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The Application of the Limit Analysis Theorem and the AdaptationTheorem for Determining the Failure Load of Continuous Beams

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By applying the limit theorem analysis of structures it is possible to determine the limit line load of the systems that are exposed to the load which increases proportionally to the formation failure mechanism. In the case when the linear systems are subjected to repeated load, the limit theorem does not provide an adequate solution, so alongside were developed theorems of adaptation that have enabled the determination of the safe limit loads. This paper presents a method for determining the load that leads to failure of continuous beams on two fields using the limit theorem and the theorem of adaptation and the limit of the breaking load change and breaking load incremental depending on the length of the field of the beam.

Key words: **failure mechanic, failure mechanism, continuous beam, beam bending, bending moment, critical force, adaptation method.**

Introduction

WHEN the structure is exposed to the load of the proportional nature that gradually increases, at some point it reaches a certain critical value, at which point it comes to plastic failure of the structure (ie, unlimited increase of deformation at constant load), after which a construction is no longer able to receive further increase of the load. This critical state is called the limit state of the construction, and load that causes it is the limit load. Determination of the bearing power of structures (limit load) is an important factor in designing structures.

The limit analysis of structures is an alternative analytical method to determine the maximum load parameter or increasing load parameter, which a perfect elastic-plastic construction is able to bear . Compared to the incremental analysis (the step-by-step method), the efficiency of the limit analysis is achieved by observing the final state, state of failure, without paying attention to what was happening with the construction and load from the moment when one section of the structure was completely plasticized (formation of the first plastic joint for solid beam) or one rod lattice was completely plasticized (formation of first plastic truss rod), until the failure. Limit analysis methods are based on the theorem of plastic failure of an ideal elasto-plastic body. These theorems are known as static (lower) and kinematic (upper) theorems of the marginal analysis of structures.

It should be noted that in addition to the limit state of load there are other limit states, which may occur before the state of limit equilibrium and which can be restrictive to the transferring of an external load, such as limit states of usability, or even a marginal state of cracks in structures made of reinforced or pre-stressed concrete [8].

Although some ideas emerged in the 18th century, the marginal analysis is of more recent date. Its origins are linked to Kazincy (1914.), who calculated failure load of mutually squeezed beams and this result was confirmed experimentally. A similar concept was proposed by Kist (1917) and Grüning (1926). However, early work in this area largely relied on engineering intuition. Although the static theorem was first proposed by Kist (1917) as an intuitive axiom, it is considered that the basic theorems of the limit analysis were first presented by Gvosdev in 1936 and released two years later at a local Russian conference, but they went unnoticed by Western authors until 1960. when translated and published by Haythornthwaite. In the meantime, a formal proof of this theorem for beams and frames is presented by Horne (1949) as well as Greenberg and Prager (1951).

The application of the adaptation theory (shakedown theory), when assessing the safety of elastic-plastic structures exposed to variable and repeated load, is important and often indispensable. In this context, the "shakedown" is a term that was introduced by Prager, meaning that after the initial appearance of plastic deformation structures behave purely elastically during their further life. The contrary state that leads to the shakiness of the structure is called "nonadaptation" of the

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construction. In this case construction collapses due to perceived failure of one or both types of failure, called alternating plasticity and incremental collapse. The first type of failure occurs due to recurrence of plastic deformation of the opposite sign (with no accumulation of plastic deformation), thus causing the phenomenon of low-cycle fatigue. The second type of failure occurs due to the accumulation of plastic strain at each cycle of loading (progressive deformation), causing reduction in the durability of the construction.

The analysis of adaptation (shakedown analysis) belongs to the class of "simplified" methods that avoid the necessity of monitoring the entire flow response of the structures on a given repeated load. In addition, it represents a significant generalization of the theorem of limit analysis.

The most important tool to control the plastic behavior of structures is the application of static and kinematic theorems of adaptation (shakedown theorems) proposed by Melanie (1938) and Koitera (1960). These two theorems have been successfully applied in a large number of problems (Maier, 1969; Polizzoto 1982; König 1987; Kaliszky 1996; Weichert and Maier 2001).

The aim of this paper is to present the use of static and kinematic theorems of the limit analysis of structures when exposed to the effects of the load that grows proportionally, and the application of the adaptation theorem in determining the safe load limit of continuous beams on two fields exposed to two parameter load.

Basic settings of the limit analysis

The calculation of structures by applying the theory of plasticity allows plastification of materials, that is to say, out of the boundaries of elastic behaviour.

In the area of elastic behavior of the structure, stresses and deformations are proportionally dependent. Increasing the load affecting the structure leads to a gradual increase in stress until a stress level in the most stressed fiber (or fibers, in the case of a symmetrical section) reaches a value of the yield stress. Further increase of load leads to plasticization of the cross section, in other words, it leads to the increase of the plasticity zone, which gradually expands in height and in length of the beam, until it comes to the plasticization of the entire cross section, and therefore the formation of a plastic joint [3].

It is known that, for statically determined beams, plasticization of one section of the structure (by forming a plastic joint in the area of the maximum bending moment) is followed by the loss of load bearing capacity and the transition of a beam into a mechanism. Unlike statically determined beams, with statically indeterminate beams, the formation of a plastic joint does not lead to the formation of a mechanism of failure. The bearing capacity of an n times statically indeterminate structure will be fully depleted when $n+1$ plastic joints are formed within the structure.

For determining the limit loads, the following assumptions are introduced:

- deformations are proportional to the deviation from the neutral axis (Bernoulli hypothesis of straight sections is valid),
- an idealized elasto-plastic dependency for materials applies for tension stress as well as pressure.
- deformations are small,
- section has the necessary ductility,
- conditions of balance of the cross-section are met, of normal forces $\Sigma X=0$, as well as the bending moment $\Sigma M=0$.

In order of the limit load of a structure to be determined by applying the theory of plasticity, first it is necessary to prove that an applicable limit state will be caused by formation of the mechanism of failure, in other words, it is necessary to eliminate the occurrence of any other limit states. It is necessary to exclude the occurrence of fatigue because of the effects of variable load, then the possibility of local instability prior to reaching full plasticization and exclude the appearance of any effects that would lead to failure of the structure before the formation of a sufficient number of plastic joints for its transition into the mechanism of failure. [7]

In the theory of the limit analysis the following assumptions apply:

- sections where the bending moment is less than the moment of plasticization of the cross-section, are in the elastic range;
- section in which full plastic moment of the cross section (M_p) happened is the perfect plastic joint;
- turning of section, after reaching the plastic moment, grows without limit without further increasing the load,
- body is made of elastic-perfect plastic material with infinite surface flow.

It can be said that one beam is in a state of limit balance when the bearing capability of the construction is fully exhausted, and in a sufficient number of sections the beam behaves completely plastically [8]. Based on this we can conclude that when it comes to forming a sufficient number of plastic joints, deformities are progressive, and the beam transforms into the failure mechanism. The moment that immediately precedes the formation of the mechanism of failure represents the moment of the limit balance of the system.

Limit theorem analysis

The basic theorem of the limit analysis can be applied to all types of static systems, be they statically determined or statically indeterminate. The basic theorems of the limit analysis are:

- static theorem or the theorem on the lower edge of the limit loads and
- kinematic theorem or the theorem about the upper limit of the ultimate load.

The static theorem is based on the static equilibrium of an observed system. For a statically indeterminate system, one can assume a large distribution of bending moments that satisfy the equilibrium conditions, due to a given external load. Greenberg and Prager (1952) called this distribution statically admissible. If such a system satisfies the condition of plasticity, or in any section the bending moment did not exceed the appropriate value, it is said that it is secure. A necessary condition is that there must be at least one certain distribution of moments in the system, which is statically admissible. The static theorem states that this condition is sufficient to ensure the bearing capacity.

The static theorem can be stated as follows: *if there is any distribution of bending moment in a static system that is both safe and statically admissible due to load λP, then the value λ must be less or equal to the failure load factor λ_C, ($\lambda_C > \lambda$). The actual limit load ($\lambda_C P \le P_p$ may be equal to or greater than a given one.*

Based on this theorem it can be concluded that due to the given load λP there is no distribution of bending moment that is both safe and statically admissible, then it is greater λ than the load factor at failure λ_C. We could also conclude that a static system can really bear the limit load without breaking since the λ_C is the maximum load factor at which one cannot achieve a static equilibrium without forming plastic joints.

The kinematic theorem refers to the possible mechanism of failure. The failure mechanism comprises a kinematically unstable system to which a beam is transformed by the joints installed in the beam sections where it is possible. [3].

In the case when the failure mechanism is known, the load factor at the failure λ_C, or the limit load ($\lambda_C P$) is determined by equating the work of external forces with the work absorbed in plastic joints. In case the mechanism of failure load corresponding to the limit load is not known in advance, the equation of work can be written for each assumed failure mechanism, where the values of (λP) will be obtained and which corresponds to the presumed mechanisms of failure.

The kinematic theorem can be stated as follows: *for a given static system, which is exposed to the external load λP, the value λ corresponding to any presumed mechanism of failure must be greater than or equal to the failure load factor λ_C*, or $\lambda_C \geq \lambda$.

By combining the static and kinematic theorems a single theorem can be established. By applying both theorems we get the upper or lower limit of the area in which the load factor is at failure. The static theorem gives the value of force, or the load factor λ_C, for which the existing distribution of bending moments is both safe and statically admissible. From the kinematic theorem it is well known that there is no mechanism of failure for which the corresponding factor λ is less than λ_C.

The theorem of singularity can be expressed as follows: *for a given static system and load, if there is at least one safe and statically admissible bending moment distribution, in which the plastic moments occur in a sufficient number of cross sections, in order to establish a mechanism, the corresponding load factor of failure should be λ_C.*

Basic postulates of the adoption method

In the adaptation method all assumptions that have been introduced in the method of the marginal analysis of structures are valid, where this method allows the analysis of structural behavior of the construction that is exposed to repeated stress. In the marginal structural analysis it is known that due to structural relief there are some residual deformations which cause the appearance of residual bending moments. The distribution of the residual bending moment in the structural elements is in equilibrium when it is relieved. The marginal analysis does not allow the introduction of residual bending moments occurring in the construction into the calculation, but it is possible by the adaption method.

If the structure is made of materials in which the initial stress state is zero (mild material), the load applied to the construction leads to stress which is in some individual cross sections above the elastic limit, then the bending moment is between the moment of elastic strain M_y and the moment of full plasticity of the cross section M_p. As the structure is in the elastic-plastic area in the case of unloading, the moment-curvature relationship is linear until the change of bending moment of the cross section is $2M_y$.

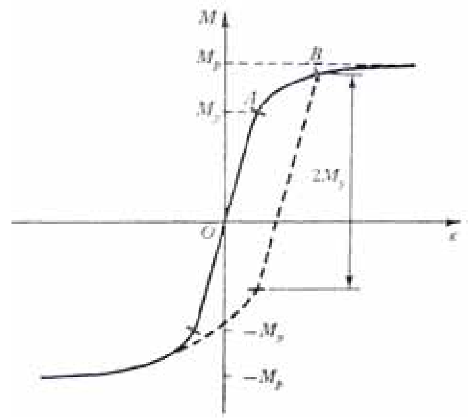

Figure 1. Relation moment–curve in the adaptation theory

In the diagram depending moment-curvature (Fig.1) shows that the size of the elastic moment of relief is $2M_y$. This diagram of depending on the moment and curvature is valid in the theory of adaptation. Values of momentum of flow M_y and the momentum of plasticity M_p, have the same value as tensile and compressive. In addition, the yield bending moment, within which occur purely elastic behavior, remains $2M_y$ regardless of the previous load history.[3]

Theorem of adaptation method

Theorems of adaptations have a role to set the main conditions under which plastic flow in the construction finally stops, no matter how often and in what order the load is applied [5]. As in the marginal analysis of structures, in the methods of adaptation there are static and kinematic theorems on the basis of which it is possible to determine the safe limit load depending on the type of variable load.

The bending moment of the observed cross section j is given by:

$$M_j = m_j + \mathcal{M}_j , \quad (1)$$

where the:

- M_j – actual bending moment of the cross section,
- $\mathcal{M}_j$ – elastic bending moment of the cross section and m_j – residual bending moment of the cross section.

Any distribution of residual bending moments, defined in this way, must be statically possible in the case when the structure is relieved, because the moments M_j and $\mathcal{M}_j$ must be balanced with the external load [4].

So we can say that the construction is adapted under the influence of variable repeated load, if at some point the condition (1) is met, and all subsequent load causes only elastic change of bending moments. Then it is possible to determine the size of the safe limit load, which, depending on the nature of repeated loads, can be:

- incremental limit load,
- alternating limit load.

Under the terms of (1) the static theorem of adaptation can be expressed in the following form: *if there is any distribution of residual bending moment in the construction m_j and if the distribution is statically possible in case when the construction is not loaded, and thereby satisfied for each cross-section j, one of the following conditions are necessary to be met*:

$$m_j + \lambda \mathrm{M}_j^{\max} \leq \left(M_p\right)_j , \quad (2)$$

$$m_j + \lambda M_j^{\min} \geq -\left(M_p\right)_j , \tag{3}$$

$$\lambda\left(M_j^{\max} - M_j^{\min}\right) \leq 2\left(M_e\right)_j . \tag{4}$$

The value λ will be equal to or less than the safe limit load factor λ_S.

A construction tends to adapt to the effect of variable repeated loads in the best possible way. So, if λ the size of λ_S, S is exceeded, there is unlimited plastic flow and in this case no distribution of residual moments is possible, which is a necessary condition for determining the safe load limit. Similarly, under the influence of the proportional load there will be structural failure when the load factor λ reaches a value of λ_C, above which a construction is not secure, and at the same time there is a possible distribution of the static bending moment. Depending on the calculated load factor λ it is possible, based on meeting some of equations (2) and (3), as well as an incremental criterion of plasticity and equation (3), as alternative terms of plasticity, to determine the safe limit load which depends on the type of variable repeated loads.

The static theorem of adaptation was first implemented by Bleich (1932), for the ram that is twice statically indeterminate. A generalization of his theorem was given by Melan (1936), and the application of Melanoma theorem for solving concrete problems was given by Symonds and Prager (1950) and Neal (1951).

Application of the static theorem of adaptation is possible only if the distribution of residual bending moments is already known [6].

Application of the static theorem is justified only in structures with a lower level of static indeterminability. As the application of kinematic theorem of adaptation is based on an assumed mechanism of failure, whose form is identical to the form of the failure mechanism in the marginal structural analysis, so it can be said that this procedure is simpler for determining the safe load limit. However, it is evident that the corresponding equations do not contain the size of the residual bending moment.

Assuming that the observed failure mechanism is known and matches the incremental failure mechanism, it is possible to observe rotations of formed plastic joints in a number of characteristic sections θ [6]. If θ is at any section positive θ^+, then we can say that the total bending moment in this section seeks to achieve value $+M_P$, and if the rotation of the formed plastic joint is negative θ^-, the bending moment tends to reach the value $-M_P$. Based on the introduced assumptions, equation (2) and (3) can be written as:

$$m_j + \lambda M_j^{\max} = \left(M_p\right)_j \quad \text{for} \quad \theta_j^+ , \tag{5}$$

$$m_j + \lambda M_j^{\min} = -\left(M_p\right)_j \quad \text{for} \quad \theta_j^- . \tag{6}$$

If equations (5) and (6) are multiplied by the appropriate rotation of the formed plastic joint in the cross section j, as follows:

$$m_j\theta_j + \lambda M_j^{max}\theta_j^+ = \left(M_p\right)_j \left|\theta_j\right| , \tag{7}$$

$$m_j\theta_j - \lambda M_j^{max}\theta_j^- = \left(M_p\right)_j \left|\theta_j\right| . \tag{8}$$

By adding equations (7) and (8) for all the plastic joints that are formed on the observed failure mechanism we get:

$$\sum m_j\theta_j + \lambda\left[\sum M_j^{\max}\theta_j^+ + \sum M_j^{\max}\theta_j^-\right] = \sum\left(M_p\right)_j \left|\theta_j\right| . \tag{9}$$

As the m is the distribution of residual moments which are in balance when the structure is relieved, and the rotation, a θ of the cross-section in which a plastic joint is formed, the principle of the virtual work equation can be written in the following format: $\sum m_j\theta_j = 0$. The basic equations of incremental failure can be written as:

$$\lambda\left[\sum M_j^{\max}\theta_j^+ + \sum M_j^{\max}\theta_j^-\right] = \sum\left(M_p\right)_j \left|\theta_j\right| . \tag{10}$$

On the basis of equation (10) the kinematic theorem of adaptation can be stated as follows: *the parameter value λ corresponding to any assumed mechanism of failure (alternating λ_a or incremental λ_I), must be greater than or equal to the value of the parameter of a safe limit load λ_S.*

The kinematic theorem of adaptation in this form is first set by Koiter (1956, 1960), although it can be said that he did so on the basis of work [9] of P. S. Symods and B. G. Neal, which was published on the first national congress of applied mechanics in Chicago, 1951. They started with the assumption that the work of all the remaining moments on the possible mechanism of failure is equal to zero. In this paper we present the method of calculating the incremental force of the failure by applying the methods of Symonds and Neal.

The limit load of continuous beam in two fields

Determination of the limit load of a continuous beam in two fields (Fig.2) will be conducted by applying the principle of virtual work by a gradual as well as direct method. A beam is once statically indeterminate, where you can see three characteristic sections (2, 3, 4) in which it is necessary to determine the bending moments. To determine the bending moment in the marked sections it is necessary to write three equations, including one compatibility equation which is obtained by applying the principle of virtual forces, while with the application of the principle of virtual displacements two equations of equilibrium are obtained.

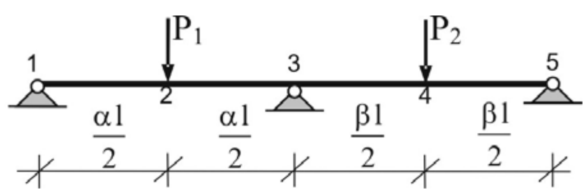

Figure 2. Continuous two–span beam loaded by concentrated forces in the middle of span

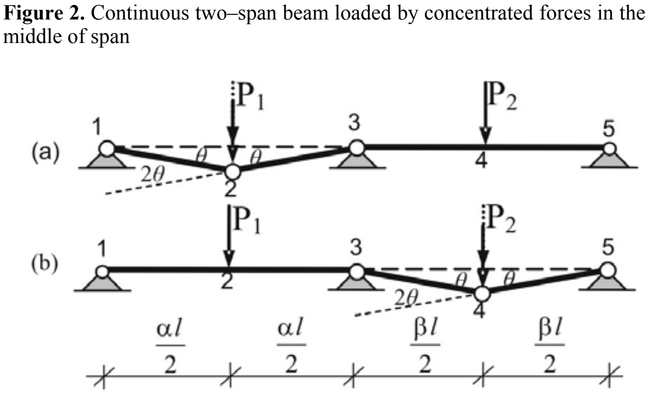

Figure 3. (a) Failure mechanism of the first field, (b) Failure mechanism of the second field

By insertion of joints in characteristic sections failure mechanisms are formed leading to virtual displacement and virtual rotation of characteristic sections of the beam. The equations of balance are obtained by considering independent mechanisms of failure, where, in this case, two independent mechanisms of failure can be seen (Fig.3 (a) and Fig.3 (b)).

Equations of balance can be presented in the following form:

$$M_2(2\theta)+M_3(-\theta)=P_1\frac{l}{2}\theta, \tag{11}$$

$$M_3(\text{-}\theta)+M_4(2\theta)=P_2\frac{l}{2}\theta. \tag{12}$$

As equations of balance of a static system do not depend on the characteristics of materials of which it is made, they must be satisfied regardless of whether the static system is in the elastic or plastic region.

Compatibility equations are obtained by applying principles of virtual forces at the moment when the observed system is unloaded. As the system is unloaded, the distribution of the bending moment does not exist, so therefore it is assumed that along the elements of the static system there is an arbitrary (virtual) distribution of bending moments (m) (Fig.4) which was first introduced by Heyman (1961.) [5, 6].

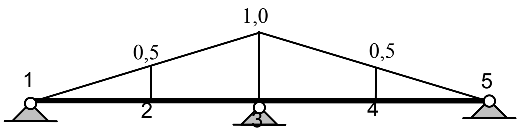

Figure 4. Assumed distribution of bending moments of the continuous beam on two fields.

A number of independent distributions of residual bending moments (Fig.4) equals the number of static indeterminacy of a static system. The assumed distribution of bending moments must satisfy equations (13) and (14) which are written by applying the principle of virtual displacement at the moment when the system is unloaded, and for the possible independent failure mechanisms:

$$2m_2-m_3=0, \tag{13}$$

$$-m_3+2m_4=0 \tag{14}$$

Under the influence of the load that affects the static system it leads to bending moment and rotation of cross sections whose distribution is given in the last two rows of Table 1. Applying the principles of virtual forces a compatibility equation is obtained, as follows:

$$3M_2+5M_3+3M_4+\frac{12\text{EI}}{l}(\theta_2+2\theta_3+\theta_4)=0\,. \tag{15}$$

Table 1. Continuous beam bending moments on two fields

virtual distribution						
Cross section		1	2	3	4	5
m	i	0	0,5	1,0	0,5	0
actual distribution						
EIκ=M		0	M_2	M_3	M_4	0_5
θ		0	θ_2	θ_3	θ_4	0

Equilibrium equations (11) and (12), as well as compatibility equations (15), make the system of equations (16) by which it is possible to determine the bending moments in the characteristic sections:

$$\begin{bmatrix}2 & -1 & 0\\ 0 & -1 & 2\\ 3 & 5 & 3\end{bmatrix}\begin{bmatrix}\Delta M_2\\ \Delta M_3\\ \Delta M_4\end{bmatrix}+\frac{12EI}{l}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}\Delta\theta_2\\ \Delta\theta_3\\ \Delta\theta_4\end{bmatrix}=\begin{bmatrix}\frac{\Delta P\alpha l}{2}\\ \frac{\Delta P\beta l}{2}\\ 0\end{bmatrix} \tag{16}$$

With the system of equations (16) it is possible to determine gradually the value of the load after the formation of each plastic joint, until development of a failure mechanism, simultaneously determining the rotations of the sections in which the plastic joints formed. In order to obtain the limit load in an one-parameter form, in the system of equations (16) forces P_1 and P_2 shall be substituted by force P.

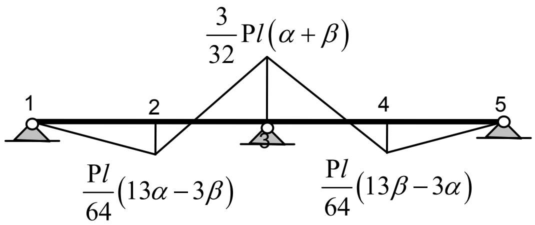

Figure 5. Elastic bending moment distribution in a function of load

Solving equation system (16) provided that the rotation of the characteristic cross-sections (θ_2, θ_3, θ_4) is equal to zero, we get the bending moment of the section when the beam is in the elastic range (Fig.5). By equating the highest values of the bending moment (section 3) and cross-sectional moment of plasticity, we get the size of the load that leads to the formation of the first plastic joint:

$$M_3=\frac{3}{32}Pl(\alpha+\beta)=M_p \;\Rightarrow P_1=\frac{32M_p}{3l(\alpha+\beta)}. \tag{17}$$

After the formation of a plastic joint in section 3, the beam becomes statically determined, and the distribution of bending moments is shown in Fig.6.

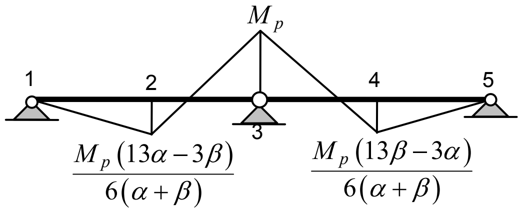

Figure 6. Distribution of bending moment after the formation of the first plastic joint

Further increasing the load, the bending moment of the section 3 is equal to the moment of plasticity of the intersection, where there is a rotation of the section. Substituting conditions (18) in the system of equations (16) we obtain the distribution of bending moments in the function of load growth after the formation of the first plastic joint (Fig.7).

$$\begin{aligned}&M_3=\text{M}_\text{p},\;\;\Delta M_3=0,\;\;\Delta\theta_3>0,\\&\Delta\theta_2=\Delta\theta_4=0.\end{aligned} \tag{18}$$

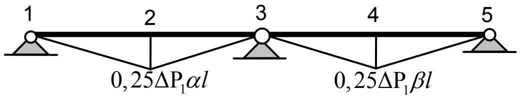

Figure 7. Distribution of the bending moment in the function of growth after the formation of the first plastic joint

Increase of the load that leads to the formation of other plastic joints in section 2 is obtained by superimposing the values of bending moments (Fig.6 and Fig.7) and:

$$-\frac{M_p(13\alpha-3\beta)}{6(\alpha+\beta)}-0,25\Delta P_1\alpha l=-M_p \Rightarrow$$
$$\Delta P_1=\frac{M_p(18\beta-14\alpha)}{3\alpha l(\alpha+\beta)}, \qquad (19)$$

while the increase stress that leads to the formation of a plastic joint section 4 is:

$$-\frac{M_p(13\beta-3\alpha)}{6(\alpha+\beta)}-0,25\Delta P_1\beta l=-M_p \Rightarrow$$
$$\Delta P_1=\frac{M_p(18\alpha-14\beta)}{3\alpha l(\alpha+\beta)}, \qquad (20)$$

a rotation of the section in which the first plastic joint is formed is:

$$\Delta\theta_3=-\frac{\Delta P_1 l^2(\alpha+\beta)}{32EI}. \qquad (21)$$

Load that leads to the formation of plastic joints of sections 3 and 2, and thus the formation mechanism of the partial failure of the first field (α≥β) is:

$$P_2=P_1+\Delta P_1=\frac{32M_p}{3l(\alpha+\beta)}+\frac{M_p(18\beta-14\alpha)}{3\alpha l(\alpha+\beta)}, \qquad (22)$$

while the load which leads to the formation of the plastic hinge section 3 and 4 and the partial failure mechanism (α≤β) is:

$$P_2=P_1+\Delta P_1=\frac{32M_p}{3l(\alpha+\beta)}+\frac{M_p(18\alpha-14\beta)}{3\alpha l(\alpha+\beta)}. \qquad (23)$$

Based on expressions (22) and (23) it is possible to display a marginal change of the breaking load in one-parameter form, depending on the change in the length of field of the beam (Fig.8). On the diagram, it can be concluded that increasing the length of the field gives a decrease of force that leads to the formation mechanism of failure. So, if $\alpha\geq\beta$ it comes to the formation of partial mechanism of failure of the first field, and when $\beta\geq\alpha$ when it comes to the formation of a partial failure mechanism of the second field of the beam, in the case when $\alpha=\beta$ leads to the formation of the mechanism of failure in both fields simultaneously.

The principle of virtual work can be applied to determine the ultimate load of the line system by the direct method. The equation of virtual work is set to the previously presumed mechanism of failure. In this case, the equation of virtual work of all external forces is performed, with the work absorbed in sections where plastic joints are assumed. The limit load is the least of all limit loads obtained on the presumed mechanisms of failure.

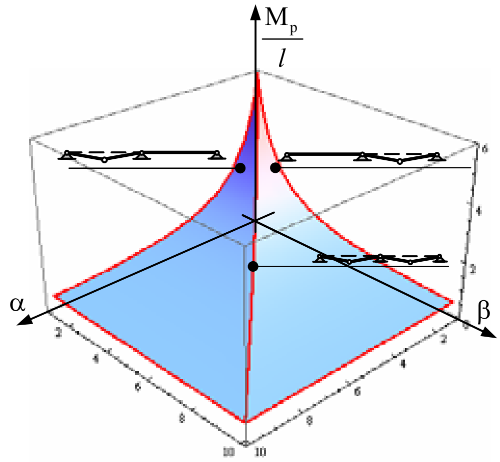

Figure 8. Change of the threshold breaking force depending on the change of α i β

For the observed beam (Fig.2), three mechanisms of failure, two independent (Fig.9.a. and 9.b.) and a combined (Fig.9.c) one, can be formed.

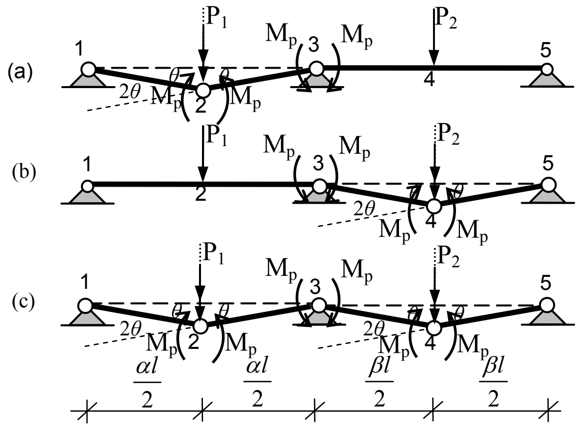

Figure 9. (a) The mechanism of failure of the first field, (b) The mechanism of failure of the second field, (c) Combined failure mechanism in both fields

Using the equation of virtual work for each of the possible mechanisms of failure marginal failure forces are obtained given by equations (15), (16) and (17):

$$M_p(2\theta)+M_p\theta=P_1\frac{\alpha l}{2}\theta,$$

$$P_1=\frac{6M_p}{\alpha l}, \qquad (24)$$

$$M_p\theta+M_p(2\theta)=P_2\frac{\beta l}{2}\theta,$$

$$P_2=\frac{6M_p}{\beta l}, \qquad (25)$$

$$M_p(2\theta)+M_p(\theta)+M_p(\theta)+M_p(2\theta)=P_1\frac{\alpha l}{2}\theta+P_2\frac{\beta l}{2}\theta,$$

$$12M_p=l(P_1\alpha+P_2\beta). \qquad (26)$$

For each of the failure mechanisms one limit breaking force was obtained, the one of which that is the lowest is at the same time the force that will lead to the formation of the mechanism of failure. The size of limit loads obtained by the direct method is equal to the threshold breaking force, which is obtained using the gradual method.

When the structure is simultaneously acted on by two independent loading systems of an arbitrary ratio, the analysis of limit loads can be made based on the interactive diagrams. Equations 24, 25 and 26, obtained by the direct proceedings principle of virtual work, .are used for the construction of the interaction diagram. The interconnection of failure mechanism and the relationship between loads is best observed in the interaction diagram.

With the interactive diagram (Fig.10) it can be concluded that the ratio of the load defined on the basis of the segment AB leads to the formation of the mechanism of failure in the second field when the ratio of the load is $P_1/P_2 \leq 1$, while for the segment bc the ratio of the load is $P_1/P_2 \geq 1$ and it leads to failure mechanism in the first field.

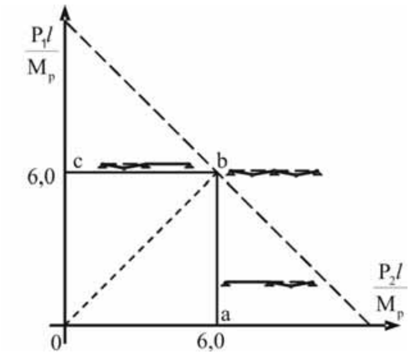

Figure 10. Interaction diagram

For any ratio of the load within the area 0abc0 the construction is safe to the occurrence of the failure mechanism. If the load ratio is such that it is defined by one of the segments, it leads to the formation of the mechanism of failure defined by that segment.

The limit load of continuous beams in two fields exposed to repeated stress

Application of the theorem of adaptation will be shown on the example of a continuous beam in two fields (Fig.11) for which in the earlier part of the work the limit breaking force is determined using static and kinematic theorems in the case when the load increases proportionally until the load limits.

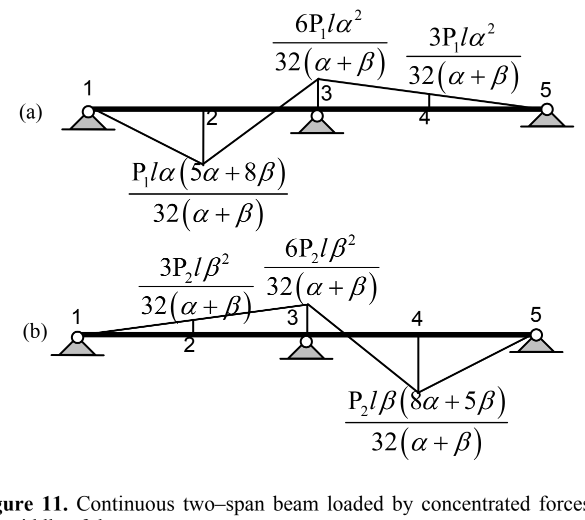

Figure 11. Continuous two–span beam loaded by concentrated forces in the middle of the span

If we apply the force P_1 to the continuous beam in the first field, and then the force P_2 in the second field, the distribution of elastic bending moments is as shown in Fig.12. As the beam system is once statically indeterminate there is only one possible distribution of the residual bending moment (Fig.13).

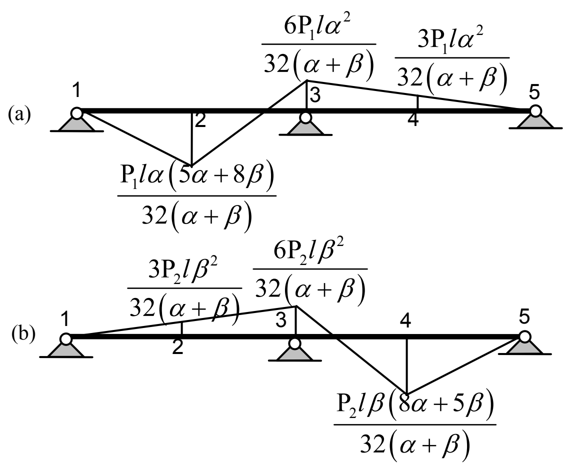

Figure 12. Elastic bending moment of the continuous two–span beam

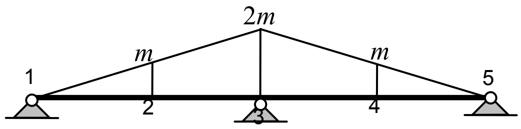

Figure 13. Possible distribution of the retained bending moment

Let the load acting on the continuous slab have values located in the following range:

$$0 \leq P_1 \leq P_1, \quad 0 \leq P_2 \leq P_2.$$

Applying the static theorem of adaptation on the basis of equations (2) and (3), as well as the incremental criterion of plasticity and equation (4) as an alternative condition of plasticity in the first field of beam, it is possible to write the following system of equations for section 2:

$$\frac{P_1 \alpha l(5\alpha+8\beta)}{32(\alpha+\beta)} - m \leq M_p, \quad (27.1)$$

$$\frac{3P_2\beta^2 l}{32(\alpha+\beta)} + m \leq M_p, \quad (27.2)$$

$$\frac{P_1 \alpha l(5\alpha+8\beta)}{32(\alpha+\beta)} - \left(-\frac{3P_2\beta^2 l}{32(\alpha+\beta)}\right) \leq 2M_e, \quad (27.3)$$

for section 3, over intermediate support:

$$\frac{6P_1\alpha^2 l}{32(\alpha+\beta)} + \frac{6P_2\beta^2 l}{32(\alpha+\beta)} + 2m \leq M_p, \quad (28.1)$$

$$0 + 2m \leq M_p, \quad (28.2)$$

$$0 - \left(-\frac{6P_1\alpha^2 l}{32(\alpha+\beta)} - \frac{6P_2\beta^2 l}{32(\alpha+\beta)}\right) \leq 2M_e, \quad (28.3)$$

while for the cross section 4, the following equations can be written

$$\frac{P_2\beta l(8\alpha+5\beta)}{32(\alpha+\beta)}-m\le M_p, \quad (29.1)$$

$$\frac{3P_1\alpha^2 l}{32(\alpha+\beta)}+m\le M_p, \quad (29.2)$$

$$\frac{P_2\beta l(8\alpha+5\beta)}{32(\alpha+\beta)}-\left(-\frac{3P_1\alpha^2 l}{32(\alpha+\beta)}\right)\le 2M_e. \quad (29.3)$$

Solving the system of equations (27.1) and (28.1) and equations (28.1) and (29.1), we get the size of the incremental limit forces that lead to the formation of the mechanism of failure, as well as the size of the corresponding residual bending moments in the two-parameter form. Applicable marginal load only depends on the relationship coefficients α i β. Thus, if $\alpha\ge\beta$, the failure mechanism forms in the first field, the size of the incremental force of failure and the residual moment are:

$$8P_1\,\alpha(\alpha+\beta)+3P_2\,\beta^2=\frac{48M_p(\alpha+\beta)}{l}, \quad (30.1)$$

$$m=\frac{P_1\,\alpha l(8\beta-\alpha)-6P_2\,\beta^2 l}{96(\alpha+\beta)}, \quad (30.2)$$

while in the case when $\beta\ge\alpha$, the failure mechanism is formed in another field, so the incremental breaking force and the residual moment are:

$$8P_2\,\beta(\alpha+\beta)+3P_1\,\alpha^2=\frac{48M_p(\alpha+\beta)}{l}, \quad (31.1)$$

$$m=\frac{P_2\,\beta l(8\alpha-\beta)-6P_1\,\alpha^2 l}{96(\alpha+\beta)}. \quad (31.2)$$

By solving equations (27.3) (28.3) and (29.3) we get the failure forces on the basis of conditions of alternative plasticity for the section in the first field, based on equation (27.3):

$$P_1\,l\alpha(5\alpha+8\beta)+3P_2\,l\beta^2=64M_e(\alpha+\beta), \quad (32.1)$$

for the section over support:

$$3P_1\,l\alpha^2+3P_2\,l\beta^2=32M_e(\alpha+\beta), \quad (33.1)$$

and for section 4 in the second field:

$$P_2\,l\beta(8\alpha+5\beta)+3P_1\,l\alpha^2=64M_e(\alpha+\beta), \quad (34.1)$$

The size of the limit of the threshold breaking force under alternative conditions of failure depends on the cross-sectional shape coefficient. In this example, we will adopt that the coefficient of the section of a rectangular shape is $\alpha_{obl}=1.50$.

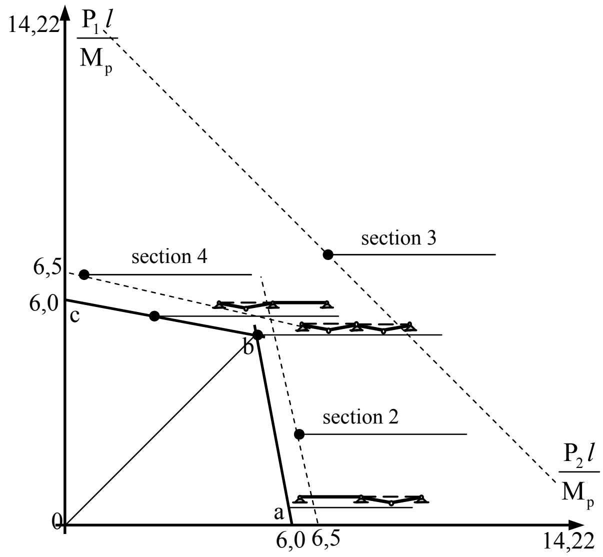

Figure 14. Interaction diagram

When two independent load systems P_1 i P_2 in an arbitrary relation simultaneously operate on the beam, the analysis of critical load can be made based on the interactive diagrams in which mutual interconnectivity of failure mechanism is observed as well as the relationship between loads. Fig.14 shows the interaction diagram in the case where the fields are the same length, $\alpha=\beta=1$,one may observe that the area 0abc0 is the limit area which is defined on the basis of incremental failure conditions.

If the load is in the range $0\le P_1\le P,\ 0\le P_2\le P$, we get the marginal load in an one-parameter form, which depends on the parameters α and β that define the field length of continuous beams.

The following incremental values of the breaking force and the residual bending moment are obtained for the observed cross-section 2 based on the equation (27.1) and (28.1):

$$P=\frac{48M_p(\alpha+\beta)}{l(8\alpha^2+8\alpha\beta+3\beta^2)},\ m=-\frac{M_p(\alpha^2-8\alpha\beta+6\beta^2)}{2(8\alpha^2+8\alpha\beta+3\beta^2)}, \quad (35)$$

while for section 4, based on equations (28.1) and (29.1), we get the incremental breaking force and the residual bending moment:

$$P=\frac{48M_p(\alpha+\beta)}{l(3\alpha^2+8\alpha\beta+8\beta^2)},\ m=-\frac{M_p(\beta^2-8\alpha\beta+6\alpha^2)}{2(8\beta^2+8\alpha\beta+3\alpha^2)}. \quad (36)$$

On the basis of expressions (35) and (36), the diagrams were constructed (Fig.15 and Fig.16) with the introduced incremental changes of the breaking load and the residual bending moment when there is $0\le\alpha\le10$ and $0\le\beta\le10$. In the diagrams, it is possible to observe the changes of the incremental failure force and residual bending moments, depending on the length of the field beam and the reliable mechanism of the failure of beams.

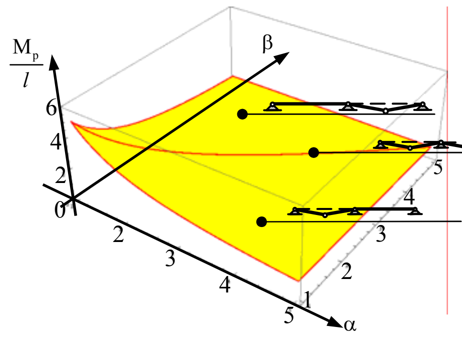

Figure 15. Incremental changes of the breaking force, depending on the α and β

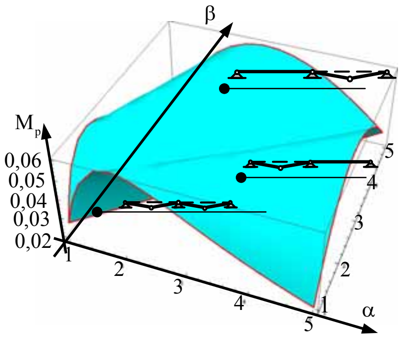

Figure 16. Change of residual bending moment depending on the α and β

An alternative breaking force in an one-parameter form for section 2 is:

$$P=\frac{64M_e}{l(5\alpha+3\beta)}, \tag{37}$$

for section 3 over support:

$$P=\frac{32M_e(\alpha+\beta)}{3l\left(\alpha^2+\beta^2\right)}, \tag{38}$$

for section 4:

$$P=\frac{64M_e}{l(3\alpha+5\beta)}. \tag{39}$$

The application of the kinematic theorem of adaptation to determine the incremental limit load will be displayed using the Symonds and Neals method presented in paper [9].

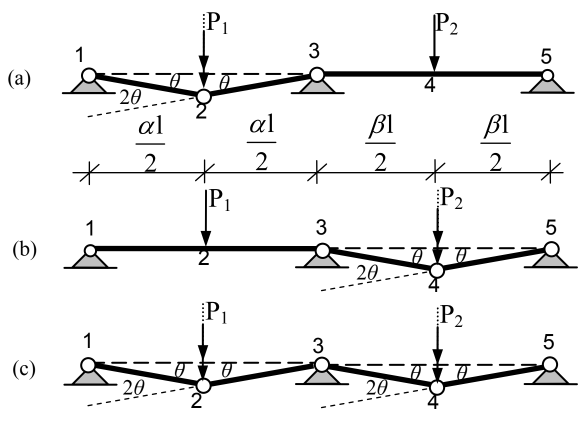

Figure 17. Possible failure mechanism of the continuous two–span beam

Based on the conditions that the residual bending moments on the possible mechanism of failure shown in Fig.17 (a) are in equilibrium, one can write the following equation:

$$m_2(2\theta)+m_3(-\theta)=0, \tag{40.1}$$

$$\left[M_p-\frac{P_1\alpha l(5\alpha+8\beta)}{32(\alpha+\beta)}\right](2\theta)+\left[-M_p-\left(-\frac{6P_1\alpha^2 l}{32(\alpha+\beta)}-\frac{6P_2\beta^2 l}{32(\alpha+\beta)}\right)\right](-\theta)=0. \tag{40.2}$$

Solving equation (40.2), we get the incremental power failure in the two-parameter form:

$$8P_1^0\alpha l(\alpha+\beta)+3P_2^0\beta^2 l=48M_p(\alpha+\beta), \tag{41.1}$$

respectively:

$$P=\frac{48M_p(\alpha+\beta)}{l\left(8\alpha^2+8\alpha\beta+3\beta^2\right)}. \tag{41.2}$$

Based on the mechanism of failure (Fig.17 (b)), it is possible to write the following equation:

$$m_3(-\theta)+m_4(2\theta)=0, \tag{42.1}$$

$$\left[-M_p-\left(-\frac{6P_1\alpha^2 l}{32(\alpha+\beta)}-\frac{6P_2\beta^2 l}{32(\alpha+\beta)}\right)\right](-\theta)+\left[M_p-\frac{P_2\beta l(8\alpha+5\beta)}{32(\alpha+\beta)}\right](2\theta)=0, \tag{42.2}$$

By solving equation (42.2) we obtain:

$$3P_1\alpha^2 l+8P_1\beta l(\beta+\alpha)=48M_p(\alpha+\beta), \tag{43.1}$$

or:

$$P=\frac{48M_p(\alpha+\beta)}{l\left(3\alpha^2+8\alpha\beta+8\beta^2\right)}. \tag{43.2}$$

For the mechanism of failure (Fig.17 (c)), it is possible to write the following equation:

$$m_2(2\theta)+m_3(-2\theta)+m_4(2\theta)=0, \tag{44.1}$$

$$\left[M_p-\frac{P_1\alpha l(5\alpha+8\beta)}{32(\alpha+\beta)}\right](2\theta)+\left[-M_p-\left(-\frac{6P_1\alpha^2 l}{32(\alpha+\beta)}-\frac{6P_2\beta^2 l}{32(\alpha+\beta)}\right)\right](-2\theta)+\left[M_p-\frac{P_2\beta l(8\alpha+5\beta)}{32(\alpha+\beta)}\right](2\theta)=0. \tag{44.2}$$

By solving the equation (44.2) we obtain the incremental force of failure:

$$P_1\alpha l(11\alpha+8\beta)+P_1\beta l(11\beta+8\alpha)=96M_p(\alpha+\beta), \tag{45.1}$$

$$P=\frac{96M_p(\alpha+\beta)}{l\left(11\alpha^2+16\alpha\beta+11\beta^2\right)}. \tag{45.2}$$

The terms obtained by static and kinematic theorems are equal, so that leads to the conclusion that the obtained solution is unique thus satisfying the theorem of uniformity. For different sizes of the coefficients α and β different sizes of the limit and an incremental loading of failure are obtained, where the applicable forms of mechanisms of failure are different.

Conclusion

The paper firstly describes the use of static and kinematic theorems of the limit analysis of structures in determining the limit load for linear support. In both cases the limit state of the beam is observed. The application of these theorems is shown in the example of a continuous beam in two fields loaded with two-parameter or one-parameter load. When the limit load is defined as a two-parameter load, dependency of load and the possible failure mechanism is shown in the diagram of interaction, whereas in the case when the limit load is defined as an one-parameter load, its changes which depend on the length of the beam field are presented. The basic advantage of the marginal analysis method based on limit theorems is reflected in the simplicity and a very rapid determination of ultimate load.

The analysis of the behavior of a linear support subjected to variable load whose intensity is in the pre-defined range is presented for the continuous beam in two fields. The beam is loaded in the middle of the field by concentrated forces. Failure load was determined using static and kinematic theorems of adaptation, as one-parameter and two-parameter load. As the load is always in the same direction, it is possible to determine only incremental load failure.

On the basis of marginal changes of the breaking force (Fig.8) and the incremental force of failure (Fig.15) in an one-parameter form, it can be concluded that the use of the methods of adaptation is justified for certain related coefficients and a continuous beam, while in some cases the load may be marginally determined by the application of the limit theorems since the difference between the limit force and the incremental force of failure is very small.

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Primena teorema granične analize i teorema adaptacije za određivanje opterećenja loma kontinualnog nosača

Primenom teorema granične analize konstrukcija moguće je odrediti granično opterećenje linijskih sistema izloženih opterećenju koje proporcionalno raste sve do formiranja mehanizma loma. U slučaju kada su linijski sistemi izloženi ponovljenom opterećenju granične teoreme ne daju adekvatna rešenja, tako su se paralelno sa njima razvijale i teoreme adaptacije koje su omogućile određivanje sigurnog graničnog opterećenja. U ovom radu je prikazan postupak određivanja opterećenja koje dovodi do loma kontinualnog nosača na dva polja primenom graničnih teorema i teorema adaptacije kao i promena granične sile loma i inkrementalne sile loma u zavisnosti od dužine polja nosača.

Ključne reči: **mehanika loma, mehanizam loma, kontinualni nosač, savijanje nosača, moment savijanja, kritična sila, adaptivna metoda.**

Application de la théorème de l’analyse marginale et la théorème d’adaptation pour la détermination de la charge de fracture de la poutrelle continue

En appliquant la théorème de l’analyse limite de la construction il est possible de déterminer la charge limite des systèmes linéaires exposés à la charge qui augmente proportionnellement jusqu’à la formation du mécanisme de

fracture. Dans le cas où les systèmes linéaires exposés à la charge répétée de la théorème limite ne donnent pas les solutions adéquates on a développé parallèlement les théorèmes d'adaptation qui ont permis la détermination de la charge critique sure. Dans ce travail on a présenté le procédé pour la détermination de la charge qui cause la fracture de la poutrelle continue à deux champs par l'application des théorèmes de limite et théorèmes d'adaptation ainsi que les changements de la force limite de fracture et la force d'incrément de fracture dépendant de la longueur du champ de porteur.

Mots clés: **mécanique de fracture, poutrelle continue, déformation de poutrelle, moment de déformation, force critique, méthode adaptable.**