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Existence of Triggers of Coupled Singularities in Nonlinear Dynamics of Mechanical Systems with Coupled Rotations

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A theorem of triggers of coupled singularities is presented as well as numerous examples of nonlinear dynamics of mechanical systems with coupled singularities in phase portraits. Abstractions of real engineering system nonlinear dynamics with rotations coupled into the model of a rigid body which performs coupled rotations around nonintersecting axes in the gravitational field shows numerous varieties of the homoclinic phase of trajectories as well as different sets of tigers of coupled singularities. A multi-parameter transformation of the phase trajectories and of the set of coupled singularities is presented. In addition, a series of triggers of coupled singularities in the phase portraits is given as well as the trigger of coupled half-one side singularities identified in the heavy mass particle oscillations/motion along a rotating rough curvilinear line and non-ideal constraints of Amontons-Coulomb friction. An example is used to show the heavy mass particle motion along rough curvilinear lines in the vertical plane, described by a corresponding differential double equation and the double equation of the phase trajectories, while more triggers of coupled half-one side singularities are identified in the phase portrait.

Key words: **mechanical system, trigger, nonlinear dynamics, coupled singularity, rotating system.**

Introduction

IN one of classical monographs [1] by *Andronov*, *Witt and Hajkin*, which has a great number of editions, some classical examples of nonlinear systems with one degree of freedom of oscillatory motion and their phase portraits except general theory of nonlinear oscillations are presented. Such examples can also be found in university books by Stoker [19] as well as by Rašković [15, 16, 17]. Especially in a monograph by *Guckenheimer and Holmes* [3] and a monograph by *Gerard and Daniel* [2], the results of research on nonlinear systems and properties of various kinds of bifurcations are pointed out.

In the papers [14], differential double equations of the heavy mass particle motion along rough curvilinear line with Coulomb's type friction are expressed in the following generalized form with double signs:

$$\begin{aligned}&\ddot{x} \pm b_{\mu}\dot{x}^2 + g\left[k, F\left(x, \mp x_{\mu}\right)\right] f\left(x, \pm x_{\mu}\right) = \\ &= 0 \quad \begin{matrix} \textit{uper sign} \\ \textit{lower sign} \end{matrix} \quad \text{for} \quad \begin{matrix} \dot{x} > 0 \\ \dot{x} < 0 \end{matrix}\end{aligned} \tag{1}$$

where b_{μ} is a coefficient depending on the Coulombs type coefficient of friction, and x_{μ} is a parameter in a coordinate dimension depending on the Coulombs type coefficient of friction. Also, a corresponding governing differential equation for ideal conservative system dynamics with one degree of freedom in the following form:

$$\ddot{x} + g\left[k, F(x)\right] f(x) = 0 \tag{2}$$

was investigated in the phase plane according to the structure of the singular point and stability of the structure of phase portrait.

Theorem of the existence of a trigger of coupled singularities

By using nonlinear dynamic analysis of systems with described nonlinear phenomenon of the trigger of coupled singularities and corresponding families of phase portraits and potential energies (see Refs. [5], [6], [8] and [14]) as well as the corresponding experimental investigations of such non-linear dynamics in mechanical engineering systems with coupled rotation motions (see Refs. [4] and [7-14]), it was easy to define and to prove the theorem of the existence of a trigger of coupled singularities in non-linear dynamical systems with a periodical structure.

Theorem: In the system whose dynamics can be described with the use of non-linear differential equation $\ddot{x} + g\left[k, F(x)\right] f(x) = 0$, *n* the form (2) (see Refs. [5], [6] and [11]) and whose potential energy is in the form:

$$\mathbf{E}_p = m\int_0^x g\left[k, F(x)\right] f(x)\, dx = \mathbf{G}\left[k, F(x)\right] \tag{3}$$

in which the functions $f(x)$ and $g(x)$ are:

$$F(x) = \int_0^x f(x)\, dx \text{ and } G(k,x) = \int_0^x g(k,x)\, dx \tag{4}$$

and satisfy the following conditions: $f(-x) = -f(x)$, $f(x + nT_0) = f(x)$, $f(0) = 0$, $f(x_s) = 0$, $x_s = sT_0$, $s = 1,2,3,4,...$ $x_r = \pm x_0 \pm rT_0$, $r = 0,1,2,3,4,...$, $|x_0| < \frac{T_0}{2}$,

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$g(k,x+nT_0)=g(k,x)$, $g(k-x)=g(k,x)$, $g[k,F(x_r)]=0$, for $k\in(k_1,k_2)\cup(k_2,k_3)...$ $g[k,F(x)]\neq 0$, for $k\notin(k_1,k_2)\cup(k_2,k_3)...$ and both functions $f(x)$ and $g(x)$ have one maximum or minimum in the interval between two zero roots: a* for parameters values $k\notin(k_{1\neq},k_2)\cup(k_2,k_3)...$, outside of the intervals $(k_{1\neq},k_2)\cup(k_2,k_3)...$, **the trigger of singularities** in the local area **does not exist;** b* for parameters values $k\in(k_{1\neq},k_2)\cup(k_2,k_3)...$, inside the intervals $(k_{1\neq},k_2)\cup(k_2,k_3)...$, **the series of triggers of coupled singularities in the local domains exists.**

A series of the theorems of a trigger of coupled singularities in the nonlinear dynamics of mechanical systems with appearance of a homoclinic orbit in the form of number eight is defined in references [5] and [6].

Example 1: Rheonomic systems with equivalent conservative systems applied to the nonlinear dynamics of the Watt's regulator

In this part, an example of rheonomic nonlinear systems which have an equivalent conservative nonlinear system by the model of the Watt's regulator will be presented. We will consider a nonlinear system with coupled rotations with two degrees of mobility, but with one degree of freedom of motion defined by one generalized coordinate, and one of two degrees of mobility is defined by the rheonomic coordinate linearly depending on time. A model of the Watt's regulator as a model of rheonomic system with coupled rotations is used to prove a theorem of the existence of homoclinic orbits (see Refs. [12] by Hedrih (Stevanovic)) in the form of number eight and the trigger of coupled singularities in the phase portrait of this nonlinear dynamics of relative motions.

For an example of a rheonomic system with coupled rotations, and with an equivalent holonomic scleronomic conservative system, we will consider a model of a mechanism of the Watt's regulator (see Fig.1.a), containing two heavy material particles moving along the corresponding symmetrically connected circles that rotate around the vertical axis with a constant angular velocity Ω in the gravitational field. The kinetic and potential energy of the simplified mechanisms of the Watt's regulator, are

$$\mathbf{E}_{\mathbf{k}}^{(sist)}=m\ell^2\left[\dot{\phi}^2+\dot{\theta}^2\left(\frac{a}{\ell}+\sin\phi\right)^2\right] \text{ and}$$
$$\mathbf{E}_{\mathbf{p}}^{(sist)}=2mg\ell(1-\cos\phi) \tag{5}$$

where ϕ is the generalized independent coordinate, $q^{o}=\theta=\Omega t$ is the rheonomic coordinate depending on time, m are masses of the balls - the material particles, and ℓ and a are constructive parameters. For a holonomic scleronomic conservative system equivalent to the considered rheonomic system, which is a mechanism of the Watt's regulator, the kinetic and potential energy in the sense of the previous definitions are (see ref. [12]):

$$\tilde{\mathbf{E}}_{\mathbf{k}}^{(sist)}=m\ell^2\dot{\phi}^2$$

and

$$\tilde{\mathbf{E}}_{\mathbf{p}}^{(sist)}=2mg\ell(1-\cos\phi)-m\ell^2\Omega^2\left(\frac{a}{\ell}+\sin\phi\right)^2.$$

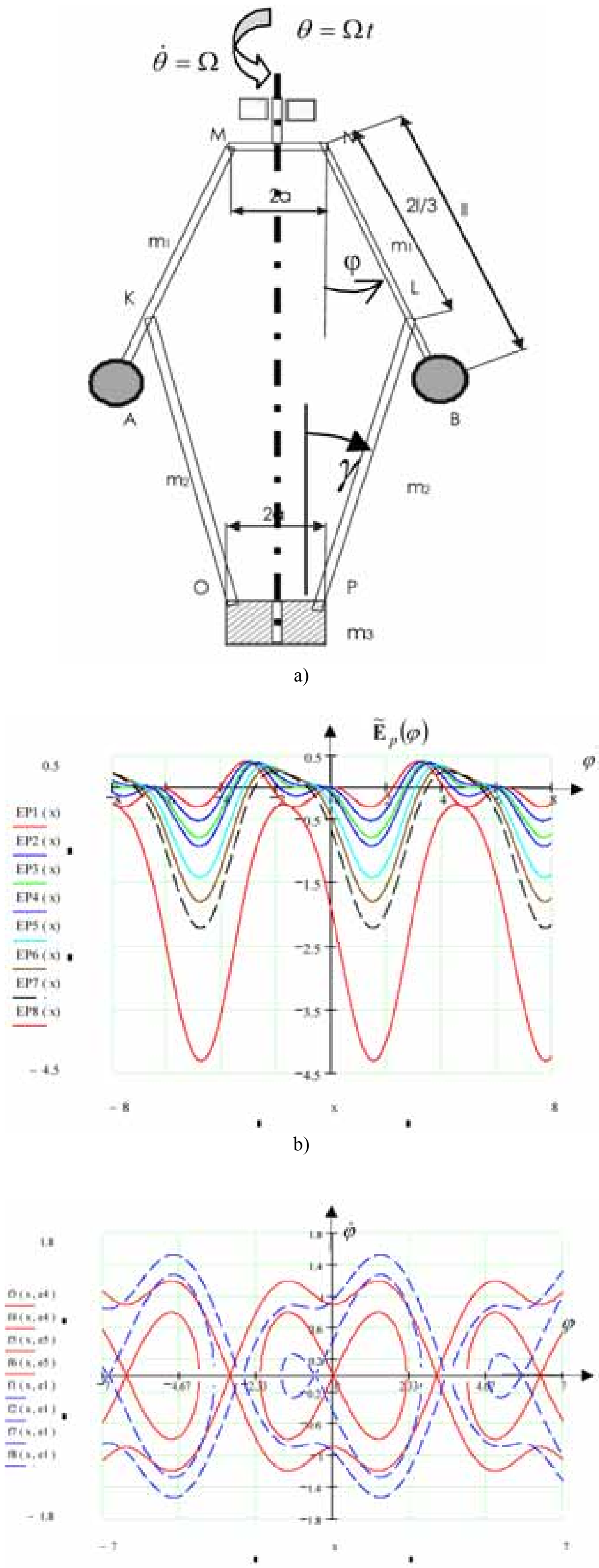
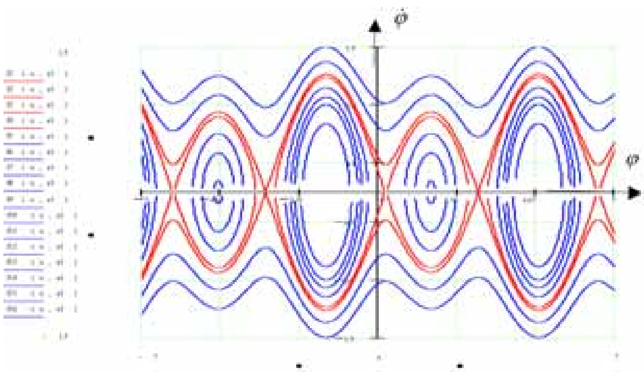
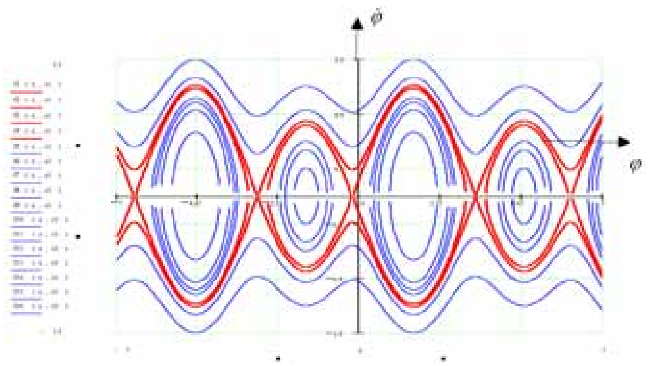


Figure 1. (a) Watt's regulator. (b) Potential energy $\tilde{\mathbf{E}}_p(\phi)$ graphs of the equivalent holonomic scleronomic conservative system to the rheonomic system (c) characteristic homoclinic orbits $\dot{\phi}(\phi)$ of nonlinear dynamics of the Watt's regulator for different system kinetic parameters: $\lambda=0.2;\ \varepsilon=0;+0.2;+0.4;+0.5;+0.8;+1;+1.2;+2;$



a)



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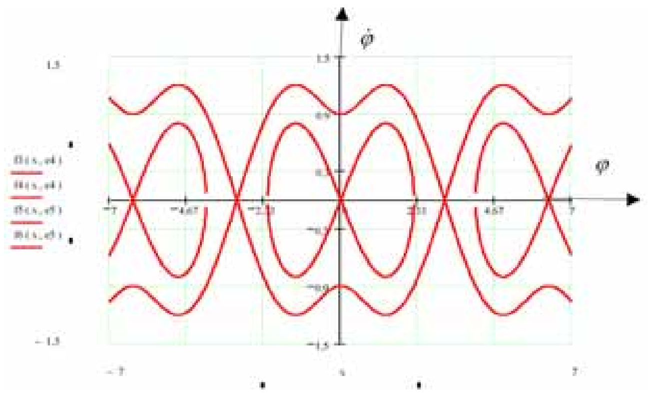

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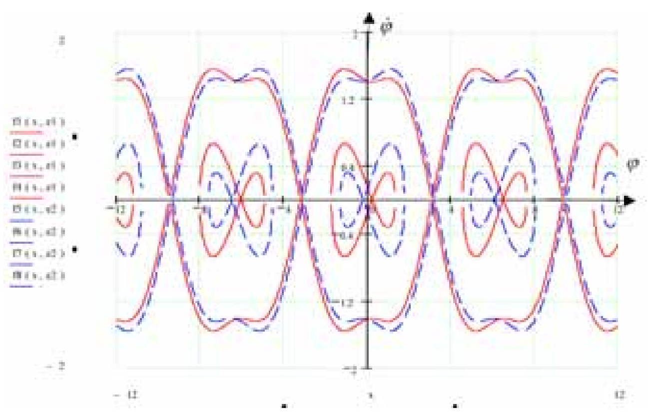

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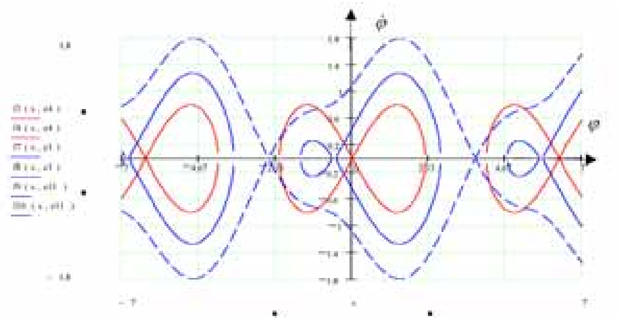


f)

Figure 2. a and b - Characteristic phase portrait of an equivalent scleronomic conservative system to the Watt's regulator nonlinear dynamics. (c, d)-(e, f) The homoclinic orbits transformations with changing of the system kinetic parameters values and the different forms of the homoclinic orbits of the equivalent holonomic scleronomic conservative system to the rheonomic system (the mechanisms of the Watt's regulator) by changing the eccentricity ε and the velocity of the support rotation $\Omega(\lambda)$.

By introducing the following notations $\dot{\theta}=\Omega$, $\lambda=\dfrac{g}{\ell\Omega^2}$, $\varepsilon=\dfrac{a}{\ell}$, we can write the following differential equations of the relative motion of the balls:

$$\ddot{\phi}+\Omega^2\left[(\lambda-\cos\phi)\sin\phi-\varepsilon\cos\phi\right]=0 \qquad (6)$$

The integral of the energy of the equivalent holonomic scleronomic conservative system to the rheonomic system, which is a mechanism of the Watt's regulator, is:

$$\begin{aligned}&\tilde{\mathbf{E}}_{\mathbf{K}}^{(sist)}+\tilde{\mathbf{E}}_{\mathbf{P}}^{(sist)}=const\Rightarrow\\&\dot{\phi}^2=\dot{\phi}_0^2+\Omega^2\left\{2\lambda(\cos\phi_0-\cos\phi)-\right.\\&\left.-\left[(\varepsilon+\sin\phi)^2-(\varepsilon+\sin\phi_0)^2\right]\right\}\end{aligned} \qquad (7)$$

where ϕ_0 and $\dot{\phi}_0$ are the initial conditions of the relative motion of the balls. Equation (7) presents also the phase trajectory of the relative motion.

In Fig.1 and 2, the results of the numerical experiment over the considered example of the rheonomic system are presented. Fig.1b, gives the potential energy graphs of the equivalent holonomic scleronomic conservative system to the rheonomic system (Fig.1a the mechanisms of the Watt's regulator) for different system kinetic parameters. From Fig.2, we can conclude that it is very suitable for the identification of homoclinic orbits in the form of number eight of nonlinear dynamics of the special class of rheonomic systems with coupled rotations and with rheonomic coordinate linearly depending on the time in the form $q^{o}=\Omega t$ to use the corresponding equivalent holonomic scleronomic conservative system and the corresponding phase portraits of this system. By using an example of the mechanisms of the Watt's regulator, we show different forms of homoclinic orbits, as well as the bifurcations of the relative equilibrium positions in the considered class of the rheonomic systems and the parametric transformation of the homoclinic orbits. We investigate existence and nonexistence of homoclinic orbits in the shape of number eight and a trigger of coupled singularities for different values of the system kinetic parameters: the eccentricity ε and the velocity of the support rotation $\Omega(\lambda)$. The examples of engineering systems such as the Watt's regulators pointed out the validity of the defined theorems of the existence of homoclinic orbits in the form of number eight and the trigger of coupled singularities in the phase portrait of its nonlinear dynamics.

Example 2: Differential double equation of the motion of a heavy mass particle along rough rotating circles

We will consider a discrete system of a heavy mass particle with the mass m along a rough rotating circle about a vertically positioned axis oriented by the unit vector $\vec{n}$, and in the case with Coulomb's type friction (see Fig.3) .The relative position of the mass particle along the rough circle with the radius ℓ is determined by the angle φ as a generalized coordinate. The rheonomic coordinate $\theta=\Omega t$ is the angle of the rotation of the circle around the vertical axis oriented by the unit vector $\vec{n}$.

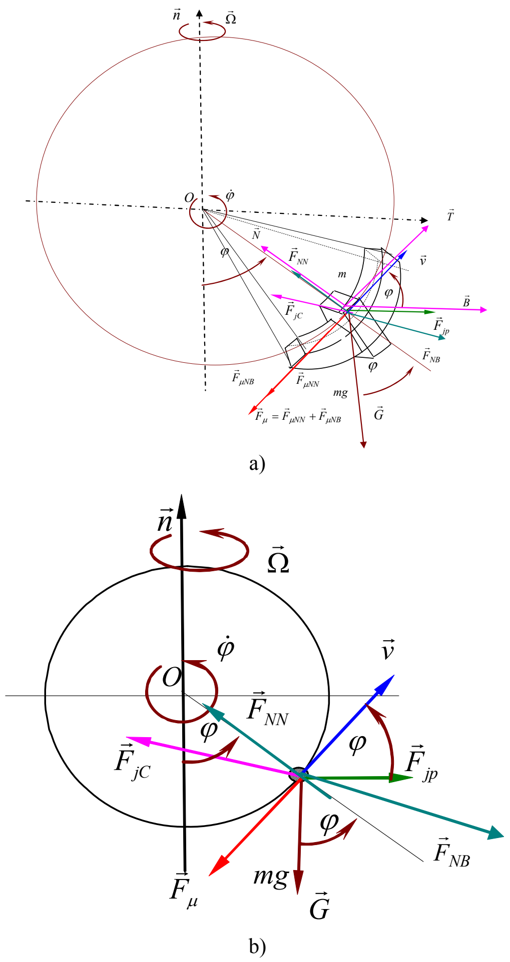


Figure 3. a - Rough surfaces around the "rough circle line" in a real construction with a corresponding coefficient of the Coulomb's type friction; b - Mass particle motion along the rough rotating circle about the vertical axis

We take into consideration the "rough circle line" with a rectangular cross section with one rough surface (or two surfaces) and one coefficient $\mu = tg\alpha_0$ of the Coulomb's type friction for the rough surfaces with a normal in the radial directions; when the surfaces with a normal in the binormal directions are ideal and without friction, then governing differential double equation is in the form:

$$\begin{gathered}\ddot{\phi} \pm \dot{\phi}^2 tg\alpha_0 + \frac{g}{\ell\cos\alpha_0}\sin(\phi \pm \alpha_0) - \\ -\frac{\Omega^2}{\cos\alpha_0}\cos(\phi \pm \alpha_0)\sin\phi = 0\end{gathered} \quad , \quad \dot{\phi} \gtrless 0 \tag{8}$$

Our research is focused on the first governing differential double equation (8) and for the beginning, we consider the simplest differential double equation in the form:

$$\begin{gathered}\ddot{\phi} + \frac{g}{\ell\cos\alpha_0}\sin(\phi \pm \alpha_0) - \\ -\frac{\Omega^2}{\cos\alpha_0}\cos(\phi \pm \alpha_0)\sin\phi = 0\end{gathered} \quad , \quad \dot{\phi} \gtrless 0 \tag{9}$$

corresponds to the differential double equation (8) in which the term $\pm\dot{\phi}^2 tg\alpha_0$ is omitted.

The first integral of the previous simplest differential double equation (9) of the corresponding fictive conservative system is obtained in the following form:

$$\begin{gathered}\dot{\phi}^2 = \frac{2g}{\ell\cos\alpha_0}\cos(\phi \pm \alpha_0) - \\ -\frac{2\Omega^2}{\cos\alpha_0}\left[\frac{1}{2}\cos(\phi \pm \alpha_0)\cos\phi \pm \frac{1}{2}\phi\sin\alpha_0\right] + C\left(\phi_0, \dot{\phi}_0\right)\end{gathered} \tag{10}$$

where $C\left(\phi_0, \dot{\phi}_0\right)$ is the integral constant depending on the initial values of the angular coordinate and the angular velocity for each trajectory branch. For the first phase trajectory branch, we take the previous equation with the upper sign, where $C\left(\phi_0, \dot{\phi}_0\right)$ is the integral constant depending on the initial conditions. For the first trajectory branch, this integral constant is in the form:

$$\begin{gathered}C_1\left(\phi_0, \dot{\phi}_0\right) = \dot{\phi}_0^2 - \frac{2g}{\ell\cos\alpha_0}\cos(\phi_0 + \alpha_0) + \\ + \frac{2\Omega^2}{\cos\alpha_0}\left[\frac{1}{2}\cos(\phi_0 + \alpha_0)\cos\phi_0 + \frac{1}{2}\phi_0\sin\alpha_0\right]\end{gathered} \tag{11}$$

The beginning of the next one, second branch, is at the kinetic state corresponding angular velocity equal to zero in the position when the friction force takes alternation of its direction. The next one, third branch, starts at the kinetic state determined by the angular velocity equal to zero at the corresponding angular coordinate. Then it is easy to write the necessary generalization using the conclusions based on the induction:

The initial (starting) kinetic state of the even $(2k)$-th branch is at the kinetic state corresponding to the angular velocity equal to zero and the alternation of the Coulomb's type friction force. Then, the initial conditions of the even $(2k)$-th trajectory branch are: $\phi_{alt,2k-1}$ and $\dot{\phi}_{alt,2k-1}^2\left(\phi_{alt,2k-1}\right)$, and the equation of the phase trajectory branch is in the following form:

$$\begin{gathered}\dot{\phi}_{2k}^2(\phi) = \frac{2g}{\ell\cos\alpha_0}\cos(\phi - \alpha_0) - \\ -\frac{2\Omega^2}{\cos\alpha_0}\left[\frac{1}{2}\cos(\phi - \alpha_0)\cos\phi - \frac{1}{2}\phi\sin\alpha_0\right] + \\ + C_{2k}\left(\phi_{alt,2k-1}, \dot{\phi}_{alt,2k-1} = 0\right)\end{gathered} \tag{12}$$

where $C_{2k}\left(\phi_{alt,2k-1}, \dot{\phi}_{alt,2k-1} = 0\right)$ is the integral constant depending on the initial conditions for the second trajectory branch and is determined by the following expression:

$$\begin{gathered}C_{2k}\left(\phi_{alt,2k-1}, \dot{\phi}_{alt,2k-1} = 0\right) - \frac{2g}{\ell\cos\alpha_0}\cos(\phi_{alt,2k-1} - \alpha_0) + \\ + \frac{2\Omega^2}{\cos\alpha_0}\left[\frac{1}{2}\cos(\phi_{alt,2k-1} - \alpha_0)\cos\phi_0 - \frac{1}{2}\phi_{alt,2k-1}\sin\alpha_0\right]\end{gathered} \tag{13}$$

Then, the initial conditions of the odd ($2k+1$)-th trajectory branch are: $\phi(t_{alt,2k})=\phi_{alt,2k}$ and $\dot{\phi}^2_{alt,2k}(\phi_{alt,2k})=0$, and $C_{2k+1}(\phi_{alt,2k},\dot{\phi}_{alt,2k}=0)$ is the integral constant depending on the initial conditions for the odd ($2k+1$)-th trajectory branch and is determined by the following expression:

$$C_{2k+1}(\phi_{alt,2k},\dot{\phi}_{alt,2k}=0)=-\frac{2g}{\ell\cos\alpha_0}\cos(\phi_{alt,2k}+\alpha_0)+ \\ +\frac{2\Omega^2}{\cos\alpha_0}\left[\frac{1}{2}\cos(\phi_{alt,2k}+\alpha_0)\cos\phi_0+\frac{1}{2}\phi_{alt,2k}\sin\alpha_0\right] \quad (14)$$

Form the previous analytical expressions of the phase trajectory branches, we can conclude that, in the fictive conservative system, an alternation of equilibrium positions, with a difference close, but not equal to $\pm\alpha_0$, for $1>>\frac{\omega_0^2}{\Omega^2}$, a very small angular velocity of circle line rotation appears. In a better approximation, this difference is approximately

$$\phi_s\approx\frac{\pm\alpha_0\frac{\omega_0^2}{\Omega^2}}{\left(1-\frac{\omega_0^2}{\Omega^2}\right)}. \quad (15)$$

In addition, we can conclude that this phenomenon is a type of two coupled equilibrium positions (with one side stable) in fictive alternations. We can conclude that a trigger of coupled singularities caused by Amontons-Coulomb's type friction forces contain two coupled one side stable singular points and one non stable singular point which in a corresponding system without friction is a unique singular point .

The stationary solutions of the governing nonlinear differential double equation (8) are the same as the stationary solutions of the corresponding fictive system described by differential double equation (9). The conditions of the relative equilibrium positions are: $\dot{\phi}=0$ and $\dot{v}=0$, and we obtain the following transcendent double equation:

$$tg(\phi\pm\alpha_0)-\frac{\Omega^2\ell}{g}\sin\phi=0 \quad (16)$$

For $\alpha_0=0$ a mechanical system is ideal, and the previous condition (16) obtained form $tg\phi-\frac{\Omega^2\ell}{g}\sin\phi=0$, and the singular points are: $\phi_s=s\pi$, $s=1,2,3,4,...$ and $\phi_s=\arccos\frac{g}{\ell\Omega^2}\pm 2s\pi$, $s=1,2,3,4,...\Rightarrow$ for the case that $\left|\frac{g}{\ell\Omega^2}\right|\le 1$. From the abovementioned, we can conclude that for $\left|\frac{g}{\ell\Omega^2}\right|\le 1$ in the phase portrait there are two forms of the separatrix phase trajectory, one of which is in the form of number "eight". With the existence of this homoclinic orbit in the form of number "eight", a trigger of coupled singularities, caused by Amontons-Coulomb's type friction forces, contain two coupled one side stable singular points and one non stable singular point which in a corresponding system without friction is unique center type singular points.

In addition, taking into our qualitative analysis that the coefficient of Coulomb's type friction is a small number, we can conclude that the roots of transcedent double equation (14) are close to the roots from the obtained set corresponding to an ideal system. Also, it is necessary to take into acount that transcedent double equation (14) contains a sign alternation, and that the obtained roots are one-side singular points corresponding to the one-side stable, or non- stable equilibrium positions.

For obtaining the roots of the transcendent double equation (14), it is necessary to use some method of approximation or a numerical method. Singular points – roots of the transcendent double equation (14) are in the intersections between two double functions: $f(\phi)=\frac{\omega_0^2}{\Omega^2}tg(\phi\pm\alpha_0)$ and $f(\phi)=\sin\phi$. The singular point in the sections between the previous two listed double functions takes into account that first function has the sign alternation $\pm\alpha_0$ of the argument. For small values of the ϕ and for the first two roots around null, it is possible to use an approximation in the form of linearization of these functions:

$$f(\phi)=\frac{\omega_0^2}{\Omega^2}tg(\phi\pm\alpha_0)\approx\frac{\omega_0^2}{\Omega^2}(\phi\pm\alpha_0) \quad (17)$$

and $f(\phi)=\sin\phi\approx\phi$. Then the first rough approximate values of two roots are:

$$\phi_s\approx\frac{\pm\alpha_0\frac{\omega_0^2}{\Omega^2}}{\left(1-\frac{\omega_0^2}{\Omega^2}\right)}\pm s\pi. \quad (18)$$

This pair of the two roots presents the first pair of the one-side stable singular points (half-center type) which corresponds to the one-side stable equilibrium positions.

The second series of the approximation of roots is possible to be obtained with the expansion of the listed functions into Taylor's series around null in the following form:

$$f(\phi)=\sin\phi\approx\phi-\frac{\phi^3}{6} \quad (19)$$

and

$$f(\phi)=\frac{\omega_0^2}{\Omega^2}tg(\phi\pm\alpha_0)\approx\frac{\omega_0^2}{\Omega^2}(\phi\pm\alpha_0). \quad (20)$$

Then we obtain the following third order nonlinear double equations:

$$\phi^3-2\phi\left(1-3\frac{\omega_0^2}{\Omega^2}\right)\pm 6\frac{\omega_0^2}{\Omega^2}\alpha_0\approx 0. \quad (21)$$

For solving the previous nonlinear cubic double equation (21) and to obtain their three roots, we applied the following formulas of roots approximations

$$x_1\approx 2\sqrt{-\frac{p}{3}}\cos\frac{\chi}{3},\ x_2\approx 2\sqrt{-\frac{p}{3}}\cos\frac{\chi+2\pi}{3}$$

and

$$x_3\approx 2\sqrt{-\frac{p}{3}}\cos\frac{\chi+4\pi}{3} \quad (22)$$

of the cubic equation in the form $x^3+px+q=0$ (for details see Rašković, P. [18]), where the condition $\Delta=\left(\frac{1}{3}p\right)^3+\left(\frac{1}{3}q\right)^3<0$ is satisfied, and the following denotation

$$\chi=\arccos\left(\frac{3q}{2p\sqrt{-\frac{p}{3}}}\right) \tag{23}$$

is introduced. The corresponding coefficients of this cubic equation (22) are in the following forms:

$$p=-2\left(1-3\frac{\omega_0^2}{\Omega^2}\right)<0\,,\ 1>3\frac{\omega_0^2}{\Omega^2}\,,\ \Delta=\left(\frac{1}{3}p\right)^3+\left(\frac{1}{3}q\right)^3<0 \tag{24}$$

and when

$$p<0\,,\ q=\pm 6\frac{\omega_0^2}{\Omega^2}\alpha_0\,,$$

$$\Delta=\left[-\frac{2}{3}\left(1-3\frac{\omega_0^2}{\Omega^2}\right)\right]^3+\left[\pm 2\frac{\omega_0^2}{\Omega^2}\alpha_0\right]^3<0\,, \tag{25}$$

$$\chi=\arccos\left(\frac{\pm 18\frac{\omega_0^2}{\Omega^2}\alpha_0}{-4\left(1-3\frac{\omega_0^2}{\Omega^2}\right)\sqrt{\frac{2\left(1-3\frac{\omega_0^2}{\Omega^2}\right)}{3}}}\right) \tag{26}$$

(see Fig.4).

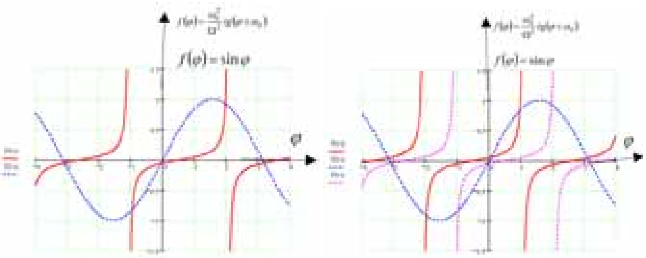
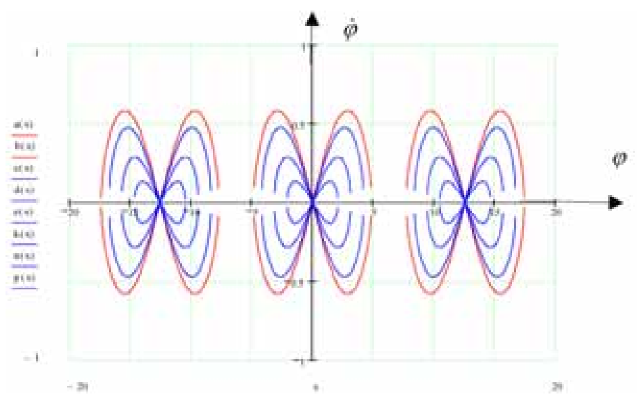
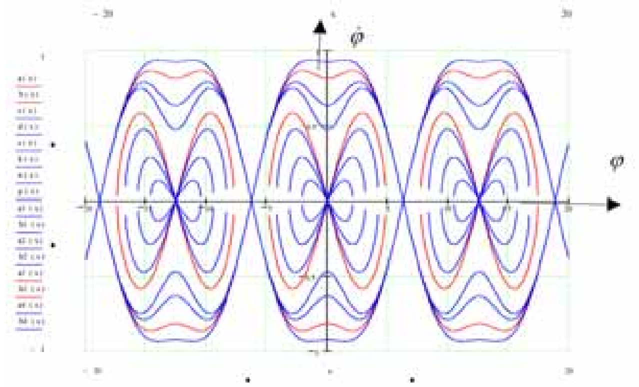


Figure 4. For $u=\alpha_0=0.03$ and non-ideal circle line, and $\frac{g}{\ell\Omega^2}=\frac{\omega_0^2}{\Omega^2}=k=0.5$, the first series of the roots $\phi_s=\phi_{s0}\pm s\pi$ with (a) left half side non stability and (b) right half side non stability, and the second series of the roots, and $\phi_{\pm 2r}=-\phi_{\pm 2r0}\pm 2s\pi$, with (a) left half side stability and (b) right half side stability.



a)



b)

$$\ddot{\phi}+\Omega^2(\lambda-\cos\phi)\sin\phi-\Omega^2\varepsilon\cos\phi=0$$

Figure 5. Homoclinic phase trajectory layering (a) and (b) for $\alpha_0=0$ and the different values of the $k=\frac{1}{\lambda}=\left|\frac{g}{\ell\Omega^2}\right|\leq 1$ and the axis eccentricity $\varepsilon=0$.

Fig.5 presents a) and b) the sets of the homoclinic phase trajectory layering, for $\alpha_0=0$ and different values of the $k=\frac{1}{\lambda}=\left|\frac{g}{\ell\Omega^2}\right|\leq 1$ and the axis eccentricity. The homoclinic orbits in the form of number eight appear and disappear with the changing of the parameter $k=\frac{1}{\lambda}=\left|\frac{g}{\ell\Omega^2}\right|\begin{matrix}\leq\\ \geq\end{matrix}1$ and the axis eccentricity $\varepsilon=0$. Two sets of singular points: $\phi_s=s\pi$, $s=1,2,3,4,...$ and $\phi_s=\arccos\frac{g}{\ell\Omega^2}\pm 2s\pi$, $s=1,2,3,4,...$ for $k=\frac{1}{\lambda}=\left|\frac{g}{\ell\Omega^2}\right|\leq 1$ exist together with homoclinic orbits – separatrix in the form of number eight.

For concluding this part and for obtaining analytical expressions or the roots of the trigonometric equation $\sin(\phi\pm\alpha_0)-k^2\cos(\phi\mp\alpha_0)\sin\phi=0$ in generalized approximation, we can use Taylor's expansion with different terms of approximations around the singular point of a corresponding ideal mechanical system to the considered non ideal with Coulomb's type friction:

1. The first rough approximation of the singular point is:

$$\phi_{0(l,d)}\approx\frac{\mp\alpha_0}{\left(1-k^2\right)}=\frac{\pm\kappa^2\alpha_0}{\left(1-\kappa^2\right)}; \tag{27}$$

2. The second approximation $\mp k^2\phi^2 tg\alpha_0+\left(1-k^2\right)\phi\pm tg\alpha_0=0$ gives the equation $\phi^2\mp\frac{\left(1-k^2\right)}{k^2 tg\alpha_0}\phi-\frac{1}{k^2}\approx 0$ and the corresponding roots in better approximations are:

$$\phi_{0(1,2),(l,d)}\approx\left[\pm\frac{\left(1-k^2\right)}{2k^2 tg\alpha_0}\right]\mp\frac{1}{k^2}\sqrt{\left[\frac{\left(1-k^2\right)}{2tg\alpha_0}\right]^2+1}\,, \tag{28}$$

$$\phi_{0(1,2),(l,d)}\approx\left[\mp\frac{\left(\kappa^2-1\right)}{2tg\alpha_0}\right]\mp\sqrt{\left[\mp\frac{\left(\kappa^2-1\right)}{2tg\alpha_0}\right]^2+\kappa^4}\,. \tag{29}$$

3. Then we can find approximate expressions of the roots of the previous transcendent equation $\sin(\phi\pm\alpha_0)-k^2\cos(\phi\mp\alpha_0)\sin\phi=0$ around the roots for the case when the circle is an ideal line. For that reason, into the previous trigonometric equation, we put a change: $\phi\Rightarrow\phi_r+\phi$ where $\phi_r=\arccos\dfrac{g}{\ell\Omega^2}\pm 2r\pi$, $r=1,2,3,4,...$ and we obtain:

$$\sin(\phi_r+\phi\pm\alpha_0)-k^2\cos(\phi_r+\phi\mp\alpha_0)\sin\phi=0 \quad (30)$$

and the an approximated expressions for the roots are in the following forms:

$$\phi_{2,3(l.d)}\approx-\frac{\kappa^2\sin(\phi_r\pm\alpha_0)-\cos(\phi_r\pm\alpha_0)\sin\phi_r}{\kappa^2\cos(\phi_r\pm\alpha_0)-\cos(2\phi_r\pm\alpha_0)}. \quad (31)$$

We can make a concluding reviews of the obtained expressions of singular points for the case of non-ideal system dynamics.

A. The singular points for the case that $\kappa^2<1$ are:

a* For $\phi_r\neq 0$, around

$$\phi_r=\pm\arccos\kappa^2\pm 2r\pi,$$

are in the forms

$$\phi_{2,3(l.d)}\approx\frac{\cos(\phi_r\pm\alpha_0)\sin\phi_r-\kappa^2\sin(\phi_r\pm\alpha_0)}{\kappa^2\cos(\phi_r\pm\alpha_0)-\cos(2\phi_r\pm\alpha_0)}, \quad (32)$$

and one side stable center type, in alternations right and left.

b* For $\phi_s=0$ and $\phi_s=\pm s\pi$, and around $\phi_s=\pm s\pi$, are in the forms

$$\phi_{0(l,d)}\approx\frac{\mp tg\alpha_0}{(1-k^2)}\approx\frac{\pm\kappa^2 tg\alpha_0}{(1-\kappa^2)} \quad (33)$$

and one side stable center type, in alternations right and left.

B. The singular points for the case that $\kappa^2>1$ and around $\phi_s=0$ $\phi_s=\pm s\pi$, are

$$\phi_{0(l,d)}\approx\frac{\mp tg\alpha_0}{(1-k^2)}\approx\frac{\pm\kappa^2 tg\alpha_0}{(1-\kappa^2)} \quad (36)$$

one side stable center type and next one side non stable saddle type, alternatively.

By introducing the following $\dot\phi^2=u$, a transformation of the nonlinear differential equation (8) gives a first order differential equation with the corresponding integral in the form:

$$\begin{aligned}\dot\phi^2(\phi)=&-\frac{2g}{\ell\cos\alpha_0}\frac{\pm tg\alpha_0\sin(\phi\pm\alpha_0)}{1+4tg^2\alpha_0}+\\&+\frac{2g}{\ell\cos^2\alpha_0}\frac{\cos\phi}{1+4tg^2\alpha_0}-\frac{\Omega^2}{2\cos^2\alpha_0}\frac{\cos 2(\phi\pm\alpha_0)}{1+tg^2\alpha_0}\mp\\&\mp\frac{\Omega^2}{2}+Ce^{\mp 2tg\alpha_0\phi}\end{aligned} \quad (37)$$

where C is the integral constant depending of initial conditions for the corresponding interval of the material particle motion and in which the upper sign is for $\dot\phi>0$ and the lower sign for $\dot\phi<0$, in accordance with alternations of the friction force directions.

Example 3. Forced nonlinear dynamics differential equation of the heavy coupled rotor dynamics in the field of turbulent damping

On the basis of the previous results and remarks, our attention is focused on the motion of the representative point on the phase trajectory in the phase plane of the forced nonlinear dynamics of multi-step coupled heavy rigid rotor dynamics in the field with and without turbulent damping (see Fig.6a*).

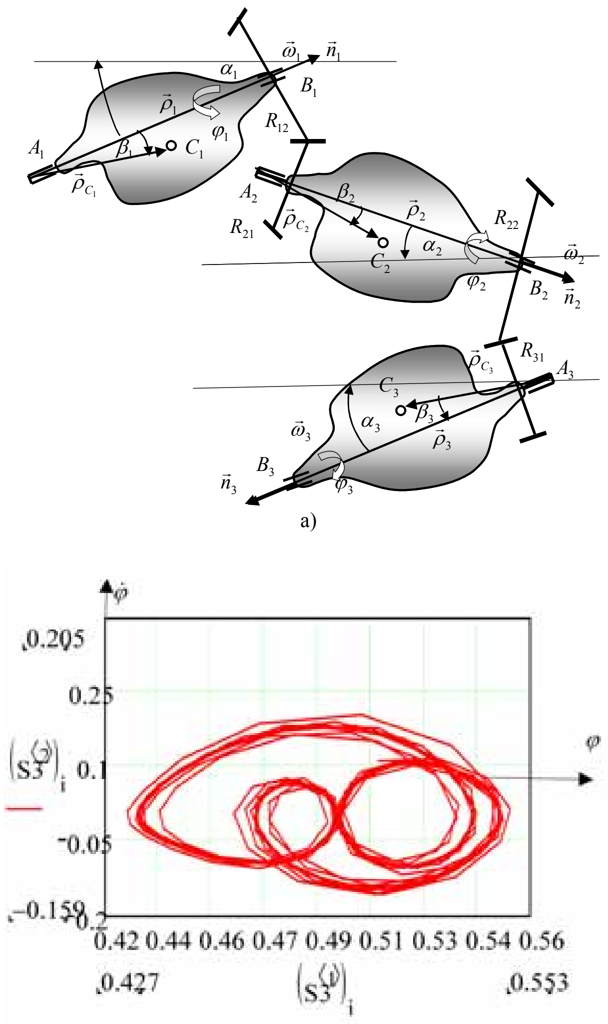


Figure 6. a) Three-step coupled heavy rotors b) forced non-linear dynamics visualizations. (reductor and multipliers) by the phase trajectory portrait of the forced nonlinear dynamics.

The corresponding main differential equation of the forced vibrations of a multi-step coupled multi-rotor system is in the following form:

$$\begin{aligned}&\ddot\phi_1+2\delta_{1M}\dot\phi_1\left|\dot\phi_1\right|+\\&+\Omega_{1M}^2\left\{\sin\phi_1+\left[\sum_{i=2}^{i=M}\frac{\prod_{j=1}^{j=i}i_{j,(j+1)}}{\prod_{j=1}^{j=i}\lambda_{j,(j+1)}}\nu_{1i}^2\sin\left(\prod_{j=1}^{j=i}i_{j,(j+1)}\phi_1\right)\right]\right\}=\\&=h_{1M}\cos\omega t\end{aligned} \quad (38)$$

For the case when the system of coupled rotors is in the field with turbulent damping, and excited by external one frequency force or one frequency couple, we can introduce damping forces proportional to the first step or square of the angular velocities of the rotor shaft. By using two phase coordinates φ and v, the main differential equation of nonlinear dynamics of coupled heavy rotors dynamics in the field with turbulent damping, for a homogeneous system, can be transformed into a system of two first order differential equations in the following form:

$$\frac{d\phi_1}{dt} = \Omega_{1M} v$$

$$\frac{dv}{dt} = -2\delta_{1M}\Omega_{1M} v - \Omega_{1M}\left\{\sin\phi_1 + \left[\sum_{k=1}^{k=M} i^k \sin\left(i^k\phi_1\right)\right]\right\} - \qquad (38)$$

$$-\frac{h_{1M}}{\Omega_{1M}}\cos\omega t$$

From the series of numerical results, we make choice of the characteristic forced processes for four step coupled heavy rotors. In the following series of the graphical presentations in Fig.7, we can see the time-history graphs and the corresponding phase trajectories of the forced rotor system dynamics.

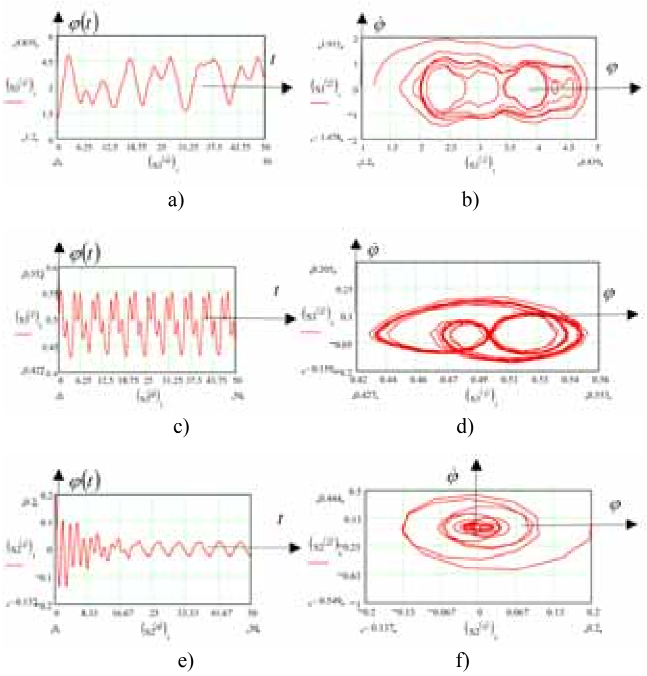


Figure 7. Three-step coupled heavy rotors forced nonlinear dynamics visualizations (reductor and multipliers). a), c) and e) Time-history curves (φ, t). b), d) and f) Phase trajectories portraits of the forced nonlinear dynamics $(\dot{\varphi}, \varphi)$

The first side series of graphs in Fig.7 a), c) and e) represent the families of time-history curves (ϕ, t) for different steps of multi-step coupled heavy rotors dynamics, as a reductor or a multiplier, of the rotate motions of the gear transmission system non-linear model. In the second series of the graphs in the same Fig.7 b), d) and f), the forced non-linear dynamics phase trajectories of the multistep coupled heavy rotors - reductor or multiplier dynamics in the phase plane are presented. Fig.7 b), d) and f) show the characteristic phase trajectories $(\dot{\phi}, \phi)$ for forced regimes, followed homoclinic orbits for multi step coupled heavy rotors - multiplier free vibrations, for different kinetic parameters of the system.

The graphical visualization of the free and forced nonlinear dynamics of gear transmission by the model leads to the concluding remarks. The phase trajectory for forced nonlinear dynamics possesses a very sensitive dependence on the initial conditions as well as on the relations between the kinetic parameters of the nonlinear model and the external excitation frequency. The behavior of the forced non-linear dynamics phase trajectory corresponds to the homoclinic orbits form for corresponding system free vibrations. The influence of the initial conditions and some system parameters with bifurcation properties which leads to the corresponding layering of the homoclinic orbits for free conservative non-linear dynamics give their sensitive dependence of nonlinear dynamics on the system initial conditions around the trigger of coupled singularities and homooclinic orbit in the form of number eight. Then, under the action of one frequency external excitation of the system, the dynamics response is not single frequency regimes but, depending on the initial conditions as well as on the relation between the kinetic parameters of the considered system and the external excitation frequency.

In some of these cases, the responses of the dynamic systems can be with different properties as they are double frequency regimes, as well as stochastic like or chaotic like regimes.

Concluding remarks

The paper presents some characteristic examples of the nonlinear dynamics with the trigger of coupled singularities in the phase portrait and with the homoclinic orbit in the form of number eight. Also, a layering of the homoclinic orbits is identified in some considered system dynamics with one or multiparameter transformation of the phase trajectories in the phase plane. The series of the theorems of the existence of the trigger of coupled singularities as well as the homoclinic orbit in the form of number eight are proven in applications to their listed examples of mechanical system dynamics with coupled rotations in the gravitation field.

In the system nonlinear dynamics with no ideal constraints introduced by Coulomb's type fiction, the appearance of the bifurcation of the equilibrium positions is identified. The series of the alternation of the directions of the Coulomb's type friction force caused the series of alternations of the one side (half) singular point which correspond to equilibrium position alternations. In the phase trajectory portraits, there appears a bifurcation of the singular points which are a special type of the trigger of coupled singularities having two half one side stable singular positions around a stable position which correspond to corresponding singular points of ideal nonlinear system dynamics. If the basic system dynamics posses, in the phase plane, a trigger of coupled three singular points caused by classical nonlinearities, then for no ideal system with Coulomb's type friction, corresponding numbers of complex triggers with three sub-triggers appear. These are the results of each singular point bifurcation into two half – one side singular points.

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Postojanje trigera spregnutih singulariteta u nelinearnoj dinamici mehaničkih sistema spregnutim rotacijama

Prikazana je teorema o postojanju trigera spregnutih singulariteta, kao i mnogobrojni primeri nelinearne dinamike sa spregnutim singularitetima u faznoj ravni. Apstrakcija nelinearne dinamike realnih inžinjerskih sistema spregnutim rotacijama do modela krutog tela koje izvodi spregnute rotacije oko mimoilaznih osa, u gravitacionom polju, pokazuje različite homokliničke fazne trajektorije kao i različite skupove trigera spregnutih singulariteta. Prikazana je višeparametarska transformacija faznih trajektorija kao i skupa spregnutih singulariteta. Takođe, je prikazana i serija trigera spregnutih singulariteta u faznoj ravni, kao i triger spregnutih jednostranih singulariteta koji su identifikovani u kretanju teške materijalne tačke po rotirajućoj hrapavoj krivoj liniji i neidealnim vezama Amontons-Coulomb-ovog tipa trenja. Koristeći primer prikazano je kretanja teške materijalne tačke po rotirajućoj hrapavoj kružnoj liniji u vertikalnoj ravni opisano dvojnom diferencijalnom jednačinom i dvojnom jednačinom faznih trajektorija i u faznoj ravni je identifikovano više trigera spregnutih jednostranih (polu) singulariteta.

Ključne reči: **mehanički sistem, triger, nelinearna dinamika, spregnuti singularitet, rotirajući sistem.**

Существование триггеров сочетаемых сингулярностей в нелинейной динамике механических систем сочетаемыми вращениями

В настоящей работе показана теорема о существовании триггеров сочетаемых сингулярностей, а также и многочисленные примеры нелинейной динамики со сочетаемыми сингулярностями на фазовой плоскости. Абстракция нелинейной динамики реальных инженерных систем связаных вращением с моделью твёрдого тела, которое осуществляет сочетании вращения вокруг непересекающихся осей, в гравитационном поле, показывает различные гомоклинические фазовые траектории и другой набор триггеров сочетаемых сингулярностей. Показана многофакторная траектория фазовых превращений и система сочетаемых сингулярностей. Также показана серия триггеров сочетаемых сингулярностей на фазовой плоскости, а в том числе и триггер сочетаемых односторонних сингулярностей, определённых в движении тяжёлой материальной точки, по вращающейся неправильной грубой линии и далёкими от идеального соединениями фрикционного типа Амонтона-Кулона. На примере показано движение тяжёлой материальной точки по вращающейся грубой кольцевой линии в вертикальной плоскости, описано двойным дифференциальным уравнением и двойственным уравнением фазовых траекторий и на фазовой плоскости определено више триггеров связанных односторонними (полу-) сингулярностями.

Ключевые слова: **механическая система, триггер, нелинейная динамика, сочетаемая сингулярность, вращающаяся система.**

Existence des déclencheurs des singularités couplées dans la dynamique non linéaire chez les systèmes mécaniques avec les rotations couplées

La théorème sur l'existence des déclencheurs des singularités couplées ainsi que de nombreux exemples de la dynamique non linéaire avec les singularités couplées sur le plan de phase ont été présentés dans cet article. L'abstraction de la dynamique non linéaire des systèmes réels d'ingénierie par les rotations couplées jusqu'au modèle du corps rigide qui effectue les rotations couplées autour des axes non croisés dans le champ de gravitation montre les différentes phases homo cliniques de la trajectoire ainsi que les différents groupes de déclencheurs de singularités couplées. On a présenté la transformation multi paramétrique des trajectoires de phase ainsi que la transformation du groupe des singularités couplées. On a présenté aussi une série de déclencheurs des singularités couplées sur le plan de phase ainsi que le déclencheur des singularités unilatérales identifiées lors du mouvement d'une lourde particule matérielle sur une ligne courbe et rêche et les relations non idéales de la friction du type Amonions-Coulomb. A l'aide de cet exemple on a présenté le mouvement de la particule lourde matérielle sur la ligne circulaire rotative sur le plan vertical, décrit par la double équation différentielle et par la double équation des trajectoires de phase. Sur le plan de phase on a identifié plusieurs déclencheurs des (demi) singularités couplées unilatérales.

Mots clés: **système mécanique, déclencheur, dynamique non linéaire, singularité couplée, système de rotation.**