paper [23] entitled "Brahistochronic motion of a variable mass system".

A mechanical system consisting of rigid bodies and material particles, some particles of which are with variable masses, is considered. Laws of variation of the masses of the points and relative velocity of particles separating from the points are well-known. The system is moving in an arbitrary field of known potential and nonpotential forces. Applying Pontryagin's Maximum Principle and the singular optimal control theory, the brachistochronic motion is determined. A two-point boundary value problem, due to the nonlinearity of equations in a general case, is needed to be solved using some of the numerical procedures. Here the Shooting method is used, where the missing boundary conditions are chosen to be the physical variables (velocity and mass). The field where they are found can be approximately estimated, which is not the case with the conjugate vector coordinates being of purely mathematical nature. The paper also presents the manner of brachistrochronic motion realization without the action of active control forces. It is realized by subsequent imposition of independent ideal holonomic mechanical constraints to the corresponding number of systems. The method is illustrated by an example of determining the brachistochronic motion of the system with three degrees of freedom and a method of its realization. The system consists of one rigid body to which two points of variable masses are attached, where the system is moving in the vertical plane. The brachistochronic motion is realized by the help of two ideal holonomic constraints.

V. Nikolić-Stanojević, Ć. Dolićanin, Lj. Veljović, M. Obradović are the authors of paper [24] *entitled "Dynamic models of buildings to mitigate fluctuations".*

A dynamic suspension system for high-rise buildings has a role to reduce the oscillations of the highest floors due to the effects of winds, earthquakes and other causes. It is a complex mechanical and mathematical problem. The suspension system, which can be active and passive, has to absorb and suppress all components of the forces that would lead to a reduced life of the structure or the disruption of comfort, stability and security. The modeling of the dynamic suspension system is reduced to the formation of physical and mathematical models that describe the real system most adequately. This paper presents a dynamic model for tall buildings. A numerical solution and appropriate comments are given.

Concluding remarks

The selected nonlinear differential equations in Chapters 2 and 3 are applicable as mathematical descriptions of nonlinear dynamics in real mechanical systems added to theoretical models for different research purposes presented here for the given authors and their papers. The papers from the mini-symposium Nonlinear dynamics organized by K.Hedrih (Stevanović), included in the program of IConSSM 2011 of the Serbian Society of Mechanics, are analysed. This paper highlights the most important nonlinear differential equations of the mechanical systems of motion of the observed real systems, the equations of the phase trajectories based on the graphs discussed, the period of oscillations, the frequency, the equation of the system response and the expressions for total mechanical energy.

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Nelinearne diferencijalne jednačine u savremenim istraživanjima nelinearne dinamike sistema

U radu su prikazane osnovne nelineare diferencijalne jednačine kojima se opisuju nelinearni fenomeni u dinamici sistema sa jednim stepenom ili sa više stepena slobode kretanja. Prikazani su najvažnji rezultati predstavljeni u radovima učesnika mini simpozijuma Nelinearna dinamika, koji je održan na Kongresu mehanike Srpskog društva za mehaniku IConSSM 2011.

Ključne reči: **nelinearna dinamika, nelinearne diferencijalne jednačine, nelinearni fenomeni, nelinearna analiza, dinamika sistema.**

Нелинейные дифференциальные уравнения в современных исследованиях по нелинейной динамике

В настоящей работе представлены основные нелинейные дифференциальные уравнения, описывающие нелинейные явления в динамике либо систем с одной степенью свободы, либо с несколькими степенями свободы. Наиболее важные результаты показаны в работах, представленных участниками небольшой конференции „Нелинейная динамика", которая состоялась в рамках Конгресса механики Сербского общества механики IConSSM 2011. года.

Ключевые слова: **нелинейная динамика, нелинейные дифференциальные уравнения, нелинейные явления, нелинейный анализ, динамика системы.**

Les équations différentielles non linéaires dans les recherches actuelles de la dynamique non linéaire de système

Dans ce papier on a présenté les équations différentielles non linéaires basiques par lesquelles on décrit les phénomènes non linéaires dans la dynamique de système à un ou plusieurs degrés de liberté de mouvement. On a cité les résultats les plus importants figurant dans les travaux des participants au mini symposium « Dynamique non linéaire » qui a eu lieu lors du Congrès de la mécanique de la Société serbe pour la mécanique IConSSM 2011.

Mots clés: **dynamique non linéaire, équations différentielles non linéaires, phénomènes non linéaires, analyse non linéaire, dynamique de système.**