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The Stability of Linear Discrete Time Delay Systems in the Sense of Lyapunov: An Overview

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This paper gives a detailed an overview of the work and the results of many authors in the area of Lyapunov stability of particular class of linear discrete time delay systems. In that sense the discrete Lyapunov equation for discrete implicit systems is of particular interest.
The stability robustness problem has been also treated.
This survey covers the period since 2002 up to nowadays and has strong intention to present the main concepts and contributions that have been derived during the mentioned period in the whole world, published in the respectable international journals or presented at workshops or prestigious conferences.

Key words: **linear system, discrete system, system stability, time delay system, Lyapunov stability, asymptotic stability**.

Introduction

THE problem of investigation of time delay systems has been exploited over many years.

Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc.

The existence of pure time lag, regardless of whether it is present in the control or/and the state, may cause undesirable system transient response, or even instability. Consequently, the problem of a stability analysis for this class of systems has been one of the main interests for many researchers. In general, the introduction of time delay factors makes the analysis much more complicated.

In the existing stability criteria, two ways of approach have been mainly adopted.

Namely, one direction is to contrive the stability condition which does not include the information on the delay, and the other is the method which takes it into account. The former case is often called the delay - independent criteria and generally provides simple algebraic conditions.

Numerous reports have been published on this matter, with particular emphasis on the application of Lyapunov's second method or on using the concept of the matrix measure *Lee, Diant* (1981), *Mori et al.* (1981), *Mori* (1985), *Hmamed* (1986), *Lee et al.* (1986), *Alastruey*, *De La Sen* (1996).

The results concerning Lyapunov stability, for non delay time systems, are well documented in a number of known references, thus for the sake of brevity are omitted here.

In discussing the problem of investigation of linear *discrete time delay* systems and their Lyapunov stability it should be pointed out that there are not too many results dealing with this problem.

Namely, *Koepcke* (1965), was the first who paid attention to this class of systems solving a synthesis problem for controling the systems governed by linear differential – difference equations. It has been shown, in the same paper, that such systems are equivalent to infinite dimensional difference equations whose matrix elements can be calculated readily by recursive formulas.

Some results, concerning stability in the sense of Lyapunov, were also derived.

The problem of finding an optimal control in linear discrete systems with time delays in both the state variables and control were studied in *Chung* (1967, 1969).

The method of orthogonal projection was used to derive the equations for optimal estimating the state of a non stationary linear discrete system with multiple delays in *Premier*, *Vacroux* (1969). A Kalman - type filter with the necessary recursive error and cross error matrix equations were also derived. The linear – quadratic tracking problem was discussed, for the first time, in *Pindyck* (1972), for a discrete – time systems with the time delay incorporating in inputs.

A more general discussion concerning different aspects of continuous and discrete time delay systems can be found in *Janusevski* (1978), with a particular attention to optimal control.

Several sufficient conditions for the asymptotic stability of linear discrete – delay systems were presented in the paper of *Mori et al.* (1982). Since these conditions are independent of delay and possess simple forms, they provide useful tools for checking system stability at the first stage.

The study of the stabilization problem for general decentralized large-scale linear continuous and discrete time delay systems using local feedback controllers were

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presented by *Lee*, *Radovic* (1987). The local feedback controls were assumed to be memoryless. In that sense, the sufficient stabilization conditions were established.

The problem of delays in interconnections, for the same class of systems, was studied later in *Lee*, *Radovic* (1988).

The paper of *Trinh*, *Aldeen* (1995) presents some new sufficient conditions for robust and *D*-stability of discrete – delay perturbed systems. It has been shown that these results are less conservative than those reported in literature, particularly in *Mori et. al* (1982).

Based on a derived algebraic inequality, a criterion to guarantee the robust stabilization and state estimation for perturbed discrete-time–delay large scale systems was proposed in *Wang*, *Mau* (1995). That criterion is independent of time delay and does not need the solution of the Lyapunov or Riccati equation.

In the first part of this overview, the asymptotic stability of a particular class of discrete time delay systems is considered. Several sufficient conditions, in the form of time delayed independent criteria, are presented.

The first group represents generalization of some previous results of *Mori et. al* (1992) and *Trinith et. al* (1995) which are concerned with the cases with only one delay.

Another group is dealing with a suitable decomposition of matrices, representing the main contribution of the paper, and it is at the same time less restrictive than other ones given in recent literature.

Basic Notations

$\mathbb{R}_+$	-All the non-negative real numbers
$\mathbb{R}^n$	-The *n*-dimensional real space
$\mathbb{Z}_+$	-Set of non-negative integers
$\mathbb{R}^{n\times m}$	-The set of all real $n\times m$ matrices
$\mathbb{C}^{n\times m}$	-The set of all complex $n\times m$ matrices
F^T	-Transpose of matrix *F*
$\det F$	-Determinant of square matrix *F*
$\lambda(F)$	-Eigenvalue of square matrix *F*
$\lvert f(\ )\rvert$	-Absolute value of $f(\ )\in\Box$
$\sigma\{F\}$	-Spectrum of matrix *F*
$\rho(F)$	-Spectral radius of matrix
$F>0$	-Positive definite matrix
$F\geq 0$	-Positive semi definite matrix
$\lVert F\rVert_2$	-Euclidean matrix norm of *F*

Notation and preliminaries

Let $\lVert\mathbf{x}\rVert_{(\cdot)}$ be any vector norm (e.g., $\cdot$ = 1, 2, ∞) and $\lVert(\cdot)\rVert$ the matrix norm induced by this vector. Here, we use $\lVert\mathbf{x}\rVert_2 \triangleq \left(\mathbf{x}^T\mathbf{x}\right)^{1/2}$ and $\lVert(\cdot)\rVert_2 = \lambda_{\max}^{1/2}\left(A^*A\right)$.

The upper indices * and *T* denote transpose conjugate and transpose, respectively. The absolute value of the matrix *A* is denoted by $\lvert A\rvert$, while ρ (*A*) and $\det A$ mean the spectral radius and the determinant of the matrix *A*.

M denotes a class of real square matrices with non positive off-diagonal elements and positive principal minors.

A linear, autonomous, multivariable discrete time-delay system can be represented by the difference equation

$$\mathbf{x}(k+1) = A_0\mathbf{x}(k) + \sum_{j=1}^{N} A_j\mathbf{x}(k-h_j), \tag{1}$$

where $\mathbf{x}(k)\in\mathbb{R}^n$, $A_j\in\mathbb{R}^{n\times n}$ and $0=h_0\leq h_1\leq h_2\leq\ldots\leq h_N$ are integers and represent the systems time delays.

System (1) can be written in another way

$$\hat{\mathbf{x}}(k+1) = A_{eq}\hat{\mathbf{x}}(k), \tag{2}$$

where:

$$\hat{\mathbf{x}}(k) = \left[\mathbf{x}(k)\quad \mathbf{x}(k-1)\quad \ldots\quad \mathbf{x}(k-h_N)\right]^T \in \Box^{\,n\times(h_N+1)} \tag{3}$$

$$A_{eq} = \begin{pmatrix} \hat{A}_0 & \hat{A}_1 & \ldots & \hat{A}_{h_N-1} & \hat{A}_{h_N} \\ I_n & 0 & \ldots & 0 & 0 \\ 0 & I_n & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I_n & 0 \end{pmatrix}$$

$$\hat{\mathbf{x}}(k) = \left[\mathbf{x}(k)\quad \mathbf{x}(k-1)\quad \ldots\quad \mathbf{x}(k-h_N)\right]^T \in \Box^{\,n\times(h_N+1)} \tag{4}$$

$$\hat{A}_i = \begin{cases} A_j\,, & i = h_j\ ,\ \mathrm{j}\ =\ 0,1,\ldots,N \\ 0\,, & i\neq h_j\ ,\ \mathrm{j}\ =\ 0,1,\ldots,N \end{cases},$$

$$\forall i = 0,\ 1,\ \ldots,\ h_N\,. \tag{5}$$

So the necessary and sufficient conditions, for the asymptotic stability of (1), are:

$$\det\left(zI_{n\times(h_N+1)} - A_{eq}\right)\neq 0\,,\quad \lvert z\rvert\geq 1\,. \tag{6}$$

Lemma 1. For any Hermite matrix $X\in\mathbb{C}^{n\times n}$ and any complex vector $\mathbf{v}\in\mathbb{C}^n/\{0\}$ it can be written

$$\lambda_{\min}(X)\leq\frac{\mathbf{v}^*X\mathbf{v}}{\mathbf{v}^*\mathbf{v}}\leq\lambda_{\max}(X), \tag{7}$$

so the lower and upper bound of this inequality can be reached if the eigenvector v corresponds to the eigenvalue $\lambda_{\min}(X)$, or $\lambda_{\max}(X)$, respectively.

Lemma 2. For any square matrix $X\in\mathbb{C}^{n\times n}$ and any complex vector $\mathbf{v}\in\mathbb{C}^n/\{0\}$, the field of values of

$$\frac{\mathbf{v}^*X\mathbf{v}}{\mathbf{v}^*\mathbf{v}}, \tag{8}$$

is always in the rectangle in the complex plane whose four vertices are given with:

$$\left[\lambda_i(H),\ j\lambda_k(K)\right],\quad i,k = \text{“}min\text{”}, \text{“}max\text{”}, \tag{9}$$

where:

$$H = \frac{1}{2}\left(X+X^T\right),\quad K = \frac{1}{2j}\left(X-X^T\right),\quad j^2\triangleq -1\,. \tag{10}$$

The matrix function

$$\begin{aligned} d(X) &\triangleq \max_{i,k}\sqrt{\lambda_i^2\left(H(X)\right)+\lambda_k^2\left(K(X)\right)} \\ &= \sqrt{\rho^2(H)+\rho^2(K)} \end{aligned}, \tag{11}$$

represents the longest distance in the complex plane between the origin and four points defined by (9).

Lemma 3. Matrix $D \in \mathbb{R}^{n\times n}$ belongs to *M – class of* matrices if and only if

$$\exists C \in \mathbb{R}^{n\times n} \geq 0 \quad \exists r \in \mathbb{R} > \rho(C) \quad D = rI_n - C\,. \quad (12)$$

Lemma 4. Let:

$$G(z) \hat{=} \left(zI_n - A\right)^{-1}, \quad (13)$$

then:

$$\left|G(z)z^{-h}\right| \leq \sum_{k=0}^{\infty}\left|G(k)\right| \hat{=} L\,,\ |z| \geq 1\,, \quad (14)$$

$G(k)$ is the pulse-response sequence matrix of $G(z)$ and $G(0) = 0$, *Trinth et al*. (1995).

Lemma 5. For any $(n\times n)$ square matrix X, the following statement is true

$$\rho(X) < 1 \quad \Rightarrow \quad \det\left(I_n - X\right) \neq 0\,. \quad (15)$$

Trinth et al. (1995).

Lemma 6. For any square matrices $X \in \mathbb{R}^{n\times n}$ and $Y \in \mathbb{R}^{n\times n}$, the following statement is true

$$|X| \leq Y \quad \Rightarrow \quad \rho(X) \leq \rho(|X|) \leq \rho(Y)\,. \quad (16)$$

Definition 1. Linear autonomous discrete time delay system (1) *is asymptotically stable* if and only if all its *zeros of characteristic equation* lie within *the unit circle*.

Asymptotic stability - approach in the complex plane [1]

Theorem 1. System (**1**) is asymptotically stable if

$$\sum_{j=0}^{N}\left\|A_j\right\| < 1\,, \quad (17)$$

Debeljkovic et al.(2002.a, 2002.b, 2003.a, 2003.b), *Stojanovic*, *Debeljkovic* (2000, 2004.a, 2004.c, 2004.i).

Note 1. It should be pointed out that the proof of *Theorem* 1 in papers *Debeljkovic et al.*(2002.a, 2002.b) is quite different from those exposed in *Debeljkovic et al.*(2003.a, 2003.b) and *Stojanovic*, *Debeljkovic* (2000, 2004.a, 2004.c, 2004.i).

Conclusion 1. If $N = 1$, condition (17) is reduced to the condition given in *Mori* (1982).

Theorem 2. System (**1**) is asymptotically stable if the following condition is satisfied

$$\sum_{j=0}^{N} d\left(A_j\right) < 1\,, \quad (18)$$

where matrix function $d(.)$ is given with (11), *Stojanovic*, *Debeljkovic* (2000, 2004.a, 2004.c, 2004.i).

Conclusion 2. If $N = 1$, from (18) follows the condition given in *Mori* (1982).

Theorem 3. If matrix D, defined with:

$$D \hat{=} I_n - \sum_{j=0}^{N}\left|A_j\right|, \quad (19)$$

$$d_{ik} \hat{=} \begin{cases} d_{ii} = 1 - \sum_{j=0}^{N}\left|a_{ii}^{j}\right| \\ d_{ik} = -\sum_{j=0}^{N}\left|a_{ik}^{j}\right| \end{cases}, \quad (20)$$

belongs to *M – class of* matrices, then system (1) is asymptotically stable, *Stojanovic*, *Debeljkovic* (2000, 2004.a, 2004.c, 2004.i).

Conclusion 3. From the basic condition of *Theorem* 3, for
$N = 1$, follows the condition given in *Mori et. al* (1982).

Conclusion 4. If one uses the norm $\|(\cdot)\|_{(\cdot)}$, $(\cdot) = 1,\ \infty$, in *Theorem* 1, then

$$1 > \sum_{j=0}^{N}\left\|A_j\right\|_{(.)} = \sum_{j=0}^{N}\left\|\,|A_j|\,\right\|_{(.)}$$

$$\geq \left\|\sum_{j=0}^{N}\left|A_j\right|\right\|_{(.)} \geq \rho\left(\sum_{j=0}^{N}\left|A_j\right|\right). \quad (21)$$

If one defines:

$$C = \sum_{j=0}^{N}\left|A_j\right| \geq 0\,, \quad (22)$$

$$r = 1 > \rho\left(\sum_{j=0}^{N}\left|A_j\right|\right) = \rho(C)\,, \quad (23)$$

$$D = I_n - \sum_{j=0}^{N}\left|A_j\right|, \quad (24)$$

then we can conclude from *Lemma* 3 that the matrix D belongs to the *M – class of* matrices.

This shows that *Theorem* 1 implies *Theorem* 3 and the condition of *Theorem* 1 is more restrictive than the condition of *Theorem* 3 when $\|(\cdot)\| = \|(\cdot)\|_1$ or $\|(\cdot)\|_\infty$.

Theorem 4. System (1) is asymptotically stable, if one of these two conditions is satisfied

$$\sum_{j=0}^{N}\rho\left(H_j\right) < 1\,, \quad (25)$$

$$\sum_{j=0}^{N}\left\|H_j\right\|_2 < 1\,, \quad (26)$$

where the matrices H_j are defined with

$$H_j = \frac{A_j + A_j^T}{2}\,,\ j = 0, 1, \ldots, N. \quad (27)$$

Stojanovic, *Debeljkovic* (2000, 2004.a, 2004.c, 2004.i).

Conclusion 5. On the basis of elementary algebra, the following conditions are fulfilled

[1] *Proofs are derived using the complex plane technique*

$$\begin{aligned}\rho(A)&=\rho(H+jK)=\max_i\left|\lambda_i(H+jK)\right|\\ &\le\max_i\left|\lambda_i(H)+j\lambda_i(K)\right|\le\max_i\left|\lambda_i(H)\right|+\max_i\left|j\lambda_i(K)\right|\\ &=\rho(H)+\rho(K)=\|H\|_2+\|K\|_2,\end{aligned}\tag{28}$$

$$\rho(H)=\|H\|_2=\frac{1}{2}\left\|A+A^T\right\|_2\le\frac{1}{2}\left(\|A\|_2+\left\|A^T\right\|_2\right)=\|A\|_2\tag{29}$$

From the Bendixsons inequality

$$\lambda_{\min}(H)\le\operatorname{Re}\lambda(A)\le\lambda_{\max}(H),\tag{30.a}$$

$$\lambda_{\min}(K)\le\operatorname{Im}\lambda(A)\le\lambda_{\max}(K),\tag{30.b}$$

It follows

$$\left|\lambda(A)\right|\le\sqrt{\lambda_H^{\,2}+\lambda_K^{\,2}},\tag{31}$$

$$\lambda_H=\max\left\{\left|\lambda_{\min}(H)\right|,\left|\lambda_{\max}(H)\right|\right\}=\max_i\left|\lambda_i(H)\right|=\rho(H),\tag{32}$$

$$\lambda_K=\max\left\{\left|\lambda_{\min}(K)\right|,\left|\lambda_{\max}(K)\right|\right\}=\max_i\left|\lambda_i(K)\right|=\rho(K),\tag{33}$$

and finally

$$\max_i\left|\lambda_i(A)\right|=\rho(A)\le\sqrt{\rho^2(H)+\rho^2(K)}\triangleq d(A).\tag{34}$$

So from (29) and (34), it follows

$$\rho(H)=\|H\|_2\le d(A),\ \rho(H)=\|H\|_2\le\|A\|_2.\tag{35}$$

Stojanovic, *Debeljkovic* (2000, 2004.a, 2004.c, 2004.i).

Conclusion 6. It is not difficult to prove, having in mind (35), that the following expressions are valid

$$\sum_{j=0}^{h_N}\rho\left(H_j\right)\le\sum_{j=0}^{h_N}d\left(A_j\right)<1,\ \sum_{j=0}^{h_N}\left\|H_j\right\|_2\le\sum_{j=0}^{h_N}\left\|A_j\right\|<1,\tag{56}$$

so the conditions given in *Theorem* 1 and *Theorem* 2 are more restrictive than those given in *Theorem* 4, *Stojanovic*, *Debeljkovic* (2000, 2004.a, 2004.c, 2004.i).

Theorem 5. System (**1**) is asymptotically stable, independent of delay, if the following conditions are satisfied:

$$\rho\left(A_0\right)<1,\tag{57.a}$$

$$\rho\left(L\sum_{j=1}^{N}\left|A_j\right|\right)<1,\tag{57.b}$$

where L is defined as in (14), and $G(k)$ is obtained from

$$G(k)=A_0^{\,k-1},\quad k=1,2,\ldots,\infty,\quad G(0)=0.\tag{58}$$

Stojanovic, *Debeljkovic* (2000, 2004.a, 2004.c, 2004.i).

Conclusion 7. The fundamental matrix of system (**1**) without delay is:

$$\Phi(z)=\left(zI_n-A_0\right)^{-1}z=G(z)z,\tag{59}$$

so:

$$\begin{aligned}&G(z)=z^{-1}\Phi(z)\ \Rightarrow\\ &G(k)=\Phi(k-1)=A_0^{\,k-1},\quad G(0)=0\end{aligned}.\tag{60}$$

If A_0 is the discrete stable matrix, $\rho\left(A_0\right)<1$, then infinite series:

$$\begin{aligned}L&=\sum_{k=0}^{\infty}\left|G(k)\right|=\sum_{k=1}^{\infty}\left|G(k)\right|=\sum_{k=0}^{\infty}\left|A_0^{\,k}\right|\\ &\le\sum_{j=0}^{\infty}\left|A_0\right|^k=\left(I_n-\left|A_0\right|\right)^{-1}\end{aligned},\tag{61}$$

is convergent, so one can find the matrix L by direct computation, *Stojanovic*, *Debeljkovic* (2000, 2004.a, 2004.c, 2004.i).

Conclusion 8. Conditions (57) are less restrictive than condition (17).

The reason is in the fact that conditions (57) take into account the matrix time delay structure A_j, whereas condition (17) takes only the norm of matrices.

Note 2. All conditions are in the form of only sufficient conditions and belong to so-called independent delay criteria.

Asymptotic stability

- Approach based on the results of tissir and hmamed [2]

We are in particular oncerned with a linear, autonomous, multivariable discrete time-delay system in the form:

$$\mathbf{x}(k+1)=A_0\mathbf{x}(k)+A_1\mathbf{x}(k-1),\tag{62}$$

Equation (62) is referred to as homogenous or the unforced state equation, $\mathbf{x}(k)$ is the state vector, A_0 and A_1 are the constant system matrices of appropriate dimensions.

It is assumed that equation (62) satisfies the adequate smoothnees requirements so that its solution exists and is unique and continuous with respect to k and the initial data and is bounded for all bounded values of its arguments.

Theorem 6. System (62) is asymptotically stable if:

$$\|A_0\|+\|A_1\|<1,\tag{63}$$

holds, *Mori et al.* (1982).

Theorem 7. Then system (62) is asymptotically stable, independent of delay, if:

$$\|A_1\|<\frac{\sigma_{\min}\left(Q^{-\frac{1}{2}}\right)}{\sigma_{\max}\left(Q^{-\frac{1}{2}}A_0^TP\right)},\tag{64}$$

where P is the solution of the *discrete Lyapunov matrix equation*:

$$A_0^TPA_0-P=-\left(2Q+A_1^TPA_1,\right)\tag{65}$$

where $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are the maximum and minimum singular values of the matrix $(\cdot)$, *Debeljkovic et al.* (2004.a, 2004.b, 2004.d, 2005.a).

[2] *Tissir*, *Hmamed* (1996).

Theorem 8. Suppose the matrix $\left(Q - A_1^T P A_1\right)$ is regular.

Then system (62) is asymptotically stable, independent of delay, if:

$$\| A_1 \| < \frac{\sigma_{\min}\left(\left(Q - A_1^T P A_1\right)^{-\frac{1}{2}}\right)}{\sigma_{\max}\left(Q^{-\frac{1}{2}} A_0^T P\right)}, \tag{66}$$

where P is the solution of the *discrete Lyapunov matrix equation*:

$$A_0^T P A_0 - P = -2Q \tag{67}$$

where $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are the maximum and minimum singular values of the matrix$(\cdot)$, *Jacic et al.* (2004), *Debeljkovic et al.* (2004.c, 2004.d, 2005.a, 2005.b).

Asymptotic stability

- Lyapunov based approach

A linear, autonomous, multivariable linear discrete time-delay system can be represented by the difference equation:

$$\begin{aligned} \mathbf{x}(k+1) &= \sum_{j=0}^{N} A_j \mathbf{x}(k-h_j) \\ \mathbf{x}(\theta) &= \boldsymbol{\psi}(\theta), \quad \theta \in \{-h_N, -h_N+1, \dots, 0\} \triangleq \Delta \end{aligned} \tag{68}$$

where:

$\mathbf{x}(k) \in \mathbb{R}^n$, $A_j \in \mathbb{R}^{n \times n}$, $0 = h_0 < h_1 < h_2 < \dots < h_N$ - are integers and represent the systems time delays.

Let $V(\mathbf{x}(k))$: $\mathbb{R}^n \to \mathbb{R}$, so that $V(\mathbf{x}(k))$ is bounded for and for which $\|\mathbf{x}\|$ is also bounded.

Lemma 7. For any two matrices of the same dimensions F and G and for some positive constant ε the following statement is true

$$(F+G)^T (F+G) \le (1+\varepsilon) F^T F + \left(1+\varepsilon^{-1}\right) G^T G, \tag{69}$$

Wang and Mau (1997).

Theorem 9. Suppose that A_0 is not a null matrix.

If for any given matrix $Q = Q^T > 0$ there exists the matrix $P = P^T > 0$ such that the following matrix equation is fulfilled

$$(1+\varepsilon_{\min}) A_0^T P A_0 + \left(1+\varepsilon_{\min}^{-1}\right) A_1^T P A_1 - P = -Q, \tag{70}$$

where

$$\varepsilon_{\min} = \frac{\|A_1\|_2}{\|A_0\|_2}, \tag{71}$$

then system (**68**) is asymptotically stable, *Stojanovic*, *Debeljkovic* (2005.b).

Corolarry 1. If for any given matrix $Q = Q^T > 0$ there exists the matrix $P = P^T > 0$ being the solution of the following Lyapunov matrix equation

$$A_0^T P A_0 - P = -\frac{\varepsilon_{\min}}{1+\varepsilon_{\min}} Q, \tag{72}$$

where $\varepsilon_{\min}$ is defined by (71) and if the following condition is satisfied

$$\sigma_{\max}(A_0) + \sigma_{\max}(A_1) < \frac{\lambda_{\min}(Q-P)}{\sigma_{\max}(A_0)\lambda_{\max}(P)} \tag{73}$$

then system (68) is asymptotically stable, *Stojanovic*, *Debeljkovic* (2005.b).

Corolarry 2. If for any given matrix $Q = Q^T > 0$ there exists the matrix $P = P^T > 0$ being the solution of the following matrix equation

$$(1+\varepsilon_{\min}) A_0^T P A_0 - P = -\varepsilon_{\min} Q, \tag{74}$$

where $\varepsilon_{\min}$ is defined by (71), and if the following condition is satisfied, too

$$\sigma_{\max}(A_0) + \sigma_{\max}(A_1) < \frac{\lambda_{\min}(Q)}{\sigma_{\max}(A_0)\lambda_{\max}(P)}, \tag{75}$$

then system (68) is asymptotically stable, *Stojanovic*, *Debeljkovic* (2005.b).

Theorem 10. If for any given matrix $Q = Q^T > 0$ there exists the matrix $P = P^T > 0$ such that the following matrix equation is fulfilled

$$2A_0^T P A_0 + 2A_1^T P A_1 - P = -Q, \tag{76}$$

then system (68) is asymptotically stable, *Stojanovic*, *Debeljkovic* (2006.a).

Corolarry 3. System (68) is asymptotically stable, independent of delay, if

$$\sigma_{\max}^2(A_1) < \frac{\lambda_{\min}(2Q-P)}{2\sigma_{\max}^2\left(P^{\frac{1}{2}}\right)}, \tag{77}$$

where, for any given matrix $Q = Q^T > 0$ there exists the matrix $P = P^T > 0$ being the solution of the following *Lyapunov matrix equation*

$$A_0^T P A_0 - P = -Q, \tag{78}$$

Stojanovic, *Debeljkovic* (2006.a).

Corolarry 4. System (68) is asymptotically stable, independent of delay, if

$$\sigma_{\max}^2(A_1) < \frac{\lambda_{\min}(Q)}{2\sigma_{\max}^2\left(P^{\frac{1}{2}}\right)}, \tag{79}$$

where, for any given matrix $Q = Q^T > 0$ there exists the matrix $P = P^T > 0$ being the solution of the following *Lyapunov matrix equation*

$$2A_0^T P A_0 - P = -Q, \tag{80}$$

Stojanovic, Debeljkovic (2006.a).

Asymptotic and exponential stability of linear and nonlinear perturbed discrete delay systems

Asymptotic stability of linear perturbed discrete delay systems – General approach

The stability robustness analysis of perturbed discrete-time systems has been well researched by many authors, *Kolla* (1989), *Rachid* (1989, 1990), *Yaz* (1989), *Chou* (1991), *Niu* (1992), *Horng* (1993), *Lee* (1992) and *Yedavalli* (1993).

Jury (1974) and *Bishop* (1975) proposed several methods for testing the stability of discrete-delay systems with no parametric perturbations. While these methods are easy enough to be used for small delays, they become troublesome as the delay increases. This is due to the fact that, in their methods, the number of the system eigenvalues increases in proportion to *n* times the delay, *n* being the order of system.

Mori et al. (1982) overcome this problem by proposing several delay-independent criteria for stability of such class of systems. These criteria are expressed in simple forms in terms of plant parameters. However, these sufficient conditions are conservative and can be applied to systems with no perturbations.

Recently, robust stability problems for linear systems with time delay attracted considerable attention and have been widely studied. In these, papers *Mohmoud*, *Al-Muthairi* (1994), *Phoojaruenchanachai*, *Furuta* (1992), *Shen et al.* (1991) and *Xie*, *de Souza* (1993) proposed the robust stability criteria independent of the size of the time-delay. On the other hand, *Niculescu et al.* (1994) and *Su*, *Huang* (1992) developed the delay-dependent robust stability criteria using the solution of an *algebraic Riccati equation* or *Lyapunov matrix equation* in order to reduce conservativeness of the delay-independent results.

Particularly, *Li*, *de Souza* (1997) proposed a delay-dependent robust stability criteria for an uncertain system with time-varying delay via LMIs (linear matrix inequalities) and their results are less conservative than those of the others.

In this section the asymptotic stability of *linear perturbed time - delay systems with multiple delays* is considered.

Several new criteria, which are independent of delay, are presented.

The first derived criterion is based on the analysis of the time-varying perturbation matrix of an equivalent system.

The second suggested criterion is based on formal matrix decomposition to a real and an imaginary part, using the comparison principle, *Mori et al.* (1981).

The criteria presented in *Trinh and Aldeen* (1995), have been generalized and used, here, as the third condition for stability.

The last criterion was based on direct application of the suggested procedure to the characteristic polynomial of the *comparison* system. In that sense it should be noted that its expression is quite simple and suitable for practical usage.

$\lambda_i(\cdot)$ denotes the eigenvalue of matrix $(\cdot)$ and Re $\lambda_i(\cdot)$ and Im $\lambda_i(\cdot)$ are real and imaginary parts, respectively.

The absolute value of the matrix A is denoted by $|A|$, while $\rho(A)$ denotes the spectral radius of matrix A.

A is said to be a *nonnegative* matrix whenever each $a_{ij} \geq 0$ and this is denoted by writing $A \geq 0$.

In general, $A \geq B$ means that each $a_{ij} \geq b_{ij}$. Similarly, A is *positive* matrix when each $a_{ij} > 0$ and this is denoted by writing $A > 0$.

Let us consider a linear perturbed discrete time delay system with multiple delays:

$$\mathbf{x}(k+1) = A_0(k)\mathbf{x}(k) + \sum_{j=1}^{N} A_j(k)\mathbf{x}(k-h_j) \tag{81}$$

where $0 = h_0 < h_1 < h_2 < \ldots < h_N$ are integers, representing the system time delays.

The time dependent perturbed matrices $A_j(k) \in \mathbb{R}^{n\times n}$, $j = 0, 1, \ldots, N$ are unknown, but the maximum deviations of their elements e.g. $\max_k \left| a_{il}^j(k) \right| \leq \alpha_{il}^j$ are known.

In comparison with *Trinh, Aldeen* (1995), where only the basic matrix $A_1(k)$ is time varying, here we make an assumption that all matrices $A_j(k)$, $0 \leq j \leq 1$, possess this property.

If we define the matrices U_j, $j = 0, 1, \ldots, N$ in the following way:

$$\alpha_j \triangleq \max_{i,l} \alpha_{il}^j,\ u_{il}^j \triangleq \frac{\alpha_{il}^j}{\alpha_j},\ 0 \leq u_{il}^j \leq 1,\ U_j = \left[u_{il}^j\right], \tag{82}$$

then:

$$\left|A_j(k)\right| \leq \alpha_j U_j, \quad j = 0, \ldots, N,\ \forall k. \tag{83}$$

Let *Lemma* 1 and *Lemma* 2 hold.

Moreover, we have

Lemma 7. For any square matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{R}^{n\times n}$ the following statement is true, *Meyer* (2001)

$$|A| \leq B \ \Rightarrow\ \rho(A) \leq \rho(|A|) \leq \rho(B). \tag{84}$$

Lemma 8. Linear time invariant discrete time delay system (81) is asymptotically stable if the following conditions are satisfied:

$$\rho(A_0) < 1,\ \rho\left(L\sum_{j=1}^{N}\left|A_j\right|\right) < 1, \tag{85}$$

$$L = \sum_{k=0}^{\infty}\left|G(k)\right| = \sum_{k=0}^{\infty}\left|A_0^{\,k}\right|. \tag{86}$$

Lemma 9. Linear time invariant discrete time delay system (81) is asymptotically stable if the following condition is satisfied:

$$\sum_{j=0}^{N}\rho\left(H_j\right) < 1,\ H_j = \frac{1}{2}\left(A_j + A_j^{\,T}\right),\ 0 \leq j \leq N. \tag{87}$$

Lemma 10. Linear time invariant discrete time delay system (81) is asymptotically stable if the following condition is satisfied:

$$\sum_{j=0}^{N}\left\|A_j\right\| < 1. \tag{88}$$

Theorem 11. System (81) is asymptotically stable if

$$\rho\left(\hat{A}_{eq}\right) < 1, \tag{89}$$

where:

$$\hat{A}_{eq} = \begin{pmatrix} \hat{\alpha}_0 \hat{U}_0 & \hat{\alpha}_1 \hat{U}_1 & \dots & \hat{\alpha}_{h_N-1} \hat{U}_{h_N-1} & \hat{\alpha}_{h_N} \hat{U}_{h_N} \\ I_n & 0 & \dots & 0 & 0 \\ 0 & I_n & \dots & 0 & 0 \\ \dots & \dots & \ddots & \dots & \dots \\ 0 & 0 & \dots & I_n & 0 \end{pmatrix}$$

$$\hat{\alpha}_i \hat{U}_i = \begin{cases} \alpha_j U_j \ , \ i = h_j \ , \mathrm{j} \ = \ 0,1,\dots,N \\ 0 \ , \quad i \neq h_j \ , \mathrm{j} \ = \ 0,1,\dots,N \end{cases} \quad 0 \le i \le h_N. \tag{90}$$

Stojanovic, *Debeljkovic* (2004.d, 2004.i)

Theorem 12. System (81) is asymptotically stable if one of the following conditions is satisfied

$$\sum_{j=0}^{N} \alpha_j \rho\left(H_j\right) < 1, \quad H_j = \tfrac{1}{2}\left(U_j + U_j^T\right), \tag{91}$$

$$\sum_{j=0}^{N} \alpha_j \left\| U_j \right\| < 1, \tag{92}$$

Stojanovic, *Debeljkovic* (2004.d, 2004.i)

Theorem 13. System (81) is asymptotically stable if the following conditions are satisfied

$$\alpha_0 \rho\left(U_0\right) < 1, \ \rho\left(L_0 \sum_{j=1}^{N} \alpha_j U_j \right) < 1, \tag{93}$$

$$L_0 = \sum_{k=0}^{\infty} \left(\alpha_0 U_0\right)^k = \left(I_n - \alpha_0 U_0\right)^{-1}, \tag{94}$$

Stojanovic, *Debeljkovic* (2004.d, 2004.i)

Theorem 14. If

$$\rho\left(\sum_{j=0}^{N} \alpha_j U_j \right) < 1, \tag{95}$$

then (81) is asymptotically stable, *Stojanovic*, *Debeljkovic* (2004.d, 2004.i)

Asymptotic stability of linear discrete delay systems with linear and nonlinear perturbations

In the analysis of dynamic systems, we are often faced with parametric uncertainties originating from identification errors, variation of operating points, etc. Therefore, the problem of robust stability analysis and robust stabilization of systems with parameter uncertainties has been of considerable interest to researchers and a number of significant results concerning this issue have been reported in the current literature.

It is well known that the location of all characteristic roots is an important indicator for the system dynamic performance of linear controlled systems. In practice, all characteristic roots cannot be assigned in fixed locations but can be only located inside some restricted regions due to the unavoidable parametric perturbations.

One such specific region for discrete systems is a disk $D(\alpha,\ r)$ centered at $(\alpha,\ 0)$ with the radius r, where $|\alpha| + r < 1$. The assignment of all poles of a system in the specific disk $D(\alpha,\ r)$ is referred to as a D-pole placement problem.

Recently, the D-stability problem which guarantees all characteristic roots of controlled systems to be located inside a specified disk in the complex plane has also become an attractive area of research for the mentioned systems.

Due to computation of data, physical properties of system elements, and signal transmission, time delay exists inherently not only in the physical, engineering, and chemical systems but also in political and economic systems, etc. Since the number of poles of the closed-loop system increases due to time delays, the introduction of a time-delay factor makes the D-pole placement problem much more complicated.

The D-stability problem for discrete time-delay systems has been discussed in *Lee et al.* (1992), *Su*, *Shyr* (1994), *Trinh*, *Aldeen* (1995), *Hsiao* (1998) and for continuous in *Le* (1995).

In this section, we consider linear discrete perturbed systems with multiple time delays.

We present robust sufficient criteria for eigenvalues of the perturbed discrete time-delay system to be located in a specified disk.

Both *structured* and *unstructured* perturbations are discussed.

A linear, autonomous, multivariable discrete perturbed time-delay system can be represented by the difference equation

$$\mathbf{x}(k+1) = \sum_{j=0}^{N} \left(A_j + \Delta A_j \right) \mathbf{x}\left(k - h_j\right), \quad 0 = h_0 < h_1 < \cdots < h_N \tag{96}$$

with an associated function of the initial state

$$\mathbf{x}(\theta) = \boldsymbol{\psi}(\theta), \quad \theta \in \{-h_N, -h_N + 1, \dots, 0\} \tag{97}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is the state vector, $A_j \in \mathbb{R}^{n \times n}$ is the constant matrix and pure system time delays are expressed by integers $h_j \in \mathbb{Z}_+$.

$\Delta A_j \in \mathbb{R}^{n \times n}$, $0 \le j \le N$ are the matrices representing perturbations in the system. In this paper we observe unstructured and structured perturbations defined by

$$\left\| \Delta A_j \right\| \le a_j \in \mathbb{R}_+ \tag{98}$$

$$\left| \Delta A_j \right| \le R_j \in \mathbb{R}_+ \tag{99}$$

respectively.

In a case when the perturbations of system (96) do not exist, e.g. $\Delta A_j = 0$, the stability of the system under consideration can be stated by the following *Theorem*.

Theorem 15. All the eigenvalues of the *non-perturbed* system (96) are inside the disc $D(\alpha,\ r)$ if the following condition is satisfied:

$$\sum_{j=0}^{N} \left\| A_j \right\| \delta^{-h_j} < \delta, \quad \delta = \min\left(|\alpha - r|, |\alpha + r| \right), \tag{100}$$

Stojanovic, *Debeljkovic* (2004.f)

In the case of the non-structured perturbations of system (96) defined by (98), $D(\alpha,\ r)$ the stability of the system under consideration can be stated by the following *Theorem*.

Theorem 16. All the eigenvalues of the perturbed systems (96) with perturbations (98) are inside the disc

$D(\alpha, r)$ if the following condition is satisfied:

$$\sum_{j=0}^{N}\|A_j\|\delta^{-h_j} < \delta - \sum_{j=0}^{N} a_j\delta^{-h_j}, \ \delta = \min(|\alpha - r|, |\alpha + r|) \quad (101)$$

Stojanovic, *Debeljkovic* (2004.f).

In the case of the structured perturbations of system (96) defined by (99), $D(\alpha, r)$ the stability of the system under consideration can be stated by the following *Theorem*.

Theorem 17. Assume that all the eigenvalues of the matrix A_0 are inside the disk $D(\alpha, r)$.

All the eigenvalues of the discrete-delay perturbed systems with perturbations are inside the disc $D(\alpha, r)$ if the following condition is satisfied

$$\rho\left(H_{\alpha,r}\left(R_0 + \sum_{j=1}^{N}(|A_j| + R_j)\,\delta^{-h_j}\right)\right) < r,$$

$$\delta = \min(|\alpha - r|, |\alpha + r|) \quad (102)$$

$$H_{\alpha,r} = \sum_{k=0}^{\infty}\left|\left(\frac{A_0 - \alpha I_n}{r}\right)^k\right| = \left(I_n - \left|\frac{A_0 - \alpha I_n}{r}\right|\right)^{-1}$$

Stojanovic, *Debeljkovic* (2004.f).

Exponential stability of linear discrete delay systems with nonlinear perturbations

The problem of exponential stability testing becomes more complicated than that of a system without time delay and/or uncertainties. *Grujic*, *Siljak* (1974), *Hmamed* (1991.a, 1991.b), addressed the stability degree testing problem for continuous time-delay systems. By testing some stability conditions and repeating the computation, they can estimate the stability degree of linear time-delay systems.

However, up to now, the same problem has been seldom treated for discrete time-delay systems *Hsien*, *Lee* (1995). This is mainly due to the fact that such systems can be transformed into augmented systems without delay. This augmentation of the systems is, however, inappropriate for systems with unknown delays or systems with time-varying delays.

The objective of this section is to investigate the exponential testing problem for linear discrete uncertain systems with time delay. Using the *Lyapunov stability approach*, a new criterion is established to ensure the exponential stability of the system under consideration.

Some sufficient conditions, in the form of *time delayed-dependent criteria*, are obtained.

A linear, autonomous, multivariable discrete perturbed time-delay system can be represented by the difference equation

$$\mathbf{x}(k+1) = \sum_{j=0}^{N}(A_j + \Delta A_j)\,\mathbf{x}(k - h_j) + \sum_{j=0}^{M} f_j(x(k - h_j), k), \quad M \le N, \ h_0 = 0 \quad (103)$$

with an associated function of the initial state

$$\mathbf{x}(\theta) = \boldsymbol{\psi}(\theta), \quad \theta \in \{-h_N, -h_N + 1, \dots, 0\}, \quad h_N = \max_i h_i \quad (104)$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is the state vector, $A_j \in \mathbb{R}^{n\times n}$ is the constant matrix and pure system time delays are expressed by integers $h_j \in \mathbb{Z}_+$.

The vector $f_j(\cdot,\cdot)$: $\mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ is a nonlinear perturbation which satisfies the condition

$$\|f_j(x(k - h_j), k)\| \le b_j\|x(k - h_j)\|, b_j \in \mathbb{R}_+ \quad (105)$$

Definition 2. System (**103**) is said to have a stability degree α (or to be exponentially stable), with $\alpha > 1$, if the state of system (103) can be written as

$$\mathbf{x}(k) = \alpha^{-k}\mathbf{p}(k) \quad (106)$$

and the system governing the state $\mathbf{p}(k)$ is globally asymptotically stable.

In this case, the parameter α is called the *convergence rate*.

Lemma 11. The Thebyshev's inequality holds for any real vector $\mathbf{v}_i$

$$\left(\sum_{i=1}^{m}\mathbf{v}_i\right)^T\left(\sum_{i=1}^{m}\mathbf{v}_i\right) \le m\sum_{i=1}^{m}\mathbf{v_i}^{\mathbf{T}}\mathbf{v_i} \quad (107)$$

Mori et al. (1982).

Lemma 12. For any two matrices *W*, *X*, *Y* and *Z* with the same dimension $(m \times n)$, if

$$W = X + Y + Z, \quad (108)$$

then for any positive square matrix $P = P^T > 0$ of the dimension *n* and the positive constants ε_1, ε_2 and ε_3 the following statement is true

$$W^T PW \le (1 + \varepsilon_1 + \varepsilon_3^{-1})X^T PX + (1 + \varepsilon_2 + \varepsilon_1^{-1})Y^T PY + (1 + \varepsilon_3 + \varepsilon_2^{-1})Z^T PZ. \quad (109)$$

Wang and Mau (1997).

Theorem 18. System (103) is asymptotically stable if

$$\|A_0\| + \left\|N\sum_{j=1}^{N}A_j^T A_j\right\|^{\frac{1}{2}} + \left((M + 1)\sum_{j=0}^{M}b_j^2\right)^{\frac{1}{2}} < 1, \quad (110)$$

Stojanovic, Debeljkovic (2006.b, 2006.d).

Theorem 19. System (103) is exponentially stable if

$$\alpha\|A_0\| + \sqrt{N\sum_{j=1}^{N}\alpha^{2(h_j+1)}\|A_j^T A_j\|} + \sqrt{(M + 1)\sum_{j=0}^{M}\alpha^{2(h_j+1)}b_j^2} < 1 \quad (111)$$

where α is the stability degree, *Stojanovic*, *Debeljkovic* (2006.b, 2006.d).

Quadratic stability of uncertain linear discrete delay systems

During the last decades, considerable attention has been devoted to the problem of the stability analysis and controller design for time-delay systems. Especially, in accordance with the advances of the robust control theory, a number of robust stability and stabilization methods have been proposed for uncertain time-delay systems.

Less attention has been drawn to the corresponding results for discrete-time delay systems, see *Verriest*, *Ivanov* (1995). This is mainly due to the fact that such systems can be transformed into augmented systems without delay. This augmentation of the systems is, however, inappropriate for systems with unknown delays or systems with time-varying delays.

One of the most popular ways to deal with the robust stability analysis and robust stabilization is the one based on the concept of quadratic stability and quadratic stabilization.

Quadratic stability means that there exists a certain *Lyapunov function* which guarantees the stability of the uncertain system.

In *Xu et al.* (2001) the conditions for quadratic stability and stabilization for uncertain linear discrete-time systems with state delay are presented in terms of nonlinear matrix inequalities, which cannot be efficiency numerically solved.

Here, in this section, we present a possibility to overcome this disadvantage by proposing new conditions of quadratic stability and stabilization in terms of linear matrix inequalities (LMI) that can be solved efficiently using recently developed convex optimization algorithms *Boyd et al.* (1994).

In this section we also present the quadratic stability analysis for uncertain linear discrete-time systems with state delay. The system under consideration involves time delay in the state and parameter uncertainties. The parameter uncertainties are assumed to be time - varying and norm - bounded.

The necessary and sufficient conditions are presented in terms of linear matrix inequalities.

Notations can be used from the previous sections.

Consider the class of uncertain linear discrete-time systems with state delay

$$\mathbf{x}(k+1) = \left(A_0 + \Delta A_0(k)\right)\mathbf{x}(k) + \left(A_1 + \Delta A_1(k)\right)\mathbf{x}(k-h) \quad (112)$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is the state and *h* is a positive integer.

A_0, and A_1 are known real constant matrices, $\Delta A_0(k)$, and $\Delta A_1(k)$ are the time-varying parameter uncertainties, and are assumed to be of the form

$$\left(\Delta A_0(k) \;\; \Delta A_1(k)\right) = MF(k)\left(N_0 \;\; N_1\right) \quad (113)$$

where M, N_0 and, N_1 are the constant matrices and $F(k) \in \mathbb{R}^{i \times j}$ is the uncertain matrix satisfying

$$F(k)F^T(k) \le I \quad (114)$$

The matrices $\Delta A_0(k)$ and $\Delta A_1(k)$ are said to be admissible if both (113) and (114) hold.

Throughout this section, we shall use the following definitions of quadratic stability and quadratic stabilizability for the uncertain time-delay system (1)-(3).

Definition 3. The uncertain discrete time-delay system (112-114) is said to be quadratically stable if there are matrices $P > 0$, $Q > 0$ and a scalar $\varepsilon > 0$ such that, for all admissible uncertainties $\Delta A_0(k)$ and $\Delta A_1(k)$, satisfies

$$\Delta V(\mathbf{x}(k)) = V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \le -\varepsilon \| \hat{\mathbf{x}}(k) \|^2 \quad (115)$$

for all pairs $\left(\hat{\mathbf{x}}(k), k\right) \in \mathbb{R}^{2n} \times \mathbb{R}$, where

$$\hat{\mathbf{x}}(k) = \left[\mathbf{x}^T(k) \;\; \mathbf{x}^T(k-h)\right]^T \quad (116)$$

$$V(\mathbf{x}(k)) = \mathbf{x}^T(k)P\mathbf{x}(k) + \sum_{j=k-h}^{k-1} \mathbf{x}^T(j)Q\mathbf{x}(j) \quad (117)$$

Theorem 20. The uncertain discrete time-delay system (112-114) is quadratically stable if and only if there are matrices $P > 0$ and $Q > 0$ and a scalar $\delta > 0$ such that the following LMI holds

$$\begin{pmatrix} Q-P & 0 & A_0^T P & \delta N_0^T & 0 \\ (*) & -Q & A_1^T P & \delta N_1^T & 0 \\ (*) & (*) & -P & 0 & PM \\ (*) & (*) & (*) & -\delta I & 0 \\ (*) & (*) & (*) & (*) & -\delta I \end{pmatrix} < 0 \quad (118)$$

Stojanovic, *Debeljkovic* (2008).

Theorem 21. The uncertain discrete time-delay system (112-114) is quadratically stable if and only if there are matrices $L > 0$ and $W > 0$ and a scalar $e > 0$ such that the following LMI holds

$$\begin{pmatrix} (W-L) & 0 & L A_0^T & L N_0^T \\ (*) & -W & L A_1^T & L N_1^T \\ (*) & (*) & \left(-L + eMM^T\right) & 0 \\ (*) & (*) & (*) & -eI \end{pmatrix} < 0, \quad (119)$$

Stojanovic, Debeljkovic (2008).

Linear LARGE SCALE discrete time delay systems

Here, in this section, we examine the so-called delay–dependent criteria based usually on advanced computational procedures.

In the existing literature, the majority of stability conditions of linear discrete large scale time delay systems were obtained during the design process of a decentralized control system, in order to stabilize the system under consideration.

To overcome the difficulties of centralized control methods, many researches have proposed as alternatives various decentralized control methods *Sandel*, *et al.* (1978).

These methods involving simplification of model descriptions, effective procedures of testing the stability and/or hierachical optimization.

Lee, *Radovic* (1987, 1988) studied the stabilization problem for time–delay large–scale systems with or without perturbations.

The aim of many previous works was, among other things, to obtain only sufficient conditions of stabilization of large scale time delay systems.

In contrast, the major contributions which will be presented in the sequel are *necessary* and *sufficient* conditions of the asymptotic stability of linear discrete *large scale time delay* systems *dependent of delay*.

The obtained conditions of stability are expressed in the form of the Lyapunov discrete matrix equation.

At that, it was necessary first to solve the system of matrix equations using an appropriate matrix entering the expression of the mentioned Lyapunov equation.

Starting from the fact that discrete large scale systems are finite-dimensional, the necessary and sufficient condition of stability were derived, independent of time delay, based on the equivalent matrix of a given large scale system.

Consider inear discrete–time large scale autonomous systems composed of N interconnected $\mathbf{S}_i$.

Each subsystem $\mathbf{S}_i$ is described as

$$\mathrm{S}_i: \quad \mathbf{x}_i(k+1) = A_i \mathbf{x}_i(k) + \sum_{j=1}^{N} A_{ij} \mathbf{x}_j(k - h_{ij}), \tag{120.a}$$

with an associated function of the initial state

$$\begin{gathered} \mathbf{x}_i(\theta) = \boldsymbol{\psi}_i(\theta) \\ \theta \in \{-h_{m_i}, -h_{m_i}+1, \ldots, 0\}, \quad 1 \le i \le N \end{gathered}, \tag{120.b}$$

$\mathbf{x}_i(k) \in \mathbb{R}^{n_i}$ is the state vector, $A_i \in \mathbb{R}^{n_i \times n_i}$ denotes the system matrix and $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ represents the interconnection matrix between the i-th and the j-th subsystems.

The constant delay h_{ij} is a positive integer and $h_{m_i} = \max_j h_{ji}$.

Let $V(\mathbf{x}(k)): \mathbb{R}^n \to \mathbb{R}$ so that $V(\mathbf{x}(k))$ is bounded for and for which $\|\mathbf{x}(k)\|$ is also bounded.

Let us observe system (120) *consisting of two subsystems,* $N=2$.

Theorem 22. Given the following system of matrix equations

$$\mathrm{A}_1^{h_{11}} Q_{11} = A_{11}, \tag{121.a}$$

$$\mathrm{A}_1^{h_{21}} Q_{21} = S A_{21}, \tag{121.b}$$

$$\mathrm{A}_1^{h_{12}} S Q_{12} = A_{12}, \tag{121.c}$$

$$\mathrm{A}_1^{h_{22}} S Q_{22} = S A_{22}, \tag{121.d}$$

$$\mathrm{A}_1 S = S \mathrm{A}_2, \tag{122.a}$$

$$\mathrm{A}_i = A_i + Q_{1i} + Q_{2i}, \ 1 \le i \le 2, \tag{122.b}$$

where A_1, A_2, A_{11}, A_{12}, A_{21} and A_{22} are the matrices of system (120) for N=2, n_i being the subsystems orders with the matrices $S \in \mathbb{C}^{n_1 \times n_2}$ and $Q_{ji} \in \mathbb{C}^{n_i \times n_i}$.

Then:

(i) There is a solution for the system of matrix equations (121–122) upon the matrix $\mathbf{A}_1 \in \mathbb{C}^{n_1 \times n_1}$.

(ii) The eigenvalues of the matrix A_1 belong to a set of roots of the characteristic equation of system (120) for $N=2$, *Stojanovic*, *Debeljkovic* (2004.b, 2004.e, 2005.d, 2007).

Theorem 23. Given the following system of matrix equations

$$\mathrm{A}_2^{h_{22}} Q_{22} = A_{22}, \quad \mathrm{A}_2^{h_{12}} Q_{12} = S A_{12}, \tag{123.a}$$

$$\mathrm{A}_2^{h_{21}} S Q_{21} = A_{21}, \quad \mathrm{A}_2^{h_{11}} S Q_{11} = S A_{11}, \tag{123.b}$$

$$\mathrm{A}_2 S = S \mathrm{A}_1, \tag{123.c}$$

$$\mathrm{A}_i = A_i + Q_{2i} + Q_{1i}, \ 1 \le i \le 2 \tag{123.d}$$

where A_1, A_2, A_{11}, A_{12}, A_{21} and A_{22} are the matrices of system (120) for $N=2$, n_i being the subsystems orders with matrices $S \in \mathbb{C}^{n_2 \times n_1}$ and $Q_{ji} \in \mathbb{C}^{n_i \times n_i}$, *Stojanovic*, *Debeljkovic* (2004.b, 2004.e, 2005.d, 2007).

Then:

(i) There is a solution of the system of matrix equations (123) upon $\mathrm{A}_2 \in \mathbb{C}^{n_2 \times n_2}$.

(ii) The eigenvalues of the matrix A_2 belong to a set of roots of the characteristic equation of system (120) for $N=2$.

Corollary 5. If system (120) is asymptotically stable, then the matrices A_1 and A_2, defined by (121–122) and

$$\begin{aligned} &\mathrm{A}_2^{h_{m2}+1} - \mathrm{A}_2^{h_{m2}} A_2 - \mathrm{A}_2^{h_{m2}-h_{22}} A_{22} \\ &- \mathrm{A}_2^{h_{m2}-h_{12}} S A_{12} = 0_{n_2} \end{aligned} \tag{124}$$

$$\begin{aligned} &\mathrm{A}_2^{h_{m1}+1} S - \mathrm{A}_2^{h_{m1}} S A_1 - \mathrm{A}_2^{h_{m1}-h_{11}} S A_{11} \\ &- \mathrm{A}_2^{h_{m1}-h_{21}} A_{21} = 0_{n_2 \times n_1} \end{aligned} \tag{125}$$

$$\mathbf{A}_2 \in \mathbb{C}^{n_2 \times n_2} \text{ and } S \in \mathbb{R}^{n_2 \times n_1}, \tag{126}$$

respectively, are discrete stable, *Stojanovic*, *Debeljkovic* (2004.b, 2004.e, 2005.d, 2007).

Theorem 24. System (120), for $N=2$, is asymptotically stable if and only if for a given matrix $R = R^* > 0$ there exists a matrix $P = P^* > 0$ as a solution of the following *discrete Lyapunov matrix equation*

$$\mathrm{A}_1^* P \mathrm{A}_1 - P = -R. \tag{127}$$

where the matrix $\mathbf{A}_1 \in \mathbb{C}^{n_1 \times n_1}$ is defined by the system of matrix equations (121–122), *Stojanovic*, *Debeljkovic* (2004.b, 2004.e, 2005.d, 2007).

Theorem 25. System (120), for $N=2$, is asymptotically stable if and only if for a given matrix $R = R^* > 0$ there exists a matrix $P = P^* > 0$ as a solution of the following *discrete Lyapunov matrix equation*

$$\mathrm{A}_2^* P \mathrm{A}_2 - P = -R. \tag{128}$$

where the matrix A_2 is defined by the system of matrix equations (125 - 126) *Stojanovic*, *Debeljkovic* (2004.b, 2004.e, 2005.d, 2007).

Linear LARGE SCALE discrete time delay interval systems

Interval systems, with *Soh* (1991) or without *Ozturk* (1988) delays, have been extensively studied in recent years. This is due not only to theoretical interests but also to a powerful tool for the robust system analysis and practical control design *Li*, *Souza* (1997).

In this section, based on the results given in *Lee*, *Hsien* (1997), the new sufficient conditions of asymptotic stability of large-scale time-delay interval systems are presented using the Gersgorin theorem.

We consider a linear composite system defined by two interconnected subsystems S_1 and S_2 with delays

$$\begin{aligned} &S_1: \\ &\mathbf{x}_1(k+1) = A_1\mathbf{x}_1(k) + A_{11}\mathbf{x}_1(k-h_{11}) + A_{12}\mathbf{x}_2(k-h_{12}), \\ &S_2: \\ &\mathbf{x}_2(k+1) = A_2\mathbf{x}_2(k) + A_{22}\mathbf{x}_2(k-h_{22}) + A_{21}\mathbf{x}_1(k-h_{21}) \end{aligned} \tag{129}$$

where $\mathbf{x}_i(k) \in \mathbb{R}^{n_i}$ represent the state of the subsystem S_i, $A_i \in \mathbb{R}^{n_i \times n_i}$ and $A_{ij} \in \mathbb{R}^{n_i \times n_j}$, $1 \le i \le 2$, $1 \le j \le 2$ are the interval matrices and $h_{ij} > 0$, which may *not be an integer*, denote the delays in the interconnections.

It is assumed that the elements $\left[a_{ij}^k\right]$ and $\left[a_{ij}^{rs}\right]$ of the matrices A_k and A_{rs}, have the following properties

$$\begin{aligned} &\underline{a}_{ij}^k \le a_{ij}^k \le \overline{a}_{ij}^k, \qquad \underline{a}_{ij}^{rs} \le a_{ij}^{rs} \le \overline{a}_{ij}^{rs}, \\ &1 \le k \le 2, \quad 1 \le r \le 2, \quad 1 \le s \le 2 \end{aligned} \tag{130}$$

where $\underline{a}_{ij}^k$, $\overline{a}_{ij}^k$, $\underline{a}_{ij}^{rs}$ and $\overline{a}_{ij}^{rs}$ are know constants.

Lemma 13. (*Gersgorin theorem*) If $M = [m_{ij}] \in \mathbb{R}^{n\times n}$, then every eigenvalue λ of the matrix M satisfies at least one of the conditions

$$|\lambda - m_{ii}| \le \sum_{\substack{j=1 \\ j\ne i}}^{n} |m_{ij}|, \; 1 \le i \le n \tag{131}$$

Lemma 24. (*Gersgorin theorem*) If $M = [m_{ij}] \in \mathbb{R}^{n\times n}$, then every eigenvalue λ of the matrix M satisfies at least one of the conditions

$$|\lambda - m_{ii}| \le \sum_{\substack{j=1 \\ j\ne i}}^{n} |m_{ji}|, \; 1 \le i \le n \tag{132}$$

Define

$$\begin{aligned} G^k = [g_{ij}^k] \in \mathbb{R}^{n_k\times n_k}, \quad 1 \le k \le 2, \quad g_{ii}^k = \overline{a}_{ii}^k, \\ g_{ij}^k = \max\left\{ |\underline{a}_{ij}^k|, |\overline{a}_{ij}^k| \right\}, \quad i \ne j, \end{aligned} \tag{133}$$

$$\begin{aligned} G^{rs} = [g_{ij}^{rs}] \in \mathbb{R}^{n_r\times n_s}, \quad 1 \le r \le 2, \quad 1 \le s \le 2, \\ g_{ij}^{rs} = \max\left\{ |\underline{a}_{ij}^{rs}|, |\overline{a}_{ij}^{rs}| \right\}, \end{aligned} \tag{134}$$

$$\begin{aligned} \mathrm{E}^k = [e_{ij}^k] \in \mathbb{R}^{n_k\times n_k}, \quad 1 \le k \le 2, \\ e_{ij}^k = \max\left\{ |\underline{a}_{ij}^k|, |\overline{a}_{ij}^k| \right\}, \end{aligned} \tag{135}$$

Theorem 26. If the following conditions hold

$$\min\{ R_r, R_c \} < 1, \tag{136}$$

$$R_r = \max\left\{ R_r^1, R_r^2 \right\}, \qquad R_c = \max\left\{ R_c^1, R_c^2 \right\}, \tag{137}$$

$$R_r^1 = \max_{1\le i\le n_1} \left\{ \sum_{j=1}^{n_1} \left(e_{ij}^1 + g_{ij}^{11} \right) + \sum_{j=1}^{n_2} g_{ij}^{12} \right\}, \tag{138}$$

$$R_r^2 = \max_{1\le i\le n_2} \left\{ \sum_{j=1}^{n_2} \left(e_{ij}^2 + g_{ij}^{22} \right) + \sum_{j=1}^{n_1} g_{ij}^{21} \right\}, \tag{139}$$

$$R_c^1 = \max_{1\le i\le n_1} \left\{ \sum_{j=1}^{n_1} \left(e_{ji}^1 + g_{ji}^{11} \right) + \sum_{j=1}^{n_2} g_{ji}^{21} \right\}, \tag{140}$$

$$R_c^2 = \max_{1\le i\le n_2} \left\{ \sum_{j=1}^{n_2} \left(e_{ji}^2 + g_{ji}^{22} \right) + \sum_{j=1}^{n_1} g_{ji}^{12} \right\} \tag{141}$$

then system (129) is asymptotically stable, *Stojanovic*, *Debeljkovic* (2005.a).

Conclusion

Different contributions, in the area of Lyapunov stability, to linear discrete time delay systems have been presented. This matter includes a particular class of a before-mentioned class of systems as well large scale systems of the same type. Some of the results derived have been successfully extended to the robustness stability consideration.

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Stabilnost linearnih diskretnih sistema sa čistim vremenskim kašnjenjem u smislu Ljapunova: Pregled radova

Ovaj rad daje detaljan pregled doprinosa mnogih autora na polju proučavanja stabilnosti u smislu Ljapunova za posebne klase linearnih diskretnih sistema sa kašnjenjem. U tom smislu diskretna Ljapunovljeva jednačina je od posebnog interesa.
U radu je, takođe, razmtaran i problem robusnosti stabilnosti.
Ovaj pregled pokriva period posle 2002. godine sve do današnjih dana i ima snažnu nameru da predstavi osnovne koncepte i doprinose koji su se pojavili tokom pomenutih godina u celom svetu a koji su obavljeni u respektabilnim međunarodnim časopisima ili saopšteni na tematskim konferencijama ili prestižnim (renomiranim) konferencijama internacionalnog značaja.

Ključne reči: **linearni sistem, diskretni sistem, stabilnost sistema, sistem sa kašnjenjem, stabilnost Ljapunova, asimptotska stabilnost.**

Ustoj~ivostx linejnwh neprerwvnwh sistem so ~istwm vremennwm zapazdwvaniem v **значении Ляпунова**: Obzor rezulxtatov

Nasto}|a} rabota daët podrobnwj obzor **вклада** mnogih avtorov v oblasti issledovani} ustoj~ivosti **в значении Ляпунова для** osobogo klassa linejnwh neprerwvnwh sistem so ~istwm vremennwm zapazdwvaniem. **В этом значении особый интерес представляет уравнение Ляпунова.**
В настоящей работе тоже рассматривана и проблема живучести устойчивости.
$tot obzor rezulxtatov ohvatwvayt period posle 2005-ogo goda do sih por i u nego vwrazitelxnoe namerenie predstavitx osnovnwe koncepcii i vkladw v &toj oblasti sozdanwe v celom mire v upom}nutom periode i opublikovannwe v peredovwh me`dunarodnwh `urnalah ili pokazanw i**ли** predstavlenw **на тематических** konferenci}h **или** na vwday|ihs} me`dunarodnwh konferenci}h.

Kly~evwe slova: Лinejna} sistema, нeprerwvna} sistema, ustoj~ivostx sistemw, sistema so zapazdwvaniem, ustoj~ivostx Л}punova, **асимптотическая** ustoj~ivostx.

Stabilité des systèmes linéaires discrets à délai temporel pur au sens de Lyapunov: Tableaux des résultats

Ce papier donne un tableau détaillé des contributions de nombreux auteurs dans le domaine des études sur la stabilité dans le sens de Lyapunov pour les classes des systèmes linéaires discrets à délai. Dans ce sens l'équation discrète de Lyapunov est de particulier intérêt. On a également considéré ici le problème de la robustesse de la stabilité. Le tableau présenté dans ce papier comprend la période après l'an 2002 jusqu'à nos jours et a pour but de présenter les concepts basiques et les contributions qui ont apparu au cours de la période citée dans le monde entier et qui sont publiés dans les revues internationales réputées ou présentés lors des conférences de prestige et d'importance internationale.

Mots clés: **système linéaire, système discret, stabilité du système, système à délai, stabilité de Lyapunov, stabilité asymptotique.**