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Eigenvalues Assignment for a Special Class of Singular Systems in Constrained Robotics

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Robotic systems in contact with environment are typical examples where external contact forces play an important role to the system dynamics. Mathematical modeling of these systems is challenging due to a variety of reasons. Mathematical models for the described class of systems contain differential equations with an associate algebraic equation, which outlines constrained system dynamics. Such a system is considered to be a singular system of differential equations (semi-state or descriptor systems). In this article, the geometric approach to the solution of singular systems with contact problem has been introduced. The mutual eigenvalues and corresponding eigenvectors assignment for the robotic systems have been investigated. In order to achieve desired dynamical system behavior the controllability conditions have been investigated as well.

Key words: **singular system, controllability, geometric approach, pole adjustment, system dynamics, mathematical model, robotics.**

Introduction

THE geometric approach is a mathematical concept developed to improve analysis and synthesis of the linear multivariable systems. Many authors consider the notion of geometry in the system theory as mutual characteristics of the matrix pencils (A, B) or (A, C) for linear systems or (A, E) for singular systems. Other authors consider the geometric aspect as a study of the characteristics of system subspaces. The descriptive definition of the geometric approach to linear singular systems could be as follows: The geometric approach (or aspect) should be understood as an approach to a study of singular systems the purpose of which is to determine and investigate characteristic subspaces which play a crucial role in the analysis of the matrix pencils (A, E).

The geometric approach was first discussed in the articles *Basile*, *Marro* (1969) and *Wonham*, *Morse* (1970). The authors discovered that dynamic behavior of time invariant linear control systems could be investigated based on the characteristics of the system invariant subspace of the matrices. As a result, system behavior could be predicted and solutions to many control problems could be tested by investigating the characteristics of the subspaces described. The basic idea of this approach was the application and calculation of subspaces on the computers using algorithms developed for that purpose. Having all this in mind, it was shown in the literature that the geometric approach can be used to solve a variety of problems, including finding a control law for systems with feedback, observability problems, disturbance localization, design of the observers, control and tracking, robust control, etc.

The geometric approach was a mathematical concept developed in order get a better understanding and to give better insight into the most important characteristics of the linear system dynamics. It is mostly represented in the state space domain and used to connect characteristics of single and multiple transfer systems.

In this article, we investigate a mathematical model of the robotic system represented as a singular system of differential equations. For some specific class of robotic systems, a mathematical model has a singular character due to a contact force which acts upon the system. In many applications it is enough to consider a contact force as a disturbance to the system. Sometimes the stochastic character and an unexpected range of the contact forces could significantly change or damage the contact surface, which could be unacceptable for some situations. Furthermore, a contact force which has an unknown value and characteristics could produce a compromised outcome. Due to the reasons given here, there is a need not only to measure the force, but also to control it and to obtain adequate control algorithms which can keep the force within acceptable limits.

The dynamic behavior of the described systems represented as a singular system of differential equations was initially investigated in *McClamroch* (1986), *Huang* (1988), and *Mills*, *Lui* (1991). The geometric approach to the robotic systems dynamic for linear non-singular systems was presented in *Dam* (1997), and partially in the article *Mills*, *Goldenberg* (1989) based on the results reported in *Cobb* (1983). *Vukobratović*, *Tuneski* (1996) presented the current state of the art in the adaptive control of single rigid robotic manipulators in constrained motion tasks. In *Stokić*, *Vukobratović* (1997), a solution for a practical stabilization problem of robots being in contact with dynamic environment has been presented. A detailed overview of the geometric approach theory used in this article can be found in *Debeljković*, *Buzurović* (2007), and *Buzurović*, *Debeljković* (2009). The method proposed here was applied

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to the prediction control of a medical robotic system *Buzurović et.al* (2010).

In this article, the geometric approach to the solution and eigenvector assignment of a singular system with a contact problem has been introduced. The necessary and sufficient conditions, for the existence and uniqueness of the solutions for robotic systems, have been investigated.

Mathematical modeling of robotic systems in contact with environment

In the article, the following notation has been used:

q – generalized coordinate
D – constrained function gradient
f – contact force
H – vector function
M – inertia matrix or matrix
I – identity matrix
G – gravitational matrix
L – space transformation
τ – torque vector
$\boldsymbol{x}$ – state space vector
J – Jacobian matrix
$\boldsymbol{u}$ – control vector
E – singular matrix
d – disturbance vector
ϕ – equation of the contact surface
K, F – matrices
λ – scalar multiplier
A, B – regular system matrices
$\mathcal{Z}$ – subspace
$\mathcal{N}(.)$ – null space of the matrix
$\mathcal{M}$ – subspace
$\mathcal{R}(.)$ – range of the matrix
$\mathcal{V}\ A$ – invariant subspace
$\mathcal{V}^*$ – maximum A-invariant subspace
a, b – coefficients
s – complex number
$\boldsymbol{x}, \boldsymbol{w}$ – state space vector components

The following introductory definitions will be used during this study.

Definition 1: The subspace S of the vector space $\Re$ is called invariant space of the linear transformation A over S if and only if $AS \subseteq S$.

Definition 2: For the subspace $\mathcal{V}$ of $\mathcal{X} = \Re^n$ is said to be A - invariant if $A\mathcal{V} \subset \mathcal{V}$.
Buzurović, *Debeljković* (2009).

The generalized coordinates vector, denoted by q, has the property $q\in\Re^n$, the contact force vector is denoted by f. The force $f\in\Re^n$ appears when the end-effector touches the constraint surface c. The constrained robotic model is represented as

$$M(q)\ddot{q}+G(q,\dot{q})=\tau+J^T(q)f\ . \tag{1}$$

$M(q)\in\Re^{nxn}$ denotes the inertia matrix function and $G(q)\in\Re^n$ is a vector function which describes Coriolis, centrifugal and gravitational effects. τ is the torque vector of the joints, $\tau\in\Re^n$. $J(q)\in\Re^{nxn}$ is defined as a Jacobian matrix function. The general dynamic equations for the robotic system in contact with environment is, as in *McClamroch* (1986),

$$\begin{bmatrix} M(q) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \ddot{q} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -G(q,\dot{q})+\tau+J^T(q)D^T\left(H(q)\right)\lambda \\ \phi\left(H(q)\right) \end{bmatrix} \tag{2}$$

Equation (2) consisted of the n differential equations and one algebraic equation with n+1 unknown value, n generalized coordinates and the scalar multiplierλ. Now it is possible to represent the equation of the robotic system (1) which is in contact with the working environment in its state space form (3) with the vector $\boldsymbol{d}$ as a disturbance

$$E\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t) + \boldsymbol{d}(t)\ , \tag{3}$$

where the corresponding matrices have been defined as in *Buzurović*, *Debeljković* (2010)

$$E = \begin{bmatrix} I & 0 & 0 \\ 0 & M(q_0) & 0 \\ 0 & 0 & 0 \end{bmatrix},$$
$$A = \begin{bmatrix} 0 & I & 0 \\ \frac{\partial}{\partial q}(G-J^TD^T\lambda)|_0 & 0 & J^TD^T|_0 \\ DJ|_0 & 0 & 0 \end{bmatrix} \tag{4}$$
$$B = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix},\quad \boldsymbol{u}(t) = \delta\tau,\quad \boldsymbol{d}(t) = \begin{bmatrix} 0 \\ \Delta\tau \\ 0 \end{bmatrix}$$

Existence of the solutions

Due to the structure of matrices (4) for system (3), it can be expressed by:

$$J_D = A\ \mathcal{N}(E) = \begin{bmatrix} 0 & J^TD^T\big|_0 & 0 \end{bmatrix}^T, \tag{5}$$

$$I_3 = \mathcal{N}\ (E) = \begin{bmatrix} 0 & 0 & I \end{bmatrix}^T \tag{6}$$

where J_D represents the influence of the contact force on the robotic system. Generally, the subscript "D" denotes a gradient, as in (4). In this case, D denotes the gradient of the contact force that acts upon the system. $\mathcal{N}$ (·) is the null space of the matrix (·) and $\mathcal{R}$ (·) is the range of the matrix (·). $\boldsymbol{x}_0$ represents the initial condition space matrix and I is the identity matrix.

The following *theorems*, *lemma* and *corollaries* are the original results obtained by using the geometric approach in the analysis of the system described by matrices (4).

***Theorem* 1**: The solution of system (3), which comprises the reactive force control, on an arbitrary time interval exists for any control vector $\boldsymbol{u}$(t) if and only if (5) and (6) are satisfied:

$$\mathcal{R}(B) \subset E\mathcal{V}^* + J_D = EM + J_D \tag{7}$$

$$\boldsymbol{x}_0 \in \mathcal{V}^* + I_3 = M + I_3 \tag{8}$$

Proof (*necessary condition*): Let us analyze arbitrary condition (9):

$$\boldsymbol{x}(t) = \boldsymbol{z}(t) + \boldsymbol{\varepsilon}(t)\ , \tag{9}$$

with

$$\boldsymbol{\varepsilon}(t) \in I_3,\ \ \boldsymbol{z}(t) \in \mathcal{Z}, \tag{10}$$

where $\mathcal{Z}$ denotes the subspace mathematically defined in the following part. Choosing the proper constraints for the subspace $\mathcal{Z}$, it is possible that $\mathcal{Z} \cap I_3 = \{0\}$; therefore, the proposed decomposition of the state space vector (9) is unique. Applying (9) to system (3) and neglecting disturbances to the system, the following equation can be written

$$E\dot{\boldsymbol{z}}(t) = A\boldsymbol{z}(t) + A\boldsymbol{\varepsilon}(t) + B\boldsymbol{u}(t)\,. \tag{11}$$

For given $\boldsymbol{u}(t)$ and $\boldsymbol{x}(t)$, $E\dot{\boldsymbol{z}}(t)$ is uniquely defined. The direct consequence for the singular system is the following geometric distribution:

$$A\mathcal{Z} \cap E\mathcal{Z} + A\, I_3, \tag{12}$$

$$\mathcal{R}\,(B) \subset E\mathcal{Z} + A\, I_3. \tag{13}$$

A further proof of the necessary condition relies on the following lemma.

Lemma 1: The maximum subspace which satisfies (12) and (13) is $\mathcal{V}^* + I_3$.

Proof: It can be shown that $\mathcal{V}^* + I_3$ satisfies (12) and (13). Let us assume that the subspace Z satisfies the same conditions. In that case, it can be written

$$\mathcal{Z} = \mathcal{V} + I_3, \tag{14}$$

as well as

$$A(\mathcal{V} + I_3) \subset E\,\mathcal{V} + J_D. \tag{15}$$

Given relations (14) and (15), it can be stated that if $\forall a \in \mathcal{V}$, $\exists \tilde{a} \in \mathcal{V}$ and $b \in \mathcal{N}(E)$, the equation $Aa = E\tilde{a} + Ab$ is satisfied. The values $\tilde{a}$ and b could be chosen so that they are linearly related to a. Let the matrix K generate $\mathcal{N}(E)$. Then a matrix exists with appropriate dimensions, so for every $a \in \mathcal{V}$, equation (34) is fulfilled:

$$Fa = E\tilde{a} + FKPa\,, \tag{16}$$

and also

$$F(I - KP)a = E\tilde{a} = E(I - KP)\tilde{a}\,. \tag{17}$$

Defining:

$$\overline{\mathcal{V}} = (I - KP)\mathcal{V}\,, \tag{18}$$

it can be stated:

$$\overline{\mathcal{V}} + \mathcal{N}\,(E) = \mathcal{V} + \mathcal{N}\,(E) = \mathcal{Z} \tag{19}$$

as well as

$$A\overline{\mathcal{V}} \subset E\overline{\mathcal{V}}\,. \tag{20}$$

The final conclusion that proves the necessary condition of *Lemma* 1 is given by equation (21):

$$\overline{\mathcal{V}} \subset \mathcal{V}^*,\ \mathcal{Z} \subset \mathcal{V}^* + I_3. \tag{21}$$

Note: In order to obtain the unique solution $z(t)$, $\mathcal{Z}$ must be chosen to satisfy $\mathcal{Z} = \mathcal{M}$, where the subspace $\mathcal{M}$ is a complement of I_3 in $\mathcal{V}^*$, *q.e.d* □.

Proof (*sufficient condition*): Let $\overline{K}$ be a matrix whose columns fully spans the null space of the matrix $E\,\mathcal{N}(E)$ and let M be a matrix chosen as $AM = EMA$. Then,

$$\boldsymbol{x}(t) = M\boldsymbol{y}(t) + \overline{K}\boldsymbol{w}(t)\,. \tag{22}$$

The conditions (7) and (8) imply the existence of the matrices $\overline{B}$ and $\overline{P}$ and

$$B = EM\overline{B} + A\overline{K}\overline{P}\,. \tag{23}$$

Now it is possible to represent system (22) in its equivalent form,

$$EM\dot{\boldsymbol{y}}(t) = EMA\boldsymbol{y}(t) + A\overline{K}\boldsymbol{w}(t) + EMN\overline{B}\boldsymbol{u}(t) + A\overline{K}\overline{P}\boldsymbol{u}(t)\,, \tag{24}$$

and the solution of system (3), together with (22), as:

$$\dot{\boldsymbol{y}}(t) = A\boldsymbol{y}(t) + \overline{B}\boldsymbol{u}(t) \tag{25}$$

$$\boldsymbol{x}(t) = M\boldsymbol{y}(t) - \overline{K}\overline{P}\boldsymbol{u}(t)\,. \tag{26}$$

Here, note that since the constraints for the matrix B are changed, it is not necessary to introduce the special relations between the matrices B and $\overline{B}$ as well as B and $\overline{P}$. Thus, equations (25) and (26) define the solution of system (3), which satisfies conditions (7) and (8), *q.e.d* □.

In the following part, another approach for the solution existence has been used. In this section, the general conditions for the existence of the solutions for system (3) have been analyzed using a strict geometric approach for the appropriate matrix pencils. Equation (3) represents the robotic system which is in contact with the working environment. Consequently, the controlled subspace $\mathcal{S}$ which consists of the position and the velocity initial elements is extended to the control of the initial contact force.

Let $\mathcal{R}\,(E)$ and $\mathcal{R}\,(B)$ be a range of the matrices E and B, given as a subspace $\mathcal{Y} = \Re^r$. The following relation is fulfilled for the linear subspace $\mathcal{V}$ belonging to $\mathcal{Y}$ for the robotic system mathematically described by equation (3):

$$A\mathcal{V} \subset E\mathcal{V} \tag{27}$$

Definition 3: A characteristic subspace of the matrix pencil (E, B) is a maximum subspace $\mathcal{V}^*$ which fulfills equation (27).

Assume that the subspace $\mathcal{S}$ is a subspace of the initial conditions.

Theorem 2: System (1), represented in the transformed form (3), has solution $\boldsymbol{x}(t) \in \Re$ for any $\boldsymbol{u}(t) \in \mathcal{S}$, on the arbitrary interval if and only if:

$$\mathcal{R}(B) \subset E\mathcal{V} \tag{28}$$

$$\boldsymbol{x}(0) \in \mathcal{V}^* \tag{29}$$

Proof (*necessary condition*): Let $\mathcal{V}$ be a subspace of the system trajectories $\boldsymbol{x}(t)$. If the velocity of the robotic joints fulfils the condition $\dot{\boldsymbol{x}}(t) \in \mathcal{V}$, then $\mathcal{V}$ has to fulfill equation (27), consequently, $\mathcal{V} \in \mathcal{V}^*$. Having the condition in mind

$$\begin{bmatrix} \hat{A} & 0 \\ \hat{C} & 0 \end{bmatrix} \boldsymbol{x}(\mathrm{t}) + \begin{bmatrix} \hat{B} & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{u}(\mathrm{t}) \subset \mathcal{R}(E), \tag{30}$$

with decomposition

$$E = \begin{bmatrix} \hat{E} & 0 \\ 0 & 0 \end{bmatrix},\ A = \begin{bmatrix} \hat{A} & 0 \\ \hat{C} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} \hat{B} & 0 \\ 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 0 \\ d \end{bmatrix} \tag{31}$$

Moreover, if (30) is fulfilled for any $\boldsymbol{u}(t) \in \Re^m$ it is also

$$\mathcal{R}(B) \subset \mathcal{R}(E), \quad \boldsymbol{x}(t) \in \mathcal{V}^0 = A^{-1}(\mathcal{V}(E)). \tag{32}$$

In order for equation (30) to be fulfilled for any $\boldsymbol{u}(t)$, in the previous instant of the recursive algorithm for the subspace calculation, which can be formally denoted by t-1, it is necessary that the following is fulfilled

$$\mathcal{R}(B) \subset E\mathcal{V}^0, \quad \boldsymbol{x}(t-1) \in \mathcal{V}^1 = A^{-1}(E\mathcal{V}). \tag{33}$$

Continuing the recursive calculation process, the sequence $\mathcal{V}^k$ has been formed according to the rule, as in (34).

$$\mathcal{V}^{k+1} = A^{-1}(E\ \mathcal{V}^k). \tag{34}$$

Taking into account that all instants during the solutions were analyzed, it can be obtained

$$(B) \subset E\mathcal{V}^k, \boldsymbol{x}(t-k) \in \mathcal{V}^k. \tag{35}$$

For the proof of the necessary condition, the following Lemma has being used.

Lemma 1: Sequence $\mathcal{V}^k$ decrease and converge to $\mathcal{V}^*$ in no more than n steps, *Bernhard* (1982).

Proof: It is clear that $A^{-1}(E\mathcal{V}^0) \in A^{-1}(\mathcal{R}(E))$, as well as $\mathcal{V}^1 \subset \mathcal{V}^0$, and so on. However, subspaces can decrease only if their dimension (rang) decrease, which cannot happen for $\Re^n$ more than n times. Denote by k the first index such as $\mathcal{V}^{k+1} \subset \mathcal{V}^k$. The sequence $\mathcal{V}^k$ becomes a stationary sequence starting from the index k and further, and consequently, equation (34) proves that $\mathcal{V}^k$ fulfils (27). It can be concluded that $\mathcal{V}^k \subset \mathcal{V}^*$ is fulfilled. This fact proves both equations (28) and (29).

Proof (*sufficient condition*): Denote by V a rectangular matrix with a full column rang, such as $\mathcal{R}(V) = \mathcal{V}^*$. Let $\dim \mathcal{V}^* = n^*$, and V: $m \times n^*$. Equations (28) and (29) imply the existence of the transformations defined as

$$\tilde{A} : A\mathcal{V} = E\ \mathcal{V}\tilde{A}, \tag{36}$$

$$\tilde{B} : B = E\ \mathcal{V}\tilde{B}, \tag{37}$$

where the matrices $\tilde{A}$ and $\tilde{B}$ are of the order $n^* \times n^*$ and $n^* \times m$, consequently. The equivalency can be established among the statement $\boldsymbol{x}(t) \in \mathcal{V}^*$ and the following value

$$\boldsymbol{\xi}(t) \in \Re^{n^*} : \boldsymbol{x}(t) = \mathcal{V}\boldsymbol{\xi}(t) \tag{38}$$

Transformed system (2) represented by matrices (4) and equation (3), is equivalent to the system

$$EV\dot{\boldsymbol{\xi}}(t) = EV(\tilde{A}\boldsymbol{\xi}(t) + \tilde{B}\,\boldsymbol{u}(t)). \tag{39}$$

The value $\boldsymbol{\xi}(t)$ together with the initial condition (30) obviously represents the solution of system (3), which was transformed to its equivalent form (40)

$$\dot{\boldsymbol{\xi}}(t) = \tilde{A}\boldsymbol{\xi}(t) + \tilde{B}\,\boldsymbol{u}(t), \boldsymbol{x}(0) = V\boldsymbol{\xi}(0). \tag{40}$$

The last statement proves the sufficient condition of *Theorem 2*, *q.e.d* □.

Definition 4: A characteristic null-subspace of the matrix pencil (E, A) is the subspace $\mathcal{N}$ defined by

$$\mathcal{N} = \mathcal{N}(E) \cap \mathcal{V}^*. \tag{41}$$

Let $\dim \mathcal{N} = n$.

Definition 5: The matrix pencil (E, A) is column-regular if n=0, i.e.

$$\mathcal{N} = \{0\}. \tag{42}$$

***Theorem* 3**: Under conditions (36) and (37), the solution of equation (3) is unique for any $\boldsymbol{u}(t)$ if and only if the matrix pencil (E, A) is column-regular.

If initial equation (2) is analyzed, condition (42) is fulfilled if any column of the matrix A is regular. The non-uniqueness of the solutions is described by an arbitrary choice of the sequence $\boldsymbol{y}(\cdot)$ from equation $E\dot{\boldsymbol{y}}(t) = EA\boldsymbol{y}(t) + EB\boldsymbol{u}(t)$. In that case, solution (38) represents all solutions of the system.

Uniqueness of the solutions

***Theorem* 4**: The solution of system (3), which comprises the reactive force control, under conditions (7) and (8) is unique for any $\boldsymbol{u}(t)$, if and only if the matrix pencil (E, A) is a pencil of full column rank.

Proof: The question concerns under which conditions equation (24) has a unique solution. Denoting the derivatives of the values $\boldsymbol{x}$ and $\boldsymbol{w}$ as $\delta\dot{\boldsymbol{x}}(t), \delta\boldsymbol{w}(t)$, the problem can be reformulated. Now it is necessary to find non-zero solutions of the following equation

$$EM\,\delta\dot{\boldsymbol{x}}(t) = AK\,\delta\boldsymbol{w}(t). \tag{43}$$

One of the possible solutions is zero if and only if the following is satisfied

$$\mathcal{N}(A) \cap \mathcal{N}(E) = \{0\} \tag{44}$$

and

$$EM \cap J_D = \{0\} \tag{45}$$

Combining equations (43) and (44) and using equation (5), it becomes clear that equation (44) is always fulfilled, because of the structure of matrices (5). This conclusion could be made based on the injectivity of the matrix EM. Consequently, for non-zero solutions, neither equations (44) nor (45) should be fulfilled. Equation (45) in fact does not have to be fulfilled. It can be concluded that a non-zero element exists in the equation for $EM \cap J_D$.

When it is $\mathcal{V}^* = A^{-1}(EM)$, then

$$\mathcal{N} = A^{-1}(EM) \cap \mathcal{N}(E) \tag{46}$$

It is now possible to verify the existence of an arbitrary linear operator A for two subspaces $\mathcal{A}$ and $\mathcal{B}$ and if the following statement is fulfilled

$$A\mathcal{A} \cap A\mathcal{B} = A[(\mathcal{A} + \mathcal{N}(A)) \cap \mathcal{B}]. \tag{47}$$

Applying statement (47) to equation (46) and applying $A^{-1}(EM) \subset \mathcal{N}(E)$, it can be shown:

$$A\mathcal{N} = EM \cap J_D. \tag{48}$$

It can also be noted that $\mathcal{N}(A) \subset \mathcal{V}^*$, and consequently that:

$$\mathcal{N} \supset \mathcal{N}(E) \cap \mathcal{N}(A) \tag{49}$$

It can be concluded from equations (48) and (49) that conditions (44) and (45) are not satisfied when the subspace $\mathcal{N}$ is nontrivial. Consequently, the system is not column regular. On the other hand, if the subspace $\mathcal{N}$ is nontrivial and if equation (44) is fulfilled, which is always true according to the previous discussion, then it can be stated that $\mathcal{N} \subset N(E)$ holds and

$$\mathcal{N} \cap N(A) = \{0\}. \qquad (50)$$

Given (50), $A\mathcal{N}$ has the same dimension as $\mathcal{N}$, and so (48) shows that (45) is not fulfilled, *q.e.d* □.

As in the previous part, in which solution uniqueness was discussed, a clear distinction between two types of the solution non-uniqueness can be made. In the case that (45) holds, but (44) does not, the non-uniqueness of $\boldsymbol{x}$ includes only $\boldsymbol{w}$ and the statement is not fulfilled at any instant of time. The solutions $\boldsymbol{x}(t)$ are unique. In this case, non-uniqueness could be named *static*. *Dynamic* non-uniqueness is the product of the non-zero element in the statement $EM \cap J_D$.

Corollary 1: The regular matrix pencil $(sE - A)$ does not have any indefinite zeroes if and only if $N(E) \cap A^{-1}(R(E))$ exists. Let us assume that the regular matrix pencil $(sE - A)$ has indefinite zeroes. In this case, the static variables do not exist if and only if $A\,N(E) \subset R(E)$.

Proof: The first part of *Corollary* 1 can be proven using *Theorem 4*, because in that case, the matrix pencil $(sE - A)$ does not have any indefinite zeroes if and only if $V_a^*= 0$, to i.e., $N(E) \cap A^{-1}(R(E)) = 0$. The second part is true because the matrix pencil $(sE - A)$ does not have indefinite zeroes and in that case $N(E) \cap A^{-1}(R(E)) \neq 0$ is fulfilled. Moreover, $N(E) \cap A^{-1}(R(E)) = N(E)$ is true, or equivalently $N(E) \subset A^{-1}(R(E)) \Leftrightarrow J_D \subset R(E)$, *q.e.d* □.

Controllability conditions

Here we introduce state-space feedback described by

$$\boldsymbol{u}(t) = K\boldsymbol{x}(t) \qquad (51)$$

where K is the matrix. Applying feedback (51) to system (25-26), the controlled system is obtained as follows

$$\begin{aligned} E\dot{\boldsymbol{x}}(t) &= (A + BK)\boldsymbol{x}(t) \\ \boldsymbol{y}(t) &= (C + DK)\boldsymbol{x}(t) \end{aligned} \qquad (52)$$

Definition 6: System (52) is controllable if the matrix pencil

$$C(s) = \begin{bmatrix} sE - A & B \end{bmatrix} \qquad (53)$$

does not have definite or indefinite zeros.

***Theorem* 5**: Systems characterized by equations (52) are controllable if and only if none of the matrix eigenvalues $C(s)$ is equal to zero. This condition is represented as

$$C(s) = \begin{bmatrix} sI & -I & 0 & 0 \\ \frac{\partial}{\partial q}(G - J^T D^T \lambda)|_0 & sM(q_0) & -J^T D^T|_0 & I \\ -DJ|_0 & 0 & 0 & 0 \end{bmatrix} \qquad (54)$$

Proof: Equation (55) of transformed system (3)

$$\begin{aligned} E\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B\boldsymbol{u}(t) \\ \boldsymbol{x}_i(t) &= C\boldsymbol{x}(t) + D\boldsymbol{u}(t) \end{aligned} \qquad (55)$$

represents the structure of the robotic system. Combining equations (55), (53) and (54) it can be concluded that condition (54) is fulfilled, *q.e.d.*◊.

Definition 7: Systems (55) is reachable if (53) is fulfilled and if

$$rang\begin{bmatrix} E & B \end{bmatrix} = n\,. \qquad (56)$$

***Theorem* 6**: Systems (55) are reachable if and only if condition (57) is fulfilled:

$$rang\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & M(q_0) & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix} = n \;\; rang\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & M(q_0) & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix} = n \qquad (57)$$

Proof: The procedure is similar to the proof of ***Theorem* 5**, using the condition in **Definition 6**, *q.e.d.*◊.

Corollary 2: Analyzing condition (57), it can be concluded that the controllability of the linearized robotic system (3), and consequently (55), in contact with environment, depends on the inertia matrix in the surroundings of the contact point.

This conclusion is significant because the design of the system with contact tasks can influence the controllability and furthermore the stability of the system. Condition (57) could be used for potential stability checking during the system design. The following definition is a direct consequence of the previous analysis.

Definition 8: Systems (55) is infinite controllable if (53) does not have indefinite zeros.

Eigenvalues assignment

The final section of this article describes the eigenvalue assignment procedure. By assigning eigenvalues and eigenvectors, the system structures can be assigned. The procedure is usually named as the eignestructure assignment.

Control system design based on the eigenvalue or pole assignment has received a great deal of attention in the literature. It is well known that for a controllable system, if state variable feedback is employed, all eigenvalues can be assigned, *Wonham*, *Morse* (1970). Also it is known that for multi-input systems, the feedback law assigning a given set of eigenvalues is not unique and that different control laws can yield identical eigenvalues while yielding radically different eigenvectors. Since the eigenvectors determine the influence of each eigenvalue on each state variable response, failure to use the multi-input design freedom fully may result in undesirable mode coupling and other poor transient behavior.

In the following part, the eigenvalue and its corresponding eigenvector assignment, using state-space feedback, has been investigated. The proposed *Theorem* represents the extension of the method presented in *Buzurović*, *Debeljković* (2004), and it is related to the special class of the control systems, described by (3) and (4).

***Theorem* 7**: Assume that the state-space system represented by (4) is controllable by the eigenvector v_i assignment. Denote by $\{\lambda_i\}$, $i \in 1, 2,.., h$, h=rang E symmetric set of n different finite complex numbers. Assume that the subspace V exists, such as V = span $\{v_i\}$, $i \in 1, 2,..., h$ and:

(i) $v_i \in H$ if λ_i is a real number, and $v_i = v_i^*$ is a complex conjugate if $\lambda_i = \lambda_i^*$.

(ii) Vectors $\{vi\}$, $i \in 1, 2,..., n$ are linearly independent and $v_i \in H_\Lambda = (\lambda_i E - A)^{-1} B = \begin{bmatrix} 0 & 0 & -(1/J^T D^T) I_n|_0 \end{bmatrix}^T$,

(iii) $\mathcal{V} \cap I_3 = 0$.

Then, a real matrix F exists, such as $(A+BF)v_i = \lambda_i Ev_i$, $i= 1, 2,..., h$, and the matrix pencil $(sE\text{-}A\text{-}BF)$ is regular.

Proof: To each complex number λ, an appropriate matrix $P_\lambda = [\lambda E - A\ B]$ can be assigned. Denote by

$$Q_\lambda = \begin{bmatrix} N_\lambda \\ R_\lambda \end{bmatrix} \tag{58}$$

a compatible matrix partition with the property that its columns span $\mathcal{N}\ (P_\lambda)$. Analyzing the structure of system (4) it can be noticed that the rang $B = m$, which implies that the columns of N_λ are linearly independent.

Because of $v_i \in \mathcal{H}_{\Lambda i} = \mathcal{N}\ (N_\lambda)$ it implies that $v_i = N_{\lambda i} K_i$, for some unique K_i

$$(\lambda_i E\text{-}A)\ N_{\lambda i} K_i + BR_{\lambda i}\ K_i = 0 \tag{59}$$

Define $F_0 \colon \mathcal{V} \to \mathcal{U}$ as

$$F_0 v_i = -\ R_{\lambda i}\ K_i\ ,\ i \in 1, 2, ..., n \tag{60}$$

Now, it is necessary to define the extension F to F_0 with a property that matrix pencil $(sE\text{-}A\text{-}BF)$ is regular. Because dim $\mathcal{V} = h$, dim $\mathcal{N}\ (E) = n\text{-}h$ and $\mathcal{V} \cap \mathcal{N}\ (E) = 0$, it implies

$$\mathcal{V} \oplus \mathcal{N}\ (E) = \mathcal{H} \tag{61}$$

Similarly, it can be derived

$$E\ \mathcal{V} = E\ (\mathcal{V} \oplus \mathcal{N}\ (E)) = E\mathcal{H}\ = \mathcal{R}\ (E) \tag{62}$$

Let $\boldsymbol{x} \in \mathcal{H}$ and assume that (61) is fulfilled. Analyzing the extension $\widetilde{F} \colon \mathcal{H} \to \mathcal{U}$ to F_0, and let $E\boldsymbol{x}$ and $(A+B\widetilde{F})\boldsymbol{x}$ are represented by the decomposition

$$E\ \mathcal{V} \oplus \mathcal{H}^* = \mathcal{H}, \tag{63}$$

where $\mathcal{H}^*$ is any subspace with the dimension $n\text{-}h$ which supplement $\mathcal{R}(E)$ to $\mathcal{H}$. In any subspace with the dimension $n\text{-}h$ which supplement $\mathcal{R}(E)$ to $\mathcal{H}$. In decompositions (61) and (63) the matrices E and $A+B\widetilde{F}$ allow the following representation:

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},\quad A + B\widetilde{F} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \tag{64}$$

Equation (64) follows from derivation using (3), where *dim I= h* and the eigenvalues of A_{11} are identical with $\{\lambda_i\}$, $i \in 1, 2, ..., h$. The matrix pencil $(sE\text{-}A\text{-}B\widetilde{F})$ is regular if and only if A_{22} is a nonsingular matrix.

Denote by Q_i the projection of $\mathcal{H}^*$ along $\mathcal{N}\ (E)$. In that case, the non-singularity of A_{22} is equivalent to the non-singularity of

$$Q_i\,(A+B\widetilde{F}) \mid \mathcal{N}\ (E) \tag{65}$$

It can be stated

$$\mathrm{Q_i(A+B}\widetilde{F}\mathrm{)} | \mathcal{N}\ \mathrm{(E) = Q_i\,A_i}\ | \mathcal{N}\ \mathrm{(E)+Q_i\,BF_i} \tag{66}$$

where the matrix $F_1 = F \mid \mathcal{N}\ (E)$ exists, such as $F_1 \colon \mathcal{N}\ (E) \to \mathcal{U}$ implying that equation (65) is nonsingular if and only if the null eigenvalues of $Q_i\,A_i \mid \mathcal{N}\ (E)$ controllable using $Q_i\,B$, i.e.

$$Q_i\ (A \mid \mathcal{N}\ (E) + B) = \mathcal{H}^* \tag{67}$$

Since system (4) is controllable, it follows that equation (67) is fulfilled. Hence, it is possible to choose F_1 such as (65) is nonsingular. Defining $F \colon \mathcal{H} \to \mathcal{U}$ s

$$F \mid \mathcal{V}\quad F_0 \text{ and } F \mid \mathcal{N}\ (E) = F_1 \tag{68}$$

Definition of a real transformation represented by the matrix F allows the eigenvector assignment, *q.e.d* □.

Comment: *Theorem* 7 allows the derivation of the opposite statement. Let $\{\lambda_i\}$, $i \in 1, 2,..., h$ represents a set of the finite eigenvalues of the matrix pencil $(sE\text{-}A\text{-}BF)$, which does not have to be regular, for this case. Then

$$(A+BF)\ v_i = \lambda_i E\ v_i \tag{69}$$

for some vector v_i , $i \in 1, 2,..., l$. It can be proved that the vectors v_i, $i \in 1, 2,..., l$ fulfill conditions (i) ÷ (iii) of *Theorem 7*, for the system given by linearized equations (3).

Moreover, it can be shown that the subspace $\mathcal{V}$ at the statement of *Theorem 7* exists or any symmetric set of different complex numbers $\{\lambda_i\}$, $i \in 01, 2,..., h$, assuring that system (4) is controllable for its finite and infinite eigenvalues. For this case, it can be concluded that the matrix F exists, with a property that the matrix pencil $(sE\text{-}A\text{-}BF)$ is regular and it has $\{\lambda_i\}$, $i \in 1, 2,..., h$ as a set of eigenvalues. That implies the existence of $v_i \in \mathcal{H}\ (\lambda_i)$, $i \in 1, 2,..., h$ such as

$$span\ \{v_i\} \cap \mathcal{N}\ (E) = 0 \tag{70}$$

A possible solution for the determination of the subspace $\mathcal{V}$ is a set of linearly independent vectors $v_i \in \mathcal{H}\ (\lambda_i)$, but with a property $v_i \notin \mathcal{N}\ (E)$. Then, $\mathcal{V} = span\ \{v_i\}$, and $\mathcal{V} \cap \mathcal{N}\ (E) = 0$. This practically means that it is possible to assign both eigenvalues and eigenvectors for system (4) using state-space feedback defined by the matrix F.

The proposed method can be used for the dynamic analysis of a class of robotic systems where the contact forces are included into the mathematical models of systems, or even in the cases where it is necessary to control the reactive force to systems.

System simulation

In this section, the simulation results for five degree-of-freedom (DOF) robotic systems are presented. The state-space values are generalized coordinates and generalized velocities of each joint. Fig.1 shows the dynamic behavior of the uncontrolled system.

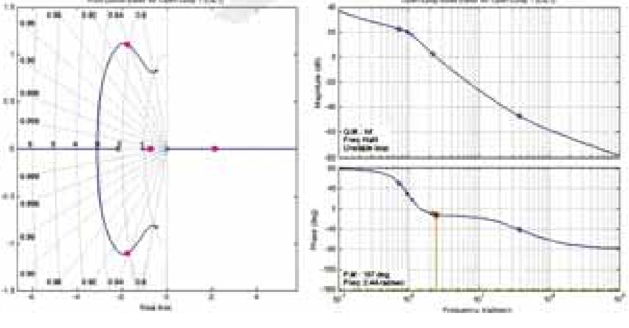

Figure 1. Pole placement and the stability analysis of an uncontrolled system

Fig.2 represents the pole positions for system (52) when the poles are adjusted by the described geometric method and control law (51).

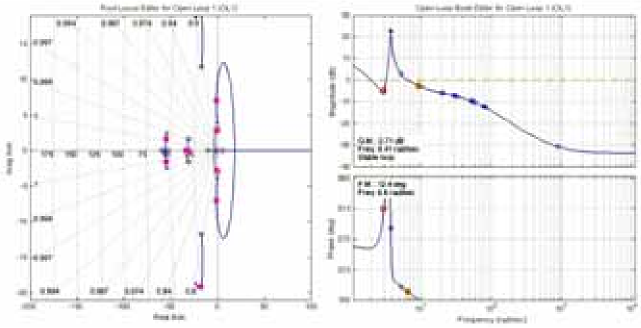

Figure 2. Pole placement and stability analysis of the controlled system with state-space feedback inside the invariant controlled subspace V

The computational programs developed for this analysis can be found in *Buzurović*, *Debeljković* (2010). The article gives the basis for determining the maximum controlled subspace, which is responsible for the negative pole position. For the purpose of this numerical example, one component of the transmission matrix that describes the dynamic contact reactive force has been shown.

Fig.3 and 4 represent the sinusoidal response of uncontrolled system (3), and then for controlled systems (52) in the invariant subspace V, when feedback (51) has been applied.

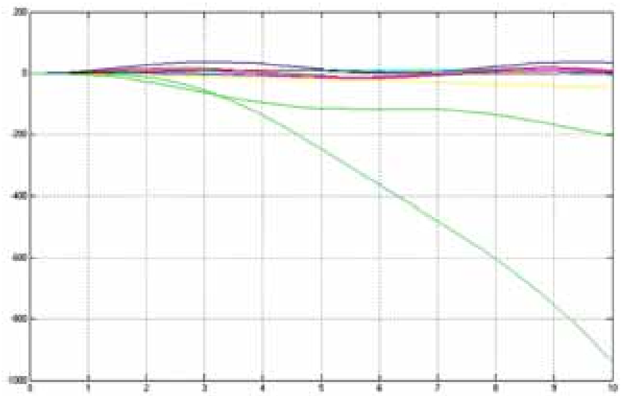

Figure 3. Sinusoidal response of the uncontrolled system

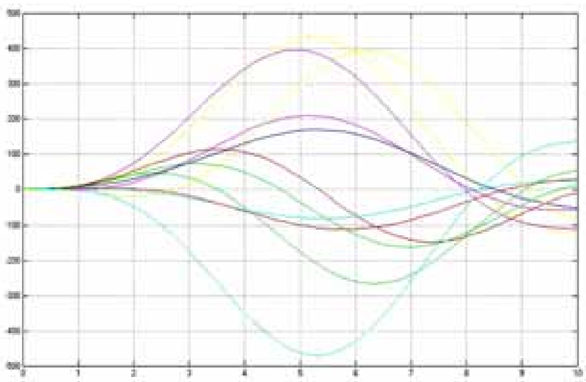

Figure 4. Sinusoidal response of the controlled system

It was shown that the controllability of system (52) depends on the structure of matrix (54). It can be noticed that the scalar multiplier λ can influence condition (54) and consequently the system stability. Using condition (54) and the adjusted pole positions calculated from *Theorem 7*, the scalar multiplier λ can be adopted in a way to guarantee the avoidance of the impulse modes in the singular system. When a robotic system is in the contact with environment, λ is not constant during the working regime. The scalar multiplier depends on the generalized coordinates q, as in Fig.5. The local maximum of λ exists when the contact force has its maximum value.

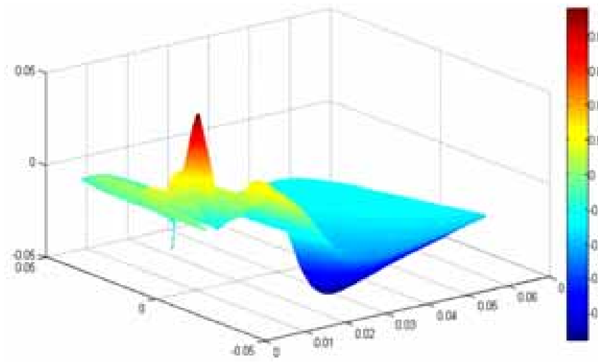

Figure 5. Scalar multiplier during the contact of a robotic system with the environment. In this case, λ depends on three active generalized coordinates.

Conclusion

The necessary and sufficient condition for the existence of the solutions for a mathematical model of one robotic system has been presented. For the investigation of the solutions, the geometric approach was used. The proposed approach could be applied to any robotic system which is in contact with the rigid frictionless surface in the working regime.

The results of system controllability have been extended to systems with contact task represented as singular (semi-state, descriptor) systems. The state controllability condition implied that it is possible to steer the states from any initial value to any final value within some time window. The derived results represented the sufficient conditions for controllability of such systems, based on the generalized Lyapunov equation derived using the geometric approach. Simple sufficient algebraic conditions were derived for controllability testing. The results could be used as a basis for further development of a similar analysis for different robotic systems with contact task as well as for nonlinear and time-variable, and time-discrete descriptor systems.

Both the eigenvalue and eigenvector assignment procedure have been analyzed in order to avoid undesirable mode coupling and other poor transient behavior due to the singular character of the mathematical model.

The presented methodology can be applied to the investigation of numerous robotic systems in contact with the environment. For such systems, the desired dynamical behavior and tasks cannot be defined solely in terms of motion of the end-effector. The robotic systems designed for scribing, deburring, grinding, writing, insertions etc., are good candidates to be analyzed using the proposed method.

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Podešavanje sopstvenih vrednosti posebne klase singularnih sistema za ograničene robotske sisteme

Robotski sistemi u kontaktu sa spoljasnjom sredinom su karakteristican primer gde kontaktne sile igraju vaznu ulogu u dinamici takvih sistema. Shodno tome, njihovo matematicko modeliranje predstavlja poseban izazov. Matematicki model opisanih sistema sadrzi kako diferencijalne, tako i algebarske jednacine koje reprezentuju ogranicenja na sistem. Poznato je da se takav sistem naziva singularni sistemi diferencijalnih jednacina. U ovom clanku prezentovan je geometrijski pristup iznalazenju resenja dinamickog sistema sa ogranicenjima. Ispitano je podesavanje sopstvenih vrednosti, kao i sopstvenih vektora sistema. Da bi sistem ostvario zeljeno dinamicko ponasanje odredjeni su i matematicki uslovi upravljivosti sistema.

Ključne reči: **singularni sistem, upravljivost, geometrijski prilaz, podešavanje polova, dinamika sistema, matematički model, robotika.**

Установка собственных значений специального класса сингулярных систем для ограниченых робототехнических систем

Роботизированные системы в контакте с внешней средой представляют типичный пример, когда контактные силы играют важную роль в динамике таких систем. Следовательно, их математическое моделирование является особым вызовом. Математическая модель описанных систем включает в себя как дифференциальные, так и алгебраические уравнения, которые представляют собой ограничения на системы. Известно, что такая система называется особой сингулярной системой дифференциальных уравнений. В данной статье представлен геометрический подход к решению динамических систем со ограничениями. Мы рассмотрели корректировки своих значений и собственных векторов системы. Для достижения желаемого динамического поведения системы, определены и математические условия управляемости системы.

Ключевые слова: **сингулярные системы, управляемость, геометрический подход, установка полюсов, динамика системы, математическая модель, робототехника.**

Le réglage des valeurs propres de classe particulière des systèmes singuliers pour les systèmes robotiques contraints

Les systèmes robotiques en contact avec l'environnement sont les exemples typiques où les forces de contact jouent un rôle important dans la dynamique de ces systèmes. C'est pourquoi leur modélisation est un défi particulier. Le modèle mathématique des systèmes décrits comprend les équations différentielles et celles algébriques qui représentent les contraintes pour le système. Il est bien connu que ce système est considéré comme le système singulier des équations différentielles. Dans cet article on a présenté l'approche géométrique servant à trouver la solution pour le système dynamique avec contraintes. On a étudié le réglage des valeurs propres du système. Pour que le système réalise le comportement dynamique voulu on a déterminé les conditions mathématiques du contrôle de système.

Mots clés: **système singulier, contrôle, approche géométrique, réglage des pôles, dynamique de système, modèle mathématique, robotique.**