UDK: 661.871:531:534:517.93
COSATI: 20-11

Free Transversal Vibrations of a Double-Membrane System

Danilo Karličić[1)]

A free vibration analysis of two parallel rectangular membranes continuously joined by a Winkler elastic layer is presented. The classical Bernoulli-Fourier method of particular integrals was used for obtaining an analytical solution of the system of two coupled partial differential equations describing transversal vibrations of a double membrane system. The solution is in a form of an infinite numbers of sets, each with two eigen circular frequencies in each of eigen amplitude modes (shapes) of transversal vibrations of a double membrane system. Some of these papers are mentioned in the introduction and references. A commercial software tool was used for a numerical analysis. The obtained numerical data are presented in graphs and tables.

Key words: **free vibrations, transversal vibrations, membranes, Bernoulli-Fourier method, numerical analysis.**

Inroduction

TRANSVERSAL vibrations of a complex system of two rectangular membranes coupled by a homogenously distributed Winkler linear elastic layer are investigated. Such systems are increasingly used in mechanical and civil engineering applications. Therefore, the issue of double membrane system vibrations is important from the practical point of view and it has a wide application in engineering practice. The vibration analysis of such a system is possible and not mathematically complex for certain particular cases of the boundary conditions; therefore, it can be carried out by using the same procedures as those used for single membranes. Free transversal vibrations of the system are described by a system of two homogeneous partial differential equations solved by using the classical Bernoulli-Fourier method (e.g. see ref. [4-7]).

The theory of vibrations of two solid objects connected with a Winkler elastically layer has been a subject of a number of papers. The transverse vibrations of an elastically connected double-beam system were considered by Seeling and Hoppmann II in [2], Zhang et al. [8] and Oniszezuk [9]. The vibrations and energy transfer problem, concerning a similar double-plate system, has been analytically solved by Hedrih (see Refs. [10-13, 15] by Hedrih and [14, 16, 17] by Hedrih and Simonović), by Oniszezuk [11] and many other researchers.

Oniszezuk [1] has discussed free transverse vibrations of two membranes connected by a Winkler elastic layer without the analysis of time functions. In addition, the mass and thickness of the elastic layer are neglected. He has performed the numerical simulation and visualization of characteristic shapes of the coupled membranes in a function of the coordinates (x, y) and the time (t).

Structural model and the formulation of the problem

The physical model of the vibrating system is composed of two parallel rectangular membranes connected by a massless linear elastic layer of Winkler type with the stiffness coefficient $\tilde{c}\left[N/m^3\right]$ per unit area of the membrane coupling. The membranes are stretched by the stresses $\sigma\left[N/m\right]$ per unit length of the corresponding membrane contours in two parallel planes. The membrane material properties are determined by: the elastic modulus E, the Poisson coefficient μ and the density of material ρ. The system is shown in Fig.1. The membrane equilibrium positions are in two parallel planes to the *x*-*y* plane. The double membrane system has negligible bending resistance, and gravitational body forces are to be neglected. It is assumed that the membranes are thin, homogeneous, perfectly elastic and of constant thickness. The membranes are uniformly tight by suitable constant tensions applied at the boundaries, marked by the mass density of the surface membrane $\rho\left[\frac{kg}{m^2}\right]$

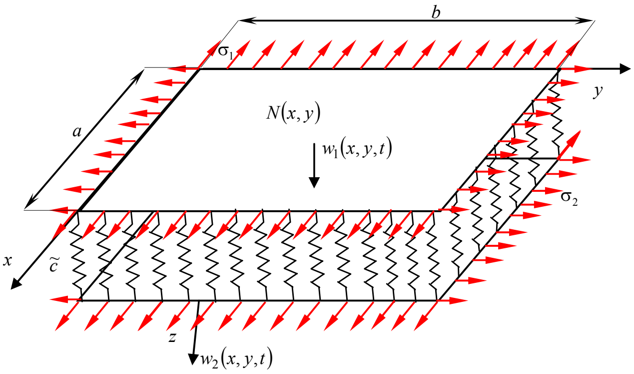

Figure 1. The physical model of the elastically connected double-membrane system

For the transversal oscillations of the system of two coupled membranes for generalized coordinates we choose two transverse displacements $w_i=(x,y,t)$, $i=1,2$ of the corresponding membrane points $N(x,y)$ and perpendicular to

[1)] Mathematical Institute SANU Belgrade, Knez Mihailova 36/III, 11000 Belgrade, SERBIA the membrane surface membrane in undeformed shapes. The small undamped free transversal vibrations of the system are analyzed. The free transverse oscillations of the double-membrane system are described by two coupled partial differential equations of the second order [1]. The transverse displacements in the direction of the Oz-axis are coupled through the elastic layer, where $w_1=(x,y,t)$ is the transverse movement of the upper membrane and $w_2=(x,y,t)$ is the transverse movement of the lower membrane.

The constitutive relation of the elastic material layer for the force-elongations F_e and

$$\Delta w(x,y,t)=\left(w_2\left(x,y,t\right)-w_1\left(x,y,t\right)\right)$$

is:

$$F_e=\tilde{c}\left(w_2\left(x,y,t\right)-w_1\left(x,y,t\right)\right) \tag{1}$$

The governing equations are formulated in terms of two unknowns: the transversal displacements $w_1(x,y,t)$ and $w_2(x,y,t)$. The coupled partial differential equations are derived using the Principle of dynamic equilibrium of a double membrane system [4] as well as decoupled subsystems in result of decomposition of the double membrane system into separate membranes using the conditions of compatibility displacements and interactions of the forces

These partial differential equations of the elastically connected membranes in the double-membrane system are:

$$\begin{aligned}\frac{\partial^2 w_1(x,y,t)}{\partial t^2}&=c_1^2\Delta w_1(x,y,t)+\\&+\frac{\tilde{c}}{\rho_1}\left(w_2\left(x,y,t\right)-w_1\left(x,y,t\right)\right)\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\partial^2 w_2(x,y,t)}{\partial t^2}&=c_2^2\Delta w_2(x,y,t)-\\&-\frac{\tilde{c}}{\rho_2}\left(w_2\left(x,y,t\right)-w_1\left(x,y,t\right)\right)\end{aligned} \tag{3}$$

where $c_i=\sqrt{\frac{\sigma_i}{\rho_i}}\left[\frac{m}{\sec}\right]$, $i=1,2$ is the propagation speed of the transverse waves of the membrane. The coefficients $\frac{\tilde{c}}{\rho}$ squared the circular frequency dimension, which corresponds to the unit [1/sec^2].

The differential operator (Laplace operator) is $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

Solution of the problem

Suppose that both membranes have the same rectangular countours and that they are parallel. Suppose also that the dimensions of the elastically connected double-membrane system are a and b and that the membrane ends are without transverse displacement.

The boundary conditions are given by

$$w_1(0,y,t)=w_1(a,y,t)=0 \tag{4a}$$

$$w_1(x,0,t)=w_1(x,b,t)=0 \tag{4b}$$

$$w_2(0,y,t)=w_2(a,y,t)=0 \tag{5a}$$

$$w_2(x,0,t)=w_2(x,b,t)=0 \tag{5b}$$

and the initial conditions are assumed as follows:

$$w_1(x,y,0)=g_1(x,y), \tag{6a}$$

$$\frac{\partial w_1(x,y,t)}{\partial t}\Big|_{t=0}=\tilde{g}_1(x,y) \tag{6b}$$

$$w_2(x,y,0)=g_2(x,y), \tag{7a}$$

$$\frac{\partial w_2(x,y,t)}{\partial t}\Big|_{t=0}=\tilde{g}_2(x,y) \tag{7b}$$

We follow the idea and the approach of K. Hedrih, shown in Ref. ([10, 12, 13, 15]), to solve the set of partial differential equations and to find the analytical solutions of a system of coupled partial differential equations for transversal vibrations of the system of coupled membranes. Following this idea, the first approach for solving the homogeneous partial differential equations (2) and (3), with the governing boundary conditions (4) and (5), is to use the Bernoulli-Fourier method and assume the solution in the form of a sum of the products of two corresponding functions $W_i(x,y)$ and $T_i(t)$.

$$w_i\left(x,y,t\right)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{i(nm)}(x,y)T_{i(nm)}(t),\quad i=1,2 \tag{8}$$

where $T_i(t)$, $i=1,2$ denotes the unknown time functions, and $W_i(x,y)$, $i=1,2$ is the known eigen amplitude (mode shape) functions for the given boundary conditions. The eigen amplitude functions are defined as

$$W_{(nm)}=\sin\frac{n\pi}{a}x\sin\frac{m\pi}{b}y\quad n,m=1,2,3,4,.....,\infty \tag{8a}$$

with characteristic numbers:

$$k_{(nm)}=\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\quad n,m=1,2,3,4,.....,\infty \tag{8b}$$

The assumed solution (8), introduced into Eqs.(2) and (3), yields

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{1(nm)}(x,y)\ddot{T}_{1(nm)}(t)=\\&=c_1^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Delta W_{1(nm)}(x,y)T_{1(nm)}(t)+\\&+a_1^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(W_{2(nm)}(x,y)T_{2(nm)}(t)-W_{1(nm)}(x,y)T_{1(nm)}(t))\end{aligned} \tag{9}$$

$$\begin{aligned}&\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{2(nm)}(x,y)\ddot{T}_{2(nm)}(t)=\\&=c_2^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\Delta W_{2(nm)}(x,y)T_{2(nm)}(t)-\\&-a_2^2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(W_{2(nm)}(x,y)T_{2(nm)}(t)-W_{1(nm)}(x,y)T_{1(nm)}(t))\end{aligned} \tag{10}$$

where $a_i^2=\frac{\tilde{c}}{\rho_i}$, $i=1,2$; [1/sec^2] are the relations between the coefficient of the elastic layer and the material density.

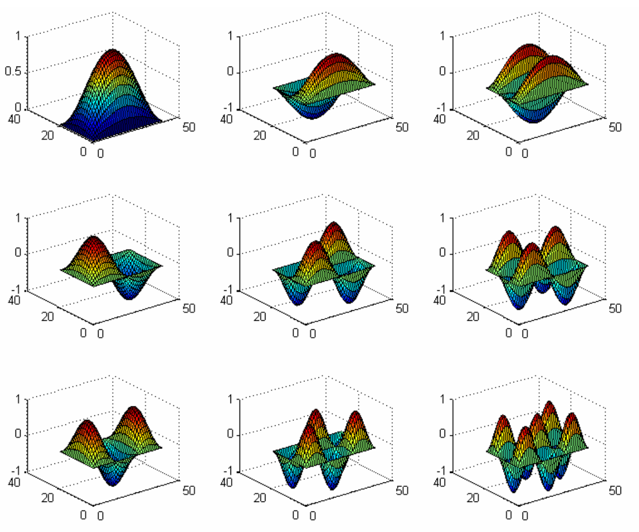

Figure 2. The eigen amplitude functions for the first nine mode shapes (mn = 11, 12, 13, 21, 22, 23, 31, 32 and 33) of membrane transversal vibrations

We compare the transverse modes of two membranes, which have an equal contour, the same boundary conditions and different density of membranes. Also, we consider different tensile forces and different stiffness of the elastic foundation which oscillates. Then, we can conclude the following: eigen amplitude functions $W_{1(nm)}(x,y)=W_{2(nm)}(x,y)=W_{nm}(x,y)$, $n,m=1,2,3,4,...,\infty$ are the same, as well as their characteristic eigen numbers k_{nm}, in both cases. We consider various appropriate time functions, $T_{1(nm)}(t)$ and $T_{2(nm)}(t)$, $n,m=1,2,3,4,...,\infty$, corresponding to each eigen amplitude functions and also sets of eigen circular frequencies $\tilde{\omega}_{01(nm)}^2=k_{(nm)}^2c_1^2+a_1^2$ and $\tilde{\omega}_{02(nm)}^2=k_{(nm)}^2c_2^2+a_2^2$, $n,m=1,2,3,4,...,\infty$.

We introduce Eqs. (8) into coupled differential equations for free double membrane system vibrations (2) and (3). Then we multiply the first and the second equation of systems (9) and (10) with $W_{1(sr)}(x,y)dxdy$, $s.r=1,2,3,3,4,...,\infty$, and the integrate along the surface takes into account the orthogonality conditions of the eigen amplitude functions $W_{1(sr)}(x,y)$ and $W_{1(nm)}(x,y)$. After using corresponding boundary conditions (4) and (5) and the ratios between eigen characteristic numbers $\tilde{\omega}_{0(nm)1}^2=k_{(nm)}^2c_1^2+a_1^2$ and $\tilde{\omega}_{02(nm)}^2=k_{(nm)}^2c_2^2+a_2^2$, $n,m=1,2,3,4,...,\infty$, we obtain infinite numbers of mn subsystems. Each of these subsystems has two coupled second order ordinary differential equations for determination of the unknown eigen time functions $T_{i(nm)}(t)$, $i=1,2$, $n,m=1,2,3,4,...,\infty$ in the following form (for details see [10-12]):

$$\ddot{T}_{1(nm)}(t)+\tilde{\omega}_{01(nm)}^2T_{1(nm)}(t)-a_1^2T_{2(nm)}(t)=0 \tag{15}$$

$$\ddot{T}_{2(nm)}(t)+\tilde{\omega}_{02(nm)}^2T_{2(nm)}(t)-a_2^2T_{1(nm)}(t)=0 \qquad n,m=1,2,3,4,...,\infty \tag{16}$$

The solutions of Eqs. (15) and (16) can be obtained by

$$T_{1(nm)}(t)=A_{1(nm)}\cos(\tilde{\omega}_{nm}t+\alpha_{nm}), \tag{17}$$

$$T_{2(nm)}(t)=A_{2(nm)}\cos(\tilde{\omega}_{nm}t+\alpha_{nm}),\ n,m=1,2,3,4,...,\infty$$

where $\tilde{\omega}_{nm}$, $n,m=1,2,3,4,...,\infty$ denotes the eigen (natural) circular frequency of the double-membrane system, and $A_{1(nm)}$ and $A_{2(nm)}$ represent the eigen amplitude coefficients of two membranes, respectively. Substituting Eqs. (17) into Eqs. (15) and (16), we obtain

$$\left(\tilde{\omega}_{01(nm)}^2-\tilde{\omega}_{nm}^2\right)A_{(1)nm}-a_1^2A_{(2)nm}=0 \tag{18}$$

$$\left(\tilde{\omega}_{02(nm)}^2-\tilde{\omega}_{nm}^2\right)A_{(2)nm}-a_2^2A_{(1)nm}=0 \tag{19}$$

When the determinant of the coefficients in Eqs. (18) and (19) vanishes, nontrivial solutions for the constants $A_{1(nm)}$ and $A_{2(nm)}$ can be obtained. That yields the following frequency (characteristic) equation:

$$\tilde{\omega}_{nm}^4-\tilde{\omega}_{nm}^2\left(\tilde{\omega}_{01(nm)}^2+\tilde{\omega}_{02(nm)}^2\right)+\tilde{\omega}_{01(nm)}^2\tilde{\omega}_{02(nm)}^2-a_1^2a_2^2=0 \tag{20}$$

Then, from the frequency equation (20), we obtain

$$\tilde{\omega}_{(I)nm}^2=\frac{(\tilde{\omega}_{01(nm)}^2+\tilde{\omega}_{02(nm)}^2)}{2}-\\-\frac{1}{2}\sqrt{(\tilde{\omega}_{01(nm)}^2+\tilde{\omega}_{02(nm)}^2)^2-4(\tilde{\omega}_{01(nm)}^2\tilde{\omega}_{02(nm)}^2-a_1^2a_2^2)} \tag{21}$$

$$\tilde{\omega}_{(II)nm}^2=\frac{(\tilde{\omega}_{01(nm)}^2+\tilde{\omega}_{02(nm)}^2)}{2}+\\+\frac{1}{2}\sqrt{(\tilde{\omega}_{01(nm)}^2+\tilde{\omega}_{02(nm)}^2)^2-4(\tilde{\omega}_{01(nm)}^2\tilde{\omega}_{02(nm)}^2-a_1^2a_2^2)} \tag{22}$$

This means that, at any nm mode of the time function $T_{1(nm)}(t)$ and $T_{2(nm)}(t)$, $n,m=1,2,3,4,...,\infty$ as dual frequency with frequencies $\tilde{\omega}_{(1)nm}$ and $\tilde{\omega}_{(2)nm}^2$, $n,m=1,2,3,4,...,\infty$, there are infinitely many eigen modes. It also means that a set of its eigen circular frequencies has its twice infinity eigen circular frequency, and that the time function in each mode is dual frequency.

For each of the natural frequencies, the associated amplitude ratio of vibration modes of the two membranes is given by

$$C_{(s)nm}=\frac{A_{1(nm)}^s}{a_1^2}=\frac{A_{2(nm)}^s}{\left(\tilde{\omega}_{01(nm)}^2-\tilde{\omega}_{(s)nm}^2\right)},\ s=I,II \qquad n,m=1,2,3,4,...,\infty \tag{23.a}$$

or

$$C_{(s)nm}=\frac{A_{1(nm)}^s}{\left(\tilde{\omega}_{02(nm)}^2-\tilde{\omega}_{(s)nm}^2\right)}=\frac{A_{2(nm)}^s}{a_2^2},\ s=I,II \qquad n,m=1,2,3,4,...,\infty \tag{23.b}$$

however, from Eqs.(23.a) we can obtain the relations for the amplitudes

$$A_{1(nm)}^s=C_{(s)nm}a_1^2$$
$$A_{2(nm)}^s=\left(\tilde{\omega}_{01(nm)}^2-\tilde{\omega}_{(s)nm}^2\right)C_{(s)nm},\ s=I,II \tag{23.c}$$
$$n,m=1,2,3,4,......,\infty$$

Solutions (17) of the system of ordinary differential equations (15) and (16), based on the previous amplitude ratio of the vibration modes, can be written in the form of:

$$T_{(1)nm}(t)=A_{(1)nm}^{(I)}\left[A_{nm}\cos(\tilde{\omega}_{nm(I)}t)+B_{nm}\sin(\tilde{\omega}_{nm(I)}t)\right]+ +A_{(1)nm}^{(II)}\left[C_{nm}\cos(\tilde{\omega}_{nm(II)}t)+D_{nm}\sin(\tilde{\omega}_{nm(II)}t)\right] \tag{24.a}$$

$$T_{(2)nm}(t)=A_{(2)nm}^{(I)}\left[A_{nm}\cos(\tilde{\omega}_{nm(I)}t)+B_{nm}\sin(\tilde{\omega}_{nm(I)}t)\right]+ +A_{(2)nm}^{(II)}\left[C_{nm}\cos(\tilde{\omega}_{nm(II)}t)+D_{nm}\sin(\tilde{\omega}_{nm(II)}t)\right] \tag{25.a}$$

$$n,m=1,2,3,4,......,\infty .$$

Introduction of Eqs.(23.c) into Eqs.(24.a) and (25.a) yields

$$T_{(1)nm}(t)=a_1^2\left[\tilde{A}_{nm}\cos(\tilde{\omega}_{nm(I)}t)+\tilde{B}_{nm}\sin(\tilde{\omega}_{nm(I)}t)\right]+ +a_1^2\left[\tilde{C}_{nm}\cos(\tilde{\omega}_{nm(II)}t)+\tilde{D}_{nm}\sin(\tilde{\omega}_{nm(II)}t)\right] \tag{24.b}$$

$$T_{(2)nm}(t)=\left(\tilde{\omega}_{01(nm)}^2-\tilde{\omega}_{nm(I)}^2\right)\left[\tilde{A}_{nm}\cos(\tilde{\omega}_{nm(I)}t)+\tilde{B}_{nm}\sin(\tilde{\omega}_{nm(I)}t)\right]+ +\left(\tilde{\omega}_{01(nm)}^2-\tilde{\omega}_{nm(II)}^2\right)\left[\tilde{C}_{nm}\cos(\tilde{\omega}_{nm(II)}t)+\tilde{D}_{nm}\sin(\tilde{\omega}_{nm(II)}t)\right] \tag{25.b}$$

where $\tilde{A}_{nm},\tilde{C}_{nm}$ and $\tilde{B}_{nm},\tilde{D}_{nm}$ are unknown constants which will be determined in the text below.

Then the transverse vibrations of the double-membrane system can be described by

$$w_1(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{nm}a_1^2\left[\tilde{A}_{nm}\cos(\tilde{\omega}_{nm(I)}t)+\tilde{B}_{nm}\sin(\tilde{\omega}_{nm(I)}t)\right]+ +\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{nm}a_1^2\left[\tilde{C}_{nm}\cos(\tilde{\omega}_{nm(II)}t)+\tilde{D}_{nm}\sin(\tilde{\omega}_{nm(II)}t)\right] \tag{26}$$

$$w_2(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{nm}\left(\tilde{\omega}_{01(nm)}^2-\tilde{\omega}_{nm(I)}^2\right)\left[\tilde{A}_{nm}\cos(\tilde{\omega}_{nm(I)}t)+\tilde{B}_{nm}\sin(\tilde{\omega}_{nm(I)}t)\right]+ +\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}W_{nm}\left(\tilde{\omega}_{01(nm)}^2-\tilde{\omega}_{nm(II)}^2\right)\left[\tilde{C}_{nm}\cos(\tilde{\omega}_{nm(II)}t)+\tilde{D}_{nm}\sin(\tilde{\omega}_{nm(II)}t)\right] \tag{27}$$

On the basis of the orthogonality property of the mode shape functions, the unknown constants $\tilde{A}_{nm}$, $\tilde{C}_{nm}$ and $\tilde{B}_{nm}$, $\tilde{D}_{nm}$ can be determined from assumed initial conditions (6) and (7). To find the final form of the transverse vibrations, the initial-value problem is solved. In this case, the classical orthogonality condition for the two-dimensional case is applied:

$$\iint_A W_{sr}(x,y)W_{nm}(x,y)dxdy=\begin{cases}\dfrac{ab}{4} & nm=sr\\ 0 & nm\neq sr\end{cases}. \tag{28}$$

The introduction of Eqs.(26) and (27) into the initial conditions (6) and (7) yields the following expressions

$$\tilde{A}_{nm}=\frac{\dfrac{4}{ab}\iint_A\left[\dfrac{1}{a_1^2}g_1(x,y)\left(\tilde{\omega}_{(01)nm}^2-\tilde{\omega}_{nm(II)}^2\right)-g_2(x,y)\right]W_{nm}(x,y)dA}{\left(\tilde{\omega}_{(01)nm}^2-\tilde{\omega}_{nm(II)}^2\right)-\left(\tilde{\omega}_{(01)nm}^2-\tilde{\omega}_{nm(I)}^2\right)}$$

$$\tilde{C}_{nm}=\frac{\dfrac{4}{ab}\iint_A\left[g_2(x,y)-\dfrac{1}{a_1^2}g_1(x,y)\left(\tilde{\omega}_{(01)nm}^2-\tilde{\omega}_{nm(I)}^2\right)-\right]W_{nm}(x,y)dA}{\left(\tilde{\omega}_{(01)nm}^2-\tilde{\omega}_{nm(II)}^2\right)-\left(\tilde{\omega}_{(01)nm}^2-\tilde{\omega}_{nm(I)}^2\right)} \tag{29}$$

$$B_{nm}=\frac{\dfrac{4}{ab}\iint_A\left[\dfrac{1}{a_1^2}\tilde{g}_1(x,y)\left(\tilde{\omega}_{(01)nm}^2-\tilde{\omega}_{nm(II)}^2\right)-\tilde{g}_2(x,y)\right]W_{nm}(x,y)dA}{\tilde{\omega}_{nm(I)}\left[\left(\tilde{\omega}_{(01)nm}^2-\tilde{\omega}_{nm(II)}^2\right)-\left(\tilde{\omega}_{(01)nm}^2-\tilde{\omega}_{nm(I)}^2\right)\right]}$$

$$D_{nm}=\frac{\dfrac{4}{ab}\iint_A\left[\tilde{g}_2(x,y)-\dfrac{1}{a_1^2}\tilde{g}_1(x,y)\left(\tilde{\omega}_{(01)nm}^2-\tilde{\omega}_{nm(I)}^2\right)\right]W_{nm}(x,y)dA}{\tilde{\omega}_{nm(II)}\left[\left(\tilde{\omega}_{(01)nm}^2-\tilde{\omega}_{nm(II)}^2\right)-\left(\tilde{\omega}_{(01)nm}^2-\tilde{\omega}_{nm(I)}^2\right)\right]}$$

Numerical experiment

The free transversal vibrations of the system of two elastically coupled identical rectangular membranes are considered. The following values of the system parameters are used in the numerical calculations:

$a=2[m], b=1[m]$, $\overline{c}=2000$[N/m^3], $\rho_1=20$[kg/m^2], $\rho_2=18$[kg/m^2], $\sigma_1=500$[N/m], $\sigma_2=300$[N/m]

In Fig.2 we can see the form of the eigen amplitude functions for the first nine shapes among the infinite family with different modes of free vibrations of the rectangular membrane and for the following *mn* pairs: 11, 12, 13, 21, 22, 23, 31, 32 and 33.

The time history diagrams of the membrane surface are presented in Fig.3. The time functions are presented together in each of the nm modes. The number of modes is infinite, but we have limited it to the first nine modes. We can see that these time functions for free vibrations are in the two-frequency regime for every shape of the modes.

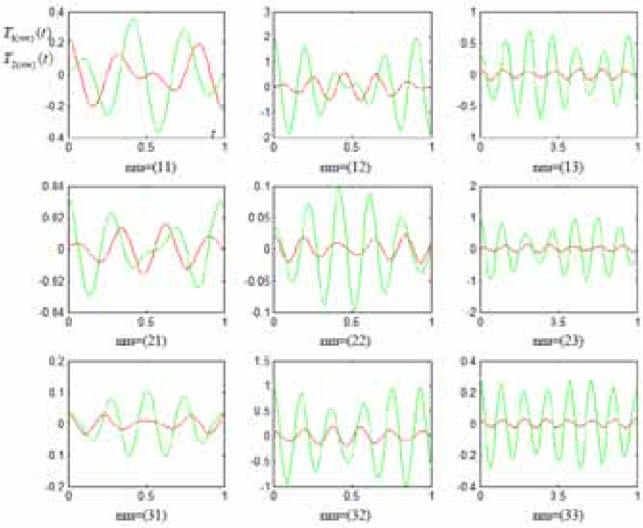

Figure 3. The time history diagrams of the membrane surface corresponding to the time functions $T_{1(nm)}(t)$ and $T_{2(nm)}(t)$ for each eigen mode (mn= 11, 12, 13, 21, 22, 23, 31, 32 and 33)

The results of the calculations of the natural frequencies are presented in Table 1.

Table 1. Natural frequencies $\tilde{\omega}_{(s)nm}$ of the double-membrane system

n	m / $\tilde{\omega}_{(s)nm}$	1 ($\tilde{\omega}_{(1)n1}$, $\tilde{\omega}_{(2)n1}$)	2 ($\tilde{\omega}_{(1)n2}$, $\tilde{\omega}_{(2)n2}$)	3 ($\tilde{\omega}_{(1)n3}$, $\tilde{\omega}_{(2)n3}$)
1	$\tilde{\omega}_{(1)1m}$	15.7362	27.9296	40.2261
	$\tilde{\omega}_{(2)1m}$	21.8524	34.3335	48.9580
2	$\tilde{\omega}_{(1)2m}$	19.6579	30.1249	41.7442
	$\tilde{\omega}_{(2)2m}$	25.4390	36.8751	50.8023
3	$\tilde{\omega}_{(1)3m}$	24.6560	33.4364	44.1550
	$\tilde{\omega}_{(2)3m}$	30.6545	40.7839	53.7385

The natural-eigen circular frequencies $\tilde{\omega}_{(s)nm}$ are evaluated from relations (21) and (22) as the functions of the tension magnitude σ_1 and σ_2. The results of the calculations for $s = I, II$ and $m,n = 1,2$ are presented in Fig.4. An evident influence of the membrane tension on the frequencies of the system is contemplated. In any case, the increase of σ_1 and σ_2 causes an increase of $\tilde{\omega}_{(s)nm}$. However, this influence of the membrane tension on the particular frequencies is different, and the effect of σ_1 and σ_2 on the frequencies $\tilde{\omega}_{(II)nm}$ is greater than on the $\tilde{\omega}_{(I)nm}$. It can be seen that the discussed combined system has an interesting feature which allows each natural frequency to change as a function of membrane tension, while other constructional and physical parameters of the system can stay the same.

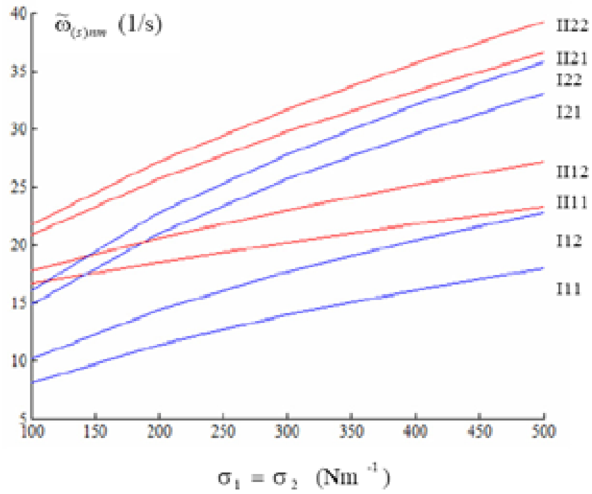

Figure 4. The natural-eigen circular frequencies of the coupled-membrane system $\tilde{\omega}_{(s)nm}$ ($s = I, II$ and $m,n = 1,2$) as a function of the membrane tension force σ_1 and σ_2

In general, an elastically-connected simply-supported double membrane system executes two fundamental kinds of vibrations. The system vibrates with lower natural frequencies $\tilde{\omega}_{(I)nm}$ and with higher frequencies $\tilde{\omega}_{(II)nm}$. It can be seen from the numerical analysis that there is a general tendency to increase the natural-eigen circular frequencies $\tilde{\omega}_{(s)nm}$ in the case of increasing the layer stiffness modulus $\tilde{c}$. The important conclusions can be drawn from expressions (21) and (22). The lower natural frequencies $\tilde{\omega}_{(I)nm}$ are not dependent on the stiffness modulus $\tilde{c}$ unlike $\tilde{\omega}_{(II)nm}$. This effect is greater for the frequencies $\tilde{\omega}_{(II)nm}$ as it has been presented in Fig.5.

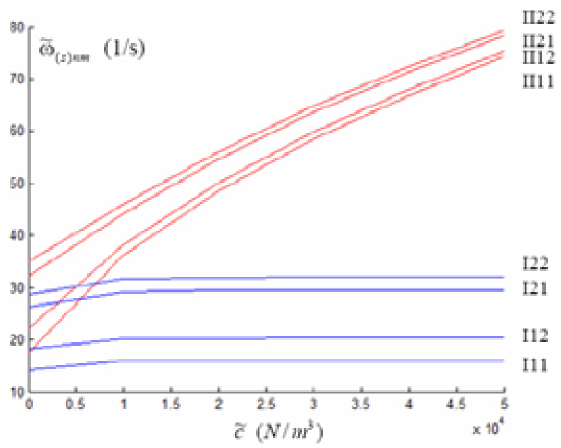

Figure 5. The natural eigen circular frequencies of the coupled-membrane system $\tilde{\omega}_{(s)nm}$ ($s = I, II$ and $m,n = 1,2$) as the function stiffness modulus $\tilde{c}$ of the Winkler elastic layer

Conclusions

In this study, the free transverse vibrations of an elastically-connected rectangular double-membrane system are analyzed theoretically. The solutions of differential equations of the motion are formulated by the Bernoulli-Fourier method of particular integrals. It is shown that one vibration mode corresponds to a two-frequency regime of free vibrations. The numerical analysis shows the effect of physical parameters of the system on the natural eigen circular frequencies. It should be noted that the natural frequencies of the system may be varied with a change of membrane tensions and a change of stiffness modulus. It is concluded that the influence of the change of membrane tensions at a high natural frequency is larger than at a lower natural frequency. Natural frequency increases with an increase of the membrane tensions and an increase of the stiffness modulus $\tilde{c}$ of the Winkler elastic layer. Moreover, higher natural frequency significantly depends on the increase of the stiffness modulus $\tilde{c}$ of the Winkler elastic layer, whereas at a lower natural frequency it is almost independent of it. Using the Matlab software tools, the corresponding visualizations of the characteristic forms of the membrane and the time functions are presented.

Acknowledgements

All my special and sincere thanks to Professor Katica R. (Stevanović) Hedrih, Leader of Project OI174001, for all her ideas, comments, valuable consultations and motivation that she gave me during my work on this paper. I acknowledge the support of the Mathematical Institute SANU and the Ministry of Education and Science of the Republic of Serbia under Project OI174001. The author is grateful to the referees for the useful remarks which helped him to improve this paper.

References

[1] ONISZCZUK,Z.: *Free vibrations of an elastically connected rectangular double-membrane compound system* , Journal of Sound and Vibration, 1999, 221(2), pp.235-250.

[2] SEELIG,J.M., HOPPMANN II,W.H.: *Normal mode vibrations of systems of elastically connected parallel bars* , Journal of the Acoustical Society of America, 1964, 36, pp.93-99.

[3] KESSEL,P.G., RASKE,T.F.: *Damped response of an elastically connected double-beam system due to a cyclic moving load,* Journal of the Acoustical Society of America, 1971, 49, pp.371-373.

[4] RAŠKOVIĆ,D.: *Teorija oscilacija*, (*Theory of oscillations*), Građevinska knjiga, Beograd, 1974, pp.503.

[5] RAŠKOVIĆ,D.: *Teorija elastičnosti* (*Theory of elasticity*), Naučna knjiga, Beograd, 1985, pp.414.

[6] HEDRIH-STEVANOVIĆ,K.: *Izabrana poglavlja TEORIJE elastičnosti* (*Chapters on Theory of elasticity*), Mašinski fakultet Niš, 1988, pp.350

[7] JANKOVIĆ,V.S., POTIĆ,V.P., HEDRIH-STEVANOVIĆ,K.: *Parcijalne diferencijalne jednačine i integralne jednačine sa primenama u inženjerstvu* (*Partial differential equations and integro differential equations with examples in engineering*), Univerzitet u Nišu, 1999, pp.347. (in Serbian).

[8] ZHANG,Y.Q., LU,Y., WANG,S.L., LIU,X.: *Vibration and buckling of a double beam system under compressive axial loading*, Journal of Sound and Vibration, 2008, 318, pp.341-352.

[9] ONISZCZUK,Z.: *Free transverse vibrations of an elastically connected simply supported double-beam complex system*, Journal of Sound and Vibration, 2000, 232(2), pp.387-403.

[10] HEDRIH-STEVANOVIĆ,K.: *Transversal vibrations of double-plate systems*, Acta Mech. Sin. 2006, 22(5), pp.487–501.

[11] ONISZCZUK,Z.: *Free transverse vibrations of an elastically connected rectangular simply supported double-plate complex system*, Journal of Sound and Vibration, 2000, 232(2), pp.387-403.

[12] HEDRIH-STEVANOVIĆ,K.: *Energy transfer in double plate system dynamics*, Acta. Mech. Sin. 2008, 24, pp.331–344.

[13] HEDRIH-STEVANOVIĆ,K., SIMONOVIĆ,J.: *Models of Hybrid Multi-Plates Systems Dynamics*, The International Conference-Mechanical Engineering in XXI Century, Niš, Serbia, 25-26 September 2010, Proceedings, pp.17-20.

[14] HEDRIH-STEVANOVIĆ,K., SIMONOVIĆ,J.: *Multi-frequency analysis of the double circular plate system non-linear dynamics*, Nonlinear Dynamics: Volume 67, Issue 3 (2012), Page 2299-2315. © Springer, NODY1915R2, DOI: 10.1007/s11071-011-0147-7

[15] HEDRIH-STEVANOVIĆ,K.: *Integrity of Dynamical Systems*, Journal of Nonlinear Analysis, 2005, 63, pp.854 – 871.

[16] HEDRIH-STEVANOVIĆ,K., SIMONOVIĆ,J.: *Transversal Vibrations of a non-conservative double circular plate system*, Facta Universitatis Series: Mechanics, Automatic Control and Robotics, 2007, Vol.6, No.1, pp.60-64.

[17] HEDRIH-STEVANOVIĆ,K., SIMONOVIĆ,J.: *Non-linear dynamics of the sandwich double circular plate system*, International Journal of Non-Linear Mechanics, , November 2010, Vol.45, Issue 9, pp.902-918.

Received: 09.02.2012.

Sopstvene transverzalne oscilacije sistema dveju spregnutih membrana

U ovom radu izvršena je analiza sopstvene transferzalne oscilacije sistema dveju paralelnih pravougaonih membrana kontinualno spojenih Winklerovim elastičnim slojem. Korišćenjem Bernoulli-Fourier-vog metoda partikularnih integrala dobijeno je analitičko rešenje sistema parcijalnih diferencijalnih jednačina pomoću kojih su opisane transverzalne oscilacije membrana. Rešenje je određeno u obliku beskonačnog niza sopstvenih amplitudnih funkcija transverzalnih oscilacija sistema za svaku od dve sopstvene frekvencije. Takođe je urađena numerička analiza sistema pomoću softverskih alata. Dobijeni numerički rezultati su prikazani u vidu dijagrama i tabela.

Ključne reči: **slobodne oscilacije, transferzalne oscilacije, membrane, Bernuli-Furijeova metoda, numerička analiza.**

Свободные поперечные колебания систем двух мембран

В данной статье анализируются свободные поперечные колебания двух параллельных прямоугольных мембран постоянно подключеных Winkler-упругим слоем. Использованием метода Бернулли-Фурье частных интегралов получается аналитическое решение систем дифференциальных уравнений в частных производных, при помощи которых описываются поперечные колебания мембран. Решение, предоставленное в виде бесконечного ряда собственных амплитудных функций поперечных колебаний системы для каждой из двух собственных частот. Также выполнен численный системный анализ с помощью программных средств. Полученные цифровые данные приведены в таблице.

Ключевые слова: **свободные колебания, поперечные колебания, мембраны, метод Бернулли-Фурье, численный анализ.**

Libres oscillations transversales du système à double membrane

Dans ce papier on a fait l'analyse de l'oscillation transversale libre du système à deux membranes rectangulaires et parallèles liées continuellement par la couche élastique de Winkler. En utilisant la méthode des intégrales particulières de Bernoulli -Fourier on a obtenu une solution analytique du système des équations différentielles partielles par lesquelles on a décrit les oscillations transversales des membranes. La solution a été déterminée sous la forme d'une série infinie des propres fonctions d'amplitude des oscillations transversales du système pour chacune des deux fréquences. On a effectué aussi l'analyse numérique du système à l'aide d'un logiciel. Les données numériques obtenues ont été exposées en forme graphique.

Mots clés: **oscillations libres, oscillations transversales , membranes, méthode Bernoulli – Fourier , analyse numérique**