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The Stability of Linear Discrete Time Delay Systems over a Finite Time Interval: An Overview

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This paper gives sufficient conditions for the practical and finite time stability of a particular class of linear discrete time delay systems. Analyzing the finite time stability concept, these new delay-independent conditions are derived using an approach based on the Lyapunov–like functions. The practical and attractive practical stability for discrete time delay systems has been investigated. The above mentioned approach was supported by the classical Lyapunov technique to guarantee the attractivity properties of the system behavior.

Key words: : **linear systems, discrete systems, system stability, time delay system, finite time stability system, Non-Lyapunov stability.**

Introduction

THE time delay systems have been investigated over many years. Time delay was often encountered in different technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc.

The existence of pure time delay, regardless if it is present in the control or/and state, may cause an undesirable system transient response or even instability. Consequently, the problem of the stability analysis of this class of systems has been one of the main interests of many researchers.

In general, the introduction of time lag factors makes the analysis more complicated.

In the existing stability criteria, two approaches have mainly been adopted. Namely, one direction (or one solution) was to contrive stability conditions which did not include any information on the delay.

The other was a method which took into account the time delay itself. The former case is often called the delay-independent criterion and, generally, provides smooth algebraic stability conditions. Numerous results have been reported on this matter, with a particular emphasis on the application of Lyapunov's second method. The other solutions were based on the idea of the matrix measure as presented in *Lee*, *Diant* (1981), *Mori et al* (1982) and *Hmammed* (1986).

From a practical point of view, the emphasis must be placed not only on the system stability (e.g. in the sense of Lyapunov), but also in the bounds of system trajectories.

A system could be stable, yet completely useless because it possesses undesirable transient performances.

Thus, it may be useful to consider the stability of such systems with respect to the certain state-space subsets which are defined a priori in a given problem.

Besides that, it is of particular significance to investigate the behavior of dynamic systems only over a finite time interval. These boundedness properties of system responses, i.e. the solution of system models, are very important from the engineering point of view. Realizing this fact, numerous definitions of the so-called technical and practical stability have been introduced. Generally speaking, these definitions are essentially based on the predefined boundaries for the perturbation of the initial conditions and allowable perturbation of the system response. In engineering applications of control systems, this fact becomes important and sometimes crucial for the purpose of characterizing in advance, in a quantitative manner including possible deviations of the system response. Thus, the analysis of these particular boundedness properties of the solutions is an important step, which precedes the design of control signals, when finite time or practical stability control is taken into account.

It should be noticed that up to now, no results have been reported concerning the aforementioned problem of the non-Lyapunov stability for discrete time delay systems. Motivated by discussions on practical stability in *La Salle*, *Lefschet* (1961) and *Weiss*, *Infante* (1965, 1967) various notations of the stability over a finite time interval for continuous-time systems and constant set trajectory bounds have been introduced so far.

Further developments of these results were presented later by other authors.

For the first time, the results of finite or practical stability for a particular class of the nonlinear singularly perturbed multiple time delay systems were introduced in *Fang*, *Hunsarg* (1996).

The definitions presented were similar to those in *Weiss*, *Infante* (1965, 1967) clearly adapted to time delay systems.

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It should be noticed that these definitions are significantly different from the definitions presented earlier, by the first named author of this article.

In the context of finite time and practical stability for linear continuous time delay systems, various results were presented in *Debeljkovic et al* (1997.a, 1997.b) and *Nenadic et al* (1997).

In *Debeljkovic et al* (1997.a) and *Nenadic et al* (1997) some basic results from the area of finite time and practical stability were extended to a particular class of linear continuous time delay systems.

Stability sufficient conditions dependent on delay expressed in terms of time delay fundamental system matrix have been derived.

Also, in certain circumstances when it was possible to establish the suitable connection between the fundamental matrices of linear time delay and the non-delay systems, the presented results enable an efficient procedure for testing the practical as well as the finite time stability of time delay systems.

For the analysis of the practical and finite time stability of linear time delayed systems, the matrix measure approach has been applied in *Debeljkovic et al* (1998, 1999).

Based on the Coppel's inequality, the matrix measure approach was introduced. It provides simple delay-dependent sufficient conditions of the practical and finite time stability. In the presented method there was no need for time delay fundamental matrix calculations.

In *Debeljkovic et al* (1997.b) this problem has been solved for forced time delay systems.

Another approach, based on the known *Bellman-Gronwall* lemma, was applied in *Debeljkovic et al* (1998). The method provided new, more efficient sufficient delay-dependent conditions for checking finite and practical stability of continuous systems with state delay.

The collection of all previous results and contributions was presented in *Debeljkovic et al* (1999) with the overall comments and a slightly modified *Bellman-Gronwall* approach.

Some of the initial results completely based on the discrete fundamental matrix of the system have been published in *Debeljkovic*, *Aleksendric* (2003). It is known that computing the discrete fundamental matrix is sometimes more difficult than to find the solution of the system of retarded difference equations.

The reported results in *Debeljkovic*, *Aleksendric* (2003) represented the extension of the concept of finite time and practical stability to the class of linear discrete time delayed systems for the first time. In discrete time delay systems, time delay can cause complicated problems in systems dynamics and in developing stability criteria as well.

In order to get a better understanding of the described discrete systems, short recapitulations and some results derived for ordinary discrete time delayed systems have been presented in the sequel.

System Description

A linear discrete system with state delay was considered. The system is described by

$$\mathbf{x}(k+1) = A_0\,\mathbf{x}(k) + A_1\,\mathbf{x}(k-1), \quad (1.a)$$

with a known vector valued function of the initial conditions:

$$\mathbf{x}(k_0) = \boldsymbol{\psi}(k_0), \quad -1 \le k_0 \le 0\,, \qquad (1.b)$$

where $\mathbf{x}(k) \in Z^n$ is a state vector and constant matrices A_0 and A_1 of the appropriate dimensions.

The time delay is constant and equal to one.

It is also assumed that (1.a) satisfies the adequate smoothness requirements. Consequently, the solution of (1.a) exists and is unique and continuous, with respect to k and the initial data. The solution is bounded for all bounded values of its arguments.

Let Z^n denote the state space of the systems given by (1) and $\|(\cdot)\|$ the Euclidean norm.

The solutions of (1) are denoted by:

$$\mathbf{x}(k, k_0, \mathbf{x}_0) \equiv \mathbf{x}(k). \qquad (2)$$

The discrete–time interval is denoted by K_N, as a set of non–negative integers:

$$\mathcal{K}_N = \{\, k : k_0 \le k \le k_0 + k_N \,\}\,. \qquad (3)$$

The quantity k_N can be a positive integer or the symbol $+\infty$, so the finite time stability and practical stability can be treated simultaneously.

Let V: $K_N \times Z^n \to Z$, so that $V(k, \mathbf{x})$ is bounded and for which $\|(\mathbf{x})\|$ is also bounded.

The total difference $\Delta V(k, \mathbf{x}(k))$ was defined along the trajectory of systems (1) as:

$$\Delta V(k,\mathbf{x}(k)) = V(k+1,\mathbf{x}(k+1)) - V(k,\mathbf{x}(k))\,. \qquad (4)$$

For time–invariant sets it is assumed that $\mathcal{S}_{(\,)}$ is a bounded open set.

Let $\mathcal{S}_\beta$ be a given set of all allowable states of the system $\forall k \in \mathcal{K}_N$. Set $\mathcal{S}_\alpha$, $\mathcal{S}_\alpha \subset \mathcal{S}_\beta$ denotes the set of all allowable initial states. Sets $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ are connected and a priori known. $\lambda(\,)$ denotes the eigenvalues of matrix. $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues, respectively.

Motivation

In the following part we have presented two different approaches to the problem of discrete time delay systems. Namely, the first result is expressed directly in terms of eigenvalues of the basic system matrices A_0 and A_1 naturally occurring in the system model.

The approach avoids the need to introduce any canonical form or transformation into the statement of the following theorems. In the second case, the geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all the solutions. This fact makes the construction of the Lyapunov and non-Lyapunov stability theory possible even for the linear continuous time delay systems in sense that the attractive property is equivalent to the existence of symmetric, positive definite solutions to a general form of the Lyapunov matrix equation.

The conditions which refer to the boundedness of solutions are incorporated into the solution.

The first method is based on a classical approach mostly used in deriving sufficient delay independent conditions for the finite time stability. In the second case a new definition is introduced, based on the attractivity properties of the system solution, which can be treated as analogous to the quasi-contractive stability as in *Weiss*, *Infante* (1965, 1967)

Moreover, a new delay dependent sufficient condition has been derived to guarantee that the system under consideration will be practically stable with the attractivity properties of its solution.

This approach can be treated as a new concept of the so-called non-Lyapunov stability. Investigating the system stability throughout the discrete fundamental matrix is cumbersome, so there is a need to find some more efficient expressions that should be based on the calculation of appropriate eigenvalues or norms of appropriate systems matrices.

The solution for this problem was proposed in this article.

Practical stability and instability

As far as we know the only result, considering and investigating the problem of non–Lyapunov analysis of linear discrete time delay systems, is one that has been mentioned in the introduction, e.g. *Debeljković*, *Aleksendrić* (2003) where this problem has been considered for the first time.

Investigating system stability throughout the discrete fundamental matrix is very cumbersome, so there is need to find some more efficient expressions that should be based on calculating appropriate eigenvalues or norms of appropriate systems matrices as it has been done in a continuous case.

Definition 1. System (1) is *attractively practically stable* with respect to $\left\{k_0, \mathcal{K}_N, \mathcal{S}_\alpha, \mathcal{S}_\beta\right\}$, $\alpha < \beta$, if and only if $\left\|\mathbf{x}(k_0)\right\|^2_{A_0^T P A_0} = \left\|\mathbf{x}_0\right\|^2_{A_0^T P A_0} < \alpha$, implies $\left\|\mathbf{x}(k)\right\|^2_{A_0^T P A_0} < \beta$, $\forall k \in \mathcal{K}_N$, with the property that: $\lim_{k\to\infty} \left\|\mathbf{x}(k)\right\|^2_{A_0^T P A_0} \to 0$.

Definition 2. System (1) is *practically stable* with respect to $\left\{k_0, \mathcal{K}_N, \mathcal{S}_\alpha, \mathcal{S}_\beta\right\}$, if and only if $\left\|\mathbf{x}_0\right\|^2 < \alpha$, implies $\left\|\mathbf{x}(k)\right\|^2 < \beta$, $\forall k \in \mathcal{K}_N$.

Definition 3. System (1) is *attractively practically unstable* with respect to $\left\{k_0, \mathcal{K}_N, \alpha, \beta, \left\|(\cdot)\right\|^2\right\}$, $\alpha < \beta$, if and only if for $\left\|\mathbf{x}_0\right\|^2_{A_0^T P A_0} < \alpha$, there exists a moment $k = k^* \in \mathcal{K}_N$, so that the condition $\left\|\mathbf{x}(k^*)\right\|^2_{A_0^T P A_0} \geq \beta$ is fulfilled with the property $\lim_{k\to\infty} \left\|\mathbf{x}(k)\right\|^2_{A_0^T P A_0} \to 0$.

Definition 4. System (1) is *practically unstable* with respect to $\left\{k_0, \mathcal{K}_N, \alpha, \beta, \left\|(\cdot)\right\|^2\right\}$, $\alpha < \beta$, if and only if for $\left\|\mathbf{x}_0\right\|^2 < \alpha$, there exists a moment $k = k^* \in \mathcal{K}_N$, such that the condition $\left\|\mathbf{x}(k^*)\right\|^2 \geq \beta$ is fulfilled for some $k = k^* \in \mathcal{K}_N$.

Definition 5. Linear discrete time delay system (1.a) is *finite time stable* with respect to $\left\{\alpha, \beta, k_0, k_N, \left\|(\cdot)\right\|\right\}$, $\alpha \leq \beta$, if every trajectory $\mathbf{x}(k)$ satisfies the initial function given by (1.b) such that $\left\|\mathbf{x}(k)\right\| < \alpha$, $k = 0, -1, -2, \cdots, -N$ which implies $\left\|\mathbf{x}(k)\right\|^2 < \beta$, $k \in K_N$, *Debeljković*, *Aleksendrić* (2003).

This Definition is analogous to the one presented in *Debeljkovic et al* (1997.a, 1997.b) and *Nenadic et al* (1997).

Some previous results

Theorem 1. For linear discrete time delay system (1) to be *finite time stable* with respect to $\left\{\alpha, \beta, M, N, \left\|(\cdot)\right\|^2\right\}$, $\alpha < \beta$, $\alpha, \beta \in Z_+$, it is sufficient that:

$$\left\|\Phi(k)\right\| < \frac{\beta}{\alpha} \cdot \frac{1}{1 + \sum_{j=1}^{M} \left\|A_j\right\|}, \quad \forall k = 0, 1, \cdots, N, \tag{5}$$

Debeljković, *Aleksendrić* (2003).

Remark 1. The matrix measure is widely used when continuous time delay systems are investigated.

The nature of discrete time delay enables using the presented approach as well as the Bellman's principle.

The solution of the stability was based only on matrix norms calculation.

This result is analogous to the one derived in *Debeljkovic et al* (1997.a), for continuous time delay systems.

Theorem 2. System (1), with $\det A_1 \neq 0$, is *attractively practically stable* with respect to $\left\{k_0, K_N, \alpha, \beta, \left\|(\cdot)\right\|^2\right\}$, $\alpha < \beta$, if there exists $P = P^T > 0$, which is the solution of:

$$2 A_0^T P A_0 - P = -Q, \tag{6}$$

where $Q = Q^T > 0$ and if the following conditions are satisfied:

$$\left\|A_1\right\| < \sigma_{\min}\left(\left(Q - A_1^T P A_1\right)^{-\frac{1}{2}}\right) \sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}} A_0^T P\right), \tag{7}$$

$$\bar{\lambda}_{\max}^{\frac{1}{2}k}(\ ) < \frac{\beta}{\alpha}, \quad \forall k \in K_N, \tag{8}$$

where:

$$\bar{\lambda}_{\max}(\ ) = \max\left\{\mathbf{x}^T(k) A_1^T P A_1 \mathbf{x}(k) : \mathbf{x}^T(k) A_0^T P A_0 \mathbf{x}(k) = 1\right\}. \tag{9}$$

Proof. The following function was used, as a possible aggregation function for the system:

$$V(\mathbf{x}(k)) = \mathbf{x}^T(k) P \mathbf{x}(k) + \mathbf{x}^T(k-1) Q \mathbf{x}(k-1), \tag{10}$$

with the matrices $P = P^T > 0$ and $Q = Q^T > 0$.

The rest of the proof is omitted here for the sake of brevity and can be found in *Debeljkovic* (2011). **Q.E.D.**

Remark 2. The assumption $\det A_1 \neq 0$ does not reduce the generality of this result, since this condition is not crucial when discrete time systems are considered.

Remark 3. The Lyapunov asymptotic stability and the finite time stability are independent concepts: a system that is finite time stable may not be Lyapunov asymptotically stable, conversely, a Lyapunov asymptotically stable system could not be finite time stable, if its motion exceeds the pre-specified bounds β during the transients.

The attractivity property is guaranteed by (6) and (7) and the system motion within pre-specified boundaries is guaranteed by condition (8).

Remark 4. For the numerical treatment of this problem $\bar{\lambda}_{\max}(\ )$ can be calculated in the following way, *Kalman, Bertram* (1960):

$$\bar{\lambda}_{\max}(\ ) = \max_{\mathbf{x}}\{\ \} = \bar{\lambda}_{\max}\left(A_1^T P A_1\left(A_0^T P A_0\right)^{-1}\right), \quad (11)$$

Remark 5. These results are in some sense analogous to those given in *Amato et al* (2003), although the results presented there have been derived for continuous time varying systems.

In the following part the delay independent criteria for the finite time stability are developed.

Reducing the demand that a basic system matrix should be a discrete stable matrix, a system does not need to be necessarily asymptotically stable.

Theorem 3. Suppose that the matrix A_1 fulfills $\left(I - A_1^T A_1\right) > 0$. The system given by (1) is *finite time stable* with respect to $\left\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\right\}$, $\alpha < \beta$, if there exists a positive real number $p, p>1$, such that

$$\left\|\mathbf{x}(k-1)\right\|^2 < p^2\left\|\mathbf{x}(k)\right\|^2, \ \forall t \in \Im, \ \forall \mathbf{x}(k) \in \mathcal{S}_\beta, \quad (12)$$

and if the following condition is satisfied

$$\lambda_{\max}^k(\ ) < \frac{\beta}{\alpha}, \quad \forall k \in K_N, \quad (13)$$

where:

$$\lambda_{\max}(\ ) = \lambda_{\max}\left(A_0^T A_1\left(I - A_1^T A_1\right)A_1^T A_0 + p^2 I\right). \quad (14)$$

Proof. System (1) was analyzed.

A function is defined:

$$V\left(\mathbf{x}(k)\right) = \mathbf{x}^T(k)\mathbf{x}(k) + \mathbf{x}^T(k-1)\mathbf{x}(k-1), \quad (15)$$

as a tentative Lyapunov–like function for the system under consideration.

The rest of the proof is omitted here for the sake of brevity and can be found in *Debeljkovic* (2011). **Q.E.D.**

Remark 6. In the case when A_1 is the null matrix result, given by (14), it reduces to that given in Debeljkovic (2001) developed for ordinary discrete time systems.

VI MAIN RESULTS

Before presenting our crucial result, we need some preliminaries, discussions and explanations, as well some additional results

The characteristic polynomial of system (1) is given by:

$$f(\lambda) \triangleq \det M(\lambda) = \sum_{j=0}^{n(h+1)} a_j \lambda^j, \ a_j \in \mathbb{R}, \qquad M(\lambda) = I_n \lambda^{h+1} - A_0\lambda^h - A_1 \quad (16)$$

Denote:

$$\Omega \triangleq \left\{\lambda \mid f(\lambda) = 0\right\} = \lambda\left(A_{eq}\right) \quad (17)$$

the set of all characteristic roots of system (1).

The number of these roots amounts to $n(h+1)$.

A root λ_m of Ω with a maximum module:

$$\lambda_m \in \Omega: \left|\lambda_m\right| = \max\left|\lambda\left(A_{eq}\right)\right| \quad (18)$$

[1]

let us call it a maximum root (eigenvalue).

If the scalar variable λ in the characteristic polynomial is replaced by the matrix $X \in \mathbb{C}^{n\times n}$ the two following monic matrix polynomials are obtained:

$$M(X) = X^{h+1} - A_0 X^h - A_1 \quad (19)$$

$$F(X) = X^{h+1} - X^h A_0 - A_1 \quad (20)$$

It is obvious that $F(\lambda) = M(\lambda)$.

A matrix $S \in \mathbb{C}^{n\times n}$ is a *right solvent* of $M(X)$, *Dennis et al.*, (1976) if:

$$M(S) = 0 \quad (21)$$

If :

$$F(R) = 0 \quad (22)$$

then $R \in \square^{n\times n}$ is a *left solvent* of $M(X)$, *Dennis et al.* (1976).

We will further use *S* to denote the right solvent and R to denote the left solvent of $M(X)$.

In the present paper the majority of presented results start from the left solvents of $M(X)$.

In contrast, in the existing literature the right solvents of $M(X)$ were mainly studied.

The mentioned discrepancy can be overcome by the following Lemma.

Lemma 1. The conjugate transpose value of the left solvent of $M(X)$ is also, at the same time, the right solvent of the following matrix polynomial:

$$M_T(X) = X^{h+1} - A_0^T X^h - A_1^T \quad (23)$$

Proof. Let *R* be the right solvent of $M(X)$.

Then it holds:

$$M_T\left(R^*\right) = \left(R^*\right)^{h+1} - A_0^T\left(R^*\right)^h - A_1^T = \left(R^{h+1} - R^h A_0 - A_1\right)^* = F^*(R) = 0 \quad (24)$$

so R^* is the right solvent of $M_T(X)$.**Q.E.D.**

Conclusion 1. Based on *Lemma* 1, all characteristics of

[1] See Appendix B.

the left solvents of $M(X)$ can be obtained by the analysis of the conjugate transpose value of the right solvents of $M_T(X)$.

The following proposed factorization of the matrix $M(\lambda)$ will help us to understand better the relationship between the eigenvalues of left and right solvents and roots of the system.

Lemma 2. The matrix $M(\lambda)$ can be factorized in the following way:

$$\begin{aligned} M(\lambda) &= \left(\lambda^h I_n + (S-A_0)\sum_{i=1}^{h}\lambda^{h-i}S^{i-1}\right)(\lambda I_n - S) \\ &= (\lambda I_n - R)\left(\lambda^h I_n + \sum_{i=1}^{h}\lambda^{h-i}R^{i-1}(R-A_0)\right) \end{aligned} \tag{25}$$

Proof.

$$\begin{aligned} M(\lambda)-M(X) &= \lambda^{h+1}I_n - X^{h+1} - A_0\left(\lambda^h I_n - X^h\right) = \\ &= \left(\sum_{i=0}^{h}\lambda^{h-i}X^i - A_0\sum_{i=0}^{h-1}\lambda^{h-1-i}X^i\right)(\lambda I_n - X) \end{aligned} \tag{26}$$

If S is a right solvent of $M(X)$, from (26) follows (25).

Similarly, if R is a left solvent of $M(X)$, from:

$$M(\lambda)-F(X) = (\lambda I_n - X)\left(\lambda^h I_n + \sum_{i=1}^{h}\lambda^{h-i}X^{i-1}(X-A_0)\right)$$

polynomial $f(\lambda)$ is an *annihilating polynomial* for the right and left solvents of $M(X)$.

The eigenvalues and eigenvectors of the matrix have a crucial influence on the existence, enumeration and characterization of solvents of the matrix equation (21), *Dennis et al*., (1976) and *Pereira* (2003).

Definition 6. Let $M(\lambda)$ be a matrix polynomial in λ.

If $\lambda_i \in \mathbb{C}$ is such that $\det\left(M(\lambda_i)\right)=0$, then we say that λ_i is a *latent root* or an *eigenvalue* of $M(\lambda)$. If a nonzero $\mathbf{v}_i \in \mathbb{C}^n$ is such that:

$$M(\lambda_i)\mathbf{v}_i = \mathbf{0} \tag{28}$$

then we say that $\mathbf{v}_i$ is a (right) *latent vector* or a (right) *eigenvector* of $M(\lambda)$, corresponding to the eigenvalue λ_i, *Dennis et al*., (1976) and *Pereira* (2003).

The eigenvalues of the matrix $M(\lambda)$ correspond to the characteristic roots of the system, i.e. eigenvalues of its block companion matrix A_{eq} *Dennis et al*., (1976)[2].

Their number is $n(h+1)$.

Since $F^*(\lambda) = M_T(\lambda^*)$ holds, it is not difficult to show that the matrices $M(\lambda)$ and $M_T(\lambda)$ have the same spectrum.

In the papers *Dennis et al*., (1976, 1978), *Kim* (2000), *Pereira* (2003) and *Lancaster*, *Tismenetsky*, (1985) some sufficient conditions for the existence, enumeration and characterization of right solvents of $M(X)$ were derived.

They show that the number of solvents can be *zero*, *finite* or *infinite*.

For the needs of system stability (1) only the so-called maximum solvents are usable, the spectrums of which contain the maximum eigenvalue λ_m.

A special case of the maximum solvent is the so-called dominant solvent *Dennis et al*., (1978) and *Kim* (2000), which, unlike maximum solvents, can be computed in a simple way.

Definition 7. Every solvent S_m of $M(X)$, whose spectrum $\sigma(S_m)$ contains the maximum eigenvalue λ_m of Ω is a *maximum solvent*.

Definition 8. The matrix A dominates the matrix B if all the eigenvalues of A are greater, in modulus, than those of B.

In particular, if the solvent S_1 of $M(X)$ dominates the solvents $S_2,\ldots,S_l$ we say it is a *dominant solvent*, *Dennis et al*., (1978) and *Kim* (2000).

(Note that a dominant solvent cannot be singular.)

Conclusion 3. The number of maximum solvents can be greater than one.

The dominant solvent is at the same time the maximum solvent too.

The dominant solvent S_1 of $M(X)$, under certain conditions, can be determined by the *Traub iteration Dennis et al*., (1978) and *Bernoulli iteration Dennis et al*., (1978) and *Kim* (2000).

The necessary and sufficient conditions for the asymptotic stability of linear discrete time-delay systems (1) are given with the following result.

Theorem 4. Suppose that there exists at least one left solvent of $M(X)$ and let R_m denote one of them. Then, linear discrete time delay system (1) is *asymptotically stable* if and only if for any matrix $Q = Q^* > 0$ there exists *Hermitian* matrix $P = P^* > 0$ such that:

$$R_m^* P R_m - P = -Q \tag{29}$$

Proof. (SuC) Define the following vector discrete functions:

$$\mathbf{x}_k = \mathbf{x}(k+\vartheta),\quad \vartheta \in \{-h, -h+1, \ldots, 0\} \tag{30}$$

$$\mathbf{z}(\mathbf{x}_k) = \mathbf{x}(k) + \sum_{j=1}^{h}\Xi(j)\mathbf{x}(k-j) \tag{31}$$

where, $\Xi(k) \in \mathbb{C}^{n\times n}$ is, in general, some time varying discrete matrix function.

The conclusion of the theorem follows immediately by defining the Lyapunov functional for the system (1), as:

$$V(\mathbf{x}_k) = \mathbf{z}^*(\mathbf{x}_k)P\mathbf{z}(\mathbf{x}_k),\ P = P^* > 0 \tag{32}$$

It is obvious that $\mathbf{z}(\mathbf{x}_k) = \mathbf{0}$ if and only if $\mathbf{x}_k = \mathbf{0}$, so it follows that $V(\mathbf{x}_k) > 0$ for $\forall \mathbf{x}_k \neq \mathbf{0}$.

The forward difference of (32), along the solutions of system (1) is:

[2] See The Appendix B.

$$\begin{aligned}\Delta V(\mathbf{x}_k) &= \Delta\mathbf{z}^*(\mathbf{x}_k)P\mathbf{z}(k)\\ &+\mathbf{z}^*(\mathbf{x}_k)P\Delta\mathbf{z}(\mathbf{x}_k)+\Delta\mathbf{z}^*(\mathbf{x}_k)P\Delta\mathbf{z}(\mathbf{x}_k)\end{aligned} \tag{33}$$

The difference of $\Delta\mathbf{z}(\mathbf{x}_k)$ can be determined in the following manner:

$$\Delta\mathbf{z}(\mathbf{x}_k)=\Delta\mathbf{x}(k)+\sum_{j=1}^{h}\Xi(j)\Delta\mathbf{x}(k-j) \tag{34}$$

with:

$$\Delta x(k)=(A_0-I_n)x(k)+A_1x(k-h) \tag{35}$$

and:

$$\begin{aligned}&\sum_{j=1}^{h}\Xi(j)\Delta\mathbf{x}(k-j)=\Xi(1)\left[\mathbf{x}(k)-\mathbf{x}(k-1)\right]+\cdots\\ &+\Xi(h)\left[\mathbf{x}(k-h+1)-\mathbf{x}(k-h)\right]\end{aligned} \tag{36}$$

Then simple manipulations lead to:

$$\begin{aligned}&\sum_{j=1}^{h}(s)(j)\Delta\mathbf{x}(k-j)=\Xi(1)\mathbf{x}(k)-\Xi(h)\mathbf{x}(k-h)\\ &+\left(\Xi(2)-\Xi(1)\right)\mathbf{x}(k-1)+\cdots\\ &+\left(\Xi(h)-\Xi(h-1)\right)\mathbf{x}(k-h+1)\end{aligned} \tag{37}$$

Define a new matrix Π by:

$$\Pi = A_0+\Xi(1) \tag{38}$$

If:

$$\Delta\Xi(h)=A_1-\Xi(h) \tag{39}$$

then $\Delta\mathbf{z}(\mathbf{x}_k)$ has a form:

$$\Delta\mathbf{z}(\mathbf{x}_k)=(\Pi-I_n)\mathbf{x}(k)+\sum_{j=1}^{h}\Delta\Xi(j)\cdot\mathbf{x}(k-j) \tag{40}$$

If one adopts:

$$\Delta\Xi(j)=(\Pi-I_n)\Xi(j),\quad j=1,2,\dots,h\,, \tag{41}$$

then $\Delta\mathbf{z}(\mathbf{x}_k)$ becomes:

$$\Delta\mathbf{z}(\mathbf{x}_k)=(\Pi-I_n)\mathbf{z}(\mathbf{x}_k)\,. \tag{42}$$

Therefore, (33) becomes:

$$\Delta V(\mathbf{x}_k)=\mathbf{z}^*(\mathbf{x}_k)\left(\Pi^*P\Pi-P\right)\mathbf{z}(x_k) \tag{43}$$

It is obvious that if the following equation is satisfied:

$$\Pi^*P\Pi-P=-Q,\qquad Q=Q^*>0 \tag{44}$$

then $\Delta V(\mathbf{x}_k)<0,\ \mathbf{x}_k\neq\mathbf{0}$.

In the Lyapunov matrix equation (44), of all possible solvents R of $M(X)$, only one of the maximum solvents is of importance, for it is the only one that contains the maximum eigenvalue $\lambda_m\in\Omega$ (*Conclusion* 2), which has dominant influence on the stability of the system.

So, (29) represents the stability *sufficient condition* (SuC) for the system given by (1).

The matrix $\Xi(1)$ can be determined in the following way.

From (41) follows:

$$\Xi(h+1)=R^h\,\Xi(1)\,, \tag{45}$$

and using (38) and (39) one can get (22), and for the sake of brevity, instead of the matrix $\Xi(1)$, one introduces a simple notation Ξ.

If a solvent which is not maximal is integrated into the Lyapunov equation, it may happen that there is a positive definite solution of Lyapunov matrix equation (29), although the system is not stable.

Conversely, if system (1) is asymptotically stable then all roots $\lambda_i\in\Omega$ are located within the unit circle. Since $\sigma(R_m)\subset\Omega$, $\rho(R_m)<1$ follows, so the positive definite solution of Lyapunov matrix equation (29) exists (*necessary condition* NcC). **Q.E.D.**

Theorem 5. Suppose that there exists at least one left solvent of $M(X)$ and let R_m denote one of them.

Then, linear discrete time delay system (1), with $\det A_1\neq 0$, is *attractively practically stable* with respect to $\left\{\alpha,\beta,M,N,\|(\cdot)\|^2\right\}$, $\alpha<\beta$, $\alpha,\ \beta\in Z_+$, if for any matrix $Q=Q^*>0$ there exists the *Hermitian* matrix $P=P^*>0$ such that:

$$R_m^*PR_m-P=-Q \tag{46}$$

as well as the following condition is satisfied:

$$\|\Phi(k)\|<\frac{\beta}{\alpha}\cdot\frac{1}{1+\sum_{j=1}^{M}\|A_j\|},\ \forall k=0,1,\cdots,N\,, \tag{47}$$

Proof. The proof is more than obvious and directly follows from the fact that the attractivity property is guaranteed by (46), *Stojanovic*, *Debeljkovic* (2008) and finite time stability by (47), *Debeljkovic*, *Aleksendric* (2003).

Theorem 6. Suppose that the matrix A_1 fulfills $\left(I-A_1^TA_1\right)>0$. A system given by (1), is *practically unstable* with respect to $\left\{k_0,K_N,\alpha,\beta,\|(\cdot)\|^2\right\}$, $\alpha<\beta$, if there exists a positive real number p, p>1, such that:

$$\|\mathbf{x}(k-1)\|^2<\wp^2\|\mathbf{x}(k)\|^2,\ \forall k\in\mathcal{K},\ \forall\mathbf{x}(k)\in\mathcal{S}_\beta\,, \tag{48}$$

if there exists a real, positive number δ, $\delta\in\,]0,\alpha[$ and time instant k, $k=k^*:\exists!\left(k^*>k_0\right)\in K_N$ for which the next condition is fulfilled:

$$\lambda_{\min}^{k^*}>\frac{\beta}{\delta},\quad k^*\in K_N\ . \tag{49}$$

Proof. Let

$$V(\mathbf{x}(k))=\mathbf{x}^T(k)\mathbf{x}(k)+\mathbf{x}^T(k-1)\mathbf{x}(k-1)\,. \tag{50}$$

Following the identical procedure as in *Theorem 3*, one can get:

$$\ln \mathbf{x}^T(k+1)\mathbf{x}(k+1) - \ln \mathbf{x}^T(k)\mathbf{x}(k) > \ln \lambda_{\min}(\ ), \quad (51)$$

where:

$$\lambda_{\min}(\ ) = \lambda_{\min}\left(A_0^T A_1 \left(I - A_1 A_1^T\right)^{-1} A_1^T A_0 + p^2 I \right). \quad (52)$$

If the summing $\sum_{j=k_0}^{k_0+k-1}$ is applied on both sides of (51) for $\forall k \in K_N$, one can obtain:

$$\begin{aligned} &\ln \mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k) \geq \quad \forall k \in K_N \\ &\geq \ln \prod_{j=k_0}^{k_0+k-1} \lambda_{\max}(\ ) \geq \ln \lambda_{\max}^{k}(\ ) + \ln \mathbf{x}^T(k_0)\mathbf{x}(k_0) \end{aligned} \quad (53)$$

It is clear that for any $\mathbf{x}_0$, $\delta < \|\mathbf{x}_0\|^2 < \alpha$ follows and for some $k^* \in K_N$, taking into account the basic condition of *Theorem* 4, eq. (45), it can be concluded:

$$\begin{aligned} &\ln \mathbf{x}^T(k_0+k^*)\mathbf{x}(k_0+k^*) > \\ &> \ln \lambda_{\min}^{k^*}(A_0, A_1, \varphi(t)) + \ln \mathbf{x}^T(k_0)\mathbf{x}(k_0) > \\ &> \ln \delta \cdot \lambda_{\min}^{k^*}(\ ) > \ln \delta \cdot \frac{\beta}{\delta} > \ln \beta, \quad \text{for some } k^* \in K_N \end{aligned} \quad (54)$$

Q.E.D.

Conclusion

New definitions and theorems have been established and proved for a particular class of the discrete time delay systems.

The conditions guarantee the practical attractivity and only practical stability within the pre-specified time-invariant sets in the state space.

Moreover, based on a classical definition, new theorems have been derived for the so-called finite time stability as well as the corresponding results for discrete time delay systems, to the ones given in *Debeljkovic et al* (2010) and *Debeljkovic*, *Nestorovic* (2010).

It is necessary to underline the difference between *Theorem* 5 and all the others.

The former belongs to the class of so-called time delay dependent conditions and all the others to the criteria which do not include the value of time delay in the final result.

The later are easier to apply for technical purposes.

The system instability was analyzed as well.

APPENDIX A - Notation

$\mathbb{R}$	Real vector space
$\mathbb{T}^+$	All the non-negative integers
$\mathbb{C}$	Complex vector space
λ^*	Conjugate of $\lambda \in \mathbb{C}$
F^*	Conjugate transpose of matrix $F \in \mathbb{C}^{n\times n}$
$F > 0$	Positive definite matrix
$\det(F)$	Determinant of matrix F
$\lambda_i(F)$	Eigenvalue of matrix F
$\lambda(F)$	$\{\lambda \mid \det(F - \lambda I) = 0\}$
$\sigma(F)$	Spectrum of matrix F
$\rho(F)$	Spectral radius of matrix F

APPENDIX B

System can be expressed with the following representation without delay, *Mori et al.,* (1982), *Malek - Zavarei*, *Jamshidi*, (1978) and *Gorecki et al*., (1989).

$$\mathbf{x}_{eq}(k) = \left[\mathbf{x}^T(k-h)\ \mathbf{x}^T(k-h+1)\ \cdots\ \mathbf{x}^T(k)\right] \in \mathbb{R}^N$$

$$\mathbf{x}_{eq}(k+1) = A_{eq}\mathbf{x}_{eq}(k), \quad N \hat{=} n(h+1) \quad (B.1)$$

$$A_{eq} = \begin{pmatrix} 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_n \\ A_1 & 0 & \cdots & A_0 \end{pmatrix} \in \mathbb{R}^{N\times N}$$

The system defined by (B.1) is called the equivalent system, while the matrix A_{eq}, the matrix of equivalent system.

APPENDIX C

Linear discrete time systems – Chronological overview of basic results

A specific concept of discrete time systems, practical stability operating on the finite time interval, was investigated by *Hurt* (1967) with a particular emphasis on the possibilities of error arising in the numerical treatment of results.

The finite time stability concept was, for the first time, extended to discrete time systems by *Michel and Wu* (1969).

Practical stability or "*set stability*", throughout estimation system trajectory behavior on the finite time interval was given by *Heinen* (1970, 1971). He was the first to give necessary and sufficient conditions for this concept of stability, using the Lyapunov approach based on the "*discrete Lyapunov functions*" application.

Even more detailed analysis of these results considering different aspects of discrete time systems practical stability as well as the questions of their realization and controllability was given by *Weiss* (1972). The same problems were treated by *Weiss and Lam* (1973), who extended them to the class of nonlinear complex discrete systems.

Efficient sufficient conditions of finite time stability of linear discrete time systems expressed through norms and/or matrices were derived by *Weiss and Lee* (1971).

Lam and Weiss (1974) were the first who applied the so–called concept of "*final stability*" on discrete time systems whose motions are scrolled within the time varying sets in the state space.

Some simple definitions connected to sets representing difference equations or at the same time discrete time systems were given by *Shanholt* (1974).

Only the sufficient conditions are given by the established theorems. These results are based on the Lyapunov stability and can be used, in a way, for a finite time stability concept, for which reason they are mentioned here.

Grippo, *Lampariello* (1976) have generalized all foregoing results and have given the necessary and sufficient conditions of different concepts of finite time stability inspired by the definitions of practical stability and instability, earlier introduced by *Heinen* (1970).

The same authors applied the before–mentioned results in the analysis of "*large–scale systems*", *Grippo*, *Lampariello* (1978).

Practical stability with settling time was for the first time introduced by *Debeljković* (1979.a) in connection with the analysis of different classes of linear discrete time systems, general enough to include time invariant and time varying systems, systems operated in free or forced operating regimes, as well as the systems whose dynamical behavior is expressed through the so–called "*functional system matrix*". In the mentioned paper, the sufficient conditions of practical instability and a discrete version of a very well known *Bellman–Gronwall* lemma have also been derived.

Other papers, *Debeljković* (1979.b, 1980.a, 1980.b, 1983) deal with the same problems and mostly represent the basic results of the PhD dissertation, *Debeljković* (1979.a).

For the particular class of discrete time systems with the functional system matrix, sufficient conditions have been derived in *Debeljković* (1993).

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Stabilnost linearnih diskretnih sistema sa čistim vremenskim kašnjenjem na konačnom vremenskom intervalu: Pregled rezultata

U ovom radu su izvedeni, dovoljni uslovi praktične stabilnosti i stabilnosti na konačnom vremenskom intervalu posebne klase linearnih dikretnih sistema sa čistim vremenskim kašnjenjem. Analizirajući koncept stabilnosti na konačnom vremenskom intervalu, ovi novi od čisto vremenskog kašnjenja nezavisni uslovi, dobijeni su prilazom koji počiva na korišćenju kvazi Ljapunovljevih funkcija, koje ne moraju da budu određene po znaku, kao ni njihovi izvodi.

Takođe, je razmatran koncept praktične stabilnosti, a po prvi put, i koncept atraktivne praktične stabilnosti. Pomenuti prilaz, jasno, oslanja se u velikoj meri na klasičnu Ljapunovljevu tehniku, kako bi se garantovala globalna osobina privlačenja kretanja sistema.

Ključne reči: **linearni sistemi, diskretni sistemi, stabilnost sistema, sistem sa kašnjenjem, sistem na konačnom vremenskom intervalu, Neljapunovska stabilnost.**

Устойчивость линейных дискретных систем с чистым временем задержки в конечном интервале времени: Обзор результатов

В данной работе получены достаточные условия практической устойчивости и устойчивости на конечном временном интервале особого класса линейных дискретных систем с чистым временем задержки. Анализируя понятие и концепции устойчивости на конечном временном интервале, эти новые независимые условия с чистым временем задержки получены путём подхода обоснованного на использовании Ляпунова-подобные функций, которые не должны определяться знаком, вместе с их функционированием. Также исследована концепция практической устойчивости, и в первый раз, концепция привлекательной и практической устойчивости.Вышеупомянутый подход, в значительной степени ясно опирается на классическую технику Ляпунова, для того, чтобы гарантировать глобальные свойства привлекательности движения системы.

Ключевые слова: **Линейные системы, дискретные системы, устойчивость системы, системные задержки, системы на конечном временном интервале, устойчивость не-Ляпунова.**

Stabilité des systèmes linéaires discrets à délai temporel sur l'intervalle temporelle finie : tableaux des résultats

Dans ce papier on présente les conditions suffisantes pour la stabilité pratique et la stabilité sur l'intervalle temporelle finie de classe particulière des systèmes linéaires discrets à pur délai temporel. Analysant le concept de stabilité sur l'intervalle temporelle finie, ces nouvelles conditions, indépendantes du délai temporel pur, ont été réalisées par l'approche basée sur les quasi fonctions de Lyapunov .On a étudié également le concept de la stabilité pratique et, pour la première fois, le concept de la stabilité pratique attractive. Evidement, l'approche citée repose en grande partie sur la technique classique de Lyapunov pour garantir la propriété globale de l'attraction du mouvement du système.

Mots clés: **système linéaire, système discret, stabilité de système, système à délai, système sur l'intervalle temporelle finie, stabilité de non Lyapunov.**