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Uncertainty Quantification of Inifinitesimal Elastoplasticity

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We analyse the stochastic finite element method for a class of mixed variational inequalities of the second kind, which arises in elastoplastic problems. The quasi-static von Mises elastoplastic rate-independent evolution problem with linear isotropic hardening is considered with the emphasis on the presence of uncertainty in the description of material parameters. Within one time-step of backward Euler discretization, the stochastic finite element method leads to a minimisation problem for smooth convex functions on discrete tensor product subspaces, whose unique minimiser is obtained via the closest point projection method. To this end, we use a description in the language of non-dissipative and dissipative operators and introduce a well-developed stochastic Newton iterative algorithm for solving coupled nonlinear systems of equations. Finally, the proposed framework is demonstrated by a numerical simulation in plane strain conditions.

Key words: **elastoplasticity, stochastic process, finite element method, stochastic method, stochastic algorithm.**

Introduction

THE deterministic description of the inelastic behaviour [3, 9] is not applicable to heterogeneous materials due to the uncertainty of corresponding characteristics at the micro-structural level. Namely, the deterministic approach has one disadvantage: the description of the material parameters is given by the first order statistical moment called a mean value or mathematical expectation. However, such representation neglects the most important property of material characteristics - their random nature. Due to this reason, we consider a mathematical model which approximates material parameters as random fields and processes in order to closely capture the real nature of the random phenomena.

The history of the stochastic elastoplasticity begins with the work of Anders and Hori [1]. They declared elastic modulus as a source of the uncertainty and treated all following subsequent uncertainties with the help of a perturbation technique. Thereafter, Jeremić [4] introduced the Fokker-Plank equation approach based on the work of Kavvas, who obtained a generic Eulerian-Lagrangian form of the Fokker-Plank equation corresponding to any nonlinear ordinary differential equation with random forcing and a random coefficient. In other words, Jeremić and his coworkers reformulated the original stochastic partial differential equations of quasi-elastoplasticity as deterministic ones. However, these methods are either mathematically very complicated to deal with or not enough accurate to be used for. Namely, the perturbation technique is limited only to the problems described by a small variation of input properties. In addition, the method experiences a "closure-problem" or the dependence of the lower-order moments on the higher-order moments. On other side, even though the Fokker-Planck method predicts the mean behaviour exactly, it over-predicts the standard deviation of the solution. The main reason for this are the Dirac delta initial conditions. In general both methods assume one uncertain parameter which is not enough to properly describe the problem.

In order to quantify uncertainties in a more appropriate manner, we introduce several material parameters as uncertain and propose spectral stochastic finite element method as a solution procedure. According to this, the paper is organized as follows: in the first section the irreversible behaviour is modelled as a quasi-static inelastic problem followed by description of material properties and solution strategy in second part. This is then accompanied by a stochastic closest point projection algorithm in the third section. Finally, the proposed method is validated on a simple numerical example in plain strain conditions by comparison with the Monte Carlo approach.

Model problem

Consider a material body occupying a bounded domain $\mathcal{G} \in \mathbb{R}^d$ with a piecewise smooth Lipschitz continuous boundary $\partial\mathcal{G}$ on which are imposed the boundary conditions in the Dirichlet and Neumann form on $\Gamma_D \subseteq \partial\mathcal{G}$ and $\Gamma_N \subset \partial\mathcal{G}$ respectively, such that $\Gamma_D \cap \Gamma_N = \varnothing$ and $\overline{\Gamma}_N \cup \overline{\Gamma}_D$. The probability space is defined as a triplet $(\Omega, \mathcal{B}, \mathbb{P})$, with Ω beign the space of all events, $\mathcal{B}$ the σ-algebra and $\mathbb{P}$ a probability measure. In such defined space the balance of the momentum localized about any point x in the domain $\mathcal{G}$ in time $t \in \mathcal{T} := [0, T]$ leads to an equilibrium equation required to hold almost surely in ω, i.e. $\mathbb{P}$-almost everywhere:

$$div\,\sigma + f = 0 \text{ on } \mathcal{G} \tag{1}$$

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The corresponding boundary conditions are specified as:

$$\sigma n = g \text{ on } \Gamma_N \tag{2}$$

$$u = 0 \text{ on } \Gamma_D \tag{3}$$

where u denotes the displacement field over $\mathcal{G}$, f the body force, σ the stress tensor, n the unit normal at $x \cup \Gamma_N$, and g a prescribed surface tension. For the sake of simplicity, we use homogeneous Dirichlet boundary conditions, and under the assumptions of small deformation theory introduce the strain

$$\varepsilon(u) = Du. \tag{4}$$

Here the linear bounded operator D is defined in a weak sense as:

$$D{:}u_1(x)u_2(\omega) \rightarrow (\nabla_S u_1(x))u_2(\omega). \tag{5}$$

However, instead of finding the solution which holds absolutely with respect to the deterministic and stochastic domain, we search for the solution of a weak counterpart of Eq. (1) with respect to the test functions. In this manner, the elastoplastic problem transforms to a stochastic variational inequality of the second kind.

Let us introduce the Hilbert space $\mathcal{Z}$ of the solution w and the convex, closed and non-empty cone $\mathcal{K}^\infty \subset \mathcal{Z}$ such that the weak formulation in the primal form reads:

Proposition 2.1. find w with $w(0) = 0$ such that for almost all $t \in \mathcal{T}$, $\dot{w} \in \mathcal{K}^\infty$ and all $z \in \mathcal{K}^\infty$ holds:

$$a(w(t), z - \dot{w}(t) + j(z) - j(\dot{w}(t)) \geq \ell(z - \dot{w}). \tag{6}$$

Here $\mathcal{Z} = \mathcal{U} \times \mathcal{E}$ represents a Hilbert space of the primal solution $w := (u, E_p)$, with u being the displacement and E_p the generalised plastic deformation containing the plastic deformation ε_p and the vector of internal variables η. Thus, the displacement $\mathcal{U}$ and the deformation $\mathcal{E}$ spaces are given as appropriate tensor products of deterministic spaces of the solution and the space of random variables with finite variance $(S) := L_2(\Omega)$. Namely, the space of the displacements is given as:

$$\mathcal{U} = H_0^1(\mathcal{G}) \otimes (S) \tag{7}$$

while the space of the generalised plastic deformation as:

$$\boldsymbol{\mathcal{E}} = L_2(\boldsymbol{\mathcal{G}}) \otimes (S) \tag{8}$$

With this notation, we may introduce the bilinear form $a : \boldsymbol{\mathcal{Z}} \times \boldsymbol{\mathcal{Z}} \rightarrow \mathbb{R}$ as:

$$a(\omega, z) = \langle\langle A : (\varepsilon(u) - \varepsilon_p), \varepsilon(v) - \varepsilon_p \rangle\rangle + \langle\langle H\eta, \mu \rangle\rangle \tag{9}$$

with $E = (\varepsilon, 0)$ being the total deformation, A elastic and H hardening constitutive tensor. Here the notation $\langle\langle \cdot, \cdot \rangle\rangle$ signifies the duality pairing:

$$\langle\langle \sigma, \varepsilon \rangle\rangle = \int_\Omega \int_{\mathcal{G}} \sigma \cdot \varepsilon \, \mathrm{d}x \, \mathbb{P}(\mathrm{d}\omega). \tag{10}$$

In similar manner, one may introduce the linear form:

$$\ell : \boldsymbol{\mathcal{Z}} \rightarrow \mathbb{R} : \ \ell(z) = \langle\langle f, z \rangle\rangle \tag{11}$$

representing the right hand side. The functional $j(z)$ is the dissipation functional assumed to be convex, positively homogenous, non-negative and lower-semi continuous. To this one adds the symmetry, $\mathcal{Z}$-ellipticity and boundness of the bilinear from a in order to show the existence and uniqueness of the solution of the primal problem [3].

Since the primal formulation is not the one we are interested in, we try to reformulate it with respect to the dual space $\mathcal{Z}^*$ to a so called mixed variational problem:

Theorem 2.1. There are unique functions, $w \in H^1(\mathcal{T}, \mathcal{Z}^*)$ and $w^* \in H^1(\mathcal{T}, \mathcal{Z}^*)$ with $w(0) = 0$ and $w^*(0) = 0$, which solve the following problem i.e. $t \in T$:

$$a(w(t), z) + \langle\langle w^*(t), z \rangle\rangle = \langle\langle f(t), z \rangle\rangle \tag{12}$$

and

$$\forall z^* \in \mathcal{K} : \langle\langle \dot{w}(t), z^* - w^*(t) \rangle\rangle \leq 0 \tag{13}$$

In this description, the first equation represents the equilibrium equation, while the second one is the flow rule describing the rate of change of the plastic deformation. Under similar conditions as in primal problem, we may show the existence and uniqueness of the mixed solution.

Material Properties

The input random fields $\kappa(x, \omega)$ (bulk and shear modulus, yield stress) are assumed to be exponential piecewise transformations of a Gaussian random field, i.e. the lognormal random fields. Their discretisation is done by a combination of the truncated Karhunen-Loeve and polynomial chaos expansion (KLE/PCE) such that one has:

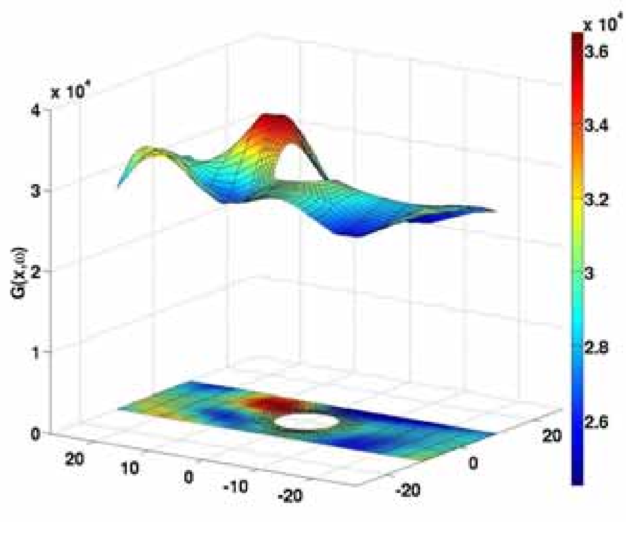

Figure 1. Realisation of shear modulus

$$\kappa(x,\theta)=\sum_{l=0}^{M}\sum_{\gamma\in\mathcal{J}}\sqrt{\lambda_l}\kappa_l^{(\gamma)}H_\gamma(\theta)\kappa_l(x) \quad (14)$$

where

$$J=\{\gamma \mid \forall j>M:\gamma_j=0,\ |\gamma|:=\sum_{j=1}^{\infty}\lambda_j\le p\} \quad (15)$$

represents the multi-index set, $\sqrt{\lambda_l}$ KL eigenvalues, $\kappa_l(x)$ the KL eigenfunctions, $\kappa_l^{(\gamma)}$ the coefficients of polynomial chaos expansion of KL - random variables and $H_\gamma(\theta)$ Hermite polynomials in Gaussian random variables θ.

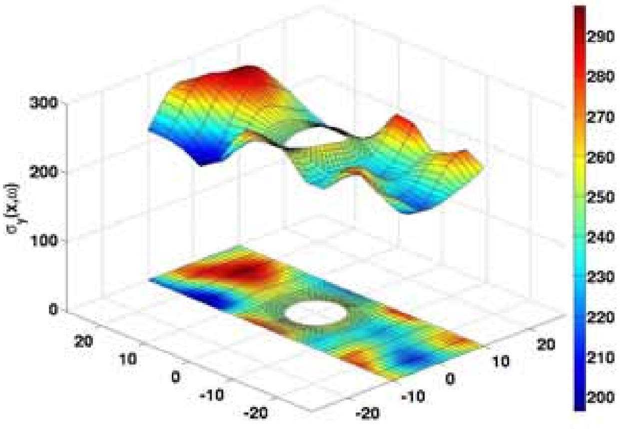

Figure 2. Realisation of yield stress

Stochastic Galerkin method

The variational inequality Eq. (12) may be equivalently formulated as a minimisation problem [3], where one has to minimise the convex functional in one time step with respect to the displacement u and the stress field Σ. After spatial discretisation, the problem can be solved on the computer via a return mapping algorithm [9].

In order to solve the mixed problem, one may observe the dependence of the generalised stress Σ on the displacement u and the generalised plastic strain E_p. Following this, we may reformulate Eq. (12) for all $v\in\mathcal{U}$ to:

$$a(v,\Sigma(u,E_p))=\langle\langle A(E_p)[u],v\rangle\rangle=\ell(t,v) \quad (16)$$

which defines a hemicontinuous operator A. In order to solve this equation numerically, we need to perform several discretisations. The first one is the discretisation with respect to time in Euler backward manner. Thus, we divide the time interval $\mathcal{T}$ into L identical steps of size $\Delta t_n=t_n-t_{n-1}$, such that the incremental quantities are represented as $\Delta(\cdot)=(\cdot)_n-(\cdot)_{n-1}$. In this way, Eq. (16) obtains incremental form further spatially discretised using the finite element method. By taking the space of the piecewise linear continuous functions $\mathcal{U}_h$ for the displacement, we may separate the deterministic part of the solution from the stochastic one:

$$u^h=\sum_{i=1}^{P}u_i(\omega)N_i(x):=Nu(\omega) \quad (17)$$

where $N=[N_1,...,N_P]$ represents the vector of shape functions, and $u=[u_1(\omega),...u_P(\omega)]^T$ the displacement vector. For the discretisation of the stress are used the piecewise constant functions; and the appropriate integrals are calculated numerically over the spatial domain $\mathcal{G}$ via quadrature rules. Following this, one search for the solution (u_n^h,Σ_n^h) which satisfies discretised equlibrium:

$$a(v^h,\Sigma_n^h)=\ell(t_n,v^h),\quad v^h\in\mathcal{U}_h \quad (18)$$

However, the previous equation is semi-discretised since both sides, left- and right-hand, depend on the parameter ω. This means that one has to discretise the infinite space (S). by taking a stochastic ansatz for the solution in a space of multivariate Hermite polynomials:

$$S^I: span\{H_\gamma \mid \gamma\in\mathcal{J}_{M,p}\}\subset S \quad (19)$$

Here represents the index set of truncated polynomial chaos expansion of order p and M random variables. Inserting the stochastic ansatz back to Eq. (18) and projecting the obtained residual in a standard Galerkin manner, we obtain a system of equations:

$$r(u)=[...,\mathbb{E}(H_\beta(f(\theta)-A(\theta,E_p(\theta))[u(\theta)])),...]=0 \quad (20)$$

which may be further solved by a stochastic counterpart of the Newton-Raphson iterative technique (Newton-Raphson, BFGS, etc.). After the linearisation of Eq. (20), the system is solved by preconditioned Krylov subspace methods.

Stochastic closest point projection

Computationally, the solution of the elastoplastic problem collapses to the (iterative) solution of a convex mathematical programming problem, with a goal to find the closest distance in the energy norm of a trial state to a convex set $\mathcal{K}$ of the elastic domain, known as the closest point projection. In other words, one search for:

$$\Sigma_n(\omega)=\underset{\Sigma(\omega)\in\mathcal{K}}{\arg\min}\,\mathcal{I}(\omega) \quad (21)$$

where $\mathcal{I}$ is given as:

$$\mathcal{I}:=\underset{\Sigma(\omega)\in\mathcal{K}}{\arg\min}\left[\frac{1}{2}\langle\langle\Sigma^{trial}-\Sigma_n,A^{-1}:(\Sigma^{trial}-\Sigma_n)\rangle\rangle\right] \quad (22)$$

in the time step n described by an implicit Euler difference scheme. Here Σ^{trial} denotes the trial stress leading to the typical operator split of the closest point projection algorithm into two steps: elastic predictor and plastic corrector.

Predictor step. The predictor step calculates the polynomial chaos expansion of the displacement u_n^k (in the iteration k) by solving the equilibrium Eq. (20) [7, 6]. The displacement is then used for the calculation of the strain increment ΔE_n^k and the trial stress $\Sigma_n^{trial,k}$, assuming the step to be purely elastic. If the stress $\Sigma_n^{trial,k}$ lies outside of the admissible region $\mathcal{K}$, we proceed with the corrector step. Otherwise, $\Sigma_n^k=\Sigma_n^{k,trial}$ represents the solution and we may move to the next step.

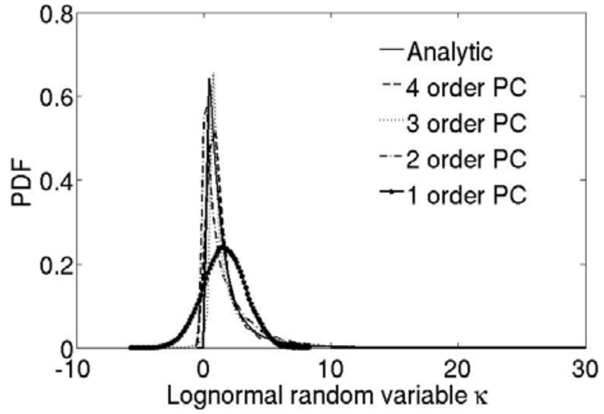

Figure 3. Approximation of the random variable by polynomial chaos expansion

Corrector step. The purpose of the corrector step is to project the stress outside of admissible region back onto point on $\mathcal{K}$. In other words one solves the minimisation problem in Eq. (21) in terms of Lagrangian:

$$\boldsymbol{\mathcal{L}}(\omega)=\boldsymbol{\mathcal{I}}(\omega)+\lambda(\omega)\varphi(\Sigma)(\omega) \tag{23}$$

where $\varphi(\Sigma)(\omega)$ represents the yield function describing the convex set $\mathcal{K}:=\{\Sigma(\omega)\in S \mid \varphi(\Sigma) \text{ a.s. in } \Omega\}$. In this formulation the unique minimiser of Eq. (23) is found by standard optimality conditions [5]:

$$0\in\partial_{\Sigma}\boldsymbol{\mathcal{L}}=\partial_{\Sigma}\boldsymbol{\mathcal{I}}(\omega)+\lambda\partial_{\Sigma}\varphi(\omega) \text{ a.s.} \tag{25}$$

The problem of the closest point projection is more complicated than its deterministic counterpart since we deal with functional representation of uncertain parameters, i.e. polynomial chaos expansions. Thus, the accuracy of the algorithm strongly depends on the truncation error in polynomial expansion. As one may see in Fig.3, the higher order of polynomial, the smaller the error.

Numerical results

The rectangular strip with a hole, Fig.4, under extension is considered. The shear and bulk modulus, yield stress and the isotropic hardening are considered as random parameters. Due to the positive definiteness of these properties, we model them as lognormal random fields, i.e. the piecewise exponential transformation of a Gaussian random field with a prescribed covariance function and correlation lengths.

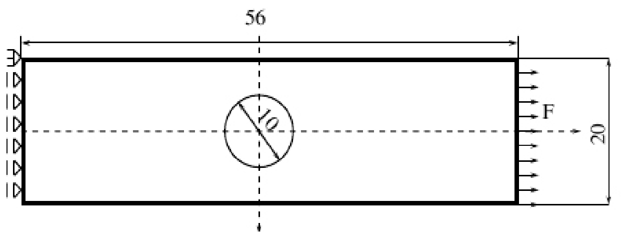

Figure 4. Geometry of the problem: plate with a hole

The extension force is of a deterministic nature, and in the initial state does not depend on the parameter ω . However, in each iteration the force gets mixed with the uncertainty of input parameters and hence becomes random. The randomness in input parameters depends on the choice of the values of the standard deviations as well as the correlation lengths. The bigger the correlation length is, the less random field oscillates. Due to this reason, the correlation lengths in this paper are chosen to be moderate, three times smaller than the dimensions of the plate.

The problem is solved in two different ways: a pure sampling technique such as Latin Hypercube sampling (denoted as MC on the plots) [2] and the method given in this paper (denoted as SG on the plots). In each case, we have used the BFGS method for solving nonlinear equations and the Krylov subspace methods with a mean based preconditioner for solving the corresponding linear system of equations.

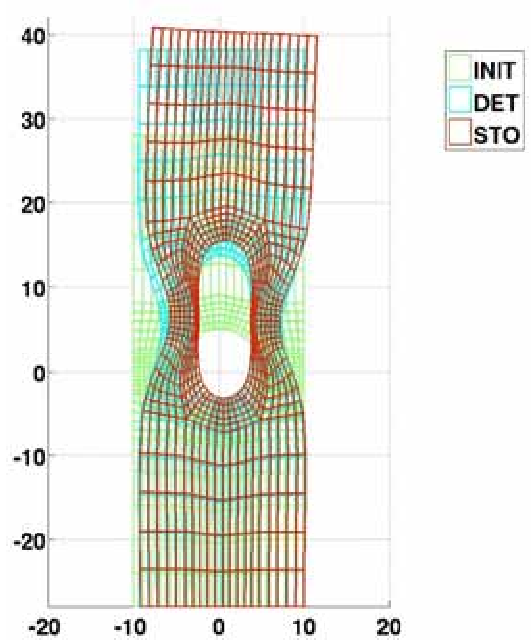

Figure 5. Comparison of the mean value of the total displacement in the stochastic configuration with the deterministic and initial value

Solving the equilibrium equation, one obtains the displacement and stresses. In Fig.5, we compared the displacement with the deterministic value obtained by running FEM with regard to the initial configuration. As one may notice, the stochastic solution is different from the corresponding deterministic one. This means that the deterministic solution is not completely reliable, since presence of moderate variation strongly influences the system response. The same phenomena may be observed in Fig.6 and Fig.7 where the first two statistical moments of the shear and von Mises stress are plotted. The mean value, as expected, has the highest value on the edge of the hole, where the plastic process begins to occur. The same is valid for the variance of the field. With respect to plotted moments, we posses much more information about reversible processes than deterministic analysis can ever provide. Namely, we may extract how much the field varies and compute the corresponding probability exceedances of the quantities.

In order to validate the method, we compare the relative error of the mean residual obtained by the Galerkin procedure with the corresponding one for the Monte Carlo simulation (100 000 realisations). In Fig.8, we may see that both methods have the same type of convergence. However, by measuring the error of higher order terms such as the variance in Fig.9, we obtain larger error by our method (10^4). This is understandable since the higher order moments are influenced by the accuracy of input approximations and the polynomial chaos algebra. Better accuracy may be obtained by taking more terms in the expansion.

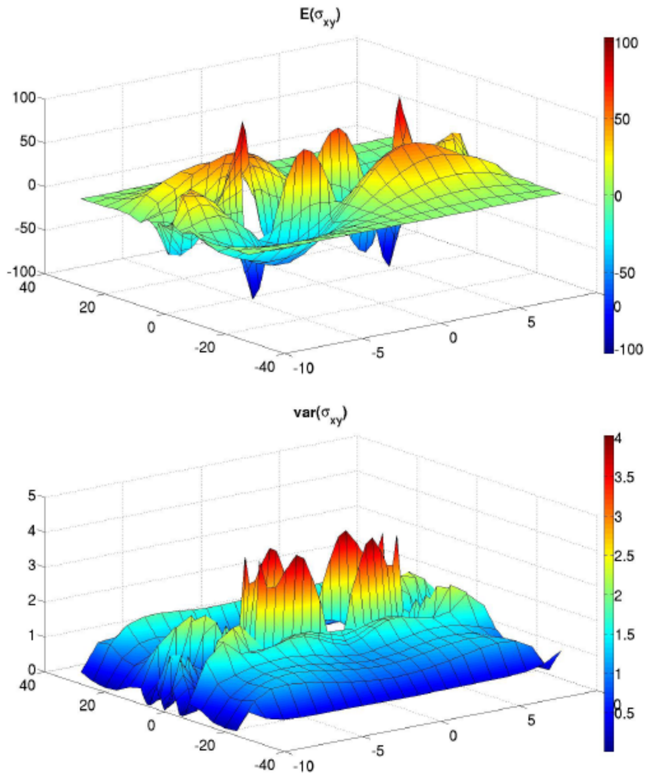

Figure 6. The shear stress σ_{xy} : the mean value and the standard deviation

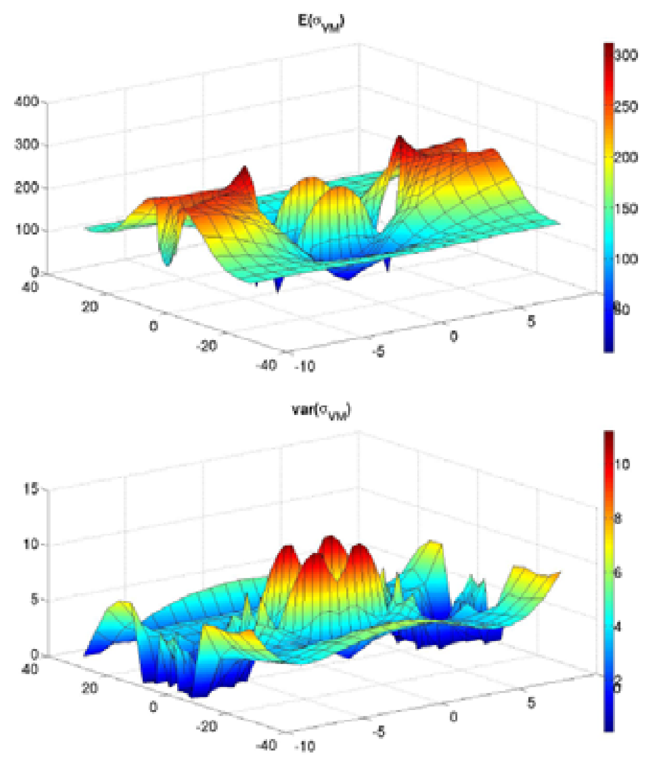

Figure 7. Von Mises stress: the mean value and the standard deviation

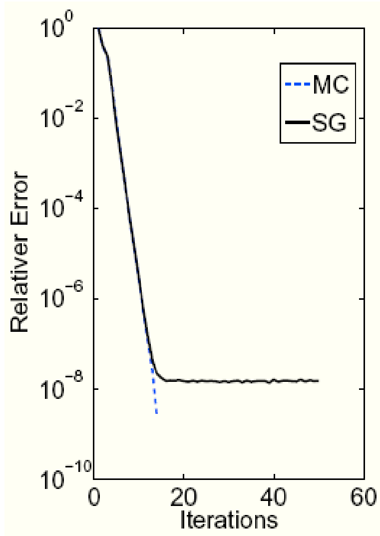

Figure 8. Comparison of the converegence of the mean value of the residual between the latin hypercube sampling (MC) and the stochastic Galerkin method (SG)

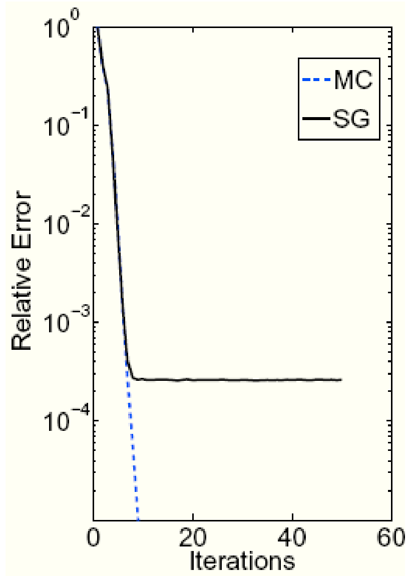

Figure 9. Comparison of the converegence of the mean value of the residual between the latin hypercube sampling (MC) and the stochastic Galerkin method (SG)

The previous comparison is maybe not completely honest, since the number of realisations taken by the MC is much larger than the number of the terms in polynomial chaos expansion. In addition, the MC computational time is around 10 times bigger than the computational time of Galerkin method.

Conclusion

The idea of random variables as functions in an infinite dimensional space approximated by the elements of finite dimensional spaces has brought a new view to the field of stochastic elastoplasticity. In this paper, we have proposed an extension of the stochastic finite element method and related numerical procedures to the resolution of inelastic stochastic problems in the context of Galerkin methods. This strategy may be understood in a sense of the model reduction technique due to the applied Karhunen Loeve and polynomial chaos expansion. A Galerkin projection minimises the error of the truncated expansion such that the resulting set of the coupled equations gives the expansion coefficients. If the smoothness conditions are met, the polynomial chaos expansion converges exponentially with the order of polynomials. In contrast to the Monte Carlo technique, the Galerkin approach, when properly implemented, can achieve fast convergence and high accuracy. With such properties the method may appear to be highly efficient in particular practical computations.

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Kvatnifikacija slučajnih veličina infinitezimalne elastičnoplastičnosti

U ovom radu analizirana je metoda stohastičkih konačnih elemenata za klasu varijacionih nejednakosti drugog reda, kojom su opisani elastoplastični problemi. Poseban fokus je stavljen na kvazistatičan proporcionalno-nezavisan evolucioni von Mises-ov problem opisan slučajnim materijalnim karakteristikama i linearnim izotropnim ojačanjem. Uz pomoć implicitne integracije stochastička metoda konačnih elemenata se svodi na minimizacijski problem u stohastičkom tenzorskom prostoru, u kome su definisane glatke konveksne funkcije slučajnog karaktera. Jedinstveno rešenje ovog minimizacijskog problema je dobijeno uz pomoć stohastičkog radial return algoritma, koji se sastoji od disipativnog (plastičnog) i elastičnog operatora, slično klasičnoj teoriji elastoplastičnosti. Model je verifikovan na primeru ploče sa otvorom u uslovima ravanskog stanja deformacije.

Ključne reči: **elastičnoplastičnost, stohastički proces, metoda konačnih elemenata, stohastička metoda, stohastički algoritam**

Расчёт неточностей в задаче эластопластичности малой деформации

В настоящей работе анализирован стохастический метод конечных элементов для класса смещённых вариационных неравенств второго порядка, которые возникают в задаче эластопластичности. Здесь рассматривается квази-статичная эластопластичная эволюционная задача вон Миза с линейным изотропичным упрочнением при условии наличия неточности в описании параметров материала. Внутри одного временного шага обратного метода Эйлера стохатический метод конечных элементов приводит к задаче минимума для гладкой выпуклой функции на тензорном продукте пространств. Уникальный минимизатор пространств получен по radial return методу. После этого мы используем описание на языке недиссипативных и диссипативных операторов и предлагаем итеративный метод Ньютона для решения связанной нелинейной системы уравнений. В заключении, предложеный метод иллюстрируется на численном двух-размерном примере.

Ключевые слова: **эластопластичность, стохастический процесс, метод конечных элементов, стохастический метод, radial return метод.**

Incertitude de la quantification des paramètres d'élasticité plastique infinitésimale

On a analysé la méthode des éléments stochastiques finis pour la classe des inégalités de variation du deuxième type par laquelle ont été décrits les problèmes d'élasticité plastique. L'accent particulier a été mis sur le problème quasi statique et indépendant proportionnellement de von Mises . Ce problème a été décrit au moyen des caractéristiques matérielles aléatoires et par le renforcement linéaire isotrope. A l'aide de l'intégration implicite la méthode stochastique se réduit au problème de minimisation dans l'espace tensorielle stochastique où les fonctions convexes plates du type aléatoires sont définies. La solution unique de ce problème de minimisation a été obtenue par l'algorithme stochastique radial return qui se compose d'un opérateur dissipatif (plastique) et élastique , pareil à la théorie classique de l'élasticité plastique. Le modèle a été vérifié au moyen d'une plaque à ouverture dans les conditions de l'état plat de déformation.

Mots clés: **élasticité plastique, problème stochastique, méthode des éléments finis, méthode stochastique, algorithme stochastique.**