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Numerical Analysis of Gear Set Dynamic Behavior

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In this paper, one dynamic model of a real gear set has been presented. In this case, the simplified model contains two discreet masses with elastic and damped connections. The basic principles of mechanics, with the initial and boundary conditions taken into account, were applied in establishing a system of differential equations describing the physicality of motion. Such system of differential equations represents a mathematical model at dynamical behaviour of gear transmission. The established system of differential equations has been solved in a numerical way. The paper presents the solution of the main form of oscillating and stress condition, using the finite element method.

Key words: **gear, gear set, dynamic behaviour, dynamic load, own oscillations, mathematical modelling, numerical analysis, system of differential equations, finite element method.**

Introduction

IN solving dynamic problems, a real engineering system is replaced by a simplified dynamic model. This model consists of continually or discretely positioned masses with elastic and damped connections among individual members.

A system of equations describing physicality of the process of motion is set by applying the basic principles of mechanics and taking into consideration the initial and boundary conditions. That system of equations represents the mathematical model of the machine or system in dynamic behaviour.

The equations of motion of dynamic systems that are analog to the equations of equilibrium in the static analysis, can be derived on the basic of D'Alambert principle, the principle of virtual displacements or Hamilton principle [1, 2, 3, 4, 11].

In order to minimize the error in modelling dynamic systems, it is necessary to represent the system with a greater number of masses, i.e., degrees of freedom, [6, 10, 13, 21].

On the other hand, an increased number of degrees of freedom make the mathematical model more complex and the solving of a given system of equations more difficult.

Formulation of equations in the finite element method

To formulate the equations of motion in the finite method, we start from Hamilton variation principle, [1, 2, 3, 4].

If the total kinetic energy of the system is denoted by E_k, the total potential energy of the internal and external forces by Π and W denoting the work of non-conservative forces including the damping forces too, then the Hamilton variation principle is expressed by the term

$$\int_{t_1}^{t_2} \delta\left(E_k - \Pi\right) dt + \int_{t_1}^{t_2} \delta W \; dt = 0 \tag{1}$$

that is

$$\delta \int_{t_1}^{t_2} \mathrm{L} \; dt = 0 \tag{2}$$

where L is functional Lagrange in the form

$$L = E_k - \Pi + W \tag{3}$$

Based on the Hamilton principle it is shown that [2, 3] "of all possible displacements which satisfy the compatibility conditions and geometry conditions on the contour, as well as the initial conditions at the time points t_1 and t_2, the real displacements are those for which the Lagrange functional has a stationary value". In the special case when $E_k = 0$ and $W = 0$ there follows the statement of stationarity of potential energy, that is

$$\delta \Pi = 0 \tag{4}$$

which is also the initial point for formulations in the finite element method.

Let us assume the following functional dependences

$$E_K = E_K(q_i, \dot{q}_i)\,;\; \Pi = \Pi(q_i);\; \delta W = Q_i \delta q_i \qquad i = 1, 1, \ldots, N \tag{5}$$

where q_i, generalized displacements, $\dot{q}_i$ generalized velocities and Q_i generalized non-conservative forces.

With the assumption that there is a solution, by applying a partial integration and taking into account that $\delta q_i\,(t_1)$ =

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$\delta q_i(t_2) = 0$, equation (2) is reduced to Lagrange equation of other type, that is

$$\frac{d}{dt}\left(\frac{\partial E_k}{\partial \dot{q}_i}\right) - \frac{\partial E_k}{\partial q_i} + \frac{\partial \Pi}{\partial q_i} = Q_i \tag{6}$$

Taking into account the equations for kinetic and potential energy, displacements of velocity, acceleration and deformation in the finite element expressed through interpolation functions, as well as the expression for damping forces, the matrix equation of motion of the finite element is obtained in the following form

$$m\,\ddot{q}_e + c\,\dot{q}_e + k\,q_e = Q_e \tag{7}$$

In this equation

$m = \int_V N^T\ \rho\ N\ dV$ matrix of mass of the element,

$c = \int_V N^T\ c\ N\ dV$ matrix of damping of the element,

$k = \int_V B^T\ D\ B\ dV$ stiffness matrix of the element,

$Q_e = \int_V N^T\ F\ dV + \int_S N^T\ F_n\ dS$ vector of generalized forces

The equation of motion for the finite element system, [2, 3], i.e., the structure, is formed on the basis of the matrix equations of motion for individual finite elements

$$M\ \ddot{q} + C\ \dot{q} + K\ q = Q \tag{8}$$

where *M*, *C* and *K* are the matrices of mass, damping and stiffness of the structure, and *Q* is the vector of generalized forces in the nodes of the finite element system.

Determination of own vibrations of structural elements is a separate problem often referred to as the problem of own values or eigen problem. The result of studying free oscillations of the system are own frequencies and own vectors. The simplest case is the system without damping and action of external forces so it is described by a differential equation

$$M\ \ddot{q} + Kq = 0 \tag{9}$$

If the vector of generalized nodal displacements is expressed in the form of

$$q = \bar{q}\ e^{i\omega t}, \quad e^{i\omega t} = \cos\ \omega t + i\sin\ \omega t$$

then the matrix equation has the form of

$$(K - \omega^2\ M)\ \bar{q} = 0 \tag{10}$$

and it is a system of algebraic equations per unknown amplitudes $\bar{q}$. In order for this system of equations to have solutions (apart from the trivial $q = 0$), it is necessary to have the determinant of the system equal to zero, i.e., [2, 3, 7]:

$$\left|K - \omega^2 M\right| = 0 \tag{11}$$

This equation is known as the characteristic equation of the system. *Solutions of this equation (roots)* ω_1^2, of which generally there are *n*, are represented by the squares of own frequencies of the system or own values of the system which we usually arrange by increasing order $\omega_1^2 < \omega_2^2 < \omega_3^2 <, \ldots, < \omega_n^2$. For each own value there is an own vector $\bar{q}$ which represents the own form or node of vibrations.

The vectors of natural or own forms possess the characteristic of modal orthogonality that can be expressed by the term

$$\begin{aligned} \bar{q}_i^T\ M\ \bar{q}_j &= \delta_{ij} = \begin{Bmatrix} 1, & i = j \\ 0, & i \neq j \end{Bmatrix} \\ \bar{q}_i^T\ K\ \bar{q}_j &= \omega_i^2\ \delta_{ij} \end{aligned} \tag{12}$$

so, if the vectors $\bar{q}_i$ are arranged in such a way that they form the columns of the matrix, then the matrix obtained represents the modal matrix which is used to transform generalized coordinates into main coordinates.

Formation of dynamic model of gear transmission

Dynamic behaviour of gear transmission is primarily the result of changes of external load as well as of occurrence of internal dynamic forces. The change of external load originates from the change of force of the power engine and also from the very fact that in engagement single and double mesh are alternated [2, 3, 4, 5, 6]. Additional internal dynamic forces depend on rigidity of teeth, displacement of profile, manufacturing errors (mesh, pitch, shape and the others) peripheral velocity, mass of rotary parts rigidity of twisting and bending of the shaft, etc. Moreover, internal dynamic forces occur due to elastic deformations of the teeth. The occurrence of internal dynamic forces has been studied for years. In the beginning, empirical formulas were used in these studies (Barth, AGMA). Later, Backingham, Tuplin and others have developed new methods for computing the internal dynamic forces, and they have used energy methods by taking into account the masses and rigidity of teeth. Further in the text a model of oscillation of a gear set will be described.

Kinetic energy of discrete mass in the presented dynamic model in case of translatory or rotary motion is expressed as

$$E_k = \frac{1}{2} m\, v^2 = \frac{1}{2} m\, \dot{q}^2 \quad or \qquad E_k = \frac{1}{2}\, J\, \dot{\phi}^2 \tag{13}$$

where *m* is the mass, *J* is the axial moment of mass inertia, *q* and φ are the generalized coordinates of movement of the points observed and $\dot{q}$ and $\dot{\varphi}$ are the generalized velocities of the corresponding coordinates of motion.

Elastic connections between the members of the dynamic model are represented in a symbolic way by means of torsion and flexion springs. The potential energy of the springs is

$$\Pi = \frac{1}{2} c\, q^2; \qquad \Pi = \frac{1}{2} c_1\, (\phi_1 - \phi_2)^2 \tag{14}$$

where c_1 and c_2 are the rigidity of the corresponding springs.

With real dynamic systems in operation there occurs energy loss as a consequence of the presence of resisting forces. Energy of losses (dissipation) which defines damping of the system in case when the damping force is in a linear dependence from velocity, is expressed as

$$W = \frac{1}{2}\, b\, \dot{q}^2 \tag{15}$$

where *b* is the coefficient of proportionality of the resisting forces (damping coefficient) and $\dot{q}$ derivatives of generalized coordinates.

On the basic of Lagrange equations of other types, differential equations of the second order are written and the solving of this system defines the dynamic process of discrete points.

Coupled gears, or single-step gear transmission, can be represented by an oscillatory model, Fig.1, [3, 4, 8, 10, 14, 17, 18, 21] , where $F_n(t)$ denotes the initiating force, while m_{r1} and m_{r2} are reduced masses of the gears.

Reduced masses of individual gears are

$$m_{r1} = \frac{J_1}{r_{b1}^2}, \qquad m_{r2} = \frac{J_2}{r_{b2}^2} \tag{16}$$

where J_1 and J_2 are the moments of inertia of rotary masses and r_{b1} and r_{b2} are the radii of base circles at gears.

The initiating force is of the following form:

$$F_n(t) = F_0 + F_1 \sin \Omega_1 t_1 . \tag{17}$$

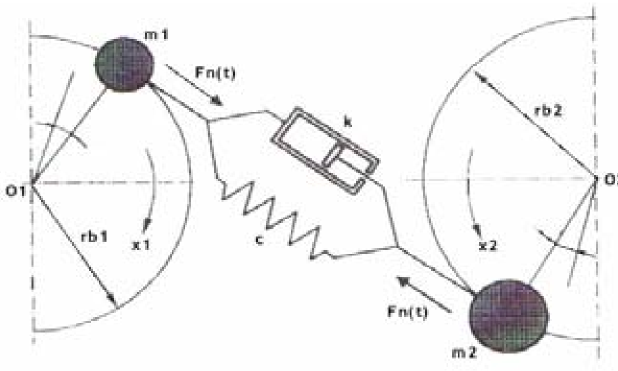

Figure 1. A simplified dynamic model of the gear set

It means that the teeth mesh is replaced by the action of a variable force at teeth flanks F_n(t) which changes in proportion to the deformation of teeth in the mesh. It is assumed here that the rigidity of the teeth in the mesh is equal to the medium rigidity c_0.

By applying the above described procedure, [3, 21], we can demonstrate that this model is described by a system of differential equations

$$\begin{aligned} m_{r1}\,\ddot{x}_1 + k\,(\dot{x}_1 - \dot{x}_2) + c_0\,(x_1 - x_2) &= F_n\,(t) \\ m_{r2}\,\ddot{x}_2 - k\,(\dot{x}_1 - \dot{x}_2) - c_0\,(x_1 - x_2) &= -F_n\,(t) \end{aligned} \tag{18}$$

In case the gear supports O_1 and O_2 are absolutely rigid, the total displacement in the direction of the contact line will be $x = x_1 - x_2$.

The corresponding matrices for the above system, i.e. the matrix of the damping coefficient **C**, the matrix of elasticity **K** and the matrix of inertia **M,** have the following forms

$$\mathbf{C} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}; \ \mathbf{K} = \begin{bmatrix} c_0 & -c_0 \\ -c_0 & c_0 \end{bmatrix};$$

$$\mathbf{M} = \begin{bmatrix} m_{r1} & \\ & m_{r2} \end{bmatrix}$$

Let us assume the homogenous part of the solution of this system in the form, [7]

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} e^{\lambda t} \tag{19}$$

which, by substituting in (18) gives the system of algebraic equations which has the characteristic polynomial

$$f(x) = \begin{vmatrix} m_{r1}\,\lambda^2 + k\,\lambda + c_0 & -c_0 - k\,\lambda \\ -k\,\lambda - c_0 & m_{r2}\,\lambda^2 + k\,\lambda + c_0 \end{vmatrix} = 0 \tag{20}$$

In this paper, the solutions of this polynomial are obtained numerically.

Presentation of the solutions

A gear set with the following geometry parameters, Table 1, is taken as an example that demonstrates the finite element method and the modal analysis.

Table 1. Parameters of the gear set

Parameter	Gear 1	Gear 2
Number of teeth	38	65
Modul	0.003 m	0.003 m
Standard angle	20º	20º
Material	Č4320	Č4320

The gear geometries and the mesh of gears have been generated automatically by using the program package NASTRAN and discretized using four node tetrahedral elements. The total number of nodes is 29777 and the total number of elements is 160491(Fig.2).

Figure 2. Gear set FME model

Boundary conditions have been defined per displacements (all displacements at gear and shaft are equal to zero), Fig.3. The load is specified in the contact points of the gears (the drive power is P = 7.5 kW and the revolution number is n = 800 min^{-1}).

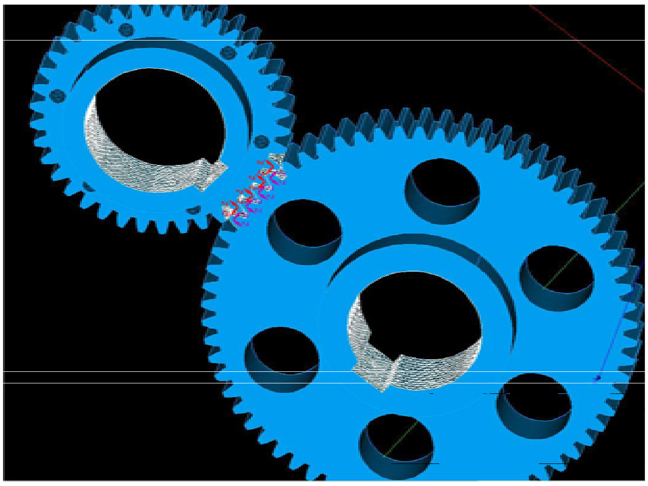

Figure 3. Displacement and force boundary conditions

The first four main shapes of oscillation of the system of two gears are shown in Fig.4, 5, 6, and 7.

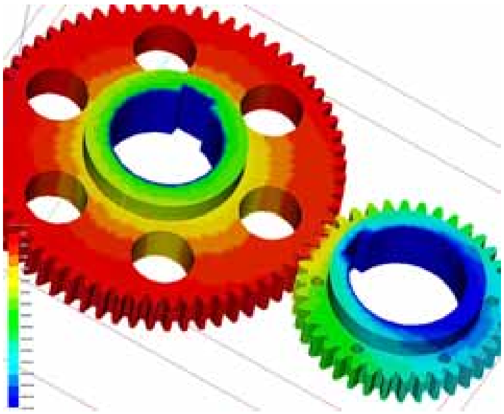

Figure 4. First mode shape (*f*=1631.48 Hz)

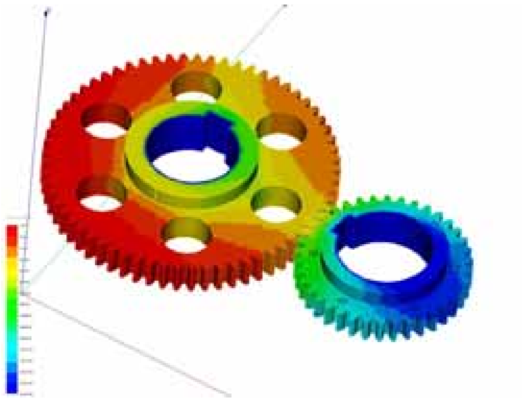

Figure 5. Second mode shape, *f*=1650.80 Hz

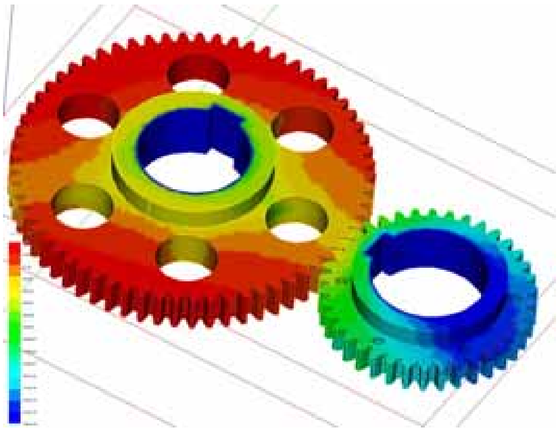

Figure 6. Third mode shape (*f*=1755.15 Hz)

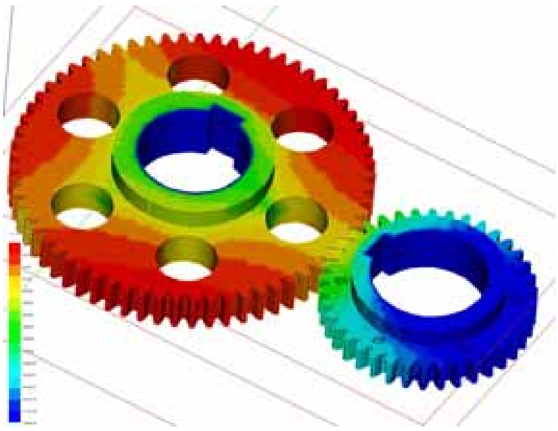

Figure 7. Fourth mode shape (1849.75 Hz)

We have further analyzed the stress and strain conditions demonstrated in the gear set, and the results are shown in Figs.8 and 9.

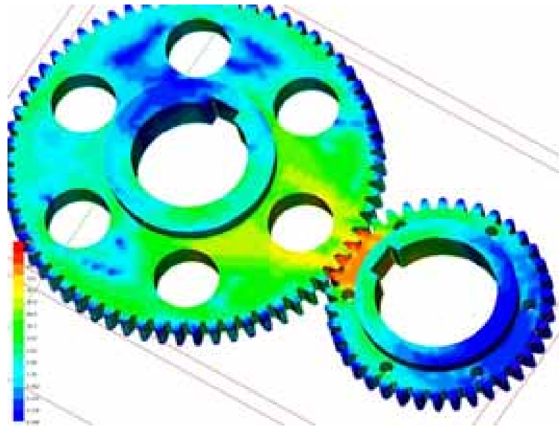

Figure 8. Strain distribution for t = 0,02

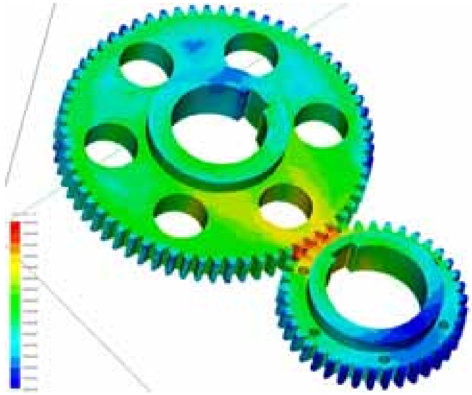

Figure 9. Stress distribution for t = 0.02 by Von Mises theory

The conclusion and guidelines for further research

On the basis of the results shown in this paper it has been concluded that the methodology developed to study the dynamic behaviour of complex systems is very efficient. It gives a lot of possibilities and can be easily upgraded for analyses of other effects.

The modal analysis and the analysis of the stress and strain conditions suggest that the system of only two gears is very complex and that it is almost impossible to include all the effects by such and similar research.

Further research should be directed at studying the effects of mutual dynamic impact of teeth in engagement, as well as at including connection between the shaft and the gear into the dynamic model and the like.

In accordance with the present trend of application of new materials, as in [19], the authors will in future studies simulate the dynamic behavior of a gear made of composite materials.

The main aim of future research will be a definition of parameters that would lead to an increase of gear service life, as in [15, 20, 21, 22], under cyclic loading.

Furthermore, future research will be focused on studying the conditions under which teeth will be fractured.

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Numerička analiza dinamičkog ponašanja zupčastih prenosnika

U radu je predstavljen uprošćeni dinamički model kojim se zamenjuje realan zupčati par. Uprošćeni model se u ovom slučaju sastoji od dve diskretne mase sa elastičnim i prigušnim vezama. Primenom osnovnih principa mehanike i uzimanjem u obzir početnih i graničnih uslova, postavljen je sistem diferencijalnih jednačina koji opisuje fizikalnost kretanja. Takav sistem jednačina predstavlja matematički model pri dinamičkom ponašanju zupčastih prenosnika. Postavljeni sistem diferencijalnih jednačina je rešen numeričkim putem. U radu je prikazano rešenje, glavnih oblika oscilovanja i naponskog stanja, dobijeno primenom metode konačnih elemenata.

Ključne reči: **zupčasti prenosnik, zpčasti par, dinamičko ponašanje, dinamičko opterećenje, sopstvene oscilacije, matematičko modelovanje, numerička analiza, sistem diferencijalnih jednačina, metoda konačnih elemenata.**

Цифровой анализ динамического поведения зубчатых переносных механизмов

В настоящей работе представлена упрощенная динамическая модель, которой заменяется реальная зубчатая пара. Упрощенная модель в даном случае состоит из двух дискретных масс с упругими и демпфирующими связями. С применением основных принципов механики и учитывая начальные и предельные условия, установлена система дифференциальных уравнений, которая описывает способность физического движения. Такая система уравнений представляет математическую модель при динамическом поведении зубчатых переносных механизмов. Установленная система дифференциальных уравнений решена цифровым способом (путём). В настоящей работе показано решение главных форм колебаний и состояний напряжения, получено применением метода конечных элементов (МКЭ).

Kly~evwe slova: **зубчатые переносные механизмы, зубчатая пара, динамическое поведение, динамическая нагрузка, собственые колебания, математическое моделирование, цифровой анализ, система дифференциальных уравнений, метод конечных элементов.**

Analyse numérique du comportement dynamique des engrenages

Un modèle dynamique simplifié qui remplace le pair d'engrenages réel est représenté dans ce papier. Dans le cas présent le modèle simplifié se compose de deux masses discrètes aux liens élastiques et étouffés. En appliquant les principes basiques de la mécanique et en considérant les conditions initiales de base on a établi un système d'équations différentielles qui décrit la physique du mouvement. Ce système d'équations représente le modèle mathématique au cours du comportement dynamique des engrenages. Le système des équations mathématique établi a été résolu par la voie numérique. Dans ce travail on a présenté la solution des formes principales de l'oscillation et de l'état de tension obtenue par l'application de la méthode des éléments finis.

Mots clés: **engrenage, couple de dents, comportement dynamique, charge dynamique, oscillations naturelles, modélisation mathématique, système des équations différentielles, analyse numérique, méthode des éléments finis.**