UDK: 66.045.1:512.2:517.9
COSATI: 20-13, 12-01

Mathematical Modelling and Robust Control of the Double Tube Heat Exchanger: the Parametric Approach

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Mathematical modeling of the heat exchangers is a very important part of the designing process in order to define a correct control strategy. Various types of heat exchangers are frequently used as a constructive element in military aircraft and their operation is usually controlled by manipulating inlet fluid temperatures or mass flow rates. In the first part of this paper, a linearized mathematical model of the double tube heat exchanger has been derived taking into account the wall dynamics, which led to its description in a form of a set of partial differential equations. In order to overcome partial differential solutions difficulties, the finite difference method is proposed. Finally, specifying the parameter values of the heat exchanger, under constant initial conditions, the step transient responses have been simulated and presented in a graphic form. In the second part of this paper, the problem of robust control of the double tube heat exchanger based on the parametric approach is considered. The differential-discrete mathematical model of the double tube heat exchanger derived in the first part of this paper containing uncertain parameters is used to determine the largest perturbation in the space of real parameters without destroying closed loop stability. The main idea is to determine the maximally robust controller in sense that it guarantees stability while maximizing the size of perturbation cenetred at nominal values.

Key words: **heat exchanger, mathematical modeling, finite difference method, robust controlling, parametric model, controller.**

Introduction

OBTAINING a sufficiently accurate mathematical model of double tube heat exchangers is very important in order to determine the transient response and implement an appropriate control strategy.

Heat exchangers are frequently used in military and civil aircraft as oil coolers and fuel heaters with heat exchangers placed inside fuel tanks. In large transport aircraft, the cabin pressurization systems cool the hot engine bleed air to a temperature suitable for use within the cabin with a heat exchanger. The air conditioning system is also based on a heat exchanger, known as an air-cycle machine, to provide air at a desired temperature to the cabin and flight deck.

Solving the problem of deriving appropriate mathematical models of double tube heat exchangers is usually based on two approaches:

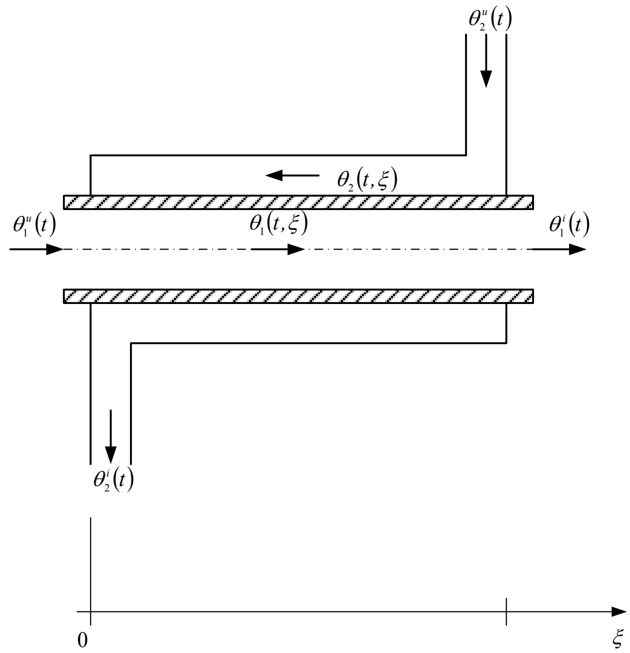
1. numerical solving of partial differential equations which pulls the convergation problems, stability, etc.
2. using the Laplace transformation which is complicated in this case and demands a numerical inversion to the original time domain.

The mentioned problems and the similar ones have motivated researchers to invent new approaches for modeling this processes in order to obtain a wider practical use and not to lose their veracity as well.

In the first part of this paper, a mathematical model of a double tube heat exchanger is derived using the finite difference method and analysing its practical significance. In the second part of this paper, the problem of the robust control of a double tube heat exchanger based on the parametric approach is considere.

Mathematical model of a double tube heat exchanger

A recuperative double tube heat exchanger is observed, shown in Fig.1, as a process with distributed parameters. The geometry of this heat exchanger consists of two tubes, an inner and an outer one.



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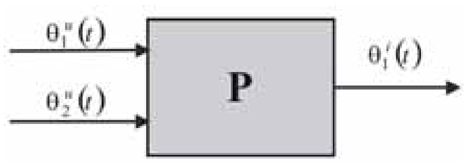

Figure 1. Symbolic-functional scheme and the diagram of the process of the cross-flow heat exchanger

The mathematical model of the double tube heat exchanger shown in Fig.1 is derived on the basis of the following assumptions:

- Fluids in the tubes are incompressible and viscous.
- Temperature field fluid in the tubes are one-dimensional.
- Temperature field of the tube wall is one-dimensional.
- Fluid entalpy can be expressed by means of a corresponding temperature.
- Specific heat of fluids and wall has constant values.
- Newton's law of heat convection determines exactly enough the amount of exchange heat in the steady state and the transient state.
- Heat transfer coefficients on the both sides of the tube-wall have constant values.
- The fluid flow in the tubes is one-phased.
- Heat loss to the environment is negligible.

On the basis of these assumptions, the mathematical model of the doble tube heat exchanger has the following form:

- *for fluid* 1:

$$\begin{aligned} c_1\rho_1 A_1 \frac{\partial \theta_1(t,\xi)}{\partial t} d\xi &= \alpha_{z1} O_{z1} \left(\theta_z(t,\xi) - \theta_1(t,\xi)\right) d\xi + \\ &+ c_1\rho_1 A_1 w_1 \theta_1(t,\xi) - c_1\rho_1 w_1 A_1 \left(\theta_1(t,\xi) + \frac{\partial \theta_1(t,\xi)}{\partial \xi} d\xi\right) \end{aligned} \tag{1}$$

- *for fluid* 2:

$$\begin{aligned} c_2\rho_2 A_2 \frac{\partial \theta_2(t,\xi)}{\partial t} d\xi &= -c_2\rho_2 w_2 A_2 \theta_2(t,\xi) + \\ &+ c_2\rho_2 w_2 A_2 \left(\theta_2(t,\xi) + \frac{\partial \theta_2(t,\xi)}{\partial \xi} d\xi\right) - \\ &- \alpha_{2z} O_{2z} \left(\theta_2(t,\xi) - \theta_z(t,\xi)\right) d\xi \end{aligned} \tag{2}$$

- *for the tube wall*:

$$\begin{aligned} c_z\rho_z A_z \frac{\partial \theta_z(t,\xi)}{\partial t} &= \alpha_{2z} O_{2z} \left(\theta_2(t,\xi) - \theta_z(t,\xi)\right) - \\ &- \alpha_{z1} O_{z1} \left(\theta_z(t,\xi) - \theta_1(t,\xi)\right) \end{aligned} \tag{3}$$

where L is the length of the tube, G_1, G_2 are the mass-flow rate of fluid 1 fluid 2, c_1, c_2, c_z are the specific heat of fluid 1, fluid 2 and the wall respectively, $\theta_1^u, \theta_1^i, \theta_2^u, \theta_2^i$ are incoming and outgoing temperatures of fluid 1 and fluid 2, α_{z1} is the heat transfer coefficient between the wall and fluid 1, α_{2z} is the heat transfer coefficient between fluid 2 and the wall, w_1, w_2 are velocites of fluid 1 and fluid 2, O_{z1}, O_{2z} are the inner and outer volume of the tube, ρ_1, ρ_2, ρ_z are the densities of fluid 1, 2 and the wall respectively, A_1 is the cross- sectional area of the inner tube, A_2 is the cross- sectional area between the outer tube and the inner tube, and A_z is the cross- sectional area of the tube wall.

After some simple transformations the mathematical model of the double tube heat exchanger is obtained in the following form:

- *for fluid* 1:

$$\begin{aligned} \frac{\partial \theta_1(t,\xi)}{\partial t} &= -\frac{G_1}{\rho_1 A_1} \frac{\partial \theta_1(t,\xi)}{\partial \xi} + \\ &+ \frac{\alpha_{z1} O_{z1}}{c_1 \rho_1 A_1} \left(\theta_z(t,\xi) - \theta_1(t,\xi)\right) \end{aligned} \tag{4}$$

- *for fluid* 2:

$$\begin{aligned} \frac{\partial \theta_2(t,\xi)}{\partial t} &= \frac{G_2}{\rho_2 A_2} \frac{\partial \theta_2(t,\xi)}{\partial \xi} - \\ &- \frac{\alpha_{2z} O_{2z}}{c_2 \rho_2 A_2} \left(\theta_2(t,\xi) - \theta_z(t,\xi)\right) \end{aligned} \tag{5}$$

- *for the tube wall*:

$$\begin{aligned} \frac{\partial \theta_z(t,\xi)}{\partial t} &= \frac{\alpha_{2z} O_{2z}}{c_z \rho_z A_z} \left(\theta_2(t,\xi) - \theta_z(t,\xi)\right) - \\ &- \frac{\alpha_{z1} O_{z1}}{c_z \rho_z A_z} \left(\theta_z(t,\xi) - \theta_1(t,\xi)\right) \end{aligned} \tag{6}$$

The finite difference method

The system of partial differential equations (4), (5) and (6) is very complicated to be solved analytically. A numerical method based on the finite differences is developed to approximate infinite-dimensional equations by finite-dimensional ones. The spatial coordinate of the double tube heat exchanger is discretized by means of the finite difference method. In this manner, the system of partial differential equations is transformed to the system of ordinary differential equations. The discretization of the spatial variable is carried out by means of substituting the partial derivation with respect to the spatial coordinate in a certain numbers of points. Physically speaking, the observed heat exchanger is divided into equal p-cells with the spatial coordinate discretization. The lenght of each cell is ℓ.

The finite difference method has a different form for the first, *k*-th (*k*= 2,3,..., p-1) and the last cell. This method has many different forms (forward difference, backward difference, central difference). Considering the one-dimensional problem, the finite difference method has the following form:

- *for the first cell*:

$$\begin{aligned} \frac{d\theta_{1,1}(t)}{dt} &= -\frac{G_1}{2l\rho_1 A_1} \left(\theta_{1,2}(t) - \theta_1^u(t)\right) + \\ &+ \frac{\alpha_{z1} O_{z1}}{c_1 \rho_1 A_1} \left(\theta_{z,1}(t) - \theta_{1,1}(t)\right) \end{aligned}$$

$$\begin{aligned} \frac{d\theta_{2,1}(t)}{dt} &= \frac{G_2}{l\rho_2 A_2} \left(\theta_{2,2}(t) - \theta_{2,1}(t)\right) - \\ &- \frac{\alpha_{2z} O_{2z}}{c_2 \rho_2 A_2} \left(\theta_{2,1}(t) - \theta_{z,1}(t)\right) \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{d\theta_{z,1}(t)}{\partial t} &= \frac{\alpha_{2z} O_{2z}}{c_z \rho_z A_z} \left(\theta_{2,1}(t) - \theta_{z,1}(t)\right) - \\ &- \frac{\alpha_{z1} O_{z1}}{c_z \rho_z A_z} \left(\theta_{z,1}(t) - \theta_{1,1}(t)\right) \end{aligned}$$

- *for the k-th cell*:

$$\begin{aligned}\frac{d\theta_{1,k}(t)}{dt} &= -\frac{G_1}{2l\rho_1 A_1}\left(\theta_{1,k+1}(t)-\theta_{1,k-1}(t)\right)+\\&+\frac{\alpha_{z1}O_{z1}}{c_1\rho_1 A_1}\left(\theta_{z,k}(t)-\theta_{1,k}(t)\right)\\ \frac{d\theta_{2,k}(t)}{dt} &= \frac{G_2}{2l\rho_2 A_2}\left(\theta_{2,k+1}(t)-\theta_{2,k-1}(t)\right)-\\&-\frac{\alpha_{2z}O_{2z}}{c_2\rho_2 A_2}\left(\theta_{2,k}(t)-\theta_{z,k}(t)\right)\\ \frac{d\theta_{z,k}(t)}{dt} &= \frac{\alpha_{2z}O_{2z}}{c_z\rho_z A_z}\left(\theta_{2,k}(t)-\theta_{z,k}(t)\right)-\\&-\frac{\alpha_{z1}O_{z1}}{c_z\rho_z A_z}\left(\theta_{z,k}(t)-\theta_{1,k}(t)\right)\end{aligned}\tag{8}$$

- *for the last cell*:

$$\begin{aligned}\frac{d\theta_{1,p}(t)}{dt} &= -\frac{G_1}{l\rho_1 A_1}\left(\theta_{1,p}(t)-\theta_{1,p-1}(t)\right)+\\&+\frac{\alpha_{z1}O_{z1}}{c_1\rho_1 A_1}\left(\theta_{z,p}(t)-\theta_{1,p}(t)\right)\\ \frac{d\theta_{2,p}(t)}{dt} &= \frac{G_2}{2l\rho_2 A_2}\left(\theta_2^u(t)-\theta_{2,p-1}(t)\right)-\\&-\frac{\alpha_{2z}O_{2z}}{c_2\rho_2 A_2}\left(\theta_{2,p}(t)-\theta_{z,p}(t)\right)\\ \frac{d\theta_{z,p}(t)}{dt} &= \frac{\alpha_{2z}O_{2z}}{c_z\rho_z A_z}\left(\theta_{2,p}(t)-\theta_{z,p}(t)\right)-\\&-\frac{\alpha_{z1}O_{z1}}{c_z\rho_z A_z}\left(\theta_{z,p}(t)-\theta_{1,p}(t)\right)\end{aligned}\tag{9}$$

Introducing relative deviations, the state and control variables in this form are:

$$\begin{aligned}\overline{\Delta\theta}_{1,k}(t) &= \frac{\theta_{1,k}(t)-\theta_{1N,k}}{\theta_{1N,k}} = x_{1,k}(t),\\ \overline{\Delta\theta}_{2,k}(t) &= \frac{\theta_{2,k}(t)-\theta_{2N,k}}{\theta_{2N,k}} = x_{2,k}(t),\\ \overline{\Delta\theta}_{z,k}(t) &= \frac{\theta_{z,k}(t)-\theta_{zN,k}}{\theta_{zN,k}} = x_{3,k}(t),\\ \overline{\Delta\theta}_1^u(t) &= \frac{\theta_1^u(t)-\theta_{1N}^u}{\theta_{1N}^u} = u_1(t),\\ \overline{\Delta\theta}_2^u(t) &= \frac{\theta_2^u(t)-\theta_{2N}^u}{\theta_{2N}^u} = u_2(t).\end{aligned}\tag{10}$$

The mathematical model for the *k*-th cell can be written in the following form:

$$\begin{aligned}\frac{dx_{1,k}(t)}{dt} &= -a_{k1}x_{1,k}(t)+a_{k2}x_{3,k}(t)+\\&+a_{k3}x_{1,k-1}(t)-a_{k4}x_{1,k+1}(t)\\ \frac{dx_{2,k}(t)}{dt} &= -b_{k1}x_{2,k}(t)+b_{k2}x_{3,k}(t)-\\&-b_{k3}x_{2,k-1}(t)+b_{k4}x_{2,k+1}(t)\\ \frac{dx_{3,k}(t)}{dt} &= -c_{k1}x_{3,k}(t)+c_{k2}x_{1,k}(t)+\\&+c_{k3}x_{2,k}(t)\end{aligned}\tag{11}$$

- *for the first cell*:

$$\begin{aligned}\frac{dx_{1,1}(t)}{dt} &= -a_{11}x_{1,1}(t)+a_{12}x_{3,1}(t)+\\&+a_{13}u_1(t)-a_{14}x_{1,2}(t)\\ \frac{dx_{2,1}(t)}{dt} &= -b_{11}x_{2,1}(t)+b_{12}x_{3,1}(t)+\\&+b_{13}x_{2,2}(t)\\ \frac{dx_{3,1}(t)}{\partial t} &= -c_{11}x_{3,1}(t)+c_{12}x_{1,1}(t)+\\&+c_{13}x_{2,1}(t)\end{aligned}\tag{12}$$

- *for the last cell:*

$$\begin{aligned}\frac{dx_{1,p}(t)}{dt} &= -a_{p1}x_{1,p}(t)+a_{p2}x_{3,p}(t)+\\&+a_{p3}x_{1,p-1}(t)\\ \frac{dx_{2,p}(t)}{dt} &= -b_{p1}x_{2,p}(t)+b_{p2}x_{3,p}(t)-\\&-b_{p3}x_{2,p-1}(t)+b_{p4}u_2(t)\\ \frac{dx_{3,p}(t)}{dt} &= -c_{p1}x_{3,p}(t)+c_{p2}x_{1,p}(t)+\\&+c_{p3}x_{2,p}(t)\end{aligned}\tag{13}$$

where $k = 1,2,\ldots,p$ and

$$\begin{gathered}a_{k1} = \frac{\alpha_{z1}O_{z1}}{c_1\rho_1 A_1},\ a_{k2} = \frac{\alpha_{z1}O_{z1}}{c_1\rho_1 A_1}\frac{\theta_{zN,k}}{\theta_{1N,k}},\\ a_{k3} = \frac{w_1}{2l}\frac{\theta_{1N,k-1}}{\theta_{1N,k}},\ a_{k4} = \frac{w_1}{2l}\frac{\theta_{1N,k+1}}{\theta_{1N,k}},\\ a_{p1} = \left(\frac{w_1}{l}+\frac{\alpha_{z1}O_{z1}}{c_1\rho_1 A_1}\right),\ a_{p3} = \frac{w_1}{l}\frac{\theta_{1N,p-1}}{\theta_{1N,p}},\\ b_{11} = \frac{w_2}{l}+\frac{\alpha_{2z}O_{2z}}{c_2\rho_2 A_2},\ b_{13} = \frac{w_2}{l}\frac{\theta_{2N,2}}{\theta_{2N,1}},\\ b_{k1} = \frac{\alpha_{2z}O_{2z}}{c_2\rho_2 A_2},\ b_{k2} = \frac{\alpha_{2z}O_{2z}}{c_2\rho_2 A_2}\frac{\theta_{zN,k}}{\theta_{2N,k}},\\ b_{k3} = \frac{w_2}{2l}\frac{\theta_{2N,k-1}}{\theta_{2N,k}},\ b_{k4} = \frac{w_2}{2l}\frac{\theta_{2N,k+1}}{\theta_{2N,k}},\\ c_{k1} = \left(\frac{\alpha_{2z}O_{2z}}{c_z\rho_z A_z}+\frac{\alpha_{z1}O_{z1}}{c_z\rho_z A_z}\right),\ c_{k2} = \frac{\alpha_{z1}O_{z1}}{c_z\rho_z A_z}\frac{\theta_{1N,k}}{\theta_{zN,k}},\\ c_{k3} = \frac{\alpha_{2z}O_{2z}}{c_z\rho_z A_z}\frac{\theta_{2N,k}}{\theta_{zN,k}},\ k = 1,2,\ldots,p\end{gathered}\tag{14}$$

Using Eqs. (10-13) the mathematical model is derived in the state space:

- *for the* k-th *cell*:

$$\dot{\mathbf{x}}_k(t) = A_k^1 \mathbf{x}_{k-1}(t) + A_k^2 \mathbf{x}_k(t) + A_k^3 \mathbf{x}_{k+1}(t) + B_k \mathbf{u}(t) \tag{15}$$

where:

$$A_k^1 = \begin{pmatrix} a_{k3} & 0 & 0 \\ 0 & -b_{k3} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \forall k = 2,3,\ldots,p-1,$$

$$A_k^2 = \begin{pmatrix} -a_{k1} & 0 & a_{k2} \\ 0 & -b_{k1} & b_{k2} \\ c_{k2} & c_{k3} & -c_{k1} \end{pmatrix}, \tag{16}$$

$$A_k^3 = \begin{pmatrix} -a_{k4} & 0 & 0 \\ 0 & b_{k4} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; B_k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

- *for the first cell*:

$$\dot{\mathbf{x}}_1(t) = A_1^2 \mathbf{x}_1(t) + A_1^3 \mathbf{x}_2(t) + B_1 \mathbf{u}(t) \tag{17}$$

where:

$$A_1^2 = \begin{pmatrix} -a_{11} & 0 & a_{12} \\ 0 & -b_{11} & b_{12} \\ c_{12} & c_{13} & -c_{11} \end{pmatrix}, \tag{18}$$

$$A_1^3 = \begin{pmatrix} -a_{14} & 0 & 0 \\ 0 & b_{13} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; B_1 = \begin{pmatrix} a_{13} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- *for the last cell:*

$$\dot{\mathbf{x}}_p(t) = A_p^1 \mathbf{x}_{p-1}(t) + A_p^2 \mathbf{x}_p(t) + B_p \mathbf{u}(t) \tag{19}$$

where:

$$A_p^1 = \begin{pmatrix} a_{p3} & 0 & 0 \\ 0 & -b_{p3} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{20}$$

$$A_p^2 = \begin{pmatrix} -a_{p1} & 0 & a_{p2} \\ 0 & -b_{p1} & b_{p2} \\ c_{p2} & c_{p3} & -c_{p1} \end{pmatrix}, \; B_p = \begin{pmatrix} 0 & 0 \\ 0 & b_{p4} \\ 0 & 0 \end{pmatrix}$$

The state equation and the output of the double tube heat exchanger are given by:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{21}$$

$$\mathbf{x}_i(t) = C\mathbf{x}(t)$$

$$A = \begin{pmatrix} A_1^2 & A_1^3 & 0 & 0 & \ldots & 0 & 0 & 0 \\ A_2^1 & A_2^2 & A_2^3 & 0 & \ldots & 0 & 0 & 0 \\ 0 & A_3^1 & A_3^2 & A_3^3 & \ldots & 0 & 0 & 0 \\ 0 & 0 & A_4^1 & A_4^2 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & A_p^1 & A_p^2 \end{pmatrix}$$

$$B = \begin{bmatrix} B_1 & | B_2 & \ldots & | B_p \end{bmatrix}^T,$$

$$C = \begin{pmatrix} 0 & 0 & 0 & \vdots & 0 & 0 & 0 & \vdots & \ldots & \vdots & 1 & 0 & 0 \end{pmatrix}$$

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_p(t) \end{bmatrix}, \; x_i(t) = \theta_{1p}(t), \; \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

Results of the simulation

For the parameters of a real double tube heat exchanger given in Table 1, the simulation is performed and appropriate step responses are obtained.

The observed heat exchanger is divided into five cells. If the results are unsatifactory, the number of cells must be increased.

Figs.2 and 4 show the step responses of the outlet temperature of fluid 1 for the step change of the fluid 1 inlet temperature and the step response of the fluid 1 outlet temperature for the step change of the fluid 2 inlet temperature. Figs. 3 and 5 show the temperature profiles in the outlet of the cells for the step change of the fluid 1 inlet temperature and for the step change of the fluid 2 inlet temperature.

Table 1.

	Parameter	Dimension	Value
1.	L	m	6
2.	d_c	m	0.0779
3.	D_c	m	0.0889
4.	d_p	m	0.1023
5.	D_p	m	0.1143
6.	w_2	m/s	0.7
7.	w_1	m/s	0.6
8.	ρ_1	kg/m^3	990.2
9.	ρ_2	kg/m^3	983.1
10.	ρ_z	kg/m^3	7870
11.	α_{2z}	$W/(m^2K)$	340
12.	α_{z1}	$W/(m^2K)$	315
13.	c_1	$J/(kgK)$	4174
14.	c_2	$J/(kgK)$	4193.4
15.	c_z	$J/(kgK)$	444
16.	θ_{f2N}^u	°C	60
17.	θ_{f1N}^u	°C	45

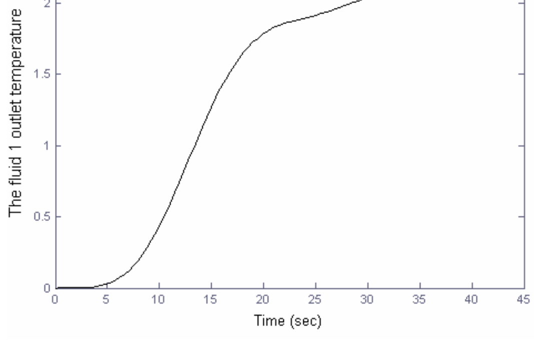

Figure 2. Step response of the fluid 1 outlet temperature for the step change in the fluid 1 inlet temperature

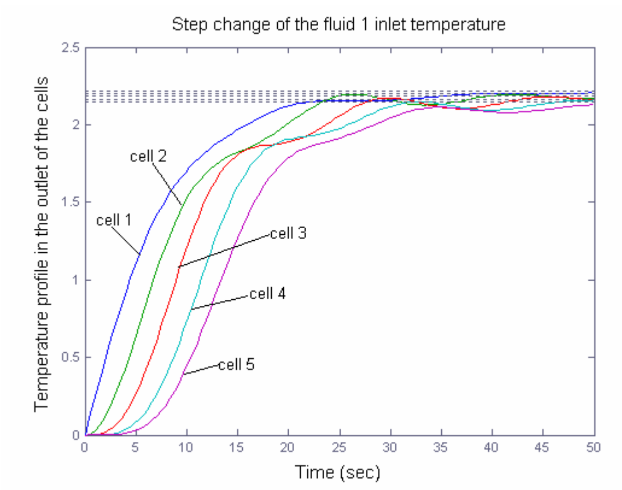

Figure 3 Temperature profile at the outlet of the cells for the step change in the fluid 1 inlet temperature

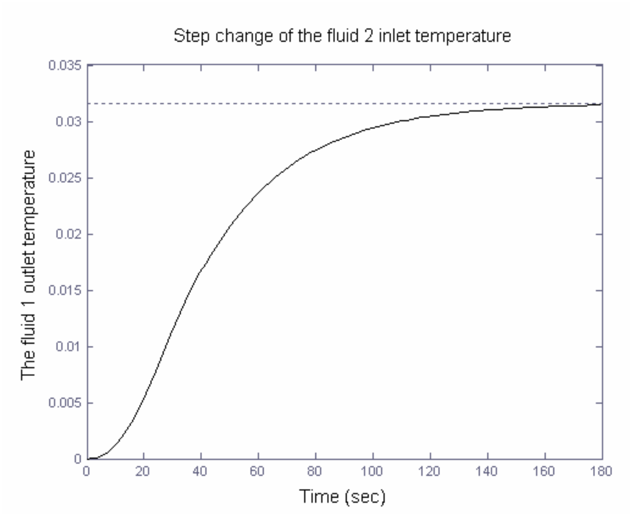

Figure 4. Step response of the luid 1 outlet temperature for the step change in the fluid 2 inlet temperature

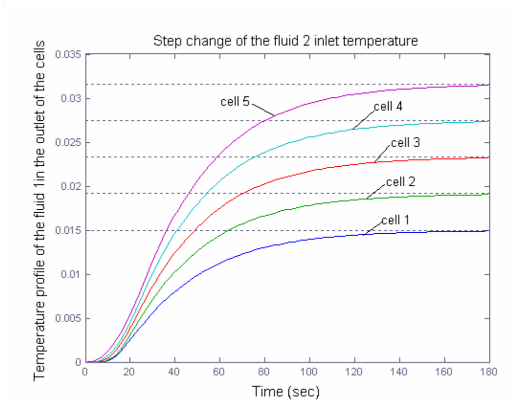

Figure 5. Temperature profile at the outlet of the cells for the step change in the fluid 2 inlet temperature

Robust control of the double tube heat exchanger based on the parametric approach

In this section the problem of robust control using the feedback proportional controller is considered. The differential-discrete mathematical model of the double tube heat exchanger derived in the first part of this paper containing uncertain parameters is used to calculate the radius of the largest stability ball in the space of real parameters under the assumption that these uncertain parameters enter the characteristic polynomial coefficients linearly or affinely [2]. Every feedback system is composed of at least two subsystems, namely a controller and a plant connected in a feedback loop.

The characteristic polynomial coefficients of such a system represent a function of plant parameters and controller parameters. The plant contains the parameters that are subject to uncontrolled variations depending on physical operating conditions, disturbances, modelling errors, etc.

Suppose that the plant transfer function contains a real parameter vector $\mathbf{p}$ and the controller is realized as a proportional feedback controller with the parameter K and suppose that $\mathbf{p}_N$ denotes the nominal value of the plant parameter vector $\mathbf{p}$ and let $\Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_N$ denote a perturbation of the plant parameter vector from the nominal value.

The main idea of this approach is to determine the largest perturbation $\Delta\mathbf{p}$ without destroying closed loop stability. The size of $\Delta\mathbf{p}$ for which closed loop stability is guaranteed provides us with a ball in the parameter space within which the parameters can freely vary without destroying closed loop stabilitymeters can freely vary without destroying closed loop stability.

Accordingly, the parametric stability margin is defined as a length of the smallest perturbation $\Delta\mathbf{p}$ which destabilizes the closed loop, [2]. It is useful in controller design as a means of comparing the performance of proposed controllers. In the design problems, one faces a task of choosing the controller parameter to maximize this margin.

We consider the case with the fixed controller parameter *K*, so the coefficients of the characteristic polynomial are the linear functions of the plan parameter $\mathbf{p}$. The characteristic polynomial in general has the following form:

$$f(s,\mathbf{p}) = a_1(s)p_1 + a_2(s)p_2 + \dots + a_l(s)p_l + b(s) \quad (21)$$

where $a_i(s)$, $i = 1,2,\dots,l$ and $b(s)$ are fixed polynomials and $\mathbf{p} = [p_1 \ \ p_2 \ \ \dots \ \ p_l]^T$ is the vector of uncertain parameters.

Introducing the perturbation vector $\Delta\mathbf{p}$ the characteristic polynomial can be written as:

$$\begin{aligned} f(s,\mathbf{p}_N + \Delta\mathbf{p}) &= \underbrace{f(s,\mathbf{p}_N)}_{f_N(s)} + \\ &+ \underbrace{a_1(s)\Delta p_1 + a_2(s)\Delta p_2 + \dots + a_l(s)\Delta p_l}_{\Delta f(s,\Delta\mathbf{p})} \end{aligned} \quad (22)$$

Let $s = s^*$ denote a point on the stability boundary ∂S. For $s^* \in \partial S$ to be root of $f(s,\mathbf{p}_N + \Delta\mathbf{p})$ it must be satisfied, [2]:

$$f(s^*,\mathbf{p}_N) + \Delta f(s^*,\Delta\mathbf{p}) = 0 \quad (23)$$

Let a_{in} denote a coefficient of the n^{th} degree in the polynomial $a_i(s)$, $i = 1,2,\dots,l$. Introducing weights $h_i > 0$ equation (23) can be written in the following form:

$$f\left(s^*,\mathbf{p}_N\right)+\frac{a_1\left(s^*\right)}{h_1}h_1\Delta p_1+...\frac{a_l\left(s^*\right)}{h_l}h_l\Delta p_l=0\,. \quad (24)$$

In (24) two cases may occur depending on whether s^* is real or complex.

If $s^*=s_r$ where s_r is real, equation (24) can be written as follows:

$$\underbrace{\begin{bmatrix}\frac{a_1(s_r)}{h_1} & \dots & \frac{a_l(s_r)}{h_l}\end{bmatrix}}_{T(s_r)}\underbrace{\begin{bmatrix}h_1\Delta p_1\\ \vdots \\ h_l\Delta p_l\end{bmatrix}}_{\mathbf{d}(s_r)}=\underbrace{-f_N\left(s_r\right)}_{\mathbf{b}(s_r)} \quad (25)$$

If $s^*=s_c$ where s_c is complex we can use this notation:

$$a_k\left(s^*\right)=a_{kr}\left(s^*\right)+ja_{ki}\left(s^*\right) \quad (26)$$

$$f_N\left(s^*\right)=f_{Nr}\left(s^*\right)+jf_{Ni}\left(s^*\right)$$

In this case equation (24) can be written as follows:

$$\underbrace{\begin{pmatrix}\frac{a_{1r}(s_c)}{h_1} & \dots & \frac{a_{lr}(s_c)}{h_1}\\ \frac{a_{1i}(s_c)}{h_1} & \dots & \frac{a_{li}(s_c)}{h_1}\end{pmatrix}}_{T(s_c)}\underbrace{\begin{bmatrix}h_1\Delta p_1\\ \vdots \\ h_l\Delta p_l\end{bmatrix}}_{\mathbf{d}(s_c)}=\underbrace{\begin{bmatrix}-f_{Nr}\left(s_c\right)\\ -f_{Ni}\left(s_c\right)\end{bmatrix}}_{\mathbf{b}(s_c)} \quad (27)$$

Equations (25) and (27) completely determine the parametric stability margin in any norm. If we denote $\mathbf{d}^*\left(s_c\right),\mathbf{d}^*\left(s_r\right)$ the minimum norm sollutions of (27) and (25) then,

$$\left\|\mathbf{d}^*\left(s_c\right)\right\|=\rho\left(s_c\right) \quad (28)$$

$$\left\|\mathbf{d}^*\left(s_r\right)\right\|=\rho\left(s_r\right)$$

In this paper we suppose that the lenght of the perturbation vector $\Delta\mathbf{p}$ is measured by a weighted l_2 norm in the following form:

$$\left\|\Delta\mathbf{p}\right\|_2^h=\sum_{i=1}^{l}\sqrt{h_i^2\Delta p_i^2} \quad (29)$$

Assuming that $T\left(s_c\right)$ has full row *rank*=2, the minimum norm sollution vector $\mathbf{d}\left(s_c\right)$ can be calculated from equation (27):

$$\mathbf{d}^*\left(s_c\right)=T^T\left(s_c\right)\left[T\left(s_c\right)T^T\left(s_c\right)\right]^{-1}\mathbf{b}\left(s_c\right) \quad (30)$$

Similarly, if $T\left(s_r\right)$ is a nonzero vector, the sollution of equation (25) can be written in the following form:

$$\mathbf{d}^*\left(s_r\right)=T^T\left(s_r\right)\left[T\left(s_r\right)T^T\left(s_r\right)\right]^{-1}\mathbf{b}\left(s_r\right) \quad (31)$$

In this paper the uncertain plant parameter of the differential-discrete mathematical model of the double tube heat exchanger is assumed to be $p=\frac{\theta_{1N,4}}{\theta_{1N,5}}$. The nominal parameter value is $p_N=0.996$.

The differential discrete mathematical model of the double tube tube heat exchanger is derived in the previous section of this paper, equation (21).

Considering the proportional output feedback controller $\mathbf{u}(t)=-Kx_i(t)$, where $K=\begin{bmatrix}0\\k\end{bmatrix},0<k<5$, the closed loop characteristic polynomial can be written as:

$$\begin{aligned}f(s,p,k)&=s^{15}+c_{37}s^{14}+\left(c_{29}p+c_{36}s\right)^{13}+\\&+\left(c_1p+c_{18}+c_{25}k\right)s^{12}+\left(c_{25}k+c_{31}+c_{42}p\right)s^{11}+\\&+\left(c_{13}k+c_{27}+c_{33}kp+c_{41}p\right)s^{10}+\left(c_{14}k+c_{28}kp+\right.\\&\left.+c_{39}p+c_{53}\right)s^9+\left(c_{12}k+c_{19}kp+c_{24}+c_{38}p\right)s^8+\\&+\left(c_{11}k+c_{21}+c_{32}kp+c_{34}p\right)s^7+\left(c_3kp+c_6+\right.\\&\left.+c_{10}k+c_{35}p\right)s^6+\left(c_2kp+c_9k+c_{28}p+c_{50}\right)s^5+\\&+\left(c_8k+c_{16}p+c_{26}kp+c_{48}\right)s^4+\left(c_7k+c_{17}p+c_{47}+\right.\\&\left.+c_{51}kp\right)s^3+\left(c_{40}p+c_{44}k+c_{45}+c_{49}kp\right)s^2+\\&+\left(c_4+c_{15}p+c_{20}k+c_{46}kp\right)s+c_2+c_5p+c_{43}kp+c_{52}\end{aligned} \quad (32)$$

where

$$c_1=0.1033\,,\ c_2=7.06\cdot10^{-8}\,,\ c_3=2.75\cdot10^{-7}\,,$$

$$c_4=6.39\cdot10^{-11}\,,\ c_5=6.73\cdot10^{-15}\,,\ c_6=3.43\cdot10^{-4}\,,$$
$$c_7=2.63\cdot10^{-10}\,,\ c_8=3.4\cdot10^{-9}$$

$$c_9=2.68\cdot10^{-8}\,,c_{10}=1.82\cdot10^{-7}\,,c_{11}=8.17\cdot10^{-7}$$

$$c_{12}=2.75\cdot10^{-6}\,,c_{13}=1.44\cdot10^{-5}\,,c_{14}=6.9\cdot10^{-6}$$

$$c_{15}=4.5\cdot10^{-12}\,,c_{16}=1.17\cdot10^{-6}\,,c_{17}=3.59\cdot10^{-8}$$

$$c_{18}=0.8\,,\ c_{19}=1.83\cdot10^{-6}\,,c_{20}=2.197\cdot10^{-3}\,,$$

$$c_{21}=0.0024\,,c_{22}=2.03\cdot10^{-5}\,,c_{23}=1.75\cdot10^{-15}$$

$$c_{24}=0.0126\,,\ c_{25}=1.72\cdot10^{-5}\,,\ c_{26}=1.219\cdot10^{-8}$$

$$c_{27}=0.103\,,\ c_{28}=2.47\cdot10^{-6}\,,\ c_{29}=0.1255\,,$$

$$c_{30}=2.16\cdot10^{-5}\,,c_{31}=0.396\,,c_{32}=7.98\cdot10^{-7}\,,$$

$$c_{33}=3.23\cdot10^{-6}\,,c_{34}=0.0013\,,c_{35}=2.25\cdot10^{-4}\,,$$

$$c_{36}=1.19\,,\ c_{37}=1.331\,,c_{38}=0.0057\,,$$

$$c_{39}=0.0185\,,c_{40}=5.96\cdot10^{-10}\,,c_{41}=0.0448\,,$$

$$c_{41}=0.089\,,c_{43}=6.993\cdot10^{-15}\,,c_{44}=1.08\cdot10^{-11}$$

$$c_{45}=4.21\cdot10^{-9}\,,c_{46}=8.769\cdot10^{-13}\,,$$

$$c_{47}=1.487\cdot10^{-7}\,,c_{48}=3.08\cdot10^{-6}\,,$$

$$c_{49}=4.29\cdot10^{-11}\,,c_{50}=3.93\cdot10^{-5}\,,c_{51}=1.025\cdot10^{-9}\,,$$
$$c_{52}=4.026\cdot10^{-13}\,,$$

$$c_{53}=0.0505\,. \quad (33)$$

The characteristic polynomial is stable for the nominal parameter $p_N=0.996$. We find that $K=\begin{bmatrix}0\\k\end{bmatrix},0<k<5$ stabilizes the nominal plant (differential-discrete mathematical model of the double tube heat exchanger). Our objective is to obtain the parameter K of the maximally robust controller and the maximal l_2 stability margin in the sense that it guarantees stability while maximizing the size of a perturbation centered at p_N.

Equation (32) can be written in the form of (21) with the weights $h_i = 1$,

$$f(s,p,k) = a_1(s,k)p + b(s,k) \quad (34)$$

where

$$\begin{aligned} a_1(s,k) = [&c_{29}s^{13} + c_1 s^{12} + c_{42}s^{11} + (c_{33} + c_{41})s^{10} + \\ &+(c_{28}k + c_{39})s^9 + (c_{19}k + c_{38})s^8 + (c_{32}k + c_{34})s^7 + \\ &+(c_3 k + c_{35})s^6 + (c_2 k + c_{30})s^5 + (c_{16} + c_{25}k)s^4 + \\ &+(c_{17} + c_{51}k)s^3 + (c_{40} + c_{49}k)s^2 + (c_{15} + c_{46}k)s + \\ &+c_5 + c_{43}k] \end{aligned} \quad (35)$$

$$\begin{aligned} b(s,k) = &s^{15} + c_{37}s^{14} + c_{36}s^{13} + (c_{18} + c_{22}k)s^{12} + \\ &+(c_{31} + c_{25}k)s^{11} + (c_{13}k + c_{27})s^{10} + \\ &+(c_{14}k + c_{53})s^9 + (c_{12}k + c_{24})s^8 + (c_{11}k + c_{21})s^7 + \\ &+(c_6 + c_{10}k)s^6 + (c_9 k + c_{50})s^5 + (c_8 k + c_{48})s^4 + \\ &+(c_{13}k + c_{47})s^3 + (c_{44}k + c_{45})s^2 + (c_4 + c_{20}k)s + \\ &+c_{23}k + c_{52} \end{aligned} \quad (36)$$

The perturbation Δp is denoted with $\Delta p = p_N + \Delta p$ so equation (25) can be written in the following form:

$$a_1(s,k)\Delta p = -a_1(s,k)p_N - b(s,k) \quad (37)$$

The aim is to find the optimum value of K such that the l_2 stability margin $\|\mathbf{d}^*\| = \rho$ with respect to the parameter p is the largest possible. Assuming $s = j\omega$ equation (37) has the following form:

$$\underbrace{\begin{bmatrix} a_{1r}(\omega,k) \\ a_{1i}(\omega,k) \end{bmatrix}}_{T(\omega,k)} \Delta p = \left[\underbrace{\begin{matrix} -a_{1r}(\omega,k)p_N - b_r(\omega,k) \\ -a_{1i}(\omega,k)p_N - b_i(\omega,k) \end{matrix}}_{b(\omega,k)} \right] \quad (38)$$

where

$$\begin{aligned} a_{1r}(\omega,k) &= R_e\{a_1(j\omega,k)\} \\ a_{1i}(\omega,k) &= I_m\{a_1(j\omega,k)\}, \\ b_r(\omega,k) &= R_e\{b(j\omega,k)\}, \\ b_i(\omega,k) &= I_m\{b(j\omega,k)\} \end{aligned} \quad (39)$$

The minimum norm sollution of equation (38) has the following form:

$$\rho(k) = \min_{\omega} \left\| T^T(\omega,k)\left[T(\omega,k)T^T(\omega,k)\right]^{-1} b(\omega,k) \right\| \quad (40)$$

The optimum value of the feedback controller parameter K that robustly stabilizes the observed plant corresponds to:

$$\rho^* = \max_k \rho(k) \quad (41)$$

Fig.6 shown the graphic of the sollution of equation (27) $\|d(k,\omega)\|_2 = T^T(\omega,k)\left[T(\omega,k)T^T(\omega,k)\right]^{-1} b(\omega,k)$ vs. k and ω.

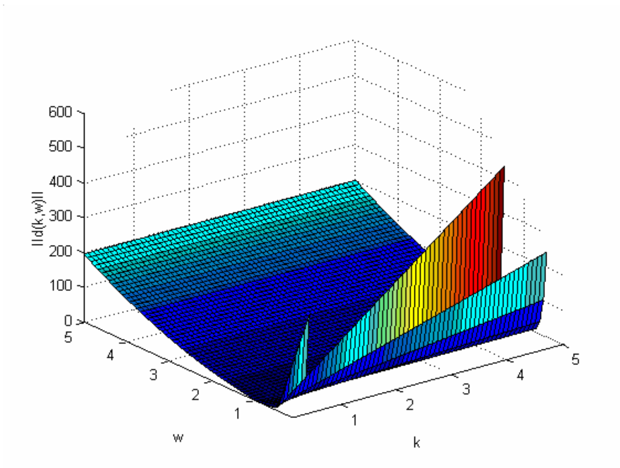

Figure 6. Norm $\|d(k,\omega)\|_2$ vs. k and ω

Fig.7 shows that he maximum value of the stability margin $\rho^* = \max_k \|d^*(k)\|_2$ is achieved for $k = 1.981$ which means that a maximally robust controller is found to be:

$$K = \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 1.981 \end{bmatrix} \quad (42)$$

The maximal l_2 stability margin obtained is:

$$\rho^* = 0.064 \quad (43)$$

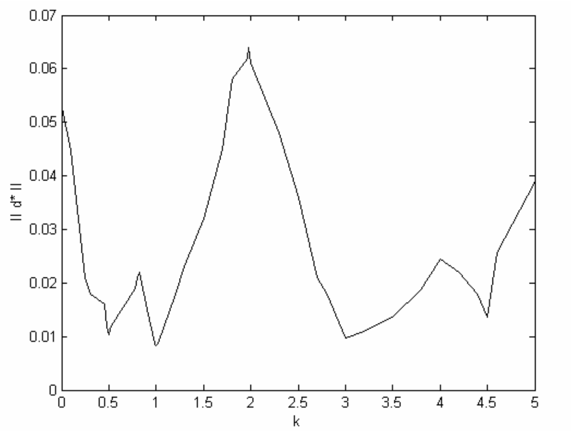

Figure 7. Change of the minimum $\|d^*\|_2$ with respect to k

Fig.8 shows the step response of the closed loop system for the calculated value of the maximally robust controller.

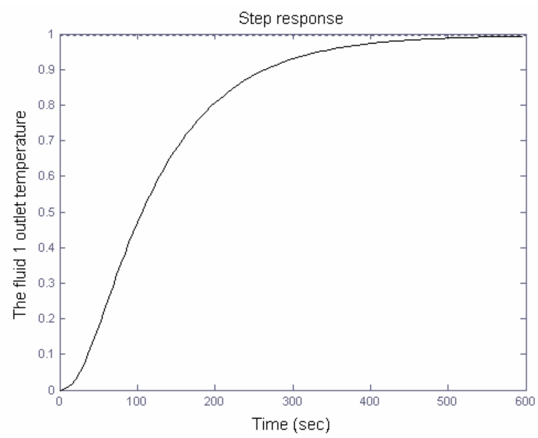

Figure 8. Step response of the closed loop system for the calculated maximally robust controller

Conclusion

In this paper the problem of mathematical modelling of the double tube heat exchanger and robust control based on the parametric approach is considered. A mathematical model of the double tube heat exchanger is derived using the finite difference method which implies dividing the observed heat exchanger into an appropriate number of cells. The spatial coordinate of the double tube heat exchanger is discretized by means of the finite difference method. The discretization of the spatial variable is carried out by substituting partial derivation with respect to the spatial coordinate in a certain number of points. As a result of this procedure, the mathematical model of the double tube heat exchanger is obtained in the form of ordinary differential equations of high order, still simple enough to be solved.

Finally, the graphical results of the simulations of the derived mathematical model of the double tube heat exchanger are presented.

In the second part of this paper, the problem of robust control of the double tube heat exchanger is considered using the parametric approach. The differential-discrete mathematical model derived in the first section of this paper is used for the application of robust control using the parametric approach.

The main idea of this approach is to determine the largest perturbation of uncertain parameters of the double tube heat exchanger without destroying closed loop stability. The uncertain plant parameter of the differential-discrete mathematical model of the double tube heat exchanger is assumed to be $p = \dfrac{\theta_{1N,4}}{\theta_{1N,5}}$.

In this section, the parameter K of the maximally robust controller and the maximal l_2 stability margin are obtained in the sense that it guarantees stability while maximizing the size of a perturbation centered at p_N.

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Received: 16.05.2011.

Matematičko modeliranje i robusno upravljanje razmenjivača toplote tipa cev u cev: parametarski prilaz

Matematičko modeliranje razmenjivača toplote je veoma važan korak u postupku njihovog projektovanja posebno sa stanovišta izbora odgovarajućeg upravljačkog sistema. Različiti tipovi razmenjivača toplote se često koriste kao konstruktivni elementi vojnih letilica i njihovim radom se najčešće upravlja promenom ulaznih temperatura ili protoka radnih fluida. U prvom delu ovog rada izveden je linearizovani matematički model razmenjivača toplote tipa cev u cev, uzimajući u obzir dinamiku zida razmenjivača, što dovodi do njegovog opisa u formi sistema parcijalnih diferencijalnih jednačina. U cilju prevazilaženja poteškoća vezanih za rešavanje sistema parcijalnih diferencijalnih jednačina primenjuje se metoda konačnih razlika. Konačno, za konkretno usvojene vrednosti parametara i početnih uslova posmatranog razmenjivača toplote izvršena je simulacija njegovog rada i dati su grafički prikazi odskočnih odziva. U drugom delu ovog rada razmatra se problem robusnog upravljanja razmenjivača toplote tipa cev u cev zasnovanog na parametarskom prilazu. Diferencijalno-diskretni matematički model razmenivača toplote tipa cev u cev dobijen u prvom delu ovog rada koji sadrži nepoznate parametre koristi se kao osnova za izračunavanje najveće dozvoljene perturbacije u prostoru realnih parametara tako da bude obezbeđena stabilnost zatvorenog sistema automatskog upravljanja. Osnovna ideja je određivanje maksimalno robusnog kontrolera koji garantuje stabilnost u isto vreme maksimizirajući veličinu perturbacije oko nominalne vrednosti.

Ključne reči: **razmenjivač toplote, matematičko modelovanje, metoda konačnih razlika, robustno upravljanje, parametraski model, kontroler.**

Математическое моделирование и надёжный робастный контроль – управление теплообменников с двойными трубами типа труба в трубе: параметрический подход

Математическое моделирование теплообменников является очень важной частью и шагом в процессе их проектирования, особенно с точки зрения выбора для определения правильной и соответствующей стратегии управления. Различные типы теплообменников часто и широко используются в качестве структурных элементов в военных самолётах и их работа, как правило, контролируется путём изменения входных температур или массового расхода рабочей жидкости. В первой части этой работы была получена линеаризованная математическая модель двойного теплообменника типа труба в трубе, с учётом динамики стены теплообменника, что приводит к его описанию в виде систем частных дифференциальных уравнений. Для того чтобы преодолеть трудности, связанные с решением систем частных дифференциальных уравнений, предлагается применнеие метода конечных разностей. Наконец, принятые значения для конкретных параметров и начальных условий наблюдаемого теплообменника изготовлены из его работ моделирования и графического представления даных ответов. Во второй части этой статьи считается проблема надёжного контроля двойного теплообменника типа труба в трубе на основе параметрического подхода. Во второй части этой статьи считается проблема надежного контроля двойного теплообменника на основе параметрического подхода. Дифференциально-дискретная математическая модель двойного теплообменника типа труба в трубе, полученная в первой части этого документа, содержащая неопределённых параметров, используется для определения крупнейших возмущений в пространстве вещественных параметров, не разрушая стабильность закрытой петли автоматического управления. Основная идея заключается в определении максимально надёжного контроллера в смысле, что он одновременно гарантирует стабильность при максимальном размере возмущения при номинальных значениях.

Ключевые слова: **двойной теплообменник, математическое моделирование, метод конечных разностей, надёжный контроль, параметрический подход, контроллер.**

La modélisation mathématique et le contrôle robuste chez l’échangeur de chaleur à double tuyau; approche paramétrique

La modélisation mathématique de l’échangeur de chaleur est un pas très important dans le processus de sa conception notamment dans le choix du système correspondant de contrôle. Les différents types des échangeurs de chaleur sont souvent utilisés comme les éléments constructifs chez les avions militaires et leur fonctionnement est le plus souvent contrôlé par le changement des températures d’entrée ou bien par le courant des fluides. Dans la première partie de ce travail on a dérivé le modèle mathématique linéarisé de double échangeur tubulaire en tenant compte du dynamique de paroi ce qui amène à sa description sous la forme du système des équations différentielles partielles. Pour dominer les difficultés liées à la solution des systèmes des équations différentielles partielles on utilise la méthode des différences finies . Finalement, pour les valeurs des paramètres adoptées et les conditions initiales de l’échangeur considéré on a effectué la simulation de son fonctionnement et donné les formes graphiques pour les réponses transitoires. Dans la seconde partie de ce travail on a traité le problème du contrôle robuste chez l’échangeur étudié en partant de l’approche paramétrique. Le modèle mathématique différentiel et discret, dérivé dans la première partie de ce travail et qui contient les paramètres inconnus, est employé comme la base pour déterminer la perturbation maximale permise dans l’espace des paramètres réels de sorte que la stabilité du système automatique fermé du contrôle soit assurée. L’idée principale est la détermination d’un contrôleur maximale robuste qui garantit la stabilité en rendant maximale la taille de la perturbation à la valeur nominale.

Mots clés: **échangeur de chaleur, modélisation mathématique, méthode des différences finies, contrôle robuste, modèle paramétrique, contrôleur.**