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# The Stability of Linear Continuous Singular and Discrete Descriptor Time Delay Systems over the Finite Time Interval: An Overview – Part II Discrete Case

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**This paper gives a detailed overview of the work and the results of many authors in the area of the Non-Lyapunov stability of the particular class of linear systems.**
**This survey covers the period since 1985 up to nowadays and has a strong intention to present the main concepts and contributions made during the mentioned period throughout the world, published in the respectable international journals or presented at workshops or prestigious conferences.**

*Key words*: **linear system, continuous system, descriptive system, singular system, time delay system, discrete system, system stability.**

## Introduction

IT should be noticed that the characters of dynamic and static states in some systems must be considered at the same time. Singular systems (also referred to as degenerate, descriptor, generalized, differential - algebraic systems or semi – state) are those the dynamics of which is governed by a mixture of algebraic and differential equations.

In recent period, many scholars have paid much attention to singular systems and have obtained many good consequences. The complex nature of singular systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control.

It is well-known that singular systems have been one of the major research fields of the control theory. During the past three decades, singular systems have attracted much attention due to the comprehensive applications in economics as the *Leontief* dynamic model *Silva*, *Lima* (2003), in electrical *Cmpbell* (1980) and mechanical models *Muller* (1997), etc.

They also arise naturally as a linear approximation of systems models, or linear system models in many applications such as electrical networks, *aircraft dynamics*, neutral delay systems, chemical, thermal and diffusion processes, large-scale systems, interconnected systems, economics, optimization problems, feedback systems, robotics, biology, etc.

The discussion on singular systems started in 1974 with the fundamental paper of *Campbell et al*. (1974) and later continued with the anthological paper of *Luenberger* (1977). Since that time, considerable progress has been made in investigating such systems (see surveys, *Lewis* (1986) and *Dai* (1989) for linear singular systems, the first results for nonlinear singular systems in *Bajic* (1992)).

The investigation of the stability of singular systems has given many results in the sense of Lyapunov stability. For example, *Bajic* (1992) and *Zhang et al.* (1999) consedered the stability of linear time-varying descriptor systems.

*Discrete descriptor systems* are those systems the dynamics of which is covered by a mixture of *algebraic* and *difference* equations.

In that sense, the question of their stability on finite and infinite time intervals deserves great attention and is tightly connected with the questions of system solution uniqueness and existence.

Moreover, the question of consistent initial conditions, time series and solutions in the state space and the phase space based on the discrete fundamental matrix also deserves specific attention.

In this case, the concept of smoothness has little meaning but the idea of consistent initial conditions being these initial conditions $\mathbf{x}_0$ that generate the solution sequence $\left(\mathbf{x}(k): k \geq 0\right)$ has a physical meaning.

Some of these questions do not exist when *normal systems* are treated.

The problem of the investigation of time delay systems has been exploited over many years. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc.

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The existence of pure time lag, regardless whether it is present in the control or/and the state, may cause an undesirable system transient response, or even instability. Consequently, the problem of the stability analysis for this class of systems has been one of the main interests of many researchers. In general, the introduction of time delay factors makes the analysis much more complicated.

We must emphasize that there are a lot of systems that have the phenomena of time delay and singular characteristics simultaneously; such systems are known as *singular differential systems with time delay*.

These systems have many special characteristics. In order to describe them more exactly, to design them more accurately and to control them more effectively, considerable effort is demanded to investigate them. In recent references, authors have discussed such systems and got some consequences. But in the study of such systems, there are still many problems to be considered. When general time delay systems are considered, with the existing stability criteria, two ways of approach have generally been adopted.

Namely, one direction is to contrive the stability condition which does not include the information on the delay, and the other is the method which takes it into account. The former case is often called the delay - independent criterion and generally provides simple algebraic conditions. In that sense, the question of their stability over finite and infinite time interval deserves great attention.

Practical matters require that we concentrate not only on the system stability (e.g. in the sense of Lyapunov), but also on the bounds of the system trajectories.

A system could be stable but still completely useless because it possesses undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain state-space subsets which are defined *a priori* in a given problem.

Besides that, it is of particular significance to concern the behavior of dynamical systems only over a finite time interval.

These boundedness properties of system responses, i.e. the solution of system models, are very important from the engineering point of view. Realizing this fact, numerous definitions of the so-called technical and practical stability were introduced. Roughly speaking, these definitions are essentially based on the predefined boundaries for the perturbation of initial conditions and allowable perturbation of a system response. In the engineering applications of control systems, this fact becomes very important and sometimes crucial, for the purpose of characterizing possible system response deviations in a quantitative manner, in advance.

Thus, the analysis of these particular boundedness properties of solutions is an important step, which precedes the design of control signals, when finite time or practical stability control is concerned.

It should be noticed that, up to nowadays, there were no results concerning the problem of **non-Lyapunov stability**, when the discrete descriptor time delay systems are considered.

In the short overview that follows, we will present the results achieved only in the area of non – Lyapunov (practical and finite time stability of *linear*, *discrete* descriptor time delay systems (LDDTDS).

We will discuss neither contributions presented in papers concerned with the problem of robustness, robust stability, stabilization and robust stabilization of this class of systems with parameter uncertainties (see the list of references) nor other questions in connection with the stability of LCSTDS being necessarily, transformed by discrete Lyapunov – Krasovski functional, to the state space model in the form of differential – integral equations, *Fridman* (2001,2002).

Moreover, in the last few years, a numerous papers have been published in the area of linear discrete descriptor time delay systems, but this discussion is out of the scope of this paper. To be familiar with this matter, see the list of references.

The geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all solutions. This fact makes possible the construction of the Lyapunov stability theory even for the LCSTDS so that asymptotic stability is equivalent to the existence of symmetric, positive definite solutions to a weak form of Lyapunov discrete algebraic matrix equation incorporating conditions which refer to the time delay term.

This paper presents a collection of the results scattered in the literature and focuses on practical and finite time stability of linear discrete descriptor time delayed systems.

However, it is not a survey in a usual sense of the word.

The paper does not try to be exhaustive and cover the vast literature concerning this problem. Our object is more to convince the reader of a practical interest of the approach and of the number and the simplicity of the results it leads to.

For each aspect, we generally give in detail only one result, which is not necessarily the most complete or the most recent one, but is the one which seems to us the most representative and illustrative.

Another aim of this paper is to give a short overview of recently published results concerning non-Lyapunov stability of a particular class of linear *discrete time delay* systems, which we can treat as a joint descriptor and time delay systems.

## Basic Notation

| | |
|---|---|
| $\mathbb{R}$ | - Real vector space |
| $\mathbb{C}$ | - Complex vector space |
| $I$ | - Unit matrix |
| $F = \left(f_{ij}\right) \in \mathbb{R}^{n\times n}$ | - real matrix |
| $F^T$ | - Transpose of matrix $F$ |
| $F > 0$ | - Positive definite matrix |
| $F \geq 0$ | - Positive semi definite matrix |
| $\Re(F)$ | - Range of matrix $F$ |
| $\aleph(F)$ | - Null space (kernel) of matrix $F$ |
| $\lambda(F)$ | - Eigenvalue of matrix $F$ |
| $\sigma_{(\,)}(F)$ | - Singular value of matrix $F$ |
| $\rho(F)$ | - Spectral radius of matrix $F$ |
| $\lVert F\rVert$ | - Euclidean matrix norm $\lVert F \rVert = \sqrt{\lambda_{\max}\left(A^T A\right)}$ |
| $F^D$ | - Drazin inverse of matrix $F$ |
| $\Rightarrow$ | - Follows |
| $\mapsto$ | - Such that |

## A short history of practical and finite time stability: chronological overview

As we consider in general some particular classes of discrete time systems within the concept of practical and finite time stability, a chronological overview of some the most significance results in this area will be presented.

### *Discrete time systems*

A specific concept of discrete time systems, practical stability operating on the finite time interval was investigated by *Hurt* (1967) with a particular emphasis on the possibilities of error arising in the numerical treatment of results.

A finite time stability concept was extended to discrete time systems by *Michel and Wu* (1969) for the first time.

Practical stability or "*set stability*" throughout the estimation system trajectory behavior on finite time interval was given by *Heinen* (1970, 1971). He was the first who gave necessary and sufficient conditions for this concept of stability, using the Lyapunov approach based on the "*discrete Lyapunov functions*" application.

Even a more detailed analysis of these results considering different aspects of discrete time systems practical stability as well as the questions of their realization and controllability was given by *Weiss* (1972). The same problems were treated by *Weiss and Lam* (1973), who extended them to the class of nonlinear complex discrete systems.

Efficient sufficient conditions of finite time stability of linear discrete time systems expressed through norms and/or matrices were derived by *Weiss and Lee* (1971).

*Lam and Weiss* (1974) were the first who applied the so–called concept of "*final stability*" on discrete time systems the motions of which are scrolled within the time varying sets in the state space.

Some simple definitions connected to sets representing, difference equations or, at the same time, discrete time systems were given by *Shanholt* (1974).

Only the sufficient conditions are given by the established theorems. These results are based on the Lyapunov stability and can be used for a finite time stability concept, thus being mentioned here.

*Grippo and Lampariello* (1976) have generalized all; foregoing results and given the necessary and sufficient conditions of different concepts of finite time stability inspired by the definitions of practical stability and instability, earlier introduced by *Heinen* (1970).

The same authors applied the before–mentioned results in the analysis of "*large–scale systems*", *Grippo, Lampariello* (1978).

Practical stability with settling time was for the first time introduced by *Debeljković* (1979.a) in a connection with the analysis of different classes of linear *discrete time* systems, general enough to include time invariant and time varying systems, systems operating in free or forced regimes as well as the systems whose dynamic behavior is expressed through the so–called "*functional system matrix*". In the mentioned paper, the sufficient conditions of practical instability and a discrete version of a very well known *Bellman–Gronwall* lemma has also been derived.

Other papers, *Debeljković* (1979.b, 1980.a, 1980.b, 1983) deal with the same problems and mostly represent the basic results of the Ph. dissertation, *Debeljković* (1979.a).

For the particular class of discrete time systems with the functional system matrix, sufficient conditions have been derived in *Debeljković* (1993).

### *Discrete time delay systems*

The initial results have been published in a paper of *Debeljković*, *Aleksendrić* (2003), completely based on the discrete fundamental matrix of a system to be considered. It is well known that computing the discrete fundamental matrix is generally more difficult than to find the concrete solution of a system of retarded difference equations.

We can admit that these results represented the first extension of the concept of finite time and practical stability to the class of linear discrete time delay systems. In order to understand better serious problems that cause existing time delay in system dynamics, but also in forming corresponding criteria, a short recapitulation of some results derived for ordinary discrete time delay systems is presented in the sequel.

Some new further contributions have been made in the paper *Debeljković et al.* (2010). Namely, in this paper several delay independent criteria for stabilty of a particular class of discrete time delay systems were derived and expressed in a simple form in terms of the system matrices $A_0$ and $A_1$.

An idea of so-called attractive practical stability, which combines the asymptotic property of systems under consideration and the boundedness of its solutions, was also introduced, parallel to the classic approach to this problem which leads to the standard form of finite time stability.

For proposed *Definitions* two corresponding *Theorems* were established in the form of only sufficient conditions.

### *Discrete descriptor systems*

The analysis of particular bound properties of solutions is an important step, which precedes the design of control signals, when *finite time* or *practical stability* control is concerned.

In the context of practical stability for linear discrete descriptor systems, various results were first obtained in *Debeljkovic* (1985) and *Owens*, *Debeljkovic* (1986). These results included a practical instability as well.

*Owens*, *Debeljkovic* (1986) derived some new results in the area of practical and finite time stability for time-invariant, linear discrete descriptor systems. These results represent the sufficient condition for the stability of such systems and are based on Lyapunov-like functions and their properties on the sub-space of consistent initial conditions. In particular, these functions need not have properties of positivity in the whole state space and negative derivatives along system trajectories. In this paper, some results developed in the area of the non-Liapunov stability theory are extended to linear, time-invariant discrete descriptor systems. Some of them are mostly analogous to those derived in *Debeljkovic*, *Owens* (1985), for a continual-time case.

A strong motivation for the previous results was found in the paper *Owens*, *Debeljkovic* (1985) where the Lyapunov stability theory for both continuous and discrete-time linear singular (descriptor) systems was also investigated. The results are expressed directly in terms of the matrices $E$ and $A$ naturally occurring in the model thus avoiding the need to introduce algebraic transformations into the statement of the theorems. It is expected that the geometric approach will give more insight into the structural properties of singular systems and problems of consistency of initial conditions as well as to make possible a basis-free description of dynamic properties.

A general extension of these results to the particular classes of linear time invariant *regular* discrete descriptor systems was given in the papers of *Debeljković et al.* (1995, 1997, 1998, 2000) and *Bajić*, *Debeljković et al.* (1998) and for irregular discrete descriptor systems in *Dihovični*, *Debebeljković et al.* (1996) and *Bajić*, *Debeljković et al.* (1998)

***Discrete descriptor time delay systems***

To the best knowledge of the authors, there is no any paper treating the problem of finite time stability for discrete descriptor time delay systems.

Only one paper has been written in the context of practical and finite time stability for *continuous* singular time delay systems, see *Yang et al.* (2006).

## Some previuos basic results

***Discrete time systems***

The systems to be considered are governed by the vector difference equation:

$$\mathbf{x}(k+1) = A(k)\mathbf{x}(k), \tag{1}$$

$$\mathbf{x}(k+1) = A\,\mathbf{x}(k), \tag{2}$$

$$\mathbf{x}(k+1) = A\,\mathbf{x}(k) + \mathbf{f}(k), \tag{3}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is the state vector and the vector function satisfies: $\mathbf{f}(k) : \mathcal{K}_N \times \mathbb{R}^n \to \mathbb{R}$.

It is assumed also that $\mathbf{f}(k)$ satisfies the adequate smoothnees requirements so that a solution of (3) exists and is unique and continuous with respect to $k$ and the initial data and is bounded for all bounded values of its arguments.

Let $\mathbb{R}^n$ denote the state space of the systems given by (1–3) and $\|(\cdot)\|$ Euclidean norm.

The solutions of (1–3) are denoted by:

$$\mathbf{x}(k, k_0, \mathbf{x}_0) \equiv \mathbf{x}(k). \tag{4}$$

The discrete-time interval is denoted with $\mathcal{K}_N$, as a set of non–negative integers:

$$\mathcal{K}_N = \{\, k : k_0 \le k \le k_0 + k_N \,\}. \tag{5}$$

The quantity $k_N$ can be a positive integer or the symbol $+\infty$, so that finite time stability and practical stability can be treated simultaneously.

Let $V : \mathcal{K}_N \times \mathbb{R}^n \to \mathbb{R}$, so that $V(k, \mathbf{x}(k))$ is bounded and for which $\|\mathbf{x}(k)\|$ is also bounded.

Define the total difference of $V(k, \mathbf{x}(k))$ along the trajectory of the systems given by (1–3), with:

$$\Delta V(k, \mathbf{x}(k)) = V(k+1, \mathbf{x}(k+1)) - V(k, \mathbf{x}(k)). \tag{6}$$

For time–invariant sets it is assumed: $\mathcal{S}_{(\ )}$ is a bounded, open set. The closure and boundary of $\mathcal{S}_{(\ )}$ are denoted by $\bar{\mathcal{S}}_{(\ )}$ and $\partial\mathcal{S}_{(\ )}$, respectively, so: $\partial\mathcal{S}_{(\ )} = \bar{\mathcal{S}}_{(\ )} \setminus \mathcal{S}_{(\ )}$.

$\bar{\mathcal{S}}^c_{(\ )}$ denotes the complement of $\mathcal{S}_{(\ )}$.

Let $\mathcal{S}_\beta$ be a given set of all allowable states of the system for $\forall k \in \mathcal{K}_N$.

Set $\mathcal{S}_\alpha$, $\mathcal{S}_\alpha \subset \mathcal{S}_\beta$ denotes the set of all allowable initial states and $\mathcal{S}_\varepsilon$ the corresponding set of allowable disturbances.

Sets $\mathcal{S}_\alpha$, $\mathcal{S}_\beta$ are connected and a priori known.

$\lambda(\ )$ denotes the eigenvalues of the matrix $(\ )$.

$\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and the minimum of the eigenvalue, respectively.

**Stability Definitions**

**Definition 1** A system, given by (1 or 2), is *finite time stable* with respect to $\{k_0, \mathcal{K}_N, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$, if and only if:

$$\|\mathbf{x}(k_0)\|^2 = \|\mathbf{x}_0\|^2 < \alpha,$$

implies:

$$\|\mathbf{x}(k)\|^2 < \beta, \quad \forall k \in \mathcal{K}_N.$$

**Definition 2** A system, given by (3), is *finite time stable* with respect to $\{k_0, \mathcal{K}_N, \mathcal{S}_\alpha, \mathcal{S}_\beta\}$, if and only if:

$$\|\mathbf{x}_0\|^2 < \alpha \quad \wedge \quad \|\mathbf{f}(k, \mathbf{x}(k))\| \le \varepsilon, \quad \forall k \in \mathcal{K}_N,$$

implies:

$$\|\mathbf{x}(k)\|^2 < \beta, \quad \forall k \in \mathcal{K}_N.$$

**Definition 3** A system given by (1 or 2) is *practically unstable* with respect to $\{k_0, \mathcal{K}_N, \alpha, \beta, \ \|(\cdot)\|^2\}$, $\alpha < \beta$, if there is:

$$\|\mathbf{x}_0\|^2 < \alpha, \quad k = k^* \in \mathcal{K}_N,$$

so that the next condition is fulfilled:

$$\|\mathbf{x}(k^*)\|^2 \ge \beta, \quad \alpha < \beta.$$

**Definition 4** A system given by (3), is *practically unstable* with respect to $\{k_0, \mathcal{K}_N, \alpha, \beta, \varepsilon \ \ \|(\cdot)\|^2\}$, $\alpha < \beta$, if there is:

$$\|\mathbf{x}_0\|^2 < \alpha \quad \wedge \quad \|\mathbf{f}(k, \mathbf{x}(k))\| \le \varepsilon, \quad \forall k \in \mathcal{K}_N,$$

so that the next condition is fulfilled:

$$\|\mathbf{x}(k^*)\|^2 \ge \beta, \quad \alpha < \beta.$$

**Stability theorems**

**Theorem 1** A system, given by (1), is *practically stable* with respect to $\{k_0, \mathcal{K}_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if the following conditions are satisfied:

$$\prod_{j=k_0}^{j=k_0+k-1} \lambda_{\max}(j) \le \frac{\beta}{\alpha}, \quad \forall k \in \mathcal{K}_N, \tag{7}$$

*Debeljković* (2001).

**Theorem 2** A system, given by (3), is *practically stable* with respect to $\left\{k_0, K_N, \alpha, \beta, \varepsilon^*, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if the next conditions are fulfilled:

$$\lambda_{\max}^{0.5k} + k \cdot \varepsilon^* \cdot \lambda_{\max}^{0.5(k-1)} \le \sqrt{\frac{\beta}{\alpha}}, \quad \forall k \in K_N \tag{8}$$

*Debeljković* (2001).

**Theorem 3** A system, given by (1), is *practically unstable* with respect to $\left\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if there exists a real, positive number $\delta$, $\delta \in \left]0, \alpha\right[$ and the time instant $k$, $k = k^* : \exists!\left(k^* > k_0\right) \in K_N$ for which the next condition is fulfilled:

$$\prod_{j=k_0}^{j=k_0+k^*-1} \lambda_{\min}(j) > \frac{\beta}{\alpha}, \quad k^* \in K_N . \tag{9}$$

*Debeljković* (2001).

**Theorem 4** A system given by (3), *is practically unstable* with respect to $\left\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if there is a real, positive number $\delta$ and $\varepsilon_0$, such that: $\delta < \|\mathbf{x}_0\|^2 < \alpha$ and $\varepsilon_0 < \|\mathbf{f}(k)\| < \varepsilon$, $\forall k \in K_N$ and the time instant $k$, $k = k^* : \exists!\left(k^* > k_0\right) \in K_N$ such that the next condition is fulfilled:

$$\left| \sqrt{\delta}\lambda_{\min}^{0,5k^*} - k^* \varepsilon \lambda_{\min}^{0,5\left(k^*-1\right)} \right| > \sqrt{\beta}, \tag{10}$$
$$k^* \in K_N .$$

*Debeljković* (2001).

*Discrete time delay systems*

Consider a linear discrete system with state delay, described by

$$\mathbf{x}(k+1) = A_0 \mathbf{x}(k) + A_1 \mathbf{x}(k-1), \tag{11a}$$

With a known vector valued function of the initial conditions:

$$\mathbf{x}(k_0) = \boldsymbol{\psi}(k_0), \quad -1 \le k_0 \le 0, \tag{11b}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is a state vector and with the constant matrices $A_0$ and $A_1$ of appropriate dimensions.

Time delay is constant and equals one.

For some other purposes, the state delay equation can be represented in the following way:

$$\mathbf{x}(k+1) = A_0 \mathbf{x}(k) + \sum_{j=1}^{M} A_j \mathbf{x}\left(k - h_j\right), \tag{12a}$$

$$\mathbf{x}(\vartheta) = \boldsymbol{\psi}(\vartheta), \quad \vartheta \in \{-h, -h+1, \dots, 0\}, \tag{12b}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$, $A_j \in \mathbb{R}^{n \times n}$, $j$ = 1, 2, $h$ – is an integer representing system time delay and $\boldsymbol{\psi}(\cdot)$ is an a priori known vector function of the initial conditions as well.

**Stability definitions**

**Definition 5** A system, given by (11), is *attractive practically stable* with respect to $\left\{k_0, K_N, S_\alpha, S_\beta\right\}$, if and only if:

$$\|\mathbf{x}(k_0)\|_{A_0^T P A_0}^2 = \|\mathbf{x}_0\|_{A_0^T P A_0}^2 < \alpha ,$$

implies:

$$\|\mathbf{x}(k)\|_{A_0^T P A_0}^2 < \beta, \quad \forall k \in K_N ,$$

with a property that:

$$\lim_{k \to \infty} \|\mathbf{x}(k)\|_{A_0^T P A_0}^2 \to 0 .$$

**Definition 6** A system, given by (11), is *practically stable* with respect to $\left\{k_0, K_N, S_\alpha, S_\beta\right\}$, if and only if:

$$\|\mathbf{x}_0\|^2 < \alpha, ,$$

implies:

$$\|\mathbf{x}(k)\|^2 < \beta, \quad \forall k \in K_N .$$

**Definition 7** A system given by (11), is *atractive practically unstable* with respect to $\left\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if for:

$$\|\mathbf{x}_0\|_{A_0^T P A_0}^2 < \alpha,$$

there exists a moment: $k = k^* \in K_N$, so that the next condition is fulfilled:

$$\left\|\mathbf{x}\left(k^*\right)\right\|_{A_0^T P A_0}^2 \ge \beta$$

with a property that:

$$\lim_{k \to \infty} \|\mathbf{x}(k)\|_{A_0^T P A_0}^2 \to 0 .$$

**Definition 8** A system given by (11), is *practically unstable* with respect to $\left\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if for:

$$\|\mathbf{x}_0\|^2 < \alpha,$$

there exists a moment: $k = k^* \in K_N$, such that the next condition is fulfilled:

$$\left\|\mathbf{x}\left(k^*\right)\right\|^2 \ge \beta ,$$

for some $k = k^* \in K_N$.

**Definition 9** A linear discrete time delay system, given by (12.a), is *finite time stable* with respect to $\{\alpha, \beta, k_0, k_N, \|(\cdot)\|\}$, $\alpha \le \beta$, if and only if for every trajectory $\mathbf{x}(k)$ satisfying the initial function, given by (12b), such that:

$$\|\mathbf{x}(k)\| < \alpha , \quad \forall\, k = 0, -1, -2, \cdots, -N ,$$

implies:

$$\left\|\mathbf{x}(k)\right\|^2 < \beta \,, \ \forall\, k \in K_N \,,$$

*Aleksendrić* (2002), *Aleksendrić, Debeljković* (2002), *Debeljković, Aleksendrić* (2003).

**Stability theorems**

**Theorem 5** A linear discrete time delay system, given by (3.2), is *finite time stable* with respect to $\left\{\alpha,\beta,M,N,\left\|(\cdot)\right\|^2\right\}$, $\alpha<\beta$, $\alpha,\beta\in\mathbb{R}_+$, if it is sufficient that:

$$\left\|\Phi(k)\right\| < \frac{\beta}{\alpha}\cdot\frac{1}{1+\sum_{j=1}^{M}\left\|A_j\right\|}, \quad \forall k = 0,1,\cdots,N., \tag{13}$$

*Aleksendrić* (2002), *Aleksendrić, Debeljković* (2002), *Debeljković, Aleksendrić* (2003).

**Theorem 6** A system given by (11), with $\det A_1 \neq 0$, is *attractive practically stable* with respect to $\left\{k_0,K_N,\alpha,\beta,\left\|(\cdot)\right\|^2\right\}$, $\alpha<\beta$, if the following condition is satisfied:

$$\bar{\lambda}_{\max}^{\frac{1}{2}k}(\ ) < \frac{\beta}{\alpha}, \quad \forall k \in K_N \,, \tag{14}$$

where:

$$\begin{aligned}&\bar{\lambda}_{\max}(\ ) = \\ &= \max\left\{\mathbf{x}^T(k)A_1^T P A_1\mathbf{x}(k) : \mathbf{x}^T(k)A_0^T P A_0\mathbf{x}(k) = 1\right\}\end{aligned} \tag{15}$$

and if there exists $P = P^T > 0$, being the solution of:

$$2A_0^T P A_0 - P = -Q \,, \tag{16}$$

where $Q = Q^T > 0$, *Debeljković et al* (2010).

**Theorem 7** Suppose the matrix $\left(I - A_1^T A_1\right) \geq 0$.

A system given by (11) is *finite time stable* with respect to $\left\{k_0,K_N,\alpha,\beta,\left\|(\cdot)\right\|^2\right\}$, $\alpha<\beta$, if the following condition is satisfied:

$$\lambda^k{}_{\max}(\ ) < \frac{\beta}{\alpha}, \quad \forall k \in K_N \,, \tag{17}$$

where:

$$\lambda_{\max}(\ ) = \lambda_{\max}\left(A_0^T\left(I - A_1^T A_1\right)A_0\right), \tag{18}$$

*Debeljković et al* (2010).

**Theorem 8** Suppose the matrix $\left(I - A_1^T A_1\right) \geq 0$.

A system given by (11) is *practically unstable* with respect to $\left\{k_0,K_N,\alpha,\beta,\left\|(\cdot)\right\|^2\right\}$, $\alpha<\beta$, if there exists a real, positive number $\delta$, $\delta\in\left]0,\alpha\right[$ and the time instant $k$, $k=k^*: \exists!\left(k^*>k_0\right)\in K_N$ for which the next condition is fulfilled:

$$\lambda_{\min}^{k^*} > \frac{\beta}{\delta}, \quad k^* \in K_N \,. \tag{19}$$

*Debeljković et al* (2010).

## Non-lyapunov stability of linear discrete descriptor time delay systems

Consider a linear discrete descriptor system with state delay, described by

$$E\,\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1)\,, \tag{20a}$$

$$\mathbf{x}(k_0) = \boldsymbol{\varphi}(k_0), \quad -1 \leq k_0 \leq 0\,, \tag{20b}$$

where $\mathbf{x}(k)\in\mathbb{R}^n$ is a state vector.

The matrix $E\in\mathbb{R}^{n\times n}$ is a necessarily singular matrix, with the property $rank\,E = r < n$ and with the matrices $A_0$ and $A_1$ of appropriate dimensions.

For a LDDTDS (20), we present the following definitions taken from *Xu et al*. (2004).

**Definition 10** The LDDTDS is said to be *regular* if $\det\left(z^2E - zA_0 - A_1\right)$ is not identical to zero.

**Definition 11** The LDDTDS is said to be *causal* if it is *regular* and

$$\deg\left(z^n\det\left(zE - A_0 - z^{-1}A_1\right)\right) = n + rang\,E\,.$$

**Definition 12** The LDDTDS is said to be *stable* if it is *regular* and $\rho\left(E,A_0,A_1\right)\subset D(0,1)$, where

$$\rho\left(E,A_0,A_1\right) = \left\{z \mid \det\left(z^2E - zA_0 - A_1\right) = 0\right\}.$$

**Definition 13** The LDDTDS is said to be *admissible* if it is *regular*, *causal* and *stable*.

**Stability definitions**

**Definition 14** A *causal* system (20) is $E$-stable if for any $\varepsilon>0$, there always exists a positive $\delta$ such that

$$\left\|E\mathbf{x}(k)\right\| < \varepsilon\,,$$

when

$$\left\|E\mathbf{x}_0\right\| < \delta\,.$$

*Liang* (2000).

**Definition 15** A *causal* system (20) is $E$-asymptotically stable if system (20) is $E$-stable and

$$\lim_{k\to+\infty} E\mathbf{x}(k) = 0\,.$$

*Liang* (2000).

**Definition 16** A *causal* system, given by (20), is *practically stable* with respect to $\left\{k_0,K_N,S_\alpha,S_\beta\right\}$, if and only if $\forall\mathbf{x}_0\in\mathcal{W}^*_{dis}$ satisfying:

$$\left\|\mathbf{x}_0\right\|^2_{E^TE} < \alpha\,,$$

implies:

$$\left\|\mathbf{x}(k)\right\|^2_{E^TE} < \beta, \quad \forall k\in K_N\,,$$

$\mathcal{W}^*_{dis}$ being the subspace of consistent initial conditions.

**Definition 17** A *causal system* given by (20), is *practically unstable* with respect to $\left\{k_0, \mathcal{K}_N, \alpha, \beta, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if and only if $\exists \mathbf{x}_0 \in \mathcal{W}^*_{dis}$ such that:

$$\|\mathbf{x}_0\|^2_{E^T E} < \alpha,$$

there exists some $k^* \in \mathcal{K}_N$, such that the next condition is fulfilled:

$$\left\|\mathbf{x}\left(k^*\right)\right\|^2_{E^T E} \geq \beta,$$

for some $k^* \in \mathcal{K}_N$.

**Definition 18** A *causal* system, given by (20), is *attractive practically stable* with respect to $\left\{k_0, \mathcal{K}_N, \mathcal{S}_\alpha, \mathcal{S}_\beta\right\}$, if and only if $\forall \mathbf{x}_0 \in \mathcal{W}^*_{dis}$ satisfying:

$$\left\|\mathbf{x}(k_0)\right\|^2_{G=E^T P E} = \left\|\mathbf{x}_0\right\|^2_{G=E^T P E} < \alpha,$$

implies:

$$\left\|\mathbf{x}(k)\right\|^2_{G=E^T P E} < \beta, \quad \forall k \in \mathcal{K}_N,$$

with property that:

$$\lim_{k \to \infty} \left\|\mathbf{x}(k)\right\|^2_{G=E^T P E} \to 0,$$

$\mathcal{W}^*_{dis}$ being the subspace of consistent initial conditions.

**Remark 1** The singularity of the matrix $E$ will ensure that solutions to (20) exist for only a special choice of $\mathbf{x}_0$.

In *Owens*, *Debeljković* (1985) the subspace of $\mathcal{W}^*_{dis}$ of consistent initial conditions is shown to be the limit of the nested subspace algorithm:

$$\begin{aligned} &\mathcal{W}^*_{dis\,0} = \mathbb{R}^n \\ &\vdots \\ &\mathcal{W}^*_{dis(k+1)} = A_0^{-1}\left(E \mathcal{W}^*_{dis\,(k)}\right)_{A_1=0}, \quad k \geq 0 \end{aligned} \tag{21}$$

Moreover, if $\mathbf{x}_0 \in \mathcal{W}^*_{dis}$ then $\mathbf{x}(k) \in \mathcal{W}^*_{dis}, \ \forall k \geq 0$ and $\left(\lambda E - A_0\right)_{A_1=0}$ is invertible for some $\lambda \in \mathbb{R}$ (a condition for uniqueness), then:

$$\mathcal{W}^*_{dis} \cap \aleph(E) = \{0\}. \tag{22}$$

**Remark 2** Note that here $G = E^T P E$, where $P = P^T > 0$ is the arbitrary matrix.

Note, also, that (23) implies that:

$$\|\mathbf{x}(k)\|_G = \sqrt{\mathbf{x}^T(k) G \mathbf{x}(k)}, \tag{23}$$

is a norm on $\mathcal{W}^*_{dis}$.

**Remark 3** We will also need the following *Definitions* of the smallest, respectively the largest eigenvalues of the matrix $R = R^T$, with respect to the subspace of the consistent initial conditions $\mathcal{W}^*_{dis}$ and the matrix $G$.

**Proposition 1** If $\mathbf{x}^T(t) R \mathbf{x}(t)$ is a quadratic form on $\mathbb{R}^n$, then it follows that there are numbers $\lambda_{\min}(R)$ and $\lambda_{\max}(R)$ satisfying: $-\infty \leq \lambda_{\min}(R) \leq \lambda_{\max}(R) \leq +\infty$, such that:

$$\begin{aligned} \lambda_{\min}(\Xi) \leq \frac{\mathbf{x}^T(k) R \mathbf{x}(k)}{\mathbf{x}^T(k) G \mathbf{x}(k)} \leq \lambda_{\max}(R), \\ \forall \mathbf{x}(k) \in \mathcal{W}_{dis} \setminus \{0\} \end{aligned} \tag{24}$$

with the matrix $R = R^T$ and the corresponding eigenvalues:

$$\begin{aligned} &\lambda_{\min}\left(R, G, \mathcal{W}^*_{dis}\right) = \\ &= \min\left\{ \begin{aligned} \mathbf{x}^T(k) R \mathbf{x}(k): \ \mathbf{x}(k) \in \mathcal{W}^*_{dis} \setminus \{0\}, \\ \mathbf{x}^T(k) G \mathbf{x}(k) = 1 \end{aligned} \right\}, \end{aligned} \tag{25}$$

$$\begin{aligned} &\lambda_{\max}\left(R, G, \mathcal{W}^*_{dis}\right) = \\ &= \max\left\{ \begin{aligned} \mathbf{x}^T(k) R \mathbf{x}(k): \ \mathbf{x}(k) \in \mathcal{W}^*_{dis} \setminus \{0\}, \\ \mathbf{x}^T(k) G \mathbf{x}(k) = 1 \end{aligned} \right\} \end{aligned} \tag{26}$$

Note that $\lambda_{\min} > 0$ if $R = R^T > 0$.

**Stability Theorems**

**Theorem 9** Suppose matrix $\left(A_1^T A_1 - E^T E\right) > 0$. *Causal s*ystem given by (20), is *finite time stable* with

respect to $\left\{k_0, \mathcal{K}_N, \alpha, \beta, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if there exist a positive real number $p$, $p > 1$, such that:

$$\begin{aligned} \|\mathbf{x}(k-1)\|^2_{A_1^T A_1} < p^2 \|\mathbf{x}(k)\|^2_{A_1^T A_1} \\ \forall k \in \mathcal{K}_N, \ \forall \mathbf{x}(k) \in \mathcal{S}_\beta, \ \forall \mathbf{x}(k) \in \mathcal{W}^*_{d,k} \setminus \{0\} \end{aligned} \tag{27}$$

and if the following condition is satisfied:

$$\bar{\lambda}^k_{\max}(\ ) < \frac{\beta}{\alpha}, \quad \forall k \in \mathcal{K}_N, \tag{28}$$

where:

$$\begin{aligned} \bar{\lambda}_{\max}(\ ) = \bar{\lambda}_{\max}\{\mathbf{x}^T(k) A_0^T (I - A_1 (A_1^T A_1 - E^T E)^{-1} A_1^T \\ + p^2 A_1^T A_1) A_0 \mathbf{x}(k), \ \mathbf{x}(k) \in \mathcal{W}_{d,k^*}, \ \mathbf{x}^T(k) E^T E \mathbf{x}(k) = 1\}. \end{aligned}$$

(29)

*Debeljković et al* (2011).

**Proof.** Define:

$$V(\mathbf{x}(k)) = \mathbf{x}^T(k)\mathbf{x}(k) + \mathbf{x}^T(k-1)\mathbf{x}(k-1). \tag{30}$$

Let $\mathbf{x}_0$ be an arbitrary consistent initial condition and $\mathbf{x}(k)$ the resulting system trajectory.

The backward difference $\Delta V(\mathbf{x}(k))$ along the trajectories of the system, yields:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) = \mathbf{x}^T(k)\left(A_0^T A_0 - E^T E + I\right)\mathbf{x}(k) \\ + 2\mathbf{x}^T(k)\left(A_0^T A_1\right)\mathbf{x}(k-1) + \mathbf{x}^T(k-1)\left(A_1^T A_1 - I\right)\mathbf{x}(k-1) \end{aligned} \tag{31}$$

From (31) one can get:

$$\begin{aligned} \mathbf{x}^T(k+1) E^T E \mathbf{x}(k+1) = \mathbf{x}^T(k)\left(A_0^T A_0\right)\mathbf{x}(k) \\ + 2\mathbf{x}^T(k)\left(A_0^T A_1\right)\mathbf{x}(k-1) + \mathbf{x}^T(k-1)\left(A_1^T A_1\right)\mathbf{x}(k-1) \end{aligned} \tag{32}$$

Using the very well known inequality, with particular choice:

$$\mathbf{x}^T(k)\Gamma\mathbf{x}(k)=\mathbf{x}^T(k)\left(A_1^TA_1-E^TE\right)\mathbf{x}(k)\geq 0 \\ \mathbf{x}(k)\in\mathcal{W}_{d,k^*},\ \forall\mathbf{x}(k)\in\mathcal{S}_\beta,\forall k\in\mathcal{K}_N \quad , \tag{33}$$

it can be obtained:

$$\begin{aligned}&\mathbf{x}^T(k+1)E^TE\,\mathbf{x}(k+1)\leq\mathbf{x}^T(k)A_0^TA_0\mathbf{x}(k)\\&-\mathbf{x}^T(k)A_0^TA_1\left(A_1^TA_1-E^TE\right)^{-1}A_1^TA_0\mathbf{x}(k)\\&+\mathbf{x}^T(k-1)\left(2A_1^TA_1-E^TE\right)\mathbf{x}(k-1)\end{aligned}\tag{34}$$

Moreover, since:

$$\|\mathbf{x}(k-1)\|_{E^TE}^2\geq 0,\ \forall k\in\mathcal{K}_N,\ \ \forall\mathbf{x}(k)\in\mathcal{W}_{d,k}^*\backslash\{0\} \tag{35}$$

and using assumption (27) it is clear that (34), reduces to:

$$\begin{aligned}&\mathbf{x}^T(k+1)E^TE\,\mathbf{x}(k+1)\\&<\mathbf{x}^T(k)A_0^T\left(I-A_1\left(A_1^TA_1-E^TE\right)^{-1}A_1^T+2p^2I\right)A_0\mathbf{x}(k)\\&<\bar{\lambda}_{\max}(\ )\mathbf{x}^T(k)E^TE\,\mathbf{x}(k)\end{aligned}\tag{36}$$

where:

$$\begin{aligned}&\bar{\lambda}_{\max}(\ )=\{\mathbf{x}^T(k)A_0^T\Upsilon A_0\mathbf{x}(k),\\&\mathbf{x}(k)\in\mathcal{W}_{dis}^*,\ \mathbf{x}^T(k)E^TE\,\mathbf{x}(k)=1\}.\ \ .\\&\Upsilon=\left(I-A_1\left(A_1^TA_1-E^TE\right)^{-1}A_1^T+2p^2I\right)\end{aligned}\tag{37}$$

Following the procedure from the previous section, it can be written:

$$\ln\mathbf{x}^T(k+1)E^TE\mathbf{x}(k+1)-\ln\mathbf{x}^T(k)E^TE\mathbf{x}(k)<\ln\bar{\lambda}_{\max}(\ ).\tag{38}$$

By applying the summing $\sum_{j=k_0}^{k_0+k-1}$ on both sides of (38) for $\forall k\in\mathcal{K}_N$, one can obtain:

$$\begin{aligned}&\ln\mathbf{x}^T(k_0+k)E^TE\mathbf{x}(k_0+k)\leq\ln\prod_{j=k_0}^{k_0+k-1}\bar{\lambda}_{\max}(\ )\\&\leq\ln\bar{\lambda}_{\max}^k(\ )+\ln\mathbf{x}^T(k_0)E^TE\,\mathbf{x}(k_0),\quad\forall k\in\mathcal{K}_N\end{aligned}\tag{39}$$

Taking into account the fact that $\|\mathbf{x}_0\|_{E^TE}^2<\alpha$ and the basic condition of *Theorem* 9, eq. (28), one can get:

$$\begin{aligned}&\ln\mathbf{x}^T(k_0+k)E^TE\mathbf{x}(k_0+k)\\&<\ln\bar{\lambda}_{\max}^k(\ )+\ln\mathbf{x}^T(k_0)E^TE\mathbf{x}(k_0)\\&<\ln\alpha\cdot\bar{\lambda}_{\max}^k(\ )<\ln\alpha\cdot\frac{\beta}{\alpha}<\ln\beta,\quad\forall k\in\mathcal{K}_N\end{aligned}\tag{40}$$

**Q.E.D.**

**Theorem 10** Suppose matrix $\left(A_1^TA_1-E^TE\right)>0$. *Causal* system (178), is *finite time unstable* with respect to $\left\{k_0,\mathcal{K}_N,\alpha,\beta,\|(\cdot)\|^2\right\},\alpha<\beta$, if there exist a positive real number $p$, $p>1$, such that:

$$\|\mathbf{x}(k-1)\|_{A_1^TA_1}^2<p^2\|\mathbf{x}(k)\|_{A_1^TA_1}^2 \\ \forall k\in K_N,\ \ \forall\mathbf{x}(k)\in\mathcal{S}_\beta,\ \ \forall\mathbf{x}(k)\in\mathcal{W}_{d,k}^*\backslash\{0\} \tag{41}$$

*and* if for $\forall\mathbf{x}_0\in\mathcal{W}_{d,k^*}$ and $\|\mathbf{x}_0\|_{G=E^TE}^2<\alpha$ there exist: real, positive number $\delta$, $\delta\in]0,\alpha[$ and time instant $k$, $k=k^*:\exists!\left(k^*>k_0\right)\in\mathcal{K}_N$, for which the next condition is fulfilled

$$\bar{\lambda}_{\min}^{k^*}(\ )>\frac{\beta}{\delta},\quad k^*\in\mathcal{K}_N\tag{42}$$

where:

$$\begin{aligned}&\bar{\lambda}_{\min}(\ )=\bar{\lambda}_{\min}\{\mathbf{x}^T(k)A_0^T\Sigma A_0\mathbf{x}(k),\\&\mathbf{x}(k)\in\mathcal{W}_{k^*}^d,\ \mathbf{x}^T(k)E^TE\,\mathbf{x}(k)=1\}.\\&\Sigma=\left(I-A_1\left(A_1^TA_1-E^TE\right)^{-1}A_1^T+2\wp(k)I\right)\end{aligned}\tag{43}$$

*Debeljković et al* (2011).

**Proof.** Let:

$$V(\mathbf{x}(k))=\mathbf{x}^T(k)\mathbf{x}(k)+\mathbf{x}^T(k-1)\mathbf{x}(k-1),\tag{44}$$

Then, following the identical procedure as in the previous *Theorem*, one can get:

$$\ln\mathbf{x}^T(k+1)\mathbf{x}(k+1)-\ln\mathbf{x}^T(k)\mathbf{x}(k)>\ln\bar{\lambda}_{\min}(\ ),\tag{45}$$

where $\bar{\lambda}_{\min}(\ )$ is given by (25).

If the summing $\sum_{j=k_0}^{k_0+k-1}$ is applied on both sides of (45) for $\forall k\in K_N$, the following can be obtained:

$$\begin{aligned}&\sum_{j=k_0}^{k_0+k-1}\ln\mathbf{x}^T(k+1)E^TE\mathbf{x}(k+1)-\ln\mathbf{x}^T(k)E^TE\,\mathbf{x}(k)\\&>\sum_{j=k_0}^{k_0+k-1}\ln\bar{\lambda}_{\min}(\ )\\&\leq\ln\prod_{j=k_0}^{k_0+k-1}\bar{\lambda}_{\min}(\ )\end{aligned}\tag{46}$$

It can be shown, in general:

$$\begin{aligned}&\sum_{j=k_0}^{k_0+k-1}\left(\ln\mathbf{x}^T(j+1)\mathbf{x}(j+1)-\ln\mathbf{x}^T(j)\mathbf{x}(j)\right)=\\&=\ln\mathbf{x}^T(k_0+1)\mathbf{x}(k_0+1)+\ln\mathbf{x}^T(k_0+2)\mathbf{x}(k_0+2)+\\&\ldots+\ldots\qquad\qquad\ldots+\ldots\\&+\ln\mathbf{x}^T(k_0+k-2+1)\mathbf{x}(k_0+k-2+1)+\ldots+\\&+\ln\mathbf{x}^T(k_0+k-1+1)\mathbf{x}(k_0+k-1+1)-\\&-\left(\ln\mathbf{x}^T(k_0)\mathbf{x}(k_0)+\ln\mathbf{x}^T(k_0+1)\mathbf{x}(k_0+1)+\ldots+\right.\\&\left.+\ldots+\ln\mathbf{x}^T(k_0+k-1)\mathbf{x}(k_0+k-1)\right)\\&=\ln\mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k)-\ln\mathbf{x}^T(k_0)\mathbf{x}(k_0)\end{aligned}\tag{47}$$

so that, for (47), it seems to be:

$$\begin{aligned}&\ln \mathbf{x}^T\left(k_0+k\right)E^TE\mathbf{x}\left(k_0+k\right)-\\&-\ln \mathbf{x}^T\left(k_0\right)E^TE\mathbf{x}\left(k_0\right)\\&>\ln\prod_{j=k_0}^{k_0+k-1}\bar{\lambda}_{\min}(\;)>\ln\bar{\lambda}_{\min}^{k}(\;),\quad\forall k\in K_N\end{aligned}\tag{48}$$

It is clear that for any $\mathbf{x}_0\in\mathcal{W}_{dis}^*$ and some $k^*\in K_N$ it follows: $\delta<\|\mathbf{x}_0\|_{E^TE}^2<\alpha$ and taking into account the basic condition of *Theorem* 10, eq. (41), one can get:

$$\begin{aligned}&\ln\mathbf{x}^T\left(k_0+k^*\right)E^TE\,\mathbf{x}\left(k_0+k^*\right)>\ln\bar{\lambda}_{\min}^{k^*}(\;)\\&+\ln\mathbf{x}^T\left(k_0\right)E^TE\,\mathbf{x}\left(k_0\right)>\ln\delta\cdot\bar{\lambda}_{\min}^{k^*}(\;)\\&>\ln\delta\cdot\frac{\beta}{\delta}>\ln\beta,\\&\text{for some } k^*\in K_N\end{aligned}\tag{49}$$

**Q.E.D**.

**Theorem 11** Suppose matrix $\left(A_1^TPA_1-E^TPE\right)\geq 0$.

*Causal s*ystem given by (178), with $\det A_0\neq 0$, is *attractive practically stable* with respect to $\left\{k_0,\mathcal{K}_N,\alpha,\beta,\|(\cdot)\|^2\right\}$, $\alpha<\beta$, if there exists a matrix $P=P^T>0$, being the solution of:

$$A_0^TPA_0-E^TPE=-2\left(Q+S\right),\tag{50}$$

with matrices $Q=Q^T>0$ and $S=S^T$, such that:

$$\mathbf{x}^T\left(k\right)\left(Q+S\right)\mathbf{x}\left(k\right)>0,\ \forall\mathbf{x}\left(k\right)\in\mathcal{W}_{d,k^*}\setminus\{0\}\tag{51}$$

is positive definite quadratic form on $\mathcal{W}_{d,k^*}\setminus\{0\}$, $p$ real number, $p>1$, such that:

$$\begin{aligned}&\|\mathbf{x}(k-1)\|_{A_1^TPA_1}^2<p^2\|\mathbf{x}(k)\|_{A_1^TPA_1}^2\\&\forall k\in\mathcal{K}_N,\ \forall\mathbf{x}\left(k\right)\in\mathcal{S}_\beta,\ \forall\mathbf{x}\left(k\right)\in\mathcal{W}_{k,d}^*\setminus\{0\}\end{aligned}\tag{52}$$

and if the following conditions are satisfied

$$\|A_1\|<\sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}E^TP\right),\tag{53}$$

and

$$\bar{\lambda}_{\max}^{k}(\;)<\frac{\beta}{\alpha},\quad\forall k\in\mathcal{K}_N,\tag{54}$$

where:

$$\begin{aligned}&\bar{\lambda}_{\max}(\;)=\max\{\mathbf{x}^T\left(k\right)A_0^TP^{\frac{1}{2}}\,\Theta\,P^{\frac{1}{2}}A_0\mathbf{x}\left(k\right):\\&\quad\mathbf{x}\left(k\right)\in\mathcal{W}_{d,k^*},\ \mathbf{x}^T\left(k\right)E^TPE\,\mathbf{x}\left(k\right)=1\}.\end{aligned}\tag{55}$$

$$\Theta=\left(I-A_1\left(A_1^TPA_1-E^TPE\right)^{-1}A_1^T+p^2I\right)$$

*Debeljković et al* (2011).

**Proof**. Let us consider the functional:

$$V\left(\mathbf{x}(k)\right)=\mathbf{x}^T\left(k\right)E^TPE\mathbf{x}\left(k\right)+\mathbf{x}^T\left(k-1\right)Q\mathbf{x}\left(k-1\right)\tag{56}$$

with matrices $P=P^T>0$ and

**Remark 4** Equations 50 – 51 are, in a modified form, taken from *Owens*, *Debeljkovic* (1985).

Note that Lemma A1 and *Theorem* A1 indicate that[1]:

$$V\left(\mathbf{x}(k)\right)=\mathbf{x}^T\left(k\right)E^TPE\mathbf{x}\left(k\right),\tag{57}$$

is a *positive quadratic form* on $W_{k^*}^d$, and it is obvious that all solutions $\mathbf{x}(k)$ evolve in $W_{k^*}^d$, so $V\left(\mathbf{x}(k)\right)$ can be used as a *Lyapunov function* for the system under consideration, *Owens*, *Debeljkovic* (1985).

It will be shown that the same argument can be used to declare the same property of another quadratic form present in (51).

For the given (51), a general backward difference is:

$$\begin{aligned}&\Delta V\left(\mathbf{x}(k)\right)=V\left(\mathbf{x}(k+1)\right)-V\left(\mathbf{x}(k)\right)=\mathbf{x}^T\left(k+1\right)E^TPE\,\mathbf{x}\left(k+1\right)\\&+\mathbf{x}^T\left(k\right)Q\mathbf{x}\left(k\right)-\mathbf{x}^T\left(k\right)E^TPE\,\mathbf{x}\left(k\right)-\mathbf{x}^T\left(k-1\right)Q\mathbf{x}\left(k-1\right).\end{aligned}\tag{58}$$

Clearly, using the equation of motion of system under consideration,, we have:

$$\begin{aligned}&\Delta V\left(\mathbf{x}(k)\right)=\mathbf{x}^T\left(k\right)\left(A_0^TPA_0-E^TPE+Q\right)\mathbf{x}\left(k\right)\\&+2\mathbf{x}^T\left(k\right)\left(A_0^TPA_1\right)\mathbf{x}\left(k-1\right)-\mathbf{x}^T\left(k-1\right)\left(Q-A_1^TPA_1\right)\mathbf{x}\left(k-1\right),\end{aligned}\tag{59}$$

or

$$\begin{aligned}&\Delta V\left(\mathbf{x}(k)\right)=\mathbf{x}^T\left(k\right)\left(A_0^TPA_0-E^TPE+2Q+2S\right)\mathbf{x}\left(k\right)\\&\quad-\mathbf{x}^T\left(k\right)Q\mathbf{x}\left(k\right)\\&\quad-2\mathbf{x}^T\left(k\right)S\mathbf{x}\left(k\right)+2\mathbf{x}^T\left(k\right)\left(A_0^TPA_1\right)\mathbf{x}\left(k-1\right)\\&\quad-\mathbf{x}^T\left(k-1\right)\left(Q-A_1^TPA_1\right)\mathbf{x}\left(k-1\right)\end{aligned}\tag{60}$$

Using (51) and (52) yields:

$$\begin{aligned}&\mathbf{x}^T\left(k+1\right)E^TPE\,\mathbf{x}\left(k+1\right)=\mathbf{x}^T\left(k\right)A_0^TPA_0\mathbf{x}\left(k\right)\\&+2\mathbf{x}^T\left(k\right)A_0^TPA_1\mathbf{x}\left(k-1\right)+\mathbf{x}^T\left(k-1\right)A_1^TPA_1\mathbf{x}\left(k-1\right).\end{aligned}\tag{61}$$

Using the very well known inequality, with particular choice:

$$\begin{aligned}&\mathbf{x}^T\left(k\right)\Gamma\,\mathbf{x}\left(k\right)=\mathbf{x}^T\left(k\right)\left(A_1^TPA_1-E^TPE\right)\mathbf{x}\left(k\right)\geq 0,\\&\mathbf{x}\left(k\right)\in\mathcal{W}_{k^*}^d,\ \forall\mathbf{x}\left(k\right)\in\mathcal{S}_\beta,\ \forall k\in\mathcal{K}_N\end{aligned}\tag{62}$$

one can get:

$$\begin{aligned}&\mathbf{x}^T\left(k+1\right)E^TPE\,\mathbf{x}\left(k+1\right)\leq\mathbf{x}^T\left(k\right)A_0^TPA_0\mathbf{x}\left(k\right)\\&-\mathbf{x}^T\left(k\right)A_0^TPA_1\left(A_1^TPA_1-E^TPE\right)^{-1}A_1^TPA_0\mathbf{x}\left(k\right)\\&+\mathbf{x}^T\left(k-1\right)\left(2A_1^TPA_1-E^TPE\right)\mathbf{x}\left(k-1\right).\end{aligned}\tag{63}$$

Moreover, since:

$$\|\mathbf{x}(k-1)\|_{E^TPE}^2\geq 0,\ \forall k\in\mathcal{K}_N,\ \ \forall\mathbf{x}\left(k\right)\in\mathcal{W}_{d,k}^*\setminus\{0\}\tag{64}$$

and using assumption (52) it is clear that (63), re-

[1] See *Apendix* A.

duces to:

$$\begin{gathered}\mathbf{x}^T(k+1)E^T PE\,\mathbf{x}(k+1) \leq \mathbf{x}^T(k)A_0^T P^{\frac{1}{2}}\,\Omega\, P^{\frac{1}{2}} A_0\mathbf{x}(k) \\ \Omega = \left(I - A_1\left(A_1^T PA_1 - E^T PE\right)^{-1} A_1^T + 2p^2 I\right)\end{gathered} \tag{65}$$

Using very well known the property of quadratic form, one can get:

$$\mathbf{x}^T(k+1)E^T PE\,\mathbf{x}(k+1) \leq \bar{\lambda}_{\max}(\ )\mathbf{x}^T(k)E^T PE\,\mathbf{x}(k) \tag{66}$$

where:

$$\begin{gathered}\bar{\lambda}_{\max}(\ ) = \{\mathbf{x}^T(k)A_0^T P^{\frac{1}{2}}\,\Pi\, P^{\frac{1}{2}} A_0\mathbf{x}(k) \\ \mathbf{x}(k) \in \mathcal{W}_{d,k^*} \setminus \{0\},\ \mathbf{x}^T(k)E^T PE\mathbf{x}(k) = 1\} \\ \Pi = \left(I - A_1(A_1^T PA_1 - E^T PE)^{-1} A_1^T + 2p^2 I\right)\end{gathered} \tag{67}$$

Then following the identical procedure as in the *Theorem*.10, one can get:

$$\ln \mathbf{x}^T(k+1)E^T PE\mathbf{x}(k+1) - \ln \mathbf{x}^T(k)E^T PE\mathbf{x}(k) < \ln \bar{\lambda}_{\max}(\ ) \tag{68}$$

where $\bar{\lambda}_{\max}(\ )$ is given by (67).

If the summing $\sum_{j=k_0}^{k_0+k-1}$ is applied to both sides of (68) for $\forall k \in \mathcal{K}_N$, one can obtain:

$$\begin{gathered}\ln \mathbf{x}^T(k_0+k)E^T PE\mathbf{x}(k_0+k) \leq \ln \prod_{j=k_0}^{k_0+k-1} \bar{\lambda}_{\max}(\ ) \\ \leq \ln \bar{\lambda}^{\,k}_{\max}(\ ) + \ln \mathbf{x}^T(k_0)E^T PE\,\mathbf{x}(k_0), \quad \forall k \in \mathcal{K}_N\end{gathered} \tag{69}$$

Taking into account the fact that $\|\mathbf{x}_0\|^2_{E^T PE} < \alpha$ and the basic condition of *Theorem* 11, (54), one can get:

$$\begin{gathered}\ln \mathbf{x}^T(k_0+k)E^T PE\mathbf{x}(k_0+k) \\ < \ln \bar{\lambda}^k_{\max}(\ ) + \ln \mathbf{x}^T(k_0)E^T PE\mathbf{x}(k_0) \\ < \ln \alpha \cdot \bar{\lambda}^k_{\max}(\ ) < \ln \alpha \cdot \frac{\beta}{\alpha} < \ln \beta, \quad \forall k \in \mathcal{K}_N\end{gathered} \tag{70}$$

*Q.E.D*

## Conclusion

This survey paper is devoted to the stability of linear discrete descriptor time delay systems (LDDTDS).

Here, we have given a number of results concerning the stability properties in the sense of Non-Lyapunov.

To assure *practical stability for* LCSTDS, it is not enough only to have the eigenvalues of the matrix pair $(E,\ A)$ somewhere in the complex plane, but also to provide an impulse-free motion (compatibile initial function) and certain conditions to be fulfilled for the system under consideration.

Some different approaches have been shown in order to construct the *Non-Lyapunov* stability theory for a particular class of autonomous LDDTDSs.

## Appendix – A

The fundamental geometric tool in the characterization of the subspace of consistent initial conditions, for *linear discrete descriptor system without delay* (20), is the subspace sequence:

$$\begin{gathered}\mathcal{W}^*_{dis\,0} = \mathbb{R}^n \\ \vdots \\ \mathcal{W}^*_{dis(j+1)} = A_0^{-1}\left(E\mathcal{W}^*_{dis\,j}\right), \quad j \geq 0\end{gathered}, \tag{A.1}$$

where $A_0^{-1}(\cdot)$ denotes the inverse image of ( ) under the operator $A_0$.

**Lemma A.1.** The subsequence $\{\mathcal{W}^*_{dis0},\ \mathcal{W}^*_{dis1},\ \mathcal{W}^*_{dis2},\ldots\}$, is nested in the sense that

$$\mathcal{W}^*_{dis0} \supset \mathcal{W}^*_{dis1} \supset \mathcal{W}^*_{dis2} \supset \mathcal{W}^*_{dis3} \supset \ldots\ldots \tag{A.2}$$

Moreover:

$$\aleph(A_0) \subset \mathcal{W}^*_{dis\,j}, \quad \forall j \geq 0, \tag{A.3}$$

and there exists an integer $k \geq 0$, such that:

$$\mathcal{W}^*_{dis(k+1)} = \mathcal{W}^*_{dis\,k}. \tag{A.4}$$

Then it is obvious that:

$$\mathcal{W}^*_{dis(k+j)} = \mathcal{W}^*_{dis\,k}, \quad \forall j \geq 1. \tag{A.5}$$

If $k^*$ is the smallest such integer with this property, then

$$\mathcal{W}^*_{dis\,k} \cap \aleph(E) = \{0\}, \quad k \geq k^*, \tag{A6}$$

provided that $(zE - A_0)$ is invertible for some $z \in \mathbb{C}$.

**Proof.** See *Owens*, *Debeljkovic* (1985).

**Theorem A.1.** Under the conditions of *Lemma* A1, $\mathbf{x}_0$ is a consistent initial condition for the system under consideration, e.g. *linear discrete singular system without delay* if and only if $\mathbf{x}_0 \in \mathcal{W}^*_{dis\,k^*}$ *.

Moreover, $\mathbf{x}_0$ generates a *discrete* solution sequence $(\mathbf{x}(k) : k \geq 0)$ such that $\mathbf{x}(k) \in \mathcal{W}^*_{dis\,k^*}, \ \forall k \geq 0$.

**Proof.** See *Owens*, *Debeljkovic* (1985).

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# Stabilnost linearnih vremenski kontinualnih singularnih i vremenski diskretnih deskriptivnih sistema sa čistim vremenskim kašnjenjem na konačnom vremenskom intervalu: Pregled rezultata - II deo Diskretan slučaj

**Ovaj rad daje detaljan pregled radova i rezultata mnogih autora u oblasti Neljapunovske stabilnosti posebne klase linearnih sistema.**

**Ovaj pregled pokriva period od 1985 godine sve do današnjih dana i ima snažnu nameru da predstavi dominirajuće koncepte i doprinose koji su stvoreni tokom pomenutog perioda i to u celom svetua koji su saopšteni na renomiranim međunarodnim konferencijama ili objavljeni u uglednim internacionalnim časopisima.**

*Ključne reči:* **linearni sistem, kontinualni sistem, deskritptivni sistem, singularni sistem, sistem sa kašnjenjem, diskretni sistem, stabilnost sistema.**

# Устойчивость линейных сингулярных непрерывных и дискретных дескриптивных систем с чистым временем задержки на конечном интервале времени: Резюме результатов дискретного случая - II часть

**В настоящей работе представлен подробный обзор результатов и вклад многих авторов в области исследования устойчивости не-Ляпунова особого класса линейных систем. Это исследование охватывает период с 1985. года по сегодняшний день и имеет твёрдое намерение представить основные и доминирующие концепци и в том числе из этого периода рассматривает роль и вклад авторов целого мира, которые опубликованы в авторитетных международных журналах или представлены на престижных международных конференциях, а некоторые из них и на престижных мировых семинарах.**

*Ключевые слова*: **линейная система, непрерывная система, дескриптивная система, сингулярная система, система с временной задержкой, дискретная система, устойчивость системы.**

# Stabilité des singuliers systèmes linéaires continus et discrets descriptifs à délai temporel pur sur l'intervalle temporelle finie: Tableau des résultats, deuxième partie - Cas discret

**Ce travail présente un tableau détaillé des travaux et des résultats de nombreux auteurs dans le domaine de la stabilité de non Lyapunov de la classe particulière des systèmes linéaires. Ce tableau comprend la période depuis 1985 jusqu'à nos jours et a pour l'intention principale de présenter les concepts dominants ainsi que les contributions réalisées pendant la période citée dans le monde entier et qui ont été présentés lors des conférences internationales réputées ou bien publiés dans les revues internationales renommées.**

*Mots clés:* **système linéaire, système continu, système descriptif, système singulier, système à délai, système discret, stabilité de système.**