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Non-Linear Dynamics of a Double-Plate System Coupled by a Layer with Viscoelastic and Inertia Properties

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The paper considers multi-frequency vibrations of a system of two isotropic circular plates interconnected by a rolling viscoelastic layer of nonlinear characteristics. The considered physical system should be of interest to many researchers in the field of vibration and acoustics absorbers. The interconnecting layer is modeled as a continually distributed layer of discrete standard rheological elements with damping properties and nonlinear elasticity.
The mathematical model of the system is derived in the form of a system of partial differential equations of transverse oscillations of a double circular plate system coupled with a layer of viscous nonlinear elastic and inertia properties, excited by external excitation continually distributed along the plate surfaces. The system of ordinary differential equations of the first order with respect to the amplitudes and the corresponding number of the phases is derived in the first asymptotic averaged approximation for different corresponding multi-frequency nonlinear vibration regimes. These equations are considered analytically and numerically in the light of stationary and non-stationary resonant regimes, as well as in the light of the interactions of nonlinear modes and the number of resonant jumps in the cases without rolling elements and in the cases with two different mass values of rolling elements.
Such an analysis proves that the presence of rolling coupling elements in the interconnecting layer of two plates causes a frequency overlap of the resonant regions of nonlinear modes, together with the increase of their interaction.

Key words: **system dynamics, nonlinear dynamics, oscillations, plate, resonant regime, resonant jumps, mathematical model, partial differential equations.**

Introduction

NOWADAYS the science of materials has a great interest in mathematical modeling of contemporary classes of materials. The better prediction of materials behavior in different dynamical surroundings is possible with the appropriate mathematical model of material characteristics. The phenomena of enlargement and jumps of amplitudes, transition processes or hysteresis in dynamics of systems may be explained by introducing the nonlinear elements in mathematical modeling of material properties. This paper will get insight in such phenomena caused not only by nonlinearity but also by the presence of rolling elements with their translation and rotation. The viscous non-linear elastic rolling element modeled in the manner of rheological models should present the Kelvin-Voigt material with added spherical material particles.

In many engineering systems with non-linearity, high frequency excitations are the sources of multi-frequency resonant regimes appearance of high as well as low frequency modes. This is visible from many experimental research results and also theoretical results (see [1] and [2]). The interaction between the amplitudes and phases of different modes in nonlinear systems with many degrees of freedom as in the deformable body with infinite numbers of frequency vibration of free and forced regimes is observed theoretically in the [3] and [4] using averaging asymptotic methods of Krilov-Bogoliyubov-Mitropolyskiy (see [5, 6]). This knowledge has great practical importance.

In the monograph [2] by Nayfeh, a coherent and unified treatment of analytical, computational, and experimental methods and concepts of modal nonlinear interactions was presented. This monograph is an obvious extension of Nayfeh's and Balachandran's well-known monograph [1] titled Applied Nonlinear Dynamics. These methods are used to explore and unfold, in a unified manner, the fascinating complexities in nonlinear dynamical systems. Through the mechanisms discussed in this monograph, energy from high-frequency sources can be transferred to the low-frequency modes of supporting structures and foundations, and the result can be harmful large-amplitude oscillations that decrease their fatigue lives. However, these mechanisms can be exploited to transfer the energy from the main examined system to the designed subsystem and hence decrease considerably the vibrations of the main system and increase its fatigue life.

An experimental and theoretical study of the response of a flexible cantilever beam to an external harmonic excitation near the beam's third natural frequency is presented in [7]. They have noted that the energy transfer between the third and first modes is very much dependent upon the closeness of the modulation (or Hopf bifurcation) frequency to the first-mode natural frequency. In earlier studies [8, 1] by Nayfeh and coworkers, the modulation frequency was close to the first-mode natural frequency, and, therefore, large first-mode swaying was observed. Nayfeh developed a reduced-order analytical model by discretising the integral partial-differential equation of motion. Identifying, evaluating, and controlling dynamical

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integrity measures in nonlinear mechanical oscillators is a topic for researchers, see [9, 10, 11, 12]. The energy transfer between coupled oscillators can be a measure of the dynamical integrity of hybrid systems as well as subsystems [10, 11, 13, 14].

The problem of detecting the homoclinic orbits applied to the dynamics of different engineering systems was investigated in the series of the papers [9, 15], which gave original research results. In [16], resonant nonlinear normal modes in the cases of two-to-one, three-to-one, and one-to-one internal resonances in undamped unforced one-dimensional systems with arbitrary linear, quadratic and cubic non-linearities are investigated for a class of shallow symmetric structural systems. The non-linear orthogonality of the modes and the activation of the associated interactions are clearly dual problems.

In the series of references, it is possible to find different approaches to solving the nonlinear dynamics of real systems, as well to discovering nonlinear phenomena or some properties of the system dynamics. There are many systems consisting of a nonlinear oscillator attached to a linear system, examples of which are nonlinear vibration absorbers, or nonlinear systems under test using shakers excited harmonically with a constant force. Paper [17] presented a study of the dynamic behavior of a specific two degree-of-freedom system representing such a system, in which the nonlinear system does not affect the vibrations of the forced linear system. The nonlinearity of the attachment was derived from a geometric configuration consisting of a mass suspended on two springs adjusted to achieve a quasi-zero stiffness characteristic with pure cubic nonlinearity. The response of the system at the frequency of excitation was found analytically by applying the method of averaging. The effects of the system parameters on the frequency-amplitude response of the relative motion are examined. It is found that closed detached resonance curves lying outside or inside the continuous path of the main resonance curve can appear as a part of the overall amplitude-frequency response. Two typical situations for the creation of the detached resonance curve inside the main resonance curve, which are dependent on the damping in the nonlinear oscillator, were discussed. The similar nonlinear phenomena were also clarified in [18], where the nonlinear dynamics of the softening and hardening lightly damped Duffing's oscillator with linear viscous damping was presented. For simple approximate non-dimensional expressions, the corresponding displacement amplitudes for the jump-up and jump-down frequencies were determined using the harmonic balance approach. These analytical expressions were validated for a range of parameters by comparing the predictions with calculations from direct numerical integration of the equation of motion. They were also compared with similar expressions derived using the perturbation method. It was shown that the jump-down frequency depends on the degree of nonlinearity and the damping in the system, whereas the jump-up frequency depends primarily upon the nonlinearity, and only weakly depends upon the damping. An expression was also given for the threshold of the excitation force and the nonlinearity that needs to be exceeded for a jump to occur. It was shown that this is only dependent upon the damping in the system.

The list of valuable research results in the connected area of the objects of the author's research is long, but in this introduction, a rather subjective choice is given.

The expressions for energy of the excited modes depending on amplitudes, phases and frequencies of different nonlinear modes are obtained by Hedrih in [10, 11, 19 and 20] and by Hedrih and Simonović in [14, 21-23] by using averaging and asymptotic methods for obtaining a system of ordinary differential equations of amplitudes and phases in first approximations. By means of these asymptotic approximations, the energy analysis of a mode interaction in the multi frequency free and forced vibration regimes of nonlinear elastic systems (beams, plates, and shells) excited by initial conditions was made and a series of resonant jumps as well as energy transfer features were identified. The excitation was considered like a perturbation of the equilibrium state of the double plate system at the initial moment, defined by the initial conditions for displacements and velocities of both plate middle surface points. In addition, for the case of an external excitation in the resonant frequency range near one of the natural eigen frequency of the basic linear system, two or more resonant energy jumps at the nonlinear modes were presented.

Using the Krilov-Bogolyubov-Mitropolskiy asymptotic method as well as the energy approach presented in the monographs by Mitopolskiy, [5, 6, 24], there are new results for a study of the elastic bodies nonlinear oscillations and the energetic analysis of elastic bodies oscillatory motions in the doctoral thesis by Stevanović (see [3] and [4]). The introduction of paper [21] presented a review survey of the original results of the author and of the researchers from the Faculty of Mechanical Engineering, University of Niš, inspired and/or obtained by the asymptotic method of Krilov-Bogolyubov-Mitropolyskiy, by a direct influence of professor Rašković [25] with his scientific instructions and by the published Mitropolskiy's papers and monographs.

The interest in the study of coupled plates as a new qualitative system dynamics has grown exponentially over the last few years because of the theoretical challenges involved in the investigation of such systems. Recent technological innovations have caused a considerable interest in the study of components and processes of hybrid dynamical systems. Hybrid systems consisting of rigid and deformable bodies (plates, beams and belts) connected with a system of discrete elements are characterized by the interaction between the dynamics of subsystem, and governed by coupled partial differential equations with boundary and initial conditions, see[10, 19, 26, 27] and [12, 14, 21-23].

In papers [11, 28, 29], through the examples of hybrid systems of a statically and dynamically coupled discrete subsystem of rigid bodies and continuous subsystem, the method for obtaining frequency equations of small oscillations was presented. In addition, series of theorems of small oscillations frequency equations were defined. The analogy between frequency equations of some classes of these systems was identified. Special cases of discretization and continualization of coupled subsystems in the light of these sets of proper circular frequencies and frequency equations of small oscillations were analyzed.

The study of transversal vibrations of both double and multi plate systems with elastic, viscous elastic of creep connections is important for both theoretical and pragmatic reason. Many important structures may be modeled from a composite structure and possess a big importance in many applications such as, e.g., in civil engineering for roofs, floors, walls, in thermo and acoustics isolation systems of wall and floor constructions, orthotropic bridge decks or for building any structural application in which the traditional method of construction uses stiffened steel. They are also applied in car, plane and ship industry for sheaths of plane wings, for inner arrangement of planes; they are suitable for building maritime vessels or for building structures such as double hull oil tankers, bulk carriers, car bodies, truck bodies or for railway vehicles.

The sandwich constructions consist of two or more facing layers structurally bonded to a core made of material with small specific weight. This type of construction provides a structural system that acts as a crack arrest layer and that can join two dissimilar metals without welding or without setting up a galvanic cell and provides equivalent in plane and transverse stiffness and strength, reduces fatigue problems, minimizes stress concentrations, improves thermal and acoustical insulation, and provides vibration control.

It is shown here that, as a model of that structure, it is possible to use a double plate system connected by a visco-elastic layer with a nonlinearity of the third order in the elastic part.

This paper will be an attempt to present the feature of the interconnecting layer introduced with rolling elements with their inertia of rolling without sliding and with the translation of mass centers. The model of a new rheological element with the properties of viscous- nonlinear elasticity and that of rolling without sliding will be presented. Such an element has different forces on its ends in motion. The presence of these elements in the model of the interconnecting layer of two plates introduces the dynamical coupling in the mathematical model of the plate system dynamics. In addition, this model with the nonlinearity of the third order in the interconnecting layer introduces the phenomenon of passing through the resonant range and the appearance of one or several resonant jumps in the amplitude–frequency and phase–frequency curves, such as multi-nonlinear mode mutual interactions between amplitudes and phases of different nonlinear modes. The analysis of the mathematical model of dynamics on double plate system with coupling layer of visco-elastic nonlinear rolling properties is going to show the interesting phenomena of nonlinear dynamics caused by the presence of viscous nonlinear elastic and rolling elements.

In systems with nonlinearity, the energy transfer between coupled subsystems is noticeable. The two or more resonant energy jumps at the nonlinear modes were investigated in paper [10] for the case of an external excitation in the resonant frequency range near one of the natural eigen frequency of the basic linear system. Also, see [20] which contains an analysis of the energy transfer in double plate system dynamics.

In the following parts of this paper, we will first present the mathematical models of the interconnecting layer and the dynamics of the double plate system coupled with viscous nonlinear elastic rolling elements continually distributed on plate surfaces. The result of that modeling will be a system of partial differential equations (PDE`s ), dynamically and statically coupled. In the third part of the paper, we will present an asymptotic approximation of the solution of PDE`s of transversal vibrations of a double circular plate system forced with two-frequency external excitations. The fourth and fifth parts consist of the analyses of stationary and no stationary regimes of transversal vibrations of a double plate system done by the series of the amplitude and phase-frequencies curves of the system. In the conclusion, we will mention all results of these analyses and point out the future use in an energy analysis of the dynamics in systems of plates connected with a layer of viscous nonlinear elastic rolling properties.

Model of the interconnecting layer and PDE`s of transversal vibrations of a double plate system

The standard rolling viscous nonlinear elastic element, presented in Figs. 1b) and 1d), introduced as a rheological model (see [30]), has the transversal displacements w_1 and w_2 on the ends, and the velocities of its ends $\dot{w}_1$ and $\dot{w}_2$. The expressions for the velocity of translation for the centre of mass C have the form: $\dot{w}_C = \frac{\dot{w}_2 + \dot{w}_1}{2}$, and for the angular velocity around the center of mass in the form: $\omega_C = \frac{\dot{w}_2 - \dot{w}_1}{2\mathrm{R}}$. Then the expressions for the kinetic energy of such an element have the following form:

$$E_{k(1,2)} = \frac{1}{2}\left(\mathrm{m}\left(\frac{\partial w_C}{\partial t}\right)^2 + \mathbf{J}_\mathbf{C}\omega_C^2\right) = = \frac{1}{8}\mathrm{m}\left\langle\left(\frac{\partial w_2}{\partial t} + \frac{\partial w_1}{\partial t}\right)^2 + \frac{\mathrm{i}_\mathrm{C}^2}{\mathrm{R}^2}\left(\frac{\partial w_2}{\partial t} - \frac{\partial w_1}{\partial t}\right)^2\right\rangle , \quad (1)$$

where $i_C^2 = \frac{\mathbf{J}_\mathbf{C}}{m}$ is the square of the radius of inertia for the rolling element. If the rolling element is the disc, then $i_C^2 = \frac{R^2}{2}$. If we introduce the notation of parameters in the form:

$$\hat{a}_{12} = \frac{m}{4} - \frac{\mathbf{J}_\mathbf{C}}{4R^2} = \frac{m}{8}, \quad \hat{a}_{11} = \frac{m}{4} + \frac{\mathbf{J}_\mathbf{C}}{4R^2} = \frac{3m}{8}$$

and

$$\hat{a}_{22} = \frac{m}{4} + \frac{\mathbf{J}_\mathbf{C}}{4R^2} = \frac{3m}{8},$$

then we have:

$$E_{k(1,2)} = \frac{1}{2}\left(\hat{a}_{11}\left(\frac{\partial w_1}{\partial t}\right)^2 + \hat{a}_{22}\left(\frac{\partial w_2}{\partial t}\right)^2 + 2\left(\frac{\partial w_1}{\partial t}\right)\left(\frac{\partial w_2}{\partial t}\right)\hat{a}_{12}\right) \quad (2)$$

The potential energy form for such an element with a nonlinearity of the third order is in the form:

$$E_{p(1,2)} = \frac{1}{2}c\left(w_2 - w_1\right)^2 + \frac{1}{4}\beta\left(w_2 - w_1\right)^4 + +\frac{1}{2}\frac{c_1}{2}\left(w_2 - w_C\right)^2 + \frac{1}{2}\frac{c_1}{2}\left(w_C - w_1\right)^2 , \quad (3)$$

it turns out:

$$E_{p(1,2)} = \frac{1}{2}c\left(w_2 - w_1\right)^2 + \frac{1}{4}\beta\left(w_2 - w_1\right)^4 + +\frac{1}{2}c_1\left(\frac{w_2 - w_1}{2}\right)^2 \quad (3a)$$

Rayleigh's function of energy dissipation is in the form:

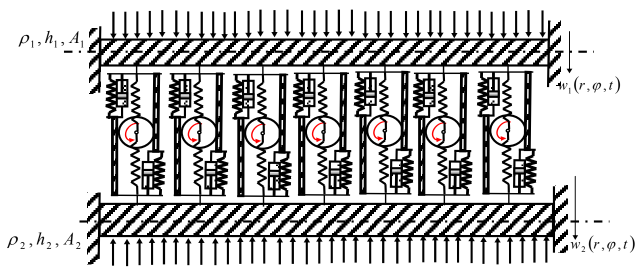
$$\Phi_{(1,2)} = \frac{1}{2}b_1\left[\frac{\partial w_2}{\partial t} - \frac{\partial w_1}{\partial t}\right]^2 \quad (4)$$

where b_1 is a known coefficient of the dissipation force.

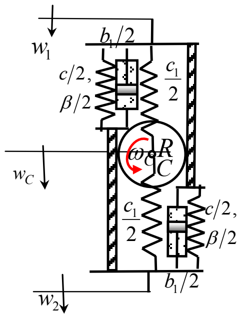
Now by using the meaning of parts in the Lagrange's equations of motions on $w_i\left(r,\phi,t\right)$, $i = 1,2$ as generalized coordinates, we may represent the inertial and elastic forces, and the force of viscous damping acting to the upper or lower plates in the following forms:

$$F_{j1} = -\left\langle \frac{d}{dt}\frac{\partial E_k}{\partial\left(\frac{\partial w_1}{\partial t}\right)} - \frac{\partial E_k}{\partial w_1}\right\rangle = -\frac{1}{4}m\frac{d}{dt}\left\langle\left(\frac{\partial w_2}{\partial t}+\frac{\partial w_1}{\partial t}\right)-\frac{i_C^2}{R^2}\left(\frac{\partial w_2}{\partial t}-\frac{\partial w_1}{\partial t}\right)\right\rangle = -\frac{1}{4}m\left\langle\left(\frac{\partial^2 w_2}{\partial t^2}+\frac{\partial^2 w_1}{\partial^2 t}\right)-\frac{i_C^2}{R^2}\left(\frac{\partial^2 w_2}{\partial t^2}-\frac{\partial^2 w_1}{\partial^2 t}\right)\right\rangle \quad (5)$$

$$F_{j2} = -\left\langle \frac{d}{dt}\frac{\partial E_k}{\partial\left(\frac{\partial w_2}{\partial t}\right)} - \frac{\partial E_k}{\partial w_2}\right\rangle = -\frac{1}{4}m\frac{d}{dt}\left\langle\left(\frac{\partial w_2}{\partial t}+\frac{\partial w_1}{\partial t}\right)+\frac{i_C^2}{R^2}\left(\frac{\partial w_2}{\partial t}-\frac{\partial w_1}{\partial t}\right)\right\rangle = -\frac{1}{4}m\left\langle\left(\frac{\partial^2 w_2}{\partial t^2}+\frac{\partial^2 w_1}{\partial^2 t}\right)+\frac{i_C^2}{R^2}\left(\frac{\partial^2 w_2}{\partial t^2}-\frac{\partial^2 w_1}{\partial^2 t}\right)\right\rangle \quad (6)$$



a)



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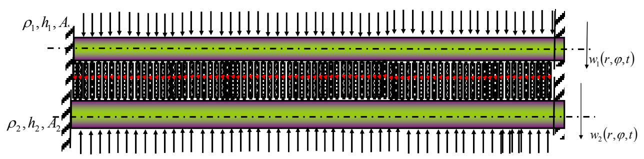

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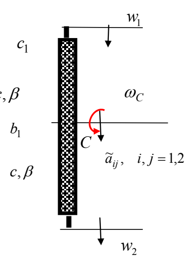


d)

Figure 1. a) Double circular plate system connected with a rolling viscous-elastical nonlinear layer; b) a rheological model of the rolling viscous nonlinear elastic discrete element; c) Model of the double circular plate system connected with a layer of the rheological elemets with properties of the nonlinear elasticity, damping and inertia of translation and rotation of the rolling part; d) a rheological scheme of the rolling viscous nonlinear elastic discrete element.

The translational dynamics of the rolling disc gives the sum of these two forces of inertia acting on upper and lower plates:

$$F_{j1}+F_{j2} = -\left\langle \frac{d}{dt}\frac{\partial E_k}{\partial\left(\frac{\partial w_1}{\partial t}\right)} - \frac{\partial E_k}{\partial w_1}\right\rangle - \left\langle \frac{d}{dt}\frac{\partial E_k}{\partial\left(\frac{\partial w_2}{\partial t}\right)} - \frac{\partial E_k}{\partial w_1}\right\rangle = -\frac{1}{2}m\left(\frac{\partial^2 w_2}{\partial t^2}+\frac{\partial^2 w_1}{\partial^2 t}\right) = -ma_C \quad (7)$$

The rotation dynamics of the rolling disc gives the difference of these two forces of inertia acting on upper and lower plates:

$$J_C\frac{d\omega_C}{dt} = \left(-F_{j1}+F_{j2}\right)R = -\left\{-\left\langle \frac{d}{dt}\frac{\partial E_k}{\partial\left(\frac{\partial w_1}{\partial t}\right)} - \frac{\partial E_k}{\partial w_1}\right\rangle\right\} R - \left\langle \frac{d}{dt}\frac{\partial E_k}{\partial\left(\frac{\partial w_2}{\partial t}\right)} - \frac{\partial E_k}{\partial w_2}\right\rangle R = -\frac{1}{2}\frac{m i_C^2}{R}\left(\frac{\partial^2 w_2}{\partial t^2}-\frac{\partial^2 w_1}{\partial^2 t}\right) \quad (8)$$

The elastic forces on upper and lower plates are:

$$F_{e1} = -F_{e2} = -\frac{\partial E_p}{\partial w_i} = c\left[w_2-w_1\right]+\beta\left[w_2-w_1\right]^3+\frac{c_1}{4}\left[w_2-w_1\right] \quad (9)$$

The forces of viscouse damping on upper and lower plates have the forms:

$$F_{w1} = -F_{w2} = -\frac{\partial\Phi}{\partial\dot{w}_i} = b\left[\frac{\partial w_2}{\partial t}-\frac{\partial w_1}{\partial t}\right] \quad (10)$$

The resulting force of the rolling visco-elastic nonlinear element on the plate middle points has the form:

$$F_i = F_{ei}+F_{wi}+F_{ji} = -\frac{\partial E_p}{\partial w_i}-\frac{\partial\Phi}{\partial\dot{w}_i}-\left\langle\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathbf{E}_k}{\partial\dot{w}_i}-\frac{\partial\mathbf{E}_k}{\partial w_i}\right\rangle, \quad i=1,2 \quad (11)$$

Hence, it follows that the resulting force of the rolling viscous nonlinear elastic element the on upper ($i=1$) plate middle point has the form:

$$F_1 = \left(\mathrm{c}+\frac{\mathrm{c}_1}{4}\right)\left[w_2-w_1\right]+\mathrm{b}\left[\frac{\partial w_2}{\partial t}-\frac{\partial w_1}{\partial t}\right]+\beta\left[w_2-w_1\right]^3-\frac{1}{4}\mathrm{m}\left\langle\left(\frac{\partial^2 w_2}{\partial t^2}+\frac{\partial^2 w_1}{\partial^2 t}\right)-\frac{\mathrm{i}_C^2}{\mathrm{R}^2}\left(\frac{\partial^2 w_2}{\partial t^2}-\frac{\partial^2 w_1}{\partial^2 t}\right)\right\rangle \quad (12)$$

and the resulting force of the rolling viscous nonlinear elastic element on the lower ($i=2$) plate middle point has the form:

$$F_2 = -\left(\mathrm{c}+\frac{\mathrm{c}_1}{4}\right)\left[w_2-w_1\right]-\mathrm{b}\left[\frac{\partial w_2}{\partial t}-\frac{\partial w_1}{\partial t}\right]-\beta\left[w_2-w_1\right]^3-\frac{1}{4}\mathrm{m}\left\langle\left(\frac{\partial^2 w_2}{\partial t^2}+\frac{\partial^2 w_1}{\partial^2 t}\right)+\frac{\mathrm{i}_C^2}{\mathrm{R}^2}\left(\frac{\partial^2 w_2}{\partial t^2}-\frac{\partial^2 w_1}{\partial^2 t}\right)\right\rangle . \quad (13)$$

The governing equations of the double plate system, Figs.1.a) and 1.c), are formulated in terms of two unknowns, [12] and [25]: the transversal displacement $w_i = w_i(r,\phi,t)$, $i=1,2$ in direction of the axis z, of the upper plate middle surface and of the lower plate middle surface. We present the interconnecting layer as a model of distributed discrete rheological rolling visco-elastic elements with nonlinearity in the elastic part of the layer as shown in Figs1.b) and 1.d). Since these elements are continually distributed on the plate surfaces, the generalized resulting forces (12) and (13) are also continually distributed onto the plate middle points. Our assumptions for the plates are: they are thin with the same contours and with an equal type of the boundary conditions and they have small transversal displacements. The system of two coupled partial differential equations is derived using d'Alembert's principle of the dynamic equilibrium in the following forms:

$$\frac{\partial^2 w_1}{\partial t^2}(1+\tilde{a}_{11})+\tilde{a}_{12(1)}\frac{\partial^2 w_2}{\partial t^2}+c_{(1)}^4\Delta\Delta w_1-2\delta_{(1)}\left[\frac{\partial w_2}{\partial t}-\frac{\partial w_1}{\partial t}\right]-$$
$$-a_{(1)}^2[w_2-w_1]=\varepsilon\beta_{(1)}[w_2-w_1]^3+\tilde{q}_{(1)}$$

$$\frac{\partial^2 w_2}{\partial t^2}(1+\tilde{a}_{22})+\tilde{a}_{12(2)}\frac{\partial^2 w_1}{\partial t^2}+c_{(2)}^4\Delta\Delta w_2+2\delta_{(2)}\left[\frac{\partial w_2}{\partial t}-\frac{\partial w_1}{\partial t}\right]+$$
$$+a_{(2)}^2[w_2-w_1]=-\varepsilon\beta_{(2)}[w_2-w_1]^3-\tilde{q}_{(2)} \quad (14)$$

where are: $\tilde{a}_{11}=\frac{\hat{a}_{11}}{\rho_1 h_1}$, $\tilde{a}_{22}=\frac{\hat{a}_{22}}{\rho_2 h_2}$, $\tilde{a}_{12(i)}=\frac{\hat{a}_{12}}{\rho_i h_i}$, and $a_{(i)}^2=\frac{1}{\rho_i h_i}\left(c+\frac{c_1}{4}\right)$, $D_i=\frac{E_i h_i^3}{12(1-\mu_i^2)}$, $c_{(i)}^4=\frac{D_i}{\rho_i h_i}$, $2\delta_i=\frac{b}{\rho_i h_i}$ and $\varepsilon\beta_{(i)}=\frac{\beta}{\rho_i h_i}$, for $i=1,2$ and for the strong nonlinear characteristic. The functions $\tilde{q}_{(i)}=\tilde{q}_{(i)}(r,\phi,t)$ are the known functions of the external continually distributed load on the plate surfaces.

Asymptotic approximation of the solution of PDE`s of transversal vibrations of a double circular plate system

The system of partial differential equations (14) describes the dynamics of the double plate - system with the rolling viscous nonlinear elastic layer. By using the Bernoulli's method of particular integrals, we suppose the solutions for that system are in the form of the eigen amplitude functions $W_{(i)nm}(r,\phi)$, $n,m=1,2,...\infty$, satisfying the same boundary conditions, expansion with time coefficients in the form of unknown time functions $T_{(i)nm}=T_{(i)nm}(t)$, and describing their time evolution (see [12]), in the form:

$$w_i(r,\phi,t)=W_{(i)nm}(r,\phi)T_{(i)nm}(t). \quad (15)$$

After substituting this solution in the system of equations (14), keeping in mind the orthogonality conditions of plate amplitude functions, it turns into a system of differential equations for the time function of one nm-mode of plates transversal oscillations :

$$\ddot{T}_{(1)nm}+\kappa_1\ddot{T}_{(2)nm}-2\tilde{\delta}_{(1)}\left(\dot{T}_{(2)nm}-\dot{T}_{(1)nm}\right)+\tilde{\omega}_{(1)nm}^2T_{(1)nm}-$$
$$-\tilde{a}_{(1)}^2T_{(2)nm}=\varepsilon\tilde{\beta}_{(1)}\aleph(W_{nm})\left[T_{(2)nm}-T_{(1)nm}\right]^3+\tilde{f}_{(1)nm}$$

$$\ddot{T}_{(2)nm}+\kappa_2\ddot{T}_{(1)nm}+2\tilde{\delta}_{(2)}\left(\dot{T}_{(2)nm}-\dot{T}_{(1)nm}\right)+\tilde{\omega}_{(2)nm}^2T_{(2)nm}-$$
$$-\tilde{a}_{(2)}^2T_{(1)nm}=-\varepsilon\tilde{\beta}_{(2)}\aleph(W_{nm})\left[T_{(2)nm}-T_{(1)nm}\right]^3-\tilde{f}_{(2)nm} \quad (16)$$

where $\tilde{\omega}_{(i)nm}^2=\frac{\omega_{(i)nm}^2}{1+\tilde{a}_{ii}}$ and $\omega_{(i)nm}^2=k_{(i)nm}^4c_{(i)nm}^4+a_{(i)nm}^2$, $i=1,2$ are the eigen circular frequencies of coupled plates, $\aleph(W_{nm})=\frac{\int_0^r\int_0^{2\pi}W_{(1)nm}^4(r,\phi)rdrd\phi}{\int_0^r\int_0^{2\pi}W_{(1)nm}^2(r,\phi)rdrd\phi}$ is the coefficient of the nonlinearity influence of the elastic layer, $f_{(i)nm}(t)=\frac{\int_0^r\int_0^{2\pi}\tilde{q}_i(r,\phi,t)W_{(i)nm}(r,\phi)rdrd\phi}{\int_0^r\int_0^{2\pi}\left[W_{(i)nm}(r,\phi)\right]^2rdrd\phi}$ are the known functions of external forces and the coefficients of reduction are: $\kappa_i=\frac{\tilde{a}_{12(i)}}{1+\tilde{a}_{ii}}$, $\tilde{a}_{(i)}^2=\frac{a_{(i)}^2}{1+\tilde{a}_{ii}}$, $2\tilde{\delta}_i=\frac{2\delta_{(i)}}{1+\tilde{a}_{ii}}$, $\tilde{\beta}_{(i)}=\frac{\beta_{(i)}}{1+\tilde{a}_{ii}}$ and $\tilde{f}_{(i)nm}(t)=\frac{f_{(i)nm}(t)}{1+\tilde{a}_{ii}}$.

Having in mind the form of solutions for the corresponding homogeneous system of (14), we suppose the solution of that system in the following form:

$$T_{(1)nm}(t)=K_{21nm}^{(1)}e^{-\hat{\delta}_{1nm}t}R_{1nm}(t)\cos\Phi_{1nm}(t)+$$
$$+K_{21nm}^{(2)}e^{-\hat{\delta}_{2nm}t}R_{2nm}(t)\cos\Phi_{2nm}(t)$$

$$T_{(2)}(t)=K_{22nm}^{(1)}e^{-\hat{\delta}_{1nm}t}R_{1nm}(t)\cos\Phi_{1nm}(t)+$$
$$+K_{22nm}^{(2)}e^{-\hat{\delta}_{2nm}t}R_{2nm}(t)\cos\Phi_{2nm}(t) \quad (17)$$

where: K_{ijnm}^{s} are the cofactors of determinant corresponding to the basic homogeneous coupled linear system (see [27], [5] and [22]), and the amplitudes $R_{inm}(t)$ and the phases $\Phi_{inm}(t)=q_i\Omega_{inm}t+\varphi_{inm}(t)$ are unknown time functions which we are going to obtain using the asymptotic Krilov-Bogolyubov-Mitropolyskiy averaging method (see [5, 6, 24]). It is taken into account that the defined task satisfies all necessary conditions for applying the asymptotic Krilov-Bogolyubov-Mitropolskiy method concerning the small parameter.

We suppose that the functions of the external excitation at nm-mode of oscillations are the two-frequency process in the form:

$$f_{(i)nm}(t)=h_{01nm}\cos\left[\Omega_{1nm}t+\varphi_{1nm}\right]+$$
$$+h_{02nm}\cos\left[\Omega_{2nm}t+\varphi_{2nm}\right], \quad (18)$$

and that the external force frequencies Ω_{1nm} and Ω_{2nm} are in the range of two corresponding eigen linear damped coupled system frequencies $\Omega_{1nm}\approx\hat{p}_{1nm}$ and $\Omega_{2nm}\approx\hat{p}_{2nm}$ of the corresponding linear and free system to system (14) and that the initial conditions of the double plate system permit the appearance of the two-frequency like vibrations regimes of the system. Also, we accept that nonlinearity is small introducing the small parameter ε. $\hat{p}_{inm}$ are the frequencies of visco-elastic coupling obtained as the imaginary parts of the solution $\lambda_{i,jnm}=-\hat{\delta}_{inm}\mp i\hat{p}_{inm}$ for the characteristic equations of system (14). For details see [26, 27] and [14, 21-23].

In addition, it is necessary to point out that all previous expressions are valid for the cases of the same plate contours, as well as for the equal boundary conditions of both plates. The previous system of equations and solutions are uniquely determined for corresponding initial conditions determining the initial middle surface of the plate forms (positions) and the corresponding initial velocities of the middle surface points.

By introducing the condition that the first derivatives $\dot{T}_{(i)nm}(t)$ have the same forms as in the case where the amplitudes $R_{inm}(t)$ and the difference of the phases $\varphi_{inm}(t)$ are constant and after introducing the first $\dot{T}_{(i)nm}(t)$ and the second $\ddot{T}_{(i)nm}(t)$ derivatives in the system of nonlinear equations (14), we obtain a system of equations in respect of the derivatives of the unknown functions $\dot{R}_{inm}(t)$ and $\dot{\varphi}_{inm}(t)$. After applying the method of averaging to the right-hand sides of that system with respect to the full phases $\Phi_{1nm}^{(i)}(t)$ and $\Phi_{2nm}^{(i)}(t)$, we obtain the first asymptotic averaged approximation of the system of differential equations for the amplitudes $R_{inm}(t)$ and the difference of the phases $\varphi_{inm}(t)$ as follows:

$$\dot{a}_{1nm}(t) = -\delta_{1nm} a_{1nm}(t) - \frac{\varepsilon P_{1nm}}{(\Omega_{1nm} + \hat{p}_{1nm})} \cos\varphi_{1nm} \tag{19}$$

$$\dot{\varphi}_{1nm}(t) = \left(\hat{p}_{nm1} - \Omega_{1nm}\right) - \frac{3}{8}\frac{\alpha_{1nm}}{\hat{p}_{nm1}} a_{1nm}^2(t) - \frac{1}{4}\frac{\beta_{1nm}}{\hat{p}_{nm1}} a_{2nm}^2(t) + \frac{\varepsilon P_{1nm}}{\left(\Omega_{1nm} + \hat{p}_{nm1}\right) a_{1nm}(t)} \sin\varphi_{1nm} \tag{20}$$

$$\dot{a}_{2nm}(t) = -\delta_{2nm} a_{2nm}(t) - \frac{\varepsilon P_{2nm}}{(\Omega_{2nm} + \hat{p}_{2nm})} \cos\varphi_{2nm} \tag{21}$$

$$\dot{\varphi}_{2nm}(t) = \left(\hat{p}_{2nm} - \Omega_{2nm}\right) - \frac{3}{8}\frac{\alpha_{2nm}}{\hat{p}_{nm2}} a_{2nm}^2(t) - \frac{1}{4}\frac{\beta_{2nm}}{\hat{p}_{nm2}} a_{1nm}^2(t) + \frac{\varepsilon P_{2nm}}{\left(\Omega_{2nm} + \hat{p}_{nm2}\right) a_{2nm}(t)} \sin\varphi_{2nm} \tag{22}$$

where $a_{inm}(t) = R_{inm}(t) e^{-\hat{\delta}_{inm} t}$ is the change of variables hence $\dot{a}_{inm}(t) = \left(\dot{R}_{inm}(t) - \hat{\delta}_{inm} R_{inm}(t)\right) \mathrm{e}^{-\hat{\delta}_{inm} t}$, and $KK_{nm} = \left(K_{22nm}^{(1)} K_{21nm}^{(2)} - K_{21nm}^{(1)} K_{22nm}^{(2)}\right)\left(1 - \frac{\tilde{a}_{12(1)}\tilde{a}_{12(2)}}{(1+\tilde{a}_{22})(1+\tilde{a}_{11})}\right)$. The coefficients δ_{inm} depend on the coupling properties via cofactors and $\tilde{\delta}_{(i)nm}$ on the damping coefficients of the viscous elastic layer too, the coefficients $\varepsilon \mathrm{P}_{1nm}$ and $\varepsilon \mathrm{P}_{2nm}$ of excited forces amplitudes, and the coefficients α_{inm}, β_{inm} of the non-linearity layer properties too, in the following forms:

$$\delta_{1nm} = \frac{\left[K_{22nm}^{(1)} - K_{21nm}^{(1)}\right]}{KK_{nm}} \left\{ \left(K_{21nm}^{(2)} + \frac{\tilde{a}_{12(1)} K_{22nm}^{(2)}}{(1+\tilde{a}_{11})} \right) \frac{\tilde{\delta}_{(2)}}{(1+\tilde{a}_{22})} + \left(K_{22nm}^{(2)} + \frac{\tilde{a}_{12(2)} K_{21nm}^{(2)}}{(1+\tilde{a}_{22})} \right) \frac{\tilde{\delta}_{(1)}}{(1+\tilde{a}_{11})} \right\} \tag{23a}$$

$$P_{1nm} = -\frac{\varepsilon \left(K_{22nm}^{(2)} + \frac{\tilde{a}_{12(2)} K_{21nm}^{(2)}}{(1+\tilde{a}_{22})} \right)}{KK_{nm}(1+\tilde{a}_{11})} \tag{23b}$$

$$\alpha_{1nm} = -\frac{\varepsilon \aleph(W_{nm}) \left(K_{22nm}^{(1)} - K_{21nm}^{(1)}\right)^3}{2KK_{nm}} \left\{ \frac{\beta_{(2)}}{(1+\tilde{a}_{22})} \left[K_{21nm}^{(2)} + \frac{\tilde{a}_{12(1)} K_{22nm}^{(2)}}{(1+\tilde{a}_{11})} \right] + \frac{\beta_{(1)}}{(1+\tilde{a}_{11})} \left(K_{22nm}^{(2)} + \frac{\tilde{a}_{12(2)} K_{21nm}^{(2)}}{(1+\tilde{a}_{22})} \right) \right\} \tag{23c}$$

$$\beta_{1nm} = -\frac{\varepsilon \aleph(W_{nm}) \left(K_{22nm}^{(1)} - K_{21nm}^{(1)}\right) \left(K_{22nm}^{(2)} - K_{21nm}^{(2)}\right)^2}{KK_{nm}} \left\{ \frac{\beta_{(2)}}{(1+\tilde{a}_{22})} \left[K_{21nm}^{(2)} + \frac{\tilde{a}_{12(1)} K_{22nm}^{(2)}}{(1+\tilde{a}_{11})} \right] + \frac{\beta_{(1)}}{(1+\tilde{a}_{11})} \left(K_{22nm}^{(2)} + \frac{\tilde{a}_{12(2)} K_{21nm}^{(2)}}{(1+\tilde{a}_{22})} \right) \right\} \tag{23d}$$

$$\delta_{2nm}(t)=\frac{\left[K_{22nm}^{(2)}-K_{21nm}^{(2)}\right]\left\{\left(K_{22nm}^{(1)}+\frac{\tilde{a}_{12(2)}K_{21nm}^{(1)}}{(1+\tilde{a}_{22})}\right)\frac{\tilde{\delta}_{(1)}}{(1+\tilde{a}_{11})}+\left(K_{21nm}^{(1)}+\frac{\tilde{a}_{12(1)}K_{22nm}^{(1)}}{(1+\tilde{a}_{11})}\right)\frac{\tilde{\delta}_{(2)}}{(1+\tilde{a}_{22})}\right\}}{KK_{nm}} \quad (23e)$$

$$P_{2nm}=-\frac{\varepsilon\left(K_{21nm}^{(1)}+\frac{\tilde{a}_{12(1)}K_{22nm}^{(1)}}{(1+\tilde{a}_{11})}\right)}{KK_{nm}(1+\tilde{a}_{22})} \quad (23f)$$

$$\alpha_{2nm}=-\frac{\varepsilon\aleph(W_{nm})\left(K_{22nm}^{(2)}-K_{21nm}^{(2)}\right)^3}{2KK_{nm}}\left\{\frac{\beta_{(1)}}{(1+\tilde{a}_{11})}\left(K_{22nm}^{(1)}+\frac{\tilde{a}_{12(2)}K_{21nm}^{(1)}}{(1+\tilde{a}_{22})}\right)+\frac{\beta_{(2)}}{(1+\tilde{a}_{22})}\left(K_{21nm}^{(1)}+\frac{\tilde{a}_{12(1)}K_{22nm}^{(1)}}{(1+\tilde{a}_{11})}\right)\right\} \quad (23g)$$

$$\beta_{2nm}=-\frac{\varepsilon\aleph(W_{nm})\left(K_{22nm}^{(1)}-K_{21nm}^{(1)}\right)^2\left(K_{22nm}^{(2)}-K_{21nm}^{(2)}\right)}{KK_{nm}}\left\{\frac{\beta_{(1)}}{(1+\tilde{a}_{11})}\left(K_{22nm}^{(1)}+\frac{\tilde{a}_{12(2)}K_{21nm}^{(1)}}{(1+\tilde{a}_{22})}\right)+\frac{\beta_{(2)}}{(1+\tilde{a}_{22})}\left(K_{21nm}^{(1)}+\frac{\tilde{a}_{12(1)}K_{22nm}^{(1)}}{(1+\tilde{a}_{11})}\right)\right\} \quad (23h)$$

We observed the case when the external distributed two-frequencies force acts at upper surfaces of the upper plate with frequencies near the circular frequencies of the coupling $\Omega_{1nm}\approx\hat{p}_{1nm}$ and $\Omega_{2nm}\approx\hat{p}_{2nm}$, and that the lower plate is free of excitation $\tilde{q}_{(2)nm}(t)=0$. This means that the passing thought the main resonant state corresponding to the frequencies of the viscous elastic coupling $\Omega_{inm}\text{»}\hat{p}_{inm}$ was observed.

Analysis of the stationary regimes of transversal vibrations of a double plate system

For the analysis of the stationary regime of oscillations, we equal the right-hand sides of differential equations (19), (21) for the amplitudes $R_{inm}(t)$ and (20), (22) for the difference of the phases $\varphi_{inm}(t)$ with null. Eliminating the phases φ_{1nm} and φ_{2nm}, we obtained a system of two algebraic equations by the unknown amplitudes a_{1nm} and a_{2nm} in the following form:

$$\left(\hat{p}_{nm1}+\Omega_{1nm}\right)^2\delta_{nm}^2a_{1nm}^2+a_{1nm}^2\left(\hat{p}_{nm1}^2-\Omega_{1nm}^2-\frac{\hat{p}_{nm1}+\Omega_{1nm}}{8\hat{p}_{nm1}}\left(3\alpha_{1nm}a_{1nm}^2+2\beta_{1nm}a_{2nm}^2\right)\right)^2-P_{1nm}^2=0 \quad (24a)$$

$$\left(\hat{p}_{nm2}+\Omega_{2nm}\right)^2\delta_{nm}^2a_{2nm}^2+a_{2nm}^2\left(\hat{p}_{nm2}^2-\Omega_{2nm}^2-\frac{\hat{p}_{nm2}+\Omega_{2nm}}{8\hat{p}_{nm2}}\left(3\alpha_{2nm}a_{2nm}^2+2\beta_{2nm}a_{1nm}^2\right)\right)^2-P_{2nm}^2=0 \quad (24b)$$

Also, with the elimination of the amplitudes a_{1nm} and a_{2nm}, we obtained the forms for the phases ϕ_{1nm} and ϕ_{2nm} in the case of two-frequencies forced oscillations in the stationary regime of one nm mode of double plate system oscillations:

$$\phi_{1nm}=\operatorname{arctg}\left(\frac{\hat{\mathrm{p}}_{nm1}-\Omega_{1nm}}{\delta_{nm}}-\frac{3\alpha_{1nm}a_{1nm}^2+2\beta_{1nm}a_{2nm}^2}{8\hat{\mathrm{p}}_{nm1}\delta_{nm}}\right) \quad (25a)$$

and

$$\varphi_{2nm}=\operatorname{arctg}\left(\frac{\hat{\mathrm{p}}_{nm2}-\Omega_{2nm}}{\delta_{nm}}-\frac{3\alpha_{2nm}a_{2nm}^2+2\beta_{2nm}a_{1nm}^2}{8\hat{\mathrm{p}}_{nm2}\delta_{nm}}\right). \quad (25b)$$

Solving systems (24a,b) and (25a,b) by the numerical Newton-Kantorovic's method in the Mathematica computer program, we obtained the stationary amplitudes and phases curves of the two-frequencies regime of one eigen nm-shape amplitude mode oscillations in the double plate system coupling with a rolling viscous nonlinear elastic layer depending on the frequencies of the external excitation force. If we fix the value of one external excitation frequency of two possible ones, we obtain amplitude-frequency curves as well as phase-frequency curves of the stationary states of the vibration regime in the following forms:

1* for the second external excitation frequency with a constant discrete value $(\Omega_{2nm}=\text{const})$ corresponding amplitude-frequency and phase-frequency curves:

$$a_{1nm}=f_1(\Omega_{1nm}),\ a_{2nm}=f_2(\Omega_{1nm}),\ \phi_{1nm}=f_3(\Omega_{1nm})$$

and

$$\phi_{2nm}=f_4(\Omega_{1nm})$$

and

2* for the first external excitation frequency with a constant discrete value $(\Omega_{1nm}=\text{const})$ corresponding amplitude-frequency and phase-frequency curves:

$$a_{1nm}=f_5(\Omega_{2nm}),\ a_{2nm}=f_6(\Omega_{2nm}),\ \phi_{1nm}=f_7(\Omega_{2nm})$$

and

$$\phi_{2nm}=f_8(\Omega_{2nm}).$$

We will present the amplitude-frequencies and phase-frequencies curves of the stationary state in a continuous exchange of fixed discrete values of external excitation frequencies and in that sense consider the system in the stationary regime. Some characteristic diagrams of these amplitide-frequency and phase-frequency curves are presented in Fig.2-17.

The following analysis considers changing the rolling element masses that affect the kinetic energy of the interconnecting layer. Since the visco-elastic part of the interconnecting elements has negligible mass, it follows that the mass of the rolling elements does not influence the part of the potential energy of the interconnecting layer. This is obvious from forms (2) and (3). For further numerical solutions, we present three cases of the interconnecting layer rolling elements. We change their masses per unit of plates surfaces from m = 240 kg and m = 100 kg to a case when we do not have rolling elements for m = 0 kg.

The numerically considered plates are with the same material characteristics with a radius of 1m, heights h_1=0,01 m and h_2=0,005m, made of still with a density of $\rho_i = 7.849 \cdot 10^3 \text{ kgm}^{-3}$, Poisson's ratio $\mu = 0.33$ and Young's modulus $E_i = 21 \cdot 10^{10} \text{Nm}^{-2}$. The plates are connected with a layer of continually distributed viscous nonlinear elastic rolling elements of stiffness $c = 2 \cdot 10^5 \text{Nm}^{-1}$ and $c_1 = 0,5 \cdot 10^5 \text{Nm}^{-1}$ and a coefficient of damping $b_1 = 0.5 \text{ kgs}^{-1}$ all per one square meter of plate surfaces. This is the case when the lower plate has the height twice greater than the upper plate, $h_2 = h_1/2$, and when we modify the mass of the rolling elements, the solutions of characteristics equations of system (19) $\lambda_{1,2nm} = -\hat{\delta}_{1nm} \mp i\hat{p}_{1nm}$ and $\lambda_{3,4nm} = -\hat{\delta}_{2nm} \mp i\hat{p}_{2nm}$ have the different values and therefore the coefficients (23a-h) have different values. The solved values of the circular frequencies of the coupling $\hat{p}_{inm}$ and the coefficients (23a-h) are presented in Table 1. Here we present the solutions for the case of the first eigen mode of the plates oscillations for n = 0 and m = 1 for which the characteristic eigen number of the clamped circular plate is $k_{11} = 3.196$. The value of the coefficient of nonlinearity influence is $\aleph(W_1) = 0.117$, and the value of the coefficient of the nonlinearity of the layer is $\beta = 5 \text{ m}^{-2}\text{s}^{-2}$, reduced values of the amplitude of excitations are $h_{0i(11)} = 10^7 \text{Nm}^{-3}$ for the value of the dimensionless parameters $\varepsilon = 10^{-2}$.

Table 1. The values of the circular frequencies of the coupling $\hat{p}_{inm}$, and the coefficients δ_{inm}, δ_{inm} and β_{inm}, P_{inm}, for $i = 1,2$ in first mode of plate system oscillations ($n = 0, m = 1$), for three different values of the rolling elements masses.

$m(\text{kg})$	$\hat{p}_1(s^{-1})$	$\hat{p}_2(\text{s}^{-1})$	δ_1	δ_2	α_1	α_2	β_1	β_2	P_1	P_2
0	108.33	174.49	11	8	12210	96220	267100	17590	2945	534
100	87.33	148.42	6.273	2.151	25480	15720	91720	17470	1402	358.5
240	71.61	126.82	3.326	0.7554	18640	3538	30310	8704	1082	289

As expected, the increasing of the mass of rolling elements has reduced the circular frequencies of the coupling $\hat{p}_{inm}$ anf the coefficients of the damping influence δ_{inm}; thus the impact of the nonlinearity should be grater. All the phenomena of the resonant transition for the stationary regime need to be more evident for the same values of the amplitude of external excitations. These are the characteristic jumps of the amplitude and the phase response in the vicinity of the resonant values $\Omega_{inm} \approx \hat{p}_{inm}$, the appearance of new stable and unstable branches which conditions the more-values responses of system and the emergence of two stable solutions of the system in the area of these new branches, the mutual interaction of the harmonics and the jumps of the system energies. All these phenomena we will present through the series of the amplitude-frequency and phase-frequency diagrams for both harmonics in the mentioned three cases of rolling element masses. Also, presented are the shapes of the corresponding branches of the amplitude-frequency and phase-frequency curves are presented for stationary non-linear vibration regimes, for the first time harmonic amplitude $a_{1nm}(\Omega_1, \Omega_2)$ and the phase $\phi_{1nm}(\Omega_1, \Omega_2)$, and for the second time harmonic amplitude $a_{2nm}(\Omega_1, \Omega_2)$ and the phase $\phi_{2nm}(\Omega_1, \Omega_2)$ in the one nm - th eigen amplitude shape mode of plate.

These shapes are the results of the modes interaction and of the particular discrete values choice of the external excitation frequencies Ω_{1nm} and Ω_{2nm} used in the resonant frequencies intervals belonging to the corresponding eigen frequencies $\hat{p}_{1nm}$ and $\hat{p}_{2nm}$ of the corresponding nm - th eigen amplitude shape mode of the plate linear system taken in the simulations. Strong interactions between the time modes in the nm - th eigen amplitude shape mode of plate pairs appear only in the case that both values of the both external excitation frequencies Ω_{1nm} and Ω_{2nm} are simultaneous in the corresponding resonant frequency interval $\Omega_{1nm} \approx \hat{p}_{1nm}$ and $\Omega_{2nm} \approx \hat{p}_{2nm}$. If one of the external excitation frequencies, Ω_{1nm} or Ω_{2nm}, is outside the corresponding resonant frequency interval, Ω_{1nm} no $\approx \hat{p}_{1nm}$ and $\Omega_{2nm} \approx \hat{p}_{2nm}$ or $\Omega_{1nm} \approx \hat{p}_{1nm}$ and Ω_{2nm} *no* $\approx \hat{p}_{2nm}$, the interactions between modes are small. For that case, a specific change of the corresponding amplitude-frequency and phase-frequency curves is not visible and looks like the one in the case of the single frequency external excitation in the corresponding resonant frequency interval, Ω_{1nm} no $\approx \hat{p}_{1nm}$ and $\Omega_{2nm} \approx \hat{p}_{2nm}$ or $\Omega_{1nm} \approx \hat{p}_{1nm}$ and Ω_{2nm} *no* $\approx \hat{p}_{2nm}$. The amplitude-frequency curve for the second time harmonics in the interval of the frequencies which are not close to the external excitation frequency, Ω_{2nm} *no* $\approx \hat{p}_{2nm}$ or Ω_{1nm} no $\approx \hat{p}_{1nm}$ and outside the resonant frequency intervals is practically a straight line, and practically has a constant value. Hence, there is no interaction between time modes in the first asymptotic approximation. This is visible from Fig.2-15 at the beginning or at the end of the external excitation frequency intervals.

The first eight Fig.2-7 present the amplitude and phase responses for both harmonics for the case of the greatest mass of the rolling elements m=240 kg per unit of plate surface.

The amplitude-frequency responses for two frequencies like stationary vibration regimes contain the amplitudes a_1 and a_2, Figures 2 and 4. The shown figures exhibit a strong characteristic as nonlinear interactions between the time modes of the two-frequency external excitation in the resonant interval of two external excitation frequencies close to the eigen linearized system frequencies.

1* In Fig.2, the amplitude-frequency curves $a_{1nm}(\Omega_1, \Omega_2 = const)$, in Fig.3 the phase-frequency curves $\phi_{1nm}(\Omega_1, \Omega_2 = const)$, in Fig.4 the amplitude-frequency response $a_{2nm}(\Omega_1, \Omega_2 = const)$ and in Fig.5 the phase-frequency curves $\phi_{2nm}(\Omega_1, \Omega_2 = const)$ for two frequency like nonlinear stationary vibration regimes are presented for the discrete values of the $\Omega_{2nm} = const$, Ω_{2nm}= $100\,s^{-1}$, $130\,s^{-1}$, $132\,s^{-1}$, $135\,s^{-1}$, $140\,s^{-1}$, $145\,s^{-1}$, $150\,s^{-1}$, $155\,s^{-1}$, for the Ω_{1nm} continuously from the interval $\Omega_{1nm} \in \left[50s^{-1}, 250s^{-1}\right]$ and the discrete values of $\Omega_{2nm} = const$ in the interval $\Omega_{2nm} \in \left[100s^{-1}, 155s^{-1}\right]$.

The amplitude-frequency and phase-frequency curves for the cases: $a_{1nm}\left(\Omega_1, \Omega_2 = 100s^{-1}\right)$, $\phi_{1nm}\left(\Omega_1, \Omega_2 = 100s^{-1}\right)$ as for $a_{1nm}\left(\Omega_1, \Omega_2 = 160s^{-1}\right)$, $\phi_{1nm}\left(\Omega_1, \Omega_2 = 160s^{-1}\right)$ presented in Figures 2. and 3 have the shapes as in the case of the corresponding single frequency amplitude-frequency and phase-frequency curves with only one pair of resonant jumps in each pair of the corresponding curves.

2* In Fig.6 the amplitude-frequency curves $a_{1nm}(\Omega_1 = const, \Omega_2)$, in Fig.7 the amplitude-frequency response $a_{2nm}(\Omega_1 = const, \Omega_2)$ for two frequencies like nonlinear stationary vibration regimes are presented for the discrete values of the $\Omega_{1nm} = const$, Ω_{1nm}= $85\,s^{-1}, 110\,s^{-1}, 120\,s^{-1}, 190\,s^{-1}, 220\,s^{-1}, 260\,s^{-1}, 300\,s^{-1}, 320\,s^{-1}$ from the interval $\Omega_{1nm} \in \left[85s^{-1}, 320s^{-1}\right]$ and the discrete values of $\Omega_{2nm} = const$ continuously in the interval $\Omega_{2nm} \in \left[60s^{-1}, 200s^{-1}\right]$. In this case, we did not present the phase-frequency diagrams because, as we notice in the previous series of the Figures, the phase transient through the resonant regime is simultaneous to that of the amplitude and gives the same quantitative conclusions.

Comparing the first and the last diagrams in Figures 2, 3 and 6, we may conclude that the amplitude and phase responses of the first harmonic have small changes after the transient regime while the amplitude and phase responses of the second harmonics have significant changes of the values and the shapes, Figures 4, 5 and 7. Therefore, we conclude that the influence of the first harmonics on the second one, in the resonant region of the frequencies Ω_{1nm} of the external excitation, is greater than vice versa in the resonant region of the frequencies Ω_{2nm} of the external excitation.

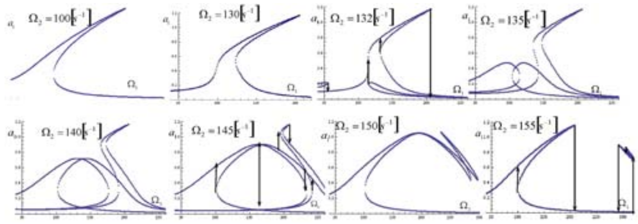


Figure 2. Amplitude-frequency characteristic curves for the amplitudes of the first time harmonics $a_{1nm} = f_1(\Omega_{1nm})$ on the different value of the excited frequency Ω_{1nm} for the discrete value of the excited frequency $\Omega_{2nm} = const$ with noted corresponding one or more resonant jumps for $m = 240\,kg$. The arrows designate the directions of the resonant jumps

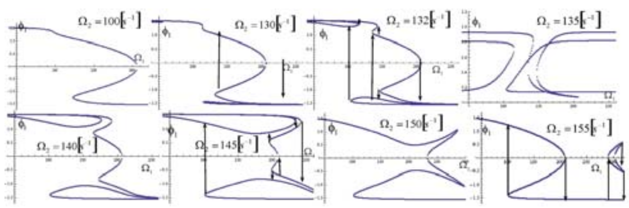


Figure 3. Phase -frequency characteristic curves for the amplitudes of the first time harmonics $\phi_{1nm} = f_3(\Omega_{1nm})$ on the different value of the excited frequency Ω_{1nm}, for the discrete value of the excited frequency $\Omega_{2nm} = const$ with noted corresponding one or more resonant jumps for $m = 240\,kg$. The arrows designate the directions of the resonant jumps

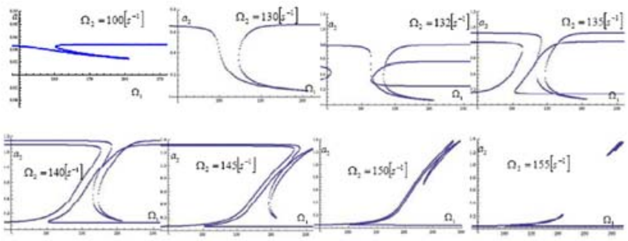


Figure 4. Amplitude-frequency characteristic curves for the amplitude of the second time harmonics $a_{2nm} = f_2(\Omega_{1nm})$ on the different value of the excited frequency Ω_{1nm} for the discrete value of the excited frequency $\Omega_{2nm} = const$ with noted corresponding one or more resonant jumps for $m = 240\,kg$

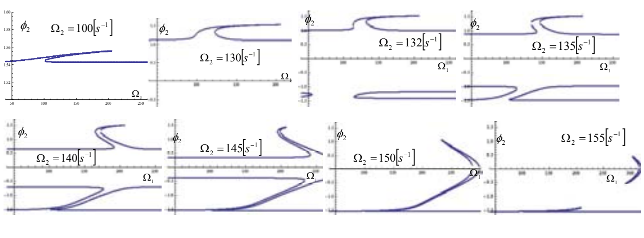


Figure 5. Phase-frequency characteristic curves for the phases of the second time harmonics $\phi_{2nm} = f_4(\Omega_{1nm})$ on the different value of the excited frequency Ω_{1nm} for the discrete value of the excited frequency $\Omega_{2nm} = const$ with noted corresponding one or more resonant jumps for $m = 240\,\text{kg}$

In the second case, we presented the amplitude-frequency diagrams for other values of rolling element masses for $m = 100\,\text{kg}$ at Figures 8 - 11.

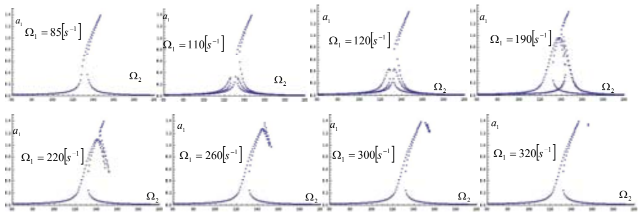


Figure 6. Amplitude-frequency characteristic curves for the amplitudes of the first time harmonics $a_{1nm} = f_5(\Omega_{1nm})$ on the different value of the excited frequency Ω_{2nm} for the discrete value of the excited frequency $\Omega_{1nm} = const$ with noted corresponding one or more resonant jumps for $m = 240\,\text{kg}$

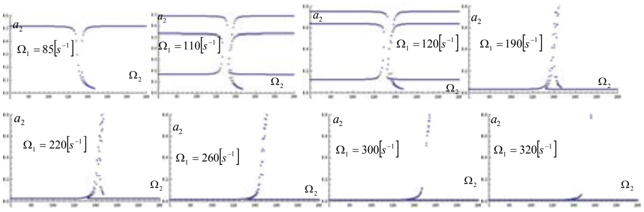


Figure 7. Amplitude-frequency characteristic curves for the phases of the second time harmonics $a_{2nm} = f_6(\Omega_{2nm})$ on the different value of the excited frequency Ω_{2nm} for the discrete value of the excited frequency $\Omega_{1nm} = const$ with noted corresponding one or more resonant jumps for $m = 240\,\text{kg}$

In Figures 8-11, for this case and select numerical particular values of the system parameters, that mutual interactions between two time modes belonging to one nm-th eigen amplitude shape mode of plates oscillation are presented at the following discrete values of the frequencies and resonant frequencies intervals:

1* Amplitude-frequency curves $a_{1nm}(\Omega_1, \Omega_2 = const)$, in Fig.8, and $a_{2nm}(\Omega_1, \Omega_2 = const)$ in Fig.9, for two frequency like nonlinear stationary vibration regimes, strong resonant interactions between time modes in the nm-the eigen shape amplitude mode are visible and presented for the discrete values of the $\Omega_{2nm} = const$, $\Omega_{2nm} = 145\,\text{s}^{-1}, 156\,\text{s}^{-1}, 158\,\text{s}^{-1}, 160\,\text{s}^{-1}, 170\,\text{s}^{-1}, 180\,\text{s}^{-1}$ from the interval $\Omega_{2nm} \in \left[145\,\text{s}^{-1}, 180\,\text{s}^{-1}\right]$ and the values of $\Omega_{1nm} = const$ continuously in the interval $\Omega_{1nm} \in \left[50\text{s}^{-1}, 250\text{s}^{-1}\right]$.

2* Amplitude-frequency curves $a_{1nm}(\Omega_1 = const, \Omega_2)$, in Fig.10, and $a_{2nm}(\Omega_1 = const, \Omega_2)$ in Fig.11 for two frequency like nonlinear stationary vibration regimes, strong resonant interactions between modes are visible and presented for the discrete values of the $\Omega_{1nm} = const$, Ω_{1nm}=$100\,\text{s}^{-1}, 120\,\text{s}^{-1}, 160\,\text{s}^{-1}, 170\,\text{s}^{-1}, 180\,\text{s}^{-1}, 190\,\text{s}^{-1}, 210\,\text{s}^{-1}$, $220\,\text{s}^{-1}$ from the interval $\Omega_{1nm} \in \left[100\text{s}^{-1}, 220\text{s}^{-1}\right]$ and the values of $\Omega_{2nm} = const$ continuously in the interval $\Omega_{2nm} \in \left[100\text{s}^{-1}, 250\text{s}^{-1}\right]$.

In this case the difference among first $\hat{p}_1 = 87.33\left(s^{-1}\right)$ and the second $\hat{p}_2 = 148.42\left(\text{s}^{-1}\right)$ frequencies is greater than in the previous case for m=240 kg. Therefore, the overlap of the resonant regions of the first $\Omega_{1nm} \in [120, 210]\left(\text{s}^{-1}\right)$ and the second $\Omega_{2nm} \in [156, 175]\left(\text{s}^{-1}\right)$ frequencies is smaller and the mutual interactions of the modes are less obvious.

The appearance of new resonant branches has the identical mechanism as in the first case. The branches appear first on the right lower side of the main resonant curve for the second resonant region at the value $\Omega_{2nm} = 156\left(\text{s}^{-1}\right)$, Figures 8 and 9, and for the first resonant region at the value $\Omega_{1nm} = 120\left(\text{s}^{-1}\right)$, Figures 10 and 11. Again, the resonant region of the first frequency of the external excitation is wider from the resonant region of the second frequency and the amplitude-frequency curves of the first harmonics are practically the same after the resonant transition, Figures 8 and 10, while the curves of the second harmonics undergo major changes by rotating, summarizing and covering a narrower frequency range, with the decrease of values, Figures 9 and 11.

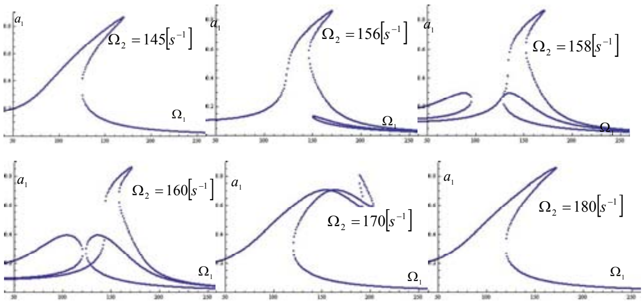


Figure 8. Amplitude-frequency characteristic curves for the phases of the first time harmonics $a_{1nm} = f_1(\Omega_{1nm})$ on the different value of the excited frequency Ω_{1nm} for the discrete value of the excited frequency $\Omega_{2nm} = const$ with noted corresponding resonant jumps for m=100 kg.

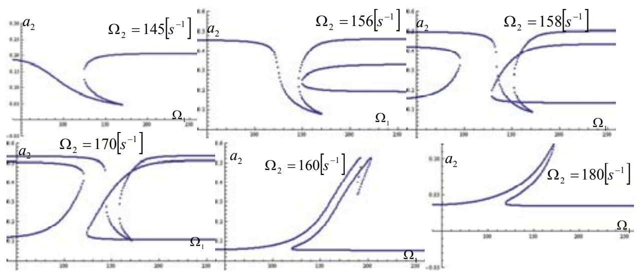


Figure 9. Amplitude-frequency characteristic curves for the phases of the second time harmonics $a_{1nm} = f_2(\Omega_{1nm})$ on the different value of the excited frequency Ω_{1nm} for the discrete value of the excited frequency $\Omega_{2nm} = \text{const}$ with noted corresponding resonant jumps for $m = 100\,\text{kg}$.

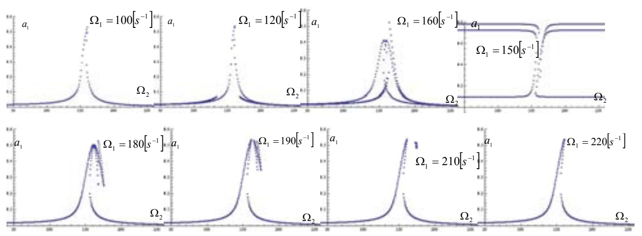


Figure 10. Amplitude-frequency characteristic curves for the phases of the first time harmonics $a_{1nm} = f_5(\Omega_{2nm})$ on the different value of the excited frequency Ω_{2nm} for the discrete value of the excited frequency $\Omega_{1nm} = \text{const}$ with noted corresponding resonant jumps for $m = 100\,\text{kg}$.

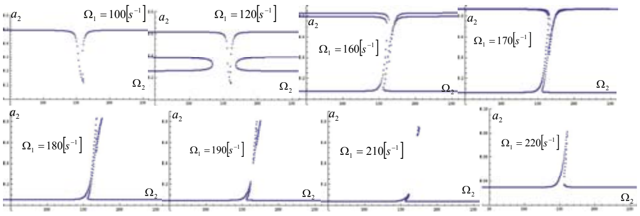


Figure 11. Amplitude-frequency characteristic curves for the phases of the second time harmonics $a_{2nm} = f_6(\Omega_{2nm})$ on the different value of the excited frequency Ω_{2nm} for the discrete value of the excited frequency $\Omega_{1nm} = \text{const}$ with noted corresponding resonant jumps for $m = 100\,\text{kg}$.

For the third, last, case, we practically consider the case without rolling elements in the connecting layer of two plates, $m = 0\,kg$. For the same selected numerical particular values of the other system parameters, the mutual interactions between two time modes in one nm -the eigen amplitude shape mode of plates vibration regimes are presented at Figs.12-15 for the following discrete values of the frequencies and the resonant frequencies intervals:

1* Amplitude-frequency curves $a_{1nm}(\Omega_1, \Omega_2 = \text{const})$, in Fig.12, and $a_{2nm}(\Omega_1, \Omega_2 = \text{const})$ in Fig.13, for two frequency like nonlinear stationary vibration regimes. The light resonant interactions between time modes in the nm - the eigen shape amplitude mode are visible and presented for the discrete values of the $\Omega_{2nm} = \text{const}$, $\Omega_{2nm} = 100\,\text{s}^{-1}, 190\,\text{s}^{-1}, 200\text{s}^{-1}, 201\text{s}^{-1}, 210\,\text{s}^{-1}$ from the interval $\Omega_{2nm} \in \left[100\text{s}^{-1}, 210\text{s}^{-1}\right]$ and the values of $\Omega_{1nm} = \text{const}$ continuously in the interval $\Omega_{1nm} \in \left[50\text{s}^{-1}, 250\text{s}^{-1}\right]$.

2* Amplitude-frequency curves $a_{1nm}(\Omega_1 = const, \Omega_2)$, in Fig.14, and $a_{2nm}(\Omega_1 = const, \Omega_2)$ in Fig.15 for two frequency like nonlinear stationary vibration regimes. The light resonant interactions between modes are visible and presented for the discrete values of the $\Omega_{1nm} = \text{const}$, $\Omega_{1nm} = 90\,\text{s}^{-1}, 100\,\text{s}^{-1}, 120\,\text{s}^{-1}$ from the interval $\Omega_{1nm} \in \left[90\text{s}^{-1}, 120\text{s}^{-1}\right]$ and the values of $\Omega_{2nm} = \text{const}$ continuously in the interval $\Omega_{2nm} \in \left[100\text{s}^{-1}, 250\text{s}^{-1}\right]$.

From Fig.12 and 13 it can be seen that the amplitude-frequency curve of the first harmonics passes through the resonant regime of the second frequency of the external excitation without characteristic resonant jumps, but the amplitude response of the second harmonics has resonant jumps in the resonant range of the second frequency of the external excitation $\Omega_{2nm} \in [185, 201]\, s^{-1}$ and after the resonant transition undergoes changes of values and shape. This leads to the conclusion that the first harmonics has more influence on the second one than in the opposite case.

In the case of the amplitude frequency curves of both harmonics in the resonant region for the discrete values of the first frequency $\Omega_{1nm} = 90\,\text{s}^{-1}, 100\,\text{s}^{-1}, 120\,\text{s}^{-1}$, Fig.14 and 15, we cannot notice the characteristic phenomenon of passing through the resonant regime; there are no resonant jumps but the curves of the amplitude for the second harmonics undergo the decrease of the value, Fig.15.

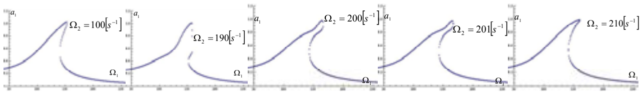


Figure 12. Amplitude-frequency characteristic curves for the phases of the first time harmonics $a_{1nm} = f_1(\Omega_{1nm})$ on the different value of the excited frequency Ω_{1nm} for the discrete value of the excited frequency $\Omega_{2nm} = \text{const}$ with noted corresponding resonant jumps for $m = 0\,\text{kg}$.

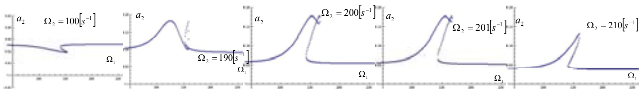


Figure 13. Amplitude-frequency characteristic curves for the phases of the second time harmonics $a_{2nm} = f_2(\Omega_{1nm})$ on the different value of the excited frequency Ω_{1nm} for the discrete value of the excited frequency $\Omega_{2nm} = const$ with noted corresponding resonant jumps for $m = 0\,\text{kg}$.

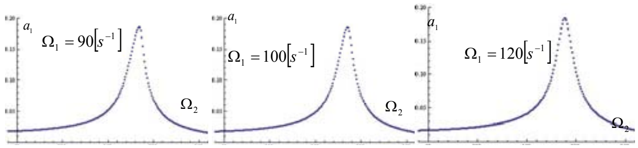


Figure 14. Amplitude-frequency characteristic curves for the phases of the first time harmonics $a_{1nm} = f_5(\Omega_{2nm})$ on the different value of the excited frequency Ω_{2nm} for the discrete value of the excited frequency $\Omega_{1nm} = const$ without noted corresponding resonant jumps for $m = 0\,\text{kg}$.

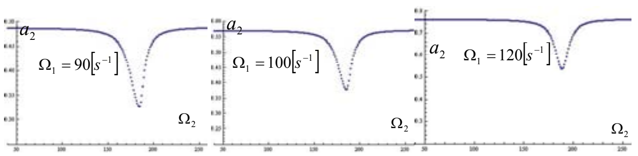


Figure 15. Amplitude-frequency characteristic curves for the phases of the second time harmonics $a_{2nm} = f_6(\Omega_{2nm})$ on the different value of the excited frequency Ω_{2nm} for the discrete value of the excited frequency $\Omega_{1nm} = \text{const}$ without noted corresponding resonant jumps for $m = 0\,\text{kg}$.

Hence, the amplitude responses in this case are similar to the case when there is no nonlinearity; we may conclude that the influence of the nonlinearity in the coupling layer is insignificant for such a selection of all other system parameters. The influence of the nonlinearity in the interconnecting layer may be less or more present depending on the parameters of the system. For example, by changing the value of the amplitude of the external excitations or the damping coefficient, we may find the same phenomena of resonant transition, resonant jumps and mutual modes interactions.

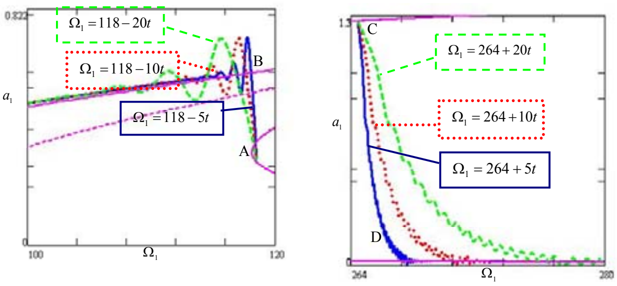
The characteristic of the presented series of the amplitude-frequency and phase-frequency curves for two-frequency like nonlinear stationary vibration regimes is that more than one pair of the resonant jumps appear, together with more than one instability branch in the corresponding amplitude-frequency and phase-frequency curves. It is visible that in the listed discrete values of the external excitation frequency from the corresponding resonant intervals two pairs plus one or three pairs with one more resonant jump appear together with a corresponding non-stable branch.

For obtaining data on stability or non-stability of the stationary amplitude and phase, it is necessary to use linearization of the system of the first approximation differential equations for two amplitudes and two phases in each discrete stationary vibration state and to compose a corresponding characteristic equation. Using real parts of the roots of the corresponding characteristic equation, it is possible to conclude whether the stationary two frequency like nonlinear vibration regimes are stable or not. If all real parts of all the roots of the characteristic equation are negative, then the regime is stable, or if only one is positive, then the regime is non-stable.

Analysis of non-stationary regimes of transversal vibrations of a double plate system

In order to investigate non-stationary regimes of transversal nonlinear oscillations of a double circular plate system as nonlinear dynamics for the presented model, we have numerically integrated the obtained system of DE's of the first approximation (19-22). We have used the Runge-Kutta's method of IV order in the MathCAD computer program and obtained the amplitudes-frequencies characteristics for the system of two circular plates connected with a viscous nonlinear elastic rolling layer shown in Fig.16. In such a manner, by changing the external excitation frequency, we can explain the passing through resonant intervals depending on the exchange velocity of external excitation frequencies $\Omega_i(t)$.

The numerical consideration of the system (19-22) of differential equations in the first asymptotic approximation for the corresponding amplitudes $R_{inm}(t)$ and the difference of phases $\varphi_{inm}(t)$ of two frequency like vibration regimes in the light of the stationary and non-stationary resonant regimes gives us the graphic results and the corresponding conclusions.



a) b)

Figure 16. Amplitude-frequency curves $a_{1nm} = f(\Omega_{1nm})$ of the passing through the resonant non-stationary kinetic states by the external excitation frequency change and for the cases with slow changing frequency with different velocities as well as by increasing and also decreasing values in the interval around the value of the eigen frequency of the corresponding linearized coupled double plate system. a) decreasing of the external excitation frequency $\Omega_1 = A - Bt$ by diferent velocities: $\Omega_{1nm} = 118 - 5\,\mathrm{t}$, $\Omega_{1nm} = 118 - 10\,\mathrm{t}$ and $\Omega_{1nm} = 118 - 15\,\mathrm{t}$; and b) increasing of the external excitation frequency $\Omega_1 = A + Bt$ by different velocities $\Omega_{1nm} = 264 + 5\,\mathrm{t}$, $\Omega_{1nm} = 264 + 10\,\mathrm{t}$ and $\Omega_{1nm} = 264 + 20\,\mathrm{t}$; of the first harmonics of the external excitation, with the initial values on the borders of the considered frequency interval and for the external excitation second frequency constant $\Omega_{2nm} = 360(\mathrm{s}^{-1})$.

Fig.16 presents the amplitude-frequency curves $a_{1nm} = f(\Omega_{1nm})$ of the passing through the resonant non-stationary kinetic states by the external excitation frequency change and for the cases with slow changing of frequency with different velocities as well as by increasing and also decreasing values in the interval around the value of the eigen frequency $\hat{p}_{inm}$ of the corresponding linearized coupled double plate system. Fig.16a) presents the amplitude-frequency curves for non-stationary regimes fot the decrease of the external excitation frequency $\Omega_1 = A - Bt$ by different velocities: $\Omega_{1nm} = 118 - 5t$, $\Omega_{1nm} = 118 - 10t$ and $\Omega_{1nm} = 118 - 15t$. Fig.16 b) presents the amplitude-frequency curves for non-stationary regimes for the increase of the external excitation frequency $\Omega_1 = A - Bt$ by different velocities $\Omega_{1nm} = 264 + 5t$, $\Omega_{1nm} = 264 + 10t$ and $\Omega_{1nm} = 264 + 20t$; of the first harmonics of the external excitation, with the initial values on the borders of the considered frequency interval and for the second external excitation second frequency constant $\Omega_{2nm} = 360(\mathrm{s}^{-1})$.

From the amplitude frequency characteristics presented in Fig.16 for the passing through the resonant interval by these different velocities of the external excitation frequency change, we can conclude that, for lower velocities of the external excitation frequency change through the resonant frequency interval, non-stationary amplitudes take larger values than in the case of higher frequency velocities.

In the same Fig.16 a) and b) the amplitude-frequency curves for the stationary resonant regimes are included. Then it is possible to compare the values of the amplitudes of the stationary and non-stationary regimes for passing through the resonant range by the external excitation frequency with different velocities. By this comparison we can conclude that non-stationary amplitudes for the passing through the resonant range frequency interval with slowchanging external excitation frequency follow the stationary stable amplitude branch and on the resonant jumps appear "resonant jumps oscillations" with jumps from higher to lower amplitudes for decreasing frequency, such as from lower to higher amplitudes for the external excitation frequency increasing on the corresponding resonant jumps frequency for stationary amplitudes.

Concluding remarks

The phenomena such as enlargement and jumps of amplitudes and phases of system oscillations, transition processes or hysteresis in the dynamics of systems, mutual interaction of modes in multi-frequency regimes may be explained by introducing the nonlinear elements in mathematical modeling of material properties of system sub elements. This paper explained these phenomena caused not only by nonlinearity but also by the presence of rolling elements with their translation and rotation without sliding. The viscous non-linear elastic rolling element modeled in the manner of rheological models presents the Kelvin-Voigt material with added spherical material particles.

We have analyzed stationary and non-stationary regimes of nonlinear oscillations for the presented model to explain the passing through the resonant regimes, amplitude and phase jumps and multi modes mutual interaction. A mathematical model of a two-plate sandwich system, a system of PDE`s (14), was used for solving an averaged asymptotic first approximation semi analytically and semi numerically. One part of the solutions was obtained numerically presenting amplitudes-frequencies and phase-frequency characteristics. These characteristics explained the interaction of the nonlinear component modes and nonlinear resonant interactions in the first asymptotic approximations of the displacement of the plate middle surface points. We concluded that the complexity in the system nonlinear response for the two frequency external excitations and the resonant range of the frequencies depends on the initial conditions and also on other system kinetic parameters and the corresponding relation between these sets of kinetic parameters.

The presented model of new features in the interconnecting layer introduced with rolling elements with their inertia of rolling without sliding and of translation of mass center is a novelty in the modeling of rheological elements. The presence of rolling elements in the interconnecting layer introduces the part of the dynamic coupling into the system of the obtained PDE's describing the nonlinear dynamics of the presented plates system. Based on the numerical comparisons, we conclude that the dynamic coupling intensifies the phenomenon of resonance transition caused by the mutual interaction of harmonics. Non-linearity is a source for the appearance of two resonant jumps in the amplitude-frequency and phase-frequency curves inside the interval of the resonant frequency. Three or five, or seven on more singular values of the stationary amplitudes and phases appear between two jumps together with alternatively stable and unstable values that build coupled singularities and trigger of coupled singularities. The trigger of coupled singularities consists of two stable amplitudes and phases around one unstable one. The unique values of the amplitudes and phases lose their stability and split into the trigger of three coupled singularities such as two stable values and one unstable passing through resonant intervals for a simple case without nonlinear interactions between time modes. But, in the case when there are resonant interactions between modes, more than one pair of the resonant jumps appear, and there are possibilities for the appearance of coupled triggers of coupled singularities containing odd number of alternating coupled stable and unstable singularities.

A trigger of coupled stationary amplitude and phase values, of two stable and one non-stable (saddle point) on the one external force frequency of two- or multi-frequency resonant regimes is presented in the considered nonlinear dynamics of the double plate system. More than one trigger of coupled stationary amplitude and phase values, of two stable and one non-stable (saddle point) on the one external force frequency are presented for the case of the interactions between resonance nonlinear modes. The coupled trigger of coupled singularities is presented as well in the resonant stationary coupled states.

The results presented in Section 5 lead to a conclusion about transient processes of system dynamics. Regimes of turning on and turning off the systems may be examples of transient processes in terms of non-stationary system regimes. For turning on regimes, it is better to use low velocities of external frequency changes because of a smaller number of amplitude jumps which happen on lower values of the external excitation frequency, Fig.16 b). For turning off regimes, in the sense of frequency decrease, it is better to use higher velocities of the external frequency changes because of a smaller number of amplitude jumps which happen on lower values of the external excitation frequency, Fig.16 a).

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Nelinearna dinamika sistema dve ploče spregnute slojem visko-elastičnih i inercijalnih svojstava

U radu su prikazane više frekventne oscilacije sistema dve izotropne kružne ploče spojene slojem kotrljajnih visko-elastičnih nelinearnih elemenata. Ovakav fizički sistem imam veliki značaj u proučavanjima vibracionih i akustičkih apsorbera. Sprežući sloj je modeliran kao kontinualno raspodeljen sloj diskretnih standardnih reoloških elemenata sa svojstvima prigušenja, nelinearne elastičnosti i inercije kotrljanja bez klizanja.

Matematički model sistema predstavljen je u obliku sistema parcijalnih diferencijalnih jednačina prinudnih transverzalnih oscilacija tačaka srednjih ravni ploča spregnitih slojem kotrljajnih visko-elastičnih elemenata pod dejstvom harmonijske kontinualno raspodeljene po površinama ploča pobude. Sistem običnih deiferencijalnih jednačina prvog reda po amplitudama i faznim kašnjenja vremenskih funkcija, odgovarajućih sopstvenih oblika oscilovanja ploča, u prvoj asimptotskoj aproksimaciji izveden je za različite odgovarajuće više frekventne režime oscilovanja. Potom je taj sistem analitički i numerički posmatran u svetlu stacionarnih i nestacionarnih rezonanatnih režima i međuinterakcija nelinearnih modova, kao i broja rezonantnih skokova i to u slučajevima kada nema kotrljajućih elemenata u sloju i za dve različite vrednosti masa kotrljajućih elemenata.

Ovakva analiza pokazuje da prisustvo kotrljajućeg elementa kao reprezenta dinamičke sprege ploča uzrokuje preklapanje rezonantnih oblasti nelinearnih modova što u isto vreme izaziva uvećanje njihove međuinterakcije.

Ključne reči: **dinamika sistema, nelinearna dinamika, oscilacije, ploča, rezonantni režim, rezonantni skokovi, matematički model, parcijalne difernecijalne jednačine.**

Анализ нелинейной динамики систем из двух пластин соединённых слоем нелинейных элементов вязкоупругого качения

В настоящей работе представлены некоторые колебания частоты систем двух изотропных круговых пластин, соединённых слоем нелинейных элементов вязкоупругого качения. Такие физические системы имеют большое значение в изучении колебательных и акустических поглотителей. Соединительный слой моделируется как сплошной слой распределённых дискретных элементов со стандартными реологическими свойствами затухания, нелинейной теории упругости и инерции качения без скольжения.

Математическая модель системы представлена в виде систем частичных дифференциальных уравнений принудительных поперечных колебаний высокой точки средних плоских панелей, соединённых слоем нелинейных элементов вязкоупругого качения, под влиянием гармонического возбуждения непрерывно распределённого в соответствии с поверхностью панели. Система обыкновенных частичных дифференциальных уравнений первого порядка по амплитуде и фазе временной задержки функции, соответствующих собственных форм пластины колебания, в первом асимптотическом приближении получается более подходящей для различных соответствующих режимов частоты колебаний. Потом система аналитически и численно рассматривана в свете стационарных и нестационарных резонантных режимов и взаимодействия нелинейных режимов, а в том числе и количества резонантных прыжков, и то при отсутствии качения элементов в слое и для двух различных значений массы тела качения.

Этот анализ показывает, что наличие подвижного элемента в качестве представителя динамической плиты связи вызывает перекрытие резонансных областей нелинейных режимов, что в то же время вызывает увеличение причины их взаимодействий.

Ключевые слова: **динамика системы, нелинейная динамика, колебания, плита, резонансный режим, резонансные прыжки, математическая модель, частичные дифференциальные уравнения.**

Analyse de la dynamique non linéaire du système de deux plaques couplées par une couche d'éléments roulants hautement élastiques

Les oscillations multi fréquentes du système de deux plaques circulaires isotropes connectées par une couche d'éléments roulants non linéaires et hautement élastiques ont été présentées dans ce travail .Ce système est très important pour les recherches des absorbeurs vibratoires et acoustiques. La couche connectée est modelée comme une couche distribuée continuellement composée des éléments discrets standards rhéologiques avec les propriétés d'étouffement, d'élasticité non linéaire et de l'inertie de roulement sans glissement.

Le modèle mathématique du système est présenté sous la forme du système des équations différentielles partielles des oscillations forcées transversales des points de moyens plans des plaques couplées par une couche d'éléments roulants hautement élastiques sous l'effet de l'excitation harmonique distribuée continuellement sur les surfaces des plaques. Le système des équations différentielles simples du premier ordre sur les amplitudes et les fonctions temporelles du délai en phase, des formes propres d'oscillations des plaques, dans la première approximation asymptotique a été dérivée pour les différents régimes d'oscillations multi fréquents. Ce système a été ensuite examiné analytiquement et numériquement sous l'aspect des régimes des résonances stationnaires et non stationnaires et des interactions des modes non linéaires. On a considéré également le nombre des sauts de résonance pour les cas où la couche ne contient pas d'éléments roulants et pour deux différentes valeurs des masses des éléments roulants.

Cette analyse démontre que la présence de l'élément roulant en tant que le représentant du couplage dynamique des plaques provoque le chevauchement des régions de résonance chez les modes non linéaires causant en même temps l'augmentation de leur interaction.

Mots clés: **dynamique de système, dynamique non linéaire, oscillations, plaque, mode de résonance, sauts de résonance, modèle mathématique, équations différentielle partielles.**