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The Stability of Linear Continuous Singular and Discrete Descriptor Time Delayed Systems in the Sense of Lyapunov: An Overview

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This paper gives a detailed overview of the work and the results of many authors in the area of Lyapunov stability of a particular class of linear systems.
This survey covers the period since 1980 up to nowadays and has strong intention to present the main concepts and contributions during the mentioned period throughout the world, published in the respectable international journals or presented at workshops or prestigious conferences.

Key words: **linear system, continuous system, singular system, discrete system, descriptive system, system stability, time delayed system, Lyapunov stability.**

Introduction

IT should be noticed that in some systems we must consider their character of dynamic and static state at the same time. Singular systems (also referred to as degenerate, descriptor, generalized, differential - algebraic systems or semi – state) are those the dynamics of which is governed by a mixture of algebraic and differential equations.

Recently, many scholars have paid much attention to singular systems and have obtained many good consequences. The complex nature of singular systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control.

It is well-known that singular systems have been one of the major research fields of the control theory. During the past three decades, singular systems have attracted much attention due to the comprehensive applications in economics as the *Leontief* dynamic model *Silva*, *Lima* (2003), in electrical *Campbell* (1980) and mechanical models *Muller* (1997), etc.

They also arise naturally as a linear approximation of systems models, or linear system models in many applications such as electrical networks, *aircraft dynamics*, neutral delay systems, chemical, thermal and diffusion processes, large-scale systems, interconnected systems, economics, optimization problems, feedback systems, robotics, biology, etc.

Discussions of singular systems originated from 1974 with the fundamental paper of *Campbell et al*. (1974) and later with the anthological paper of *Luenberger* (1977). Since that time, considerable progress has been made in investigating such systems - see surveys, *Lewis* (1986) and *Dai* (1989) for linear singular systems, the first results for nonlinear singular systems in *Bajic* (1992).

In the investigation of stability of singular systems, many results in sense of Lyapunov stability have been derived. For example, *Bajic* (1992) and *Zhang et al.* (1999) considered the stability of linear time-varying descriptor systems.

Discrete descriptor systems are the systems the dynamics of which is covered by a mixture of *algebraic* and *difference* equations.

In that sense, the question of their stability deserves great attention and is tightly connected with the questions of system solution uniqueness and existence.

Moreover, the question of consistent initial conditions, time series and solution in the state space and phase space based on a discrete fundamental matrix also deserve a specific attention.

In this case, the concept of smoothness has little meaning but the idea of consistent initial conditions being these initial conditions $\mathbf{x}_0$ that generate solution sequence $\left(\mathbf{x}(k): k \geq 0\right)$ has a physical meaning.

Some of these question do not exist when the *normal systems* are treated.

The problem of investigation of time delay systems has been exploited over many years. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time lag, regardless of the fact that it is present in the control or/and the state, may cause an undesirable system transient response, or even instability. Consequently, the problem of the stability analysis for this class of systems has been one of the main interests for many researchers. In general, the introduction of time delay factors makes the analysis much more complicated.

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It must be emphasized that there are many systems which have the phenomena of time delay and singular simultaneously, such systems are called *the singular differential systems with time delay*.

These systems have many special characteristics.

If we want to describe them more exactly, to design them more accurately and to control them more effectively, we must work hard to investigate them, which proves to be obviously very difficult. In recent references, the authors have discussed such systems and got some consequences. However, in the study of such systems, there are still many problems to be considered. When the general time delay systems are considered, in the existing stability criteria, mainly two ways of approach have been adopted.

Namely, one direction is to contrive the stability condition which does not include the information on the delay, and the other is the method which takes it into account. The former case is often called the delay - independent criterion and generally provides simple algebraic conditions.

Therefore, the question of their stability deserves great attention.

We must emphasize that there are many systems that have the phenomena of time delay and singular characteristics simultaneously.

We denote such systems as singular (*descriptor*) differential (*difference*) systems with time delay.

These systems have many special properties. If we want to describe them more exactly, to design them more accurately and to control them more effectively, we must work hard to investigate them, which proves to be obviously very difficult. In recent references, the authors have discussed such systems and got some consequences. However, in the study of such systems, there are still many problems to be considered.

In the short overview that follows, we will present the results achieved, both in the area of Lyapunov stability of *linear, continuous singular time delay systems* (LCSTDS) and *linear, discrete descriptor time delay systems* (LDDTDS). We will not discuss contributions presented in papers concerned with problem of robust stability, stabilization of this class of systems with parameter uncertainty (see the list of references, as well as with other questions in connection with the stability of LCSTDS being necessarily transformed by Lyapunov – Krasovski functional to the state space model in the form of differential – integral equations, *Fridman* (2001, 2002)).

Moreover, in the last few years, a numerous papers have been published in the area of linear discrete descriptor time delay systems, but this discussion is out of the scope of this paper. To get deeper into this matter, see the list of references.

To the best of our knowledge, some attempts in the stability investigation of LCSTDS were due to *Saric* (2001, 2002) where sufficient conditions for convergence of an appropriate fundamental matrix were established.

Recently, in the paper of *Xu et al*. (2002) the problem of robust stability and stabilization for uncertain LCSTDS was addressed and necessary and sufficient conditions were obtained in terms of strict LMI. Moreover, in the same paper, using suitable canonical description of LCSTDS, rather simple criteria for asymptotic stability testing were also proposed.

In this paper, besides the contribution of some other authors, we present quite another approach to this problem established by the authors of this paper.

Namely, these results are expressed directly in terms of matrices E, A_0 and A_1 naturally occurring in the system models and avoiding the need to introduce any canonical form into the statement of the *Theorem.*

The geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all solutions.

This fact enables the construction of the Lyapunov stability theory even for the LCSTDS and LDDTDS bearing in mind that asymptotic stability is equivalent to the existence of symmetric, positive definite solutions to a weak form of Lyapunov matrix equation incorporating condition which refer to a time delay term.

This paper presents, in a unified way, a collection of results spread out in the literature which focuses on stability of linear *continuous singular* and *discrete descriptor* time delayed systems

This paper is not a survey in its usual sense.

We do not try to be exhaustive regarding the vast literature about this problem.

Our object is more to convince the reader of the practical interest of the approach and of the number and the simplicity of the results it leads to.

For each aspect we generally give in detail only one result, which is not necessarily the most complete or the most recent one, but is the one which seems to us the most representative and illustrative.

Also, another aim of this paper is to present an overview of the results concerning *asymptotic* stability of a particular class of linear *continuous singular* and linear *discrete time delay* systems published in last ten years and nowadays.

Basic notation

$\mathbb{R}$	– Real vector space
$\mathbb{C}$	– Complex vector space
I	– Unit matrix
$F=\left(f_{ij}\right)\in\mathbb{R}^{n\times n}$	– real matrix
F^T	– Transpose of matrix F
$F>0$	– Positive definite matrix
$F\geq 0$	– Positive semi definite matrix
$\Re(F)$	– Range of matrix F
$\mathbb{N}(F)$	– Null space (kernel) of matrix F
$\lambda(F)$	– Eigenvalue of matrix F
$\sigma_{(\ )}(F)$	– Singular value of matrix F
$\rho(F)$	– Spectral radius of matrix F
$\lVert F\rVert$	– Euclidean matrix norm $\lVert F\rVert=\sqrt{\lambda_{\max}\left(A^TA\right)}$
F^D	– Drazin inverse of matrix F
$\Rightarrow$	– Follows
$\mapsto$	– Such that

Asymptotic stability of Linear continuous singular TIME DELAYED systems

Generally, *the linear continuous singular differential control systems with time delay* can be written as:

$$\begin{gathered}E(t)\,\dot{\mathbf{x}}(t)=\mathbf{f}\left(t,\mathbf{x}(t),\mathbf{x}(t-\tau),\mathbf{u}(t)\right),\quad t\geq 0\\ \mathbf{x}(t)=\boldsymbol{\varphi}(t),\ -\tau\leq t\leq 0\end{gathered}\quad, \tag{1}$$

where $\mathbf{x}(t)\in\mathbb{R}^n$ is a state vector, $\mathbf{u}(t)\in\mathbb{R}^l$ is a control vector, $E(t)\in\mathbb{R}^{n\times n}$ is a singular matrix, $\boldsymbol{\varphi}\in\mathcal{C}=\mathcal{C}([-\tau,0],\mathbb{R}^n)$ is an admissible initial state functional, $\mathcal{C}=\mathcal{C}([-\tau,0],\mathbb{R}^n)$ is the *Banach space* of continuous functions mapping the interval $[-\tau,\ 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence.

Linear continuous singular time invariant time delayed systems - some previous results

Consider a linear continuous singular system with state delay, described by

$$E\,\dot{\mathbf{x}}(t)=A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau)\,, \tag{2.a}$$

with a known compatible vector valued function of initial conditions

$$\mathbf{x}(t)=\boldsymbol{\varphi}_x(t),\ \ -\tau\le t\le 0\,, \tag{2.b}$$

where A_0 and A_1 are constant matrices of appropriate dimensions.

Time delay is constant, e.g. $\tau\in\mathbb{R}_+$

Moreover, we shall assume that $rank\,E=r<n$.

Definition 1. The matrix pair $(E,\ A_0)$ is said to be regular if $\det(sE-A_0)$ is not identically zero, *Xu et al*. (2002).

Definition 2. The matrix pair $(E,\ A_0)$ is said to be impulse free if $\deg\left(\det(sE-A_0)\right)=rang\,E$, *Xu et al*. (2002).

The linear continuous singular time delay system (2) may have an impulsive solution; however, the regularity and the absence of impulses of the matrix pair $(E,\ A_0)$ ensure the existence and uniqueness of an impulse free solution to the system under consideration, which is defined in the following *Lemma*.

Lemma 1. Suppose that the matrix pair $(E,\ A_0)$ is *regular* and *impulsive free* and *unique* on $[0,\infty)$, *Xu et al* (2002).

Necessity for system stability investigation creates a need for establishing a proper stability definition. Therefore, one can get:

Definition 3. Linear continuous singular time delay system, (2) is said to be *regular* and *impulsive free* if the matrix pair $(E,\ A_0)$ is regular and impulsive free.

Stability definitions

Definition 4. If $\forall t_0\in T$ and $\forall\varepsilon>0$, there always exists $\delta(t_0,\varepsilon)$, such that $\forall\boldsymbol{\varphi}_x\in\mathbb{S}_\delta(0,\delta)\cap\mathbb{S}(t_0,\ t^*)$, the solution $\mathbf{x}(t,\ t_0,\ \boldsymbol{\varphi}_x)$ to (2) satisfies that $\|\mathbf{q}(t,\mathbf{x}(t))\|\le\varepsilon$, $\forall t\in(t_0,\ t^*)$, then the zero solution to (2.3) is said to be stable on $\{\mathbf{q}(t,\mathbf{x}(t)),T\}$, where $T=[0,+t^*]$, $0<t^*\le+\infty$ and also $\mathbb{S}_\delta(0,\delta)=\left\{\boldsymbol{\varphi}_x\in\mathcal{C}([-\tau,0],\mathbb{R}^n),\|\boldsymbol{\varphi}_x\|<\delta,\ \delta>0\right\}$.

$\mathbb{S}_*(t_0,t^*)$ is a set of all consistency initial functions and for $\forall\boldsymbol{\psi}\in\mathbb{S}_*(t_0,\ t^*)$, there exists a continuous solution to (2) in $[t_0-\tau,\ t^*[$ through $(t_0,\ \boldsymbol{\varphi}_x)$ at least, *Li*, *Liu* (1997, 1998).

Definition 5. If δ is only related to ε and has nothing to do with t_0, then the zero solution is said to be uniformly stable on $\{\mathbf{q}(t,\mathbf{x}(t)),\ T\}$, *Li*, *Liu* (1997, 1998).

Definition 6. Linear continuous singular time delay system, (2), is said to be stable if for any $\varepsilon>0$ there exists a scalar $\delta(\varepsilon)>0$ such that, for any compatible initial conditions $\boldsymbol{\varphi}_x(t)$, satisfies the condition: $\sup\limits_{-\tau\le t\le 0}\|\boldsymbol{\varphi}_x(t)\|\le\delta(\varepsilon)$, the solution $\mathbf{x}(t)$ of system (2) satisfies $\|\mathbf{x}(t)\|\le\varepsilon,\ \forall t\ge 0$.

Moreover, if $\lim\limits_{t\to\infty}\|\mathbf{x}(t)\|\to 0$, a system is said to be *asymptotically stable*, *Xu et al* (2002).

Stability theorems

Theorem 1. Suppose that for any solution $\mathbf{x}(t)$ to (2), $\dot{\mathbf{q}}(t,\mathbf{x}(t))$ is bounded when $\mathbf{q}(t,\mathbf{x}(t))$ is bounded and that there exist two wedge functions $u(\cdot)$ and $v(\cdot)$, and a nonnegative and non-decreasing function $\wp(\cdot)$, and continuous $V(t,\boldsymbol{\varphi}_x)$ functional

$$V(t,\boldsymbol{\varphi}_x)\colon\ [0,\ +\infty)\times\mathcal{C}([-\tau,\ 0],\ \mathbb{R}^n)\to\mathbb{R}\,, \tag{3}$$

which satisfies

$$\text{(i)}\ \ u(\|\mathbf{q}(t,\mathbf{x}(t))\|)\le v(t,\mathbf{x}_t)\le V(\|\mathbf{x}_t\|)\,, \tag{4}$$

$$\text{(ii)}\ \ D^+V(\|\mathbf{x}_t\|)\le\wp(\|\mathbf{q}(t,\mathbf{x}(t))\|)\,, \tag{5}$$

then the zero solution to eq.(2) is uniformly stable on $\{\mathbf{q}(t,\mathbf{x}(t)),\ T\}$ where $t^*\le+\infty$.

If $\wp(\kappa)>0$ when $\kappa>0$, then the zero solution to (2) is asymptotically stable on $\{\mathbf{q}(t,\mathbf{x}(t)),\ [0,\ +\infty]\}$, *Li*, *Liu* (1997, 1998).

Theorem 2. Suppose a regular singular delay free system (2) is asymptotically stable.

If there are positive definite matrices W and V satisfying

$$A_0^TV\,E+E^TVA_0=-E^TW\,E\,, \tag{6}$$

$$rang\,E^TV\,E=rank\,V=r\,, \tag{7}$$

and a positive real number γ fulfilling the following conditions:

a) The solution $\mathbf{x}(t)$ to (2) satisfies

$$\|A_1\mathbf{x}(t)\|\le\gamma\|E\mathbf{x}(t)\|, \tag{8}$$

b) $W-V^2-\delta I>0$ is a positive definite matrix, then the zero solution to (2) is asymptotically stable, *Liang* (2001).

Theorem 3. System (2) is regular, impulse free and stable if there are a matrix $Q>0$ and a matrix P such that

$$EP^T = PE^T \geq 0\,, \tag{9}$$

$$A_0P^T + PA_0^T + A_1P^TQ^{-1}PA_1^TQ < 0\,, \tag{10}$$

Xu et al. (2002).

Theorem 4. Suppose a regular singular delay free system (2) is regular and impulse free.

If there are matrices $P>0$, $Q>0$, $T>0$, $X>0$, $Z>0$ and a symmetric matrix Y and a real positive number μ fulfills the following LMI:

$$\Xi = \begin{pmatrix} \Theta & P\bar{A}_1 - Y + \mu\bar{A}_0^T Z\bar{A}_1 & -\mu\bar{A}_0^T ZC - \bar{A}_0^T TC - PC \\ * & \mu\bar{A}_1^T Z\bar{A}_1 + \bar{A}_1^T T\bar{A}_1 - Q & -\mu\bar{A}_1^T ZC - \bar{A}_1^T TC \\ * & * & \mu C^T ZC + C^T TC - T \end{pmatrix} < 0 \tag{11}$$

where

$$\begin{aligned}\Theta &= P\bar{A}_0 + \bar{A}_0^T P + \mu X \\ &+ Y + Y^T + \bar{A}_0^T(\mu Z + T)\bar{A}_0 + Q\end{aligned}, \tag{12}$$

and

$$\begin{aligned}&\bar{A}_0 = \begin{pmatrix} A_{01} & 0 \\ 0 & -I_{n-r}\end{pmatrix},\ \bar{A}_1 = \begin{pmatrix} A_{11} & A_{12} \\ -A_{13} & -A_{14}\end{pmatrix} \\ &\begin{pmatrix} X & Y \\ Y^T & Z\end{pmatrix} \geq 0,\ C = \begin{pmatrix} 0 & 0 \\ A_{13} & A_{14}\end{pmatrix}\end{aligned}, \tag{13}$$

then system (2) is asymptotically stable for any $\tau \in [0,\ \mu]$, *Zhang et al.* (2003).

Let us consider the case when the subspace of consistent initial conditions for a *singular time delay* and a *singular nondelay system* coincide.

Theorem 5. Suppose that the matrix pair $(E,\ A_0)$ is *regular* with the system matrix A_0 being nonsingular., e.i. $\det A_0 \neq 0$.

The system (2) is *asymptotically stable*, independent of delay, if there exists a symmetric positive definite matrix $P = P^T > 0$, being the solution of Lyapunov matrix equation

$$A_0^T PE + E^T PA_0 = -2(S+Q), \tag{14}$$

with the matrices $Q = Q^T > 0$ and $S = S^T$, such that:

$$\mathbf{x}^T(t)(S+Q)\mathbf{x}(t) > 0, \quad \forall \mathbf{x}(t) \in \mathcal{W}_{k^*} \setminus \{0\}\,, \tag{15}$$

is positive definite quadratic form on $\mathcal{W}_{k^*} \setminus \{0\}$, $\mathcal{W}_{k^*}$ being the subspace of consistent initial conditions[1], and if the following condition is satisfied:

$$\| A_1 \| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}E^T P\right), \tag{16}$$

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are the maximum and minimum singular values of the matrix $(\cdot)$ respectively, *Debeljkovic et al.* (2003, 2004.c, 2006, 2007).

Remark 1. (12-13) are, in a modified form, taken from *Owens*, *Debeljkovic* (1985).

Remark 2. If the system under consideration is just ordinary time delay, e.g. $E = I$, the result is identical to that presented in *Tissir*, *Hmamed* (1996).

Remark 3. Let us discuss first the case when the time delay *is absent.*

Then the *singular* (weak) Lyapunov matrix (12) is a natural generalization of the classical Lyapunov theory.

a)If is a *nonsingular matrix*, then the system is asymptotically stable if and only if $A = E^{-1}A_0$ *Hurwitz* matrix.

(12) can be written in the form:

$$A^T E^T PE + E^T PEA = -Q\,, \tag{17}$$

with the matrix Q being symmetric and positive definite, in the whole state space, since then $\mathcal{W}_{k^*} = \Re\left(E^{k^*}\right) = \mathbb{R}^n$.

In these circumstances $E^T PE$ is a Lyapunov function for the system.

b) The matrix A_0 by necessity is nonsingular and hence the system has the form

$$E_0\dot{\mathbf{x}}(t) = \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0\,. \tag{18}$$

Then for this system to be stable, (3.3) must hold as well, and have the well-known Lyapunov structure

$$E_0^T P + PE_0 = -Q\,, \tag{19}$$

where Q is a symmetric matrix but only required to be positive definite on $\mathcal{W}_{k^*}$.

Remark 4. There is no need for the system under consideration to possess properties given in *Definition* 2, since this is obviously guaranteed by the demand that all smooth solutions $\mathbf{x}(t)$ evolve in $\mathcal{W}_{k^*}$.

Remark 5. The idea and approach are based upon the papers of *Owens*, *Debeljkovic* (1985) and *Tissir*, *Hmamed* (1996).

Theorem 6. Suppose that the system matrix A_0 is nonsingular, e.i. $\det A_0 \neq 0$.

Then we can consider system (2) with a known compatible vector valued function of initial conditions and we shall assume that $rank\, E_0 = r < n$.

The matrix E_0 is defined in the following way $E_0 = A_0^{-1}E$.

System (2) is asymptotically stable, independent of delay, if

$$\| A_1 \| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}E_0^T P\right), \tag{20}$$

and if there exists:

$(n \times n)$ matrix P, being the solution of the Lyapunov matrix:

$$E_0^T P + PE_0 = -2I_\Omega\,, \tag{21}$$

with the following properties:

a) $P = P^T$ (22.a)

[1] W_{k^*} subspace of consistent initial conditions, *Owens, Debeljković* (1985).

b) $P\mathbf{q}(t)=\mathbf{0}, \quad \mathbf{q}(t)\in\Lambda$ (22.b)

c) $\mathbf{q}^T(t)P\mathbf{q}(t)>0, \quad \mathbf{q}(t)\neq\mathbf{0}, \quad \mathbf{q}(t)\in\Omega$, (22.c)

where:

$$\Omega=\mathbb{N}\left(I-EE^D\right), \tag{23}$$

$$\Lambda=\mathbb{N}\left(EE^D\right), \tag{24}$$

with the matrix I_Ω representing the generalized operator on $\mathbb{R}^n$ and the identity matrix on the subspace Ω and the zero operator on the subspace Λ, the matrix Q being any positive definite matrix.

Moreover, the matrix P is symmetric and positive definite on the subspace of consistent initial conditions.

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are the maximum and minimum singular values of the matrix $(\cdot)$, respectively, *Debeljkovic et al.* (2005.b, 2005.c, 2006.a).

Remark 6. The basic idea and approach are based upon the paper of *Pandolfi* (1980) and *Tissir*, *Hmamed* (1996).

(21-24) are estimated in the light of the idea of *Pandolfi* (1980).

Linear continuous singular time varying time delayed systems

Consider the singular system with time delay represented by

$$E\dot{\mathbf{x}}(t)=\mathbf{f}\left(t,\mathbf{x}(t),\mathbf{x}(t-\tau)\right), \tag{25.a}$$

$$\mathbf{x}_{t_0}=\boldsymbol{\varphi}_x, \qquad \boldsymbol{\varphi}_x\in\mathcal{C}\left([-\tau,\ 0],\ \mathbb{R}^n\right), \tag{25.b}$$

where E is a $(n\times n)$ constant matrix, $t\geq t_0$, t_0 is the initial moment, $\mathbf{x}(t)\in\mathbb{R}^n$, $f(t,\mathbf{x},\mathbf{y})$ is an n-dimensional continuous vector function on $\left[0,+\infty\right[\times\mathbb{R}^n\times\mathbb{R}^n$, $r>0$, (25.b) is the initial condition of (25.a).

The solution to (25) is written by $\mathbf{x}_k(t,\,t_0,\,\boldsymbol{\varphi}_x)$ or $\mathbf{x}_k(t)$.

Stability definitions

Definition 7. If $\forall t_0\in T$ and $\forall\varepsilon>0$, there always exists $\delta(t_0,\varepsilon)$, such that $\forall\boldsymbol{\varphi}\in\mathbb{S}_\delta(\boldsymbol{\varphi}_x,\delta)\cap\mathbb{S}^*(t_0,\,t^*)$, the solution $\mathbf{x}(t,\,t_0,\,\boldsymbol{\varphi}_x)\in Q(t,\varepsilon)$, $\forall t\in\left[t_0,t^*\right[$, then the zero solution to (25), is said to be stable on $\left\{\mathbf{q}\left(t,\mathbf{x}(t)\right),\ T\right\}$, where $\mathbb{S}_\delta(\boldsymbol{\varphi}_x,\delta)=\left\{\boldsymbol{\varphi}_x\in\mathcal{C}\left([-\tau,0],\mathbb{R}^n\right),\ \|\boldsymbol{\varphi}-\boldsymbol{\varphi}_x\|<\delta,\ \delta>0\right\}$ $Q(t,\varepsilon)=\left\{\mathbf{x}\in\mathbb{R}^n,\left\|\mathbf{q}\left(t,\mathbf{x}(t)\right)-\mathbf{q}\left(t,\mathbf{x}_k(t)\right)\right\|<\varepsilon,\ \varepsilon>0\right\}$ *Liu*, *Li* (1997).

Definition 8. If δ is only related to ε and has nothing to do with t_0, then the zero solution is said to be *uniformly* stable on $\left\{\mathbf{q}\left(t,\mathbf{x}(t)\right),\ T\right\}$ *Liu*, *Li* (1997).

Stability theorems

Theorem 7. Suppose that there exists a $V(\,\cdot\,)$ function

$$\begin{aligned}&V\left(t,\mathbf{q}\left(t,\mathbf{x}(t)\right)\right):\ \Im\times\mathbb{R}^n\rightarrow\mathbb{R}^-\\&V\left(t,\mathbf{q}\left(t,\mathbf{x}(t)\right)\right)\in\mathcal{C}\left(\Im\times\mathbb{R}^n\right)\end{aligned}, \tag{26}$$

and a wedge function $\Phi(\,\cdot\,)$ fulfilling the following conditions:

(i) $V\left(t,\mathbf{q}\left(t,\mathbf{x}(t)\right)\right)\equiv0,\ \ \forall t\in\left[t_0,\,t^*\right[$ (27)

(ii) $\begin{aligned}&\Phi\left(\left\|\mathbf{q}\left(t,\mathbf{x}(t)\right)-\mathbf{q}\left(t,\mathbf{x}_k(t)\right)\right\|\right)\leq V\left(t,\mathbf{q}\left(t,\mathbf{x}(t)\right)\right)\\&\forall(t,\mathbf{x})\in\Im\times\mathbb{R}^n\end{aligned}$ (28)

(iii) The derivative of the solution to (25) on $V(\,\cdot\,)$ fulfills

$$V\left(t,\mathbf{q}\left(t,\mathbf{x}\left(t,\,t_0,\,\Psi\right)\right)\right)\leq0\ \text{ for any }\ \Psi\in\mathbb{S}^*\left(t_0,\,t^*\right), \tag{29}$$

then $\mathbf{x}(t,\,t_0,\,\Psi)$ is stable on $\{\mathbf{q}(t,\mathbf{x}(t)),T\}$, *Liu*, *Li* (1997).

Theorem 8. Suppose that there exists a $V(\,\cdot\,)$ function

$$\begin{aligned}&V\left(t,\mathbf{q}\left(t,\mathbf{x}(t)\right)\right):\ \Im\times\mathbb{R}^n\rightarrow\mathbb{R}^-\\&V\left(t,\mathbf{q}\left(t,\mathbf{x}(t)\right)\right)\in\mathcal{C}\left(\Im\times\mathbb{R}^n\right)\end{aligned}, \tag{30}$$

and two wedge functions Φ and Φ_1 fulfilling the following conditions

(i) $\begin{aligned}&\Phi\left(\left\|\mathbf{q}\left(t,\mathbf{x}(t)\right)-\mathbf{q}\left(t,\mathbf{x}_k\left(t,t_0,\boldsymbol{\varphi}_x\right)\right)\right\|\right)\leq V\left(t,\mathbf{q}\left(t,\mathbf{x}(t)\right)\right)\\&\leq\Phi_1\left(\left\|\mathbf{x}-\mathbf{x}_k\left(t,t_0,\boldsymbol{\varphi}_x\right)\right\|\right),\forall t\in\left[t_0,t^*\right[\end{aligned}$,(31)

(i) The derivative of the solution to eq.(3.23) on $V(\,\cdot\,)$ fulfills

$$V\left(t,\mathbf{q}\left(t,\mathbf{x}\left(t,t_0,\boldsymbol{\varphi}_x\right)\right)\right)\leq0\ \text{ for any }\ \Psi\in s_k\left(t_0,t_k\right), \tag{32}$$

then $\mathbf{x}(t,\,t_0,\,\boldsymbol{\varphi}_x)$ is uniformly stable on $\left\{\mathbf{q}\left(t,\mathbf{x}(t)\right),T\right\}$, *Liu*, *Li* (1997).

Consider the continuous singular system with time delay represented by:

$$E\dot{\mathbf{x}}(t)=A\mathbf{x}_t(t)+A_d\mathbf{x}_{t-\tau}(t) \tag{33.a}$$

$$\mathbf{x}_t(t)=\boldsymbol{\varphi}_x(t),\quad -\tau\leq t\leq0, \tag{33.b}$$

where $\mathbf{x}_t(t)\in\mathbb{R}^n$ represents the state and the control input of the system, $\boldsymbol{\varphi}_x(t)\in\mathcal{C}([0,\ 1])$ is the initial condition, $E,\,A,\,A_d\in\mathbb{R}^{n\times n}$ are given constant matrices and τ denotes the time-delay in the state.

Theorem 9. If there exist symmetric and positive-definite matrices $X_1\in\mathbb{R}^{n\times n}$, $X_2\in\mathbb{R}^{n_2\times n_2}$, R_1, $R_2\in\mathbb{R}^{n_1\times n_1}$ and a positive scalar $\wp$ such that

$$\Xi=\begin{pmatrix}J_1 & E^TX_1A_dPI_1 & E^TX_1A_dPI_0^T & E^TX_1A_dPI_0^T\\ I_1^TP^TA_d^TX_1E & -X_2 & 0 & 0\\ I_0P^TA_d^TX_1E & 0 & -\frac{1}{\tau}R_1 & 0\\ I_0P^TA_d^TX_1E & 0 & 0 & -\frac{1}{\tau}R_2\end{pmatrix}<0 \tag{34}$$

where

$$\begin{aligned} J_1 &= E^T X_1 A + A^T X_1 E + E^T X_1 A_d P I_0^T I_0 P^{-1} \\ &+ \left(P^{-1}\right)^T I_0^T I_0 P^T A_d^T X_1 E + \tau A^T Q^T I_0^T R_1 I_0 Q A \\ &+ \tau A_d^T Q^T I_0^T R_2 I_0 Q A_d + \left(P^{-1}\right)^T \begin{pmatrix} 0 & 0 \\ 0 & X_2 \end{pmatrix} P^{-1} \\ &+ \wp \left(P^{-1}\right)^T \left(I_0^T M I_0 - I_1 I_1^T\right) P^{-1} \\ I_0 &= \begin{pmatrix} I & 0 \end{pmatrix}, \quad I_1 = \begin{pmatrix} 0 \\ I \end{pmatrix} \end{aligned} \tag{35}$$

holds for some symmetric and positive semi - definite matrix $M^T = M \geq 0$, then the system (33) is asymptotically stable, *Boukas*, *Liu* (2003).

Linear continuous singular time invariant time delayed systems – New results

Theorem 10. Suppose that the matrix pair $(E,\, A_0)$ is *regular* with the system matrix A_0 being nonsingular., e.i. $\det A_0 \neq 0$. System (2) is *asymptotically stable*, independent of delay, if there exists a symmetric positive definite matrix $P = P^T > 0$, being the solution of the Lyapunov matrix equation

$$A_0^T P E + E^T P A_0 = -2(S + Q), \tag{36}$$

with the matrices $Q = Q^T > 0$ and $S = S^T$, such that:

$$\mathbf{x}^T(t)(S + Q)\mathbf{x}(t) > 0, \quad \forall \mathbf{x}(t) \in \mathcal{W}_{k^*} \setminus \{0\}, \tag{37}$$

is a positive definite quadratic form on $\mathcal{W}_{k^*} \setminus \{0\}$, $\mathcal{W}_{k^*}$ being the subspace of consistent initial conditions, and if the following condition is satisfied:

$$\| A_1 \| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right) \sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}} E^T P\right), \tag{38}$$

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are the maximum and minimum singular values of the matrix $(\cdot)$, respectively, *Debeljkovic et al.* (2003, 2004.c, 2006, 2007).

Proof. Let us consider the functional:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t) E^T P E \mathbf{x}(t) + \int_{t-\tau}^{t} \mathbf{x}^T(\vartheta) Q \mathbf{x}(\vartheta) d\kappa. \tag{39}$$

Note that (*Owens, Debeljković* 1985) indicates that:

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t) E^T P E \mathbf{x}(t), \tag{40}$$

Is a *positive quadratic form* on $\mathcal{W}_{k^*}$, and it is obvious that all smooth solutions $\mathbf{x}(t)$ evolve in $\mathcal{W}_{k^*}$, so $V(\mathbf{x}(t))$ can be used as a *Lyapunov function* for the system under consideration, *Owens*, *Debeljkovic* (1985).

It will be shown that the same argument can be used to declare the same property of another quadratic form present in (39).

Clearly, using the equation of motion of (2), we have:

$$\begin{aligned} \dot V(\mathbf{x}(t)) &= \mathbf{x}^T(t)\left(A_0^T P E + E^T P A_0 + Q\right)\mathbf{x}(t) \\ &+ 2\mathbf{x}^T(t)\left(E^T P A_1\right)\mathbf{x}(t-\tau) - \mathbf{x}^T(t-\tau) Q \mathbf{x}(t-\tau) \end{aligned} \tag{41}$$

and after some manipulations, the following expression is obtained:

$$\begin{aligned} \dot V(\mathbf{x}(t)) &= \mathbf{x}^T\left(A_0^T P E + E^T P A_0 + 2Q + 2S\right)\mathbf{x} \\ &+ 2\mathbf{x}^T(t)\left(E^T P A_1\right)\mathbf{x}(t-\tau) - \mathbf{x}^T(t) Q \mathbf{x}(t) \\ &- 2\mathbf{x}^T(t) S \mathbf{x}(t) - \mathbf{x}^T(t-\tau) Q \mathbf{x}(t-\tau) \end{aligned} \tag{42}$$

From (36) and the fact that the choice of the matrix S, can be done, such that:

$$\mathbf{x}^T(t) S\, \mathbf{x}(t) \geq 0, \quad \forall \mathbf{x}(t) \in \mathcal{W}_{k^*} \setminus \{0\}, \tag{43}$$

one obtains the following result:

$$\begin{aligned} \dot V(\mathbf{x}(t)) &\leq 2\mathbf{x}^T(t)\left(E^T P A_1\right)\mathbf{x}(t-\tau) \\ &- \mathbf{x}^T(t) Q \mathbf{x}(t) - \mathbf{x}^T(t-\tau) Q \mathbf{x}(t-\tau) \end{aligned}, \tag{44}$$

and based on the well known inequality:

$$\begin{aligned} 2\mathbf{x}^T(t) E^T P A_1 \mathbf{x}(t-\tau) &= 2\mathbf{x}^T(t) E^T P A_1 Q^{-\frac{1}{2}} Q^{\frac{1}{2}} \mathbf{x}(t-\tau) \\ &\leq \mathbf{x}^T(t) E^T P A_1 Q^{-1} A_1^T P E^T \mathbf{x}(t) + \mathbf{x}^T(t-\tau) Q \mathbf{x}(t-\tau) \end{aligned}, \tag{45}$$

and by substituting into (45), it yields:

$$\begin{aligned} \dot V(\mathbf{x}(t)) &\leq -\mathbf{x}^T(t) Q \mathbf{x}(t) + \mathbf{x}^T(t) E^T P A_1 Q^{-1} A_1^T P E \mathbf{x}(t) \\ &\leq -\mathbf{x}^T(t) Q^{\frac{1}{2}} \Gamma Q^{\frac{1}{2}} \mathbf{x}(t)(t) \end{aligned}, \tag{46}$$

with the matrix Γ defined by:

$$\Gamma = \left(I - Q^{-\frac{1}{2}} E^T P A_1 Q^{-\frac{1}{2}} Q^{-\frac{1}{2}} A_1^T P E Q^{-\frac{1}{2}}\right) \tag{47}$$

$\dot V(\mathbf{x}(t))$ is negative definite, if:

$$1 - \lambda_{\max}\left(Q^{-\frac{1}{2}} E^T P A_1 Q^{-\frac{1}{2}} Q^{-\frac{1}{2}-1} A_1^T P E Q^{-\frac{1}{2}}\right) > 0, \tag{48}$$

which is satisfied, if:

$$1 - \sigma_{\max}^2\left(Q^{-\frac{1}{2}} E^T P A_1 Q^{-\frac{1}{2}}\right) > 0. \tag{49}$$

Using the properties of the singular matrix values, *Amir - Moez* (1956), the condition (49), holds if:

$$1 - \sigma_{\max}^2\left(Q^{-\frac{1}{2}} E^T P\right) \sigma_{\max}^2\left(A_1 Q^{-\frac{1}{2}}\right) > 0, \tag{50}$$

which is satisfied if:

$$1 - \sigma_{\min}^{-2}\left(Q^{\frac{1}{2}}\right)\left(\| A_1 \|^2 \sigma_{\max}^2\left(Q^{-\frac{1}{2}} E^T P\right)\right) > 0. \tag{51}$$

Q.E.D.

Remark 7. (36-37) are, in a modified form, taken from *Owens & Debeljkovic* (1985).

Remark 8. If the system under consideration is just ordinary time delay, e.g. $E = I$, the result is identical to that presented in *Tissir & Hmamed* (1996).

Remark 9. Let us discuss first the case when the time delay *is absent.*

a) Then the *singular* (weak) Lyapunov matrix (36) is the natural generalization of the classical Lyapunov theory. In particular:

b) If E is a *nonsingular matrix*, then the system is asymptotically stable if and only if $A = E^{-1}A_0$ *Hurwitz* matrix. (33) can be written in the form:

$$A^T E^T PE + E^T PEA = -(Q+S), \tag{52}$$

with the matrix Q being symmetric and positive definite, in the whole state space, hence $\mathcal{W}_{k^*} = \Re\left(E^{k^*}\right) = \mathbb{R}^n$. In these circumstances $E^T PE$ is a Lyapunov function for the system.

c) The matrix A_0 by necessity is nonsingular and hence the system has the form:

$$E_0\dot{\mathbf{x}}(t) = \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 . \tag{53}$$

Then for this system to be stable, (49) must hold as well and have the well-known Lyapunov structure:

$$E_0^T P + PE_0 = -Q , \tag{54}$$

where Q is a symmetric matrix but only required to be positive definite on $\mathcal{W}_{k^*}$.

Remark 10. There is no need for the system under consideration to possess properties given in *Definition* 2, since this is obviously guaranteed by the demand that all smooth solutions $\mathbf{x}(t)$ evolve in $\mathcal{W}_{k^*}$.

Remark 11. The idea and approach are based upon the papers of *Owens & Debeljkovic* (1985) and *Tissir & Hmamed* (1996).

Theorem 11. Suppose that the system matrix A_0 is nonsingular., e.i. $\det A_0 \neq 0$. Then we can consider system (2) with a known compatible vector valued function of initial conditions and we shall assume that $rank\, E_0 = r < n$.

The matrix E_0 is defined in the following way $E_0 = A_0^{-1}E$. System (26) is *asymptotically stable*, independent of delay, if:

$$\| A_1 \| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}E_0^T P\right), \tag{55}$$

and if there exists $(n \times n)$ matrix P, being the solution of the Lyapunov matrix:

$$E_0^T P + PE_0 = -2I_{\mathcal{W}_k}, \tag{56}$$

with the usual properties.

Moreover, the matrix P is symmetric and positive definite on the subspace of consistent initial conditions.

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are the maximum and minimum singular values of the matrix $(\cdot)$, respectively *Debeljkovic et al.* (2005.b, 2005.c, 2006.a).

For the sake of brevity, the proof is here omitted and is completely identical to that of the preceeding *Theorem.*

Remark 12. The basic idea and approach are based upon the paper of *Pandolfi* (1980) and *Tissir*, *Hmamed* (1996).

Asymptotic stability of linear discrete descriptor TIME DELAYED systems

Consider a linear discrete descriptor system with state delay (LDDTDS), described by

$$E\,\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1), \tag{57.a}$$

$$\mathbf{x}(k_0) = \boldsymbol{\psi}(k_0), \quad -1 \le k_0 \le 0, \tag{57.b}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is a state vector.

The matrix $E \in \mathbb{R}^{n\times n}$ is a necessarily singular matrix, with a property $rank\, E = r < n$ and with the matrices A_0 and A_1 of appropriate dimensions.

For an LDDTDS, (57), we present the following definitions taken from, *Xu et al*. (2004).

Definition 9. The LDDTDS is said to be *regular* if $\det\left(z^2E - zA_0 - A_1\right)$, is not identically zero.

Definition 10. The LDDTDS is said to be *causal* if it is *regular* and

$$\deg\left(z^n \det\left(zE - A_0 - z^{-1}A_1\right)\right) = n + rang\, E .$$

Definition 11. The LDDTDS is said to be *stable* if it is *regular* and $\rho\left(E, A_0, A_1\right) \subset D(0,1)$, where

$$\rho\left(E, A_0, A_1\right) = \left\{z \mid \det\left(z^2E - zA_0 - A_1\right) = 0\right\}.$$

Definition 12. The LDDTDS is said to be *admissible* if it is *regular*, *causal* and *stable*.

Stability definitions

Definition 13. System (57) is E-stable if for any $\varepsilon > 0$, there always exists a positive δ such that

$$\|E\mathbf{x}(k)\| < \varepsilon ,$$

when

$$\|E\mathbf{x}_0\| < \delta .$$

Liang (2000).

Definition 14. System (57) is E-asymptotically stable if system (57) is E-stable and

$$\lim_{k\to+\infty} E\mathbf{x}(k) = 0 .$$

Liang (2000).

Linear discrete descriptor time invariant time delayed systems –New results

Stability theorems

Theorem 12. Suppose that there exist a positive definite matrix W_1, a semi-positive definite matrix V_1, $rank\left(E^T V_1 E\right) = rank\, E = r$ and a scalar $\gamma,\ \varepsilon > 0$ such that:

$$\begin{aligned} &A_0^T V_1 A_0 - E^T V_1 E + (1+\gamma) A_1^T A_1 + \\ &+\frac{1}{\gamma} A_0^T V_1^2 A_0 \ \le\ -\varepsilon E^T W_1 E \end{aligned}, \tag{58}$$

then the zero solution to system (57) is E-asymptotically stable, *Liang* (2000).

Theorem 13. Suppose $|z| \geq 1$, $rank\left(zE - A_0,\ \ A_1\right) = n$ holds for any complex number z and system (4.1) is causal and without time delay, then the zero solution to system (4.1) is asymptotically stable, *Liang* (2000).

Theorem 14. System (57) is *admissible* if there exist a matrix $Q > 0$ and an invertible symmetric matrix $P = P^T$ such that

(i) $E^T PE \geq 0$ (59)

(ii)
$$\begin{aligned} &A_0^T PA_0 - E^T PE + \\ &+ A_0^T PA_1 \left(Q - A_1^T PA_1\right)^{-1} A_1^T PA_0 + Q < 0 \end{aligned} \quad (60)$$

(iii) $Q - A_1^T PA_1 > 0$, (61)

Xu et al. (2004).

Theorem 15. Suppose that system (57) is *regular* and *causal* with the system matrix A_0 being nonsingular, e.i.

$$\det A_0 \neq 0. \quad (62)$$

System (57) is *asymptotically stable*, independent of delay, if

$$\| A_1 \| < \frac{\sigma_{\min}\left(Q^{\frac{1}{2}}\right)}{\sigma_{\max}\left(Q^{-\frac{1}{2}} A_0^T P\right)}, \quad (63)$$

and if there exists a symmetric positive definite matrix P on the whole state space, being the solution of the *discrete* Lyapunov *matrix equation*

$$A_0^T PA_0 - E^T PE = -2\left(S + Q\right), \quad (64)$$

with the matrices $Q = Q^T > 0$ and $S = S^T$, such that

$$\mathbf{x}^T\left(k\right)\left(S + Q\right)\mathbf{x}\left(k\right) > 0, \ \ \forall \mathbf{x}\left(k\right) \in \mathcal{W}_{k^*}^d \setminus \{0\}, \quad (65)$$

is a positive definite quadratic form on $\mathcal{W}_{k^*}^d \setminus \{0\}$, $\mathcal{W}_{k^*}^d$ being the subspace of consistent initial conditions.

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are the maximum and minimum singular values of the matrix $(\cdot)$, respectively, *Debeljkovic et al.* (2004.c, 2004.d, 2005.e, 2005.f).

Remark 13. (64-65) are, in a modified form, taken from *Owens*, *Debeljkovic* (1985).

Remark 14. If the system under consideration is just ordinary time delay, e.g. $E = I$, the result is identical to that presented in *Debeljkovic et al.* (2004.a – 2004.d, 2005.a, 2005.b).

Remark 15. The idea and approach are based upon the papers of *Owens*, *Debeljkovic* (1985) and *Tissir*, *Hmamed* (1996).

Theorem 16. Suppose that system (57) is *regular* and *causal.*

Moreover, suppose the matrix $\left(Q_\lambda - A_1^T P_\lambda A_1\right)$ is regular, with $Q_\lambda = Q_\lambda^T > 0$.

System (57) is *asymptotically stable*, independent of delay, if:

$$\| A_1 \| < \frac{\sigma_{\min}\left(\left(Q_\lambda - A_1^T P_\lambda A_1\right)^{\frac{1}{2}}\right)}{\sigma_{\max}\left(Q_\lambda^{-\frac{1}{2}}\left(A_0 - \lambda E\right)^T P_\lambda\right)}, \quad (66)$$

and if there is a real positive scalar $\lambda^* > 0$ such that for all λ within the range $0 < |\lambda| < \lambda^*$ there exists a symmetric positive definite matrix P_λ, being the solution of the *discrete* Yaupon *matrix equation*:

$$\left(A_0 - \lambda E\right)^T P_\lambda \left(A_0 - \lambda E\right) - E^T P_\lambda E = -2\left(S_\lambda + Q_\lambda\right), \quad (67)$$

with the matrix $S_\lambda = S_\lambda^T$, such that:

$$\mathbf{x}^T\left(k\right)\left(S_\lambda + Q_\lambda\right)\mathbf{x}\left(k\right) > 0, \quad \forall \mathbf{x}\left(k\right) \in W_{k^*}^d \setminus \{0\}, \quad (68)$$

is a positive definite quadratic form on $\mathcal{W}_{k^*}^d \setminus \{0\}$, $\mathcal{W}_{k^*}^d$ being the subspace of consistent initial conditions for both *time delay* and *non-time delay* discrete descriptor systems.

Such conditions we call compatible consistent initial conditions

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are the maximum and minimum singular values of the matrix $(\cdot)$, respectively, *Debeljkovic et al.* (2007).

Conclusion

This survey paper is devoted to the stability of linear discrete descriptor time delayed systems (LDDTDS). Here, we gave a number of results concerning stability properties in the sense of Lyapunov and analyzed the relationship between them. To assure *asymptotical stability for* LCSTDS, it is not enough only to have the eigenvalues of the matrix pair $(E,\ A)$ in the left half of the complex plane, but also to provide an impulse-free motion (compatible initial function) and some conditions to be fulfilled for the system under consideration.

Some different approaches have been shown in order to construct a *Lyapunov* stability theory for a particular class of autonomous LDDTDS.

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Stabilnost linearnih kontinualnih i diskretnih deskriptivnih sistema sa čistim vremenskim kašnjenjem u smislu Ljapunova: Pregled rezultata

U ovom radu izlaže se pregled rezultata i doprinosa mnogih autora na polju izučavanja ljapunoske stabilnosti posebnih klasa linearnih sistema.
Ovaj pregled pokriva period počev od 1997. godine pa sve do današnjih dana i razmatra doprinose autora širom sveta koji su objavljeni u repektabilnim međunarodnim časopisima ili saopšteni na renomiranim internacionalnim konferencijama a neki od njih i na prestižnim svetskim workšopovima.

Key words: **linearni sistem, kontinualni sistem, singularni sistem, diskretni sistem, deskirptivni sistem, stabilnost sistema, sistem sa kašnjenjem, stabilnost Ljapunova.**

Устойчивость линейных сингулярных непрерывных и дискретных дескриптивных систем с чистым временем задержки в смысле Ляпунова: Подведение итогов

В настоящей работе представлен подробный обзор результатов и вклад многих авторов в области исследования устойчивости Ляпунова особого класса линейных систем. Это исследование охватывает период с 1997 года по сегодняшний день и рассматривает роль и вклад авторов целого мира, которые опубликованы в авторитетных международных журналах или представлены на престижных международных конференциях, а некоторые из них и на престижных мировых семинарах.

Ключевые слова: **линейная система, непрерывная система, сингулярная система, дискретная система, дескриптивная система, устойчивость системы, системная задержка, устойчивость Ляпунова.**

Stabilité des systèmes linéaires continus discrets et descriptifs à délai temporel pur dans le sens de Lyapunov: Tableau des résultats

Dans ce papier on présente le tableau des résultats et des contributions de nombreux auteurs dans le domaine des recherches sur la stabilité de Lyapunov des classes particulières des systèmes linéaires . Ce tableau comprend la période à partir de 1997 jusqu'à nos jours et considère les contributions des auteurs du monde entier dont les travaux ont été publiés dans les revues internationales renommées ou bien présentés au cours des conférences et des ateliers de prestige.

Mots clés: **système linéaire, système continu, système singulier, système discret, système descriptif, stabilité du système, système à délai, stabilité de Lyapunov.**