UDK: 681.511.2.013:517.938 COSATI: 12-01, 14-07

Further Results on Stability of Linear Discrete Time Delay Systems Over the Finite Time Interval: A Quite New Approach

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This paper gives sufficient conditions for the practical and finite time stability of of a particular class of linear discrete time delay systems. $\mathbf{x}(k+1) = A_0 \mathbf{x}(k) + A_1 \mathbf{x}(k-1)$. When we consider the finite time stability concept, these new, delay independent conditions are derived using an approach based on the Lyapunov – like functions. When the practical and attractive practical stability are considered, the above mentioned approach is combined and supported by a classical Lyapunov technique to guarantee *atractivity* properties of the system behavior.

Key words: linear system, discrete system, time delay system, system over the finite time interval, system stability, Non-Lyapunov stability, asymptotic stability.

Introduction

THE problem of investigation of time delay systems has been exploited over many years. Delay is very often encountered in different technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time delay, regardless of whether it is present in the control or/and state, may cause an undesirable system transient response, or generally, even an instability. Consequently, the problem of stability analysis of this class of systems has been one of the main interest of many researchers. In general, the introduction of time lag factors makes the analysis much more complicated. In the existing stability criteria, mainly two ways of approach have been adopted. Namely, one direction is to contrive the stability condition which does not include information on the delay, and the other is the method which takes it into account. The former case is often called the delay-independent criteria and generally provides nice algebraic conditions. Numerous reports have been published on this matter, with particular emphasis on the application of Lyapunov's second method, or on using the idea of matrix measure Lee, Diant (1981), Mori (1985), Mori et al. (1981), Hmamed (1986), Lee et al. (1986).

Practical matters require that we concentrate not only on the system stability (e.g. in the sense of Lyapunov), but also on bounds of system trajectories.

A system could be stable but still completely useless because it possesses undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain subsets of the state-space which are defined a priori in a given problem.

Besides that, it is of particular significance to consider the behavior of dynamical systems only over a finite time interval. These boundedness properties of system responses, i.e. the solution of system models, are very important from the engineering point of view.

Due to this fact, numerous definitions of the so-called technical and practical stability were introduced. Roughly speaking, these definitions are essentially based on the predefined boundaries for the perturbation of initial conditions and allowable perturbation of the system response. In engineering applications of control systems, this fact becomes very important and sometimes crucial, for the purpose of characterizing in advance, in a quantitative manner, possible deviations of the system response.

Thus, the analysis of these particular boundedness properties of solutions is an important step, which precedes the design of control signals, when finite time or practical stability control is concerned.

It should be noticed that up to nowadays, there were no results concerning that problem of non–Lyapunov stability, when the discrete time delay systems are considered.

Some of initial results have been published inthe paper of *Debeljković*, *Aleksendrić* (2003), completely based on the discrete fundamental matrix of a system to be considered. It is well-known that computing the discrete fundamental matrix is generally more difficult than to find the concrete solution of a system of retarded difference equations.

We can admit that these results represented the first extension of the concept of finite time and practical stability to the class of the linear discrete time delayed system. In order to understand better serious problems that cause existing time delay in systems dynamics, but also in forming corresponding criteria, some short recapitulation of some results, derived for ordinary discrete time delayed systems, is presented in the sequel.

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Linear discrete time systems

A specific concept of discrete time systems, practical stability operating on the finite time interval, was investigated by *Hurt* (1967) with a particular emphasis on the possibilities of error arising in the numerical treatment of results.

A finite time stability concept was, for the first time, extended to discrete time systems by *Michel and Wu* (1969).

Practical stability or "set stability", throughout the estimation system trajectory behavior on the finite time interval was given by *Heinen* (1970, 1971). He was the the first who gave necessary and sufficient conditions for this concept of stability, using the Lyapunov approach based on the "discrete Lyapunov functions" application.

Even more detailed analysis of these results considering different aspects of discrete time systems practical stability as well as the questions of their realization and controllability was given by *Weiss* (1972). The same problems were treated by *Weiss and Lam* (1973) who extended them to the class of nonlinear complex discrete systems.

Efficient sufficient conditions of finite time stability of linear discrete time systems expressed through norms and/or matrices were derived by *Weiss and Lee* (1971).

Lam and Weiss (1974) were the first to apply the socalled concept of "*final stability*" to discrete time systems whose motions are scrolled within the time varying sets in the state space.

Some simple definitions connected to sets, representing difference equations or at the same time discrete time systems, were given by *Shanholt* (1974).

Only the sufficient conditions are given by the established theorems. These results are based on the Lyapunov stability and can be used, in a way, for a finite time stability concept, for which reason they are mentioned here.

Grippo and Lampariello (1976) have generalized all; foregoing results and given the necessary and sufficient conditions of different concepts of finite time stability inspired by definitions of practical stability and instability, earlier introduced by *Heinen* (1970).

The same authors applied the before-mentioned results in the analysis of "*large-scale systems*", *Grippo*, *Lampariello* (1978).

Practical stability with settling time was for the first time introduced by *Debeljković* (1979.a) in connection with the analysis of different classes of linear discrete time systems, general enough to include time invariant and time varying systems, systems operated in free or forced operating regimes, as well as the systems the dynamical behavior of which is expressed through the so-called "*functional system matrix*". In the mentioned paper, the sufficient conditions of practical instability and a discrete version of a very well known Bellman–Gronwall lemma have also been derived.

Other papers, *Debeljković* (1979.b, 1980.a, 1980.b, 1983) deal with the same problems and mostly represent the basic results of the PhD. dissertation, *Debeljković* (1979.a).

For the particular class of discrete time systems with the functional system matrix, sufficient conditions have been derived in *Debeljković* (1993).

System description

Systems to be considered are governed by the vector difference equation:

$$\mathbf{x}(k+1) = A(k)\mathbf{x}(k), \qquad (1)$$

$$\mathbf{x}(k+1) = A \,\mathbf{x}(k)\,,\tag{2}$$

$$\mathbf{x}(k+1) = A(k, \mathbf{x}(k))\mathbf{x}(k), \qquad (3)$$

$$\mathbf{x}(k+1) = A(\mathbf{x}(k))\mathbf{x}(k), \qquad (4)$$

$$\mathbf{x}(k+1) = A \mathbf{x}(k) + \mathbf{f}(k), \qquad (5)$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is the state vector and the vector function satisfies: $\mathbf{f} : \mathcal{K}_N \times \mathbb{R}^r \to \mathbb{R}$.

It is assumed also that f() satisfies the adequate smoothness requirements so that the solution of (2) exists and is unique and continuous with respect to k and initial data and is bounded for all bounded values of its arguments.

Let \mathbb{R}^n denote the state space of the systems given by (1–5) and $\|(\cdot)\|$ Euclidean norm.

The solutions of (1-5) are denoted by:

$$\mathbf{x}(k,k_0,\mathbf{x}_0) \equiv \mathbf{x}(k) \,. \tag{6}$$

The discrete-time interval is denoted with K_N , as a set of non-negative integers:

$$K_N = \{ k : k_0 \le k \le k_0 + k_N \}.$$
(7)

The quantity k_N can be positive integer or the symbol $+\infty$, so that finite time stability and practical stability can be treated simultaneously.

 $k_s, k_s \in \{0, k_N\}$ is the prespecified settling time.

 K_N^s denotes the discrete-time interval as follows:

$$K_N^s = \{ k : (k_0 + k_s) < k < (k_0 + k_N) \}.$$
(8)

The set difference is denoted by: $K_N \setminus K_N^s$.

Let $V: K_N \times \mathbb{R}^n \to \mathbb{R}$, so that $V(k, \mathbf{x})$ is bounded for and for which $||\mathbf{x}||$ is also bounded.

Define the total difference of $V(k, \mathbf{x}(k))$ along the trajectory of the systems given by (1–5) with:

$$\Delta V(k, \mathbf{x}(k)) = V(k+1, \mathbf{x}(k+1)) - V(k, \mathbf{x}(k)).$$
(9)

For the time-invariant sets, it is assumed: $S_{()}$ is a bounded, open set.

The closure and the boundary of $S_{()}$ are denoted by $\overline{S}_{()}$ and $\forall k \in K_N \setminus K_N^s$, respectively, so; $\partial S_{()} = \overline{S}_{()} \setminus S_{()}$.

 $\overline{S}_{()}^{c}$ denotes the complement of $S_{()}$.

Let S_{β} be a given set of all allowable states of the system for $\forall k \in K_N \setminus K_N^s$ and S_{γ} is a set of all allowable states of the system for $\forall k \in K_N$, $S_{\gamma} \subset S_{\beta}$.

Set S_{α} , $S_{\alpha} \subset S_{\beta}$ denotes set of all allowable initial states and S_{ε} corresponding set of disturbances.

Sets S_{α} , S_{β} , S_{γ} are connected and a priori known. λ () denotes the eigenvalues of the matrix ().

 λ_{max} is the maximum eigenvalue¹.

Definition of practical stability and practical instability

Definition 1. The systems given by (1), (3), and (4) are *practically stable* with respect to $\{k_0, K_N, S_\alpha, S_\beta\}$, if and only if:

$$\left\|\mathbf{x}(k_0)\right\|^2 = \left\|\mathbf{x}_0\right\|^2 < \alpha , \qquad (10)$$

implies:

$$\left\|\mathbf{x}(k)\right\|^{2} < \beta, \quad \forall k \in K_{N}.$$
(11)

Definition 2. A system, given by (2), is *practically* stable with respect to $\{k_0, K_N, S_\alpha, S_\beta\}$, if and only if:

$$\left\|\mathbf{x}_{0}\right\|^{2} < \alpha \wedge \left\|\mathbf{f}\left(k, \mathbf{x}\left(k\right)\right)\right\| \leq \varepsilon, \,\forall k \in K_{N}, \qquad (12)$$

implies:

$$\|\mathbf{x}(k)\|^2 < \beta, \quad \forall k \in K_N.$$
(13)

Definition 3. A system given by (1), is *practically unstable* with respect to $\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if there is:

$$\left\|\mathbf{x}_{0}\right\|^{2} < \alpha, \quad k = k^{*} \in \mathcal{K}_{N}, \qquad (14)$$

so that the next condition is fulfilled:

$$\left\|\mathbf{x}\left(\boldsymbol{k}^{*}\right)\right\|^{2} \geq \boldsymbol{\beta}, \quad \boldsymbol{\alpha} < \boldsymbol{\beta} .$$
(15)

Definition 4. A system given by (2), is *practically unstable* with respect to $\{k_0, \mathcal{K}_N, \alpha, \beta, \varepsilon, \|(\cdot)\|^2\}$, $\alpha < \beta$, if there is:

$$\|\mathbf{x}_0\|^2 < \alpha \wedge \|\mathbf{f}(k, \mathbf{x}(k))\| \le \varepsilon, \forall k \in \mathcal{K}_N, \qquad (16)$$

so that the next condition is fulfilled:

$$\left\|\mathbf{x}\left(\boldsymbol{k}^{*}\right)\right\|^{2} \geq \boldsymbol{\beta}, \quad \boldsymbol{\alpha} < \boldsymbol{\beta} .$$
(17)

Some previous results

Theorem 1. A system, given by (1), is *practically stable* with respect to $\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if the following conditions are satisfied:

$$\prod_{j=k_0}^{k_0+k-1} \lambda_{\max}(j) \leq \frac{\beta}{\alpha}, \quad \forall k \in \mathcal{K}_N,$$
(18)

Debeljković (2001).

Theorem 2. A system, given by (2), is *practically stable* with respect to $\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if the next conditions are fulfilled:

$$\lambda_{\max}^{0.5k} + k \cdot \varepsilon^* \cdot \lambda_{\max}^{0.5(k-1)} \le \sqrt{\frac{\beta}{\alpha}}, \quad \forall k \in \mathcal{K}_N ,$$
 (19)

Debeljković (2001).

Theorem 3. A system, given by (1), is *practically unstable* with respect to $\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if there exists a real, positive number $\delta, \delta \in]0, \alpha[$ and a time instant $k, k = k^* : \exists ! (k^* > k_0) \in K_N$ for which the next condition is fulfilled:

$$\prod_{j=k_0}^{j=k_0+k^*-1} \lambda_{\min}(j) > \frac{\beta}{\alpha}, \quad k^* \in \mathcal{K}_N,$$
(20)

Debeljković (2001).

Theorem 4. A system given by (2), is *practically unstable* with respect to $\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if there exists a real, positive number δ and ε_0 , such that: $\delta < \|\mathbf{x}_0\|^2 < \alpha$ and $\varepsilon_0 < \|\mathbf{f}(k)\| < \varepsilon$, $\forall k \in K_N$ and a time instant $k, k = k^* : \exists ! (k^* > k_0) \in K_N$ such that the next condition is fulfilled:

$$\left|\sqrt{\delta}\lambda_{\min}^{0,5k^*} - k^* \varepsilon \lambda_{\min}^{0,5(k^*-1)}\right| > \sqrt{\beta}, \ k^* \in \mathcal{K}_N, \qquad (21)$$

Debeljković (2001).

Theorem 5. A system, given by (3), is *practically stable* with respect to $\{k_0, \mathcal{K}_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if there exists a real, positive number η and if the next conditions are fulfilled²:

$$\|A(\mathbf{x})\| < \eta, \quad \forall \mathbf{x} \in \overline{\kappa}_{\beta}, \quad \forall k \in \mathcal{K}_{N-1},$$
(22)

$$\eta^k < \sqrt{\frac{\beta}{\alpha}}, \quad \forall k \in \mathcal{K}_N,$$
 (23)

Debeljković (2001).

Theorem 6. A system, given by (4), is *practical stable* with respect to $\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if there exists a real, scalar function $\xi(k)$, which is bounded for and if the following conditions are satisfied:

$$\|A(k,\mathbf{x})\| < \xi(k), \ \forall \mathbf{x} \in \overline{\kappa}_{\beta}, \ \forall k \in K_{N-1}, \qquad (24)$$

$$\prod_{j=k_0}^{j=k_0+k-1} \xi(j) < \sqrt{\frac{\beta}{\alpha}}, \quad \forall k \in \mathcal{K}_N,$$
(25)

Debeljković (2001).

Linear discrete time delay systems

As far as we know, the only result, considering and investigating the problem of the non-Lyapunov analysis of linear discrete time delay systems, is one that has been mentioned in the introduction, e.g. *Debeljković*, *Aleksendrić*

¹ See Apendix A.

² See Apendix B.

(2003), where this problem has been considered for the first time.

Investigating system stability throughout the discrete fundamental matrix is very cumbersome, so there is a need to find some more efficient expressions that should be based on calculation appropriate eigenvalues or a norm of appropriate systems matrices as it has been done in a continuous case.

System description

Consider a linear discrete system with state delay, described by:

$$\mathbf{x}(k+1) = A_0 \mathbf{x}(k) + A_1 \mathbf{x}(k-1),$$
 (26.a)

with the known vector valued function of initial conditions:

$$\mathbf{x}(k_0) = \mathbf{\psi}(k_0), \quad -1 \le k_0 \le 0,$$
 (26.b)

where $\mathbf{x}(k) \in \mathbb{R}^n$ is a state vector and with the constant matrices A_0 and A_1 of appropriate dimensions.

The time delay is constant and equals one.

For some other purposes, the state delay equation can be represented in the following way:

$$\mathbf{x}(k+1) = A_0 \mathbf{x}(k) + \sum_{j=1}^{M} A_j \mathbf{x}(k-h_j), \qquad (27.a)$$

$$\mathbf{x}(\boldsymbol{\vartheta}) = \mathbf{\psi}(\boldsymbol{\vartheta}), \quad \boldsymbol{\vartheta} \in \{-h, -h+1, \dots, 0\}, \quad (27.b)$$

where $\mathbf{x}(k) \in \mathbb{R}^n$, $A_j \in \mathbb{R}^{n \times n}$, j = 1, 2, h- is an integer representing the system time delay and $\boldsymbol{\psi}(\cdot)$ is an apriori known vektor function of the initial conditions as well.

Definition of practical stability and practical instability

Definition 5. A system, given by (26), is *attractive* practically stable with respect to $\{k_0, K_N, S_\alpha, S_\beta\}$, if and only if:

$$\left\|\mathbf{x}(k_{0})\right\|_{A_{0}^{T}PA_{0}}^{2} = \left\|\mathbf{x}_{0}\right\|_{A_{0}^{T}PA_{0}}^{2} < \alpha , \qquad (28)$$

implies:

$$\left\|\mathbf{x}(k)\right\|_{A_0^T P A_0}^2 < \beta, \quad \forall k \in \mathcal{K}_N,$$
(29)

with a property that:

$$\lim_{k \to \infty} \left\| \mathbf{x}(k) \right\|_{A_0^T P A_0}^2 \to 0.$$
(30)

Definition 6. A system, given by (26), is *practically* stable with respect to $\{k_0, K_N, S_\alpha, S_\beta\}$, if and only if:

$$\left\|\mathbf{x}_{0}\right\|^{2} < \alpha , \qquad (31)$$

implies:

$$\|\mathbf{x}(k)\|^2 < \beta, \quad \forall k \in K_N.$$

Definition 7. A system, given by (26), is *atractive* practically unstable with respect to $\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\}$,

 $\alpha < \beta$, if for:

$$\left\|\mathbf{x}_{0}\right\|_{A_{0}^{T}PA_{0}}^{2} < \alpha , \qquad (33)$$

there exists a moment: $k = k^* \in K_N$, so that the next condition is fulfilled:

$$\left\|\mathbf{x}\left(\boldsymbol{k}^{*}\right)\right\|_{\boldsymbol{A}_{0}^{T}\boldsymbol{P}\boldsymbol{A}_{0}}^{2} \geq \boldsymbol{\beta}, \qquad (34)$$

with a property that:

$$\lim_{k \to \infty} \left\| \mathbf{x}(k) \right\|_{A_0^T P A_0}^2 \to 0 .$$
(35)

Definition 8. A system given by (2) is *practically unstable* with respect to $\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if for:

$$\mathbf{x}_0 \|^2 < \alpha , \qquad (36)$$

there exists a moment: $k = k^* \in K_N$, such that the next condition is fulfilled:

$$\left\|\mathbf{x}\left(k^{*}\right)\right\|^{2} \geq \beta , \qquad (37)$$

for some $k = k^* \in K_N$.

Definition 9. The linear discrete time delay system, given by (27.a) is *finite time stable* with respect to $\{\alpha, \beta, k_0, k_N, \|(\cdot)\|\}, \alpha \leq \beta$, if and only if for every trajectory $\mathbf{x}(k)$ satisfying initial function, given by (26.b) such that:

$$\|\mathbf{x}(k)\| < \alpha, \quad k = 0, -1, -2, \dots, -N$$
, (38)

implies:

$$\left\|\mathbf{x}(k)\right\|^{2} < \beta, \quad k \in \mathcal{K}_{N}, \tag{39}$$

Aleksendrić (2002), Aleksendrić, Debeljković (2002), Debeljković, Aleksendrić (2003).

This Definition is analogous to that presented, for the first time, in *Debeljković et al.* (1997.a, 1997.b, 1997.c, 1997.d) and *Nenadic et al.* (1997).

Some previous results

Theorem 7. The linear discrete time delay system, given by (27), is *finite time stable* with respect to $\{\alpha, \beta, M, N, \|(\cdot)\|^2\}$, $\alpha < \beta$, $\alpha, \beta \in \mathbb{R}_+$, it is sufficient that:

$$\left\|\Phi\left(k\right)\right\| < \frac{\beta}{\alpha} \cdot \frac{1}{1 + \sum_{j=1}^{M} \left\|A_{j}\right\|}, \forall k = 0, 1, \cdots, N, \qquad (40)$$

Aleksendrić (2002), Aleksendrić, Debeljković (2002), Debeljković, Aleksendrić (2003).

Proof. The solution of (27.a), with initial condition (27.b) can be expressed in terms of the fundamental matrix, as it is written below:

$$\mathbf{x}(k) = \Phi(k)\mathbf{x}(0) + \Phi(k)A_1\mathbf{x}(-1) + \cdots + \Phi(k)A_M\mathbf{x}(-N)$$
(41)

Remark 1. The matrix measure is widely used when continuous time delay systems are investigated, *Coppel* (1965), *Desoer*, *Vidysagar* (1975).

The nature of discrete time delay enables one to use this approach as well as Bellmans principle, so the problem must be attacked from the point of view which is based only on norms.

So one can get:

$$\mathbf{x}(k) \| = \| \Phi(k) \mathbf{x}(0) + \Phi(k) A_1 \mathbf{x}(-1) + \\ + \cdots \Phi(k) A_M \mathbf{x}(-N) \| \\ \leq \| \Phi(k) \| \left(\| \mathbf{x}(0) \| + \sum_{j=1}^M \| A_j \| \cdot \| \mathbf{x}(-i) \| \right)$$
(42)
$$< \| \Phi(k) \| \left(\alpha + \alpha \sum_{j=1}^M \| A_1 \| \right) \\ \leq \alpha \cdot \| \Phi(k) \| \left(1 + \sum_{i=1}^M \| A_i \| \right)$$

where the first condition of Definition 9 has been used.

To obtain the final result, one has to use (40), so it can be written:

$$\|\mathbf{x}(k)\| < \alpha \cdot \frac{\frac{\beta}{\alpha}}{1 + \sum_{j=1}^{N} \|A_j\|} \left(1 + \sum_{j=1}^{N} \|A_j\|\right) < \beta$$

$$\forall k = 0, 1, 2, \cdots, N$$
(43)

what has to be proved. Q.E.D.

This result is analogous to that one, for the first time derived, in *Debeljković et al.* (1997.a) for continuous time delay systems.

Main results:

Practical and Finite Time Stability

Theorem 8. A system given by (26), with det $A_1 \neq 0$, is *attractive practically stable* with respect to $\{k_0, K_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if the following condition is satisfied:

$$\overline{\lambda}_{\max}^{\frac{1}{2}k}\left(\right) < \frac{\beta}{\alpha}, \quad \forall k \in \mathcal{K}_N, \qquad (44.a)$$

where:

$$\overline{\lambda}_{\max}(\) = \max\left\{\mathbf{x}^{T}(k)A_{1}^{T}PA_{1}\mathbf{x}(k): \\ \mathbf{x}^{T}(k)A_{0}^{T}PA_{0}\mathbf{x}(k) = 1\right\},$$
(44.b)

and if there exists $P = P^T > 0$, being a solution of:

$$2A_0^T P A_0 - P = -Q, (44.c)$$

where $Q = Q^T > 0$, such that:

$$\|A_1\| < \frac{\sqrt{2}}{2} \frac{\sigma_{\min}\left(Q^{\frac{1}{2}}\right)}{\sigma_{\max}^2\left(P^{\frac{1}{2}}\right)}$$
(44.d)

Proof. Let us use a functional, as a possible aggregation function, for the system to be considered:

$$V(\mathbf{x}(k)) = \mathbf{x}^{T}(k)P\mathbf{x}(k) + \mathbf{x}^{T}(k-1)Q\mathbf{x}(k-1), \quad (45)$$

with the matrices $P = P^T > 0$ and $Q = Q^T > 0$.

Clearly, using the equation of motion of (26.a), we have:

$$\Delta V(\mathbf{x}(k)) = V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)), \qquad (46)$$

or:

$$\Delta V(\mathbf{x}(k)) =$$

$$= \mathbf{x}^{T} (k+1) P \mathbf{x}(k+1) - \mathbf{x}^{T} (k) P \mathbf{x}(k)$$

$$+ \mathbf{x}^{T} (k) Q \mathbf{x}(k) - \mathbf{x}^{T} (k-1) Q \mathbf{x}(k-1)$$

$$= \mathbf{x}^{T} (k) (A_{0}^{T} P A_{0} + Q - P) \mathbf{x}(k)$$

$$+ 2 \mathbf{x}^{T} (k) A_{0}^{T} P A_{1} \mathbf{x}(k-1)$$

$$- \mathbf{x}^{T} (k-1) (Q - A_{1}^{T} P A_{1}) \mathbf{x}(k-1).$$

It has been shown, Debeljković et al. (2004), that if:

$$2A_0^T P A_0 - P = -Q, (48)$$

where $P = P^T > 0$ and $Q = Q^T > 0$ then for:

$$V(\mathbf{x}(k)) = \mathbf{x}^{T}(k)P\mathbf{x}(k) + \mathbf{x}^{T}(k-1)Q\mathbf{x}(k-1), \quad (49)$$

the backward difference along the trajectories of the systems is:

$$\Delta V (\mathbf{x}(k)) = V (\mathbf{x}(k+1)) - V (\mathbf{x}(k))$$

= $\mathbf{x}^{T} (k) (A_{0}^{T} P A_{0} - P + Q) \mathbf{x}(k)$
+ $\mathbf{x}^{T} (k-1) (A_{1}^{T} P A_{1} - Q) \mathbf{x}(k-1)$ (50)
+ $\mathbf{x}^{T} (k) A_{0}^{T} P A_{1} \mathbf{x}(k-1)$
+ $\mathbf{x}^{T} (k-1) A_{1}^{T} P A_{0} \mathbf{x}(k),$

or:

$$\Delta V (\mathbf{x}(k)) =$$

$$= \mathbf{x}^{T} (k) (2A_{0}^{T}PA_{0} - P + Q)\mathbf{x}(k)$$

$$+ \mathbf{x}^{T} (k-1) (2A_{1}^{T}PA_{1} - Q)\mathbf{x}(k-1)$$

$$+ \mathbf{x}^{T} (k) A_{0}^{T}PA_{1} \mathbf{x}(k-1)$$

$$+ \mathbf{x}^{T} (k-1)A_{1}^{T}PA_{0} \mathbf{x}(k)$$

$$- \mathbf{x}^{T} (k) A_{0}^{T}PA_{0} \mathbf{x}(k)$$

$$- \mathbf{x}^{T} (k-1) A_{1}^{T}PA_{1} \mathbf{x}(k-1),$$
(51)

and since we have to take into account (49), one can get:

$$\Delta V(\mathbf{x}(k)) =$$

$$= \mathbf{x}^{T} (k-1) (2A_{1}^{T} P A_{1} - Q) \mathbf{x} (k-1) - (52)$$

$$- [A_{0} \mathbf{x}(k) - A_{1} \mathbf{x} (k-1)]^{T} P \cdot [A_{0} \mathbf{x}(k) - A_{1} \mathbf{x} (k-1)].$$

Since the matrix $P = P^T > 0$, it is more than obviuos that:

$$\Delta V(\mathbf{x}(k)) < \mathbf{x}^{T}(k-1)(2A_{1}^{T}PA_{1}-Q)\mathbf{x}(k-1).$$
(53)

If one equalize the rights sides of (47) and (53), it yields:

$$\mathbf{x}^{T}(k) (A_{0}^{T}PA_{0} - P + Q) \mathbf{x}(k) + 2 \mathbf{x}^{T}(k) A_{0}^{T}PA_{1} \mathbf{x}(k-1) - \mathbf{x}^{T}(k-1) (Q - A_{1}^{T}PA_{1}) \mathbf{x}(k) < \mathbf{x}^{T}(k-1) (2A_{1}^{T}PA_{1} - Q) \mathbf{x}(k-1),$$
(54)

or:

$$\Delta V (\mathbf{x}(k)) =$$

$$= \mathbf{x}^{T} (k) (A_{0}^{T} P A_{0} + Q - P) \mathbf{x}(k)$$

$$+ 2 \mathbf{x}^{T} (k) A_{0}^{T} P A_{1} \mathbf{x}(k-1)$$

$$< \mathbf{x}^{T} (k-1) (A_{1}^{T} P A_{1}) \mathbf{x}(k-1),$$
(55)

Using the very well-known inequality³, with a particular choice:

$$\Gamma = \frac{1}{2} \left(A_1^T P A_1 \right), \tag{56}$$

it can be obtained:

$$\mathbf{x}^{T}(k) \Big(A_{0}^{T} P A_{0} + Q - P + A_{0}^{T} P A_{1} \Big) \Big)^{-1} A_{1}^{T} P A_{0} \Big) \mathbf{x}(k)$$

+ $\frac{1}{2} \mathbf{x}^{T} (k-1) \Big(A_{1}^{T} P A_{1} \Big) \mathbf{x}(k-1)$ (57)
< $\mathbf{x}^{T} (k-1) \Big(A_{1}^{T} P A_{1} \Big) \mathbf{x}(k-1)$

or:

$$\mathbf{x}^{T}(k) \Big(2A_{0}^{T}PA_{0} + Q - P + A_{0}^{T}PA_{0} \Big) \mathbf{x}(k) < \frac{1}{2} \mathbf{x}^{T}(k-1) \Big(A_{1}^{T}PA_{1} \Big) \mathbf{x}(k-1).$$
(58)

Since:

$$2A_0^T P A_0 + Q - P = 0, (59)$$

it is finally obtained:

$$\mathbf{x}^{T}(k)A_{0}^{T}PA_{0}\mathbf{x}(k) < \frac{1}{2}\mathbf{x}^{T}(k-1)(A_{1}^{T}PA_{1})\mathbf{x}(k-1), (60)$$

or:

$$\mathbf{x}^{T}(k) A_{0}^{T} P A_{0} \mathbf{x}(k) < < \frac{1}{2} \overline{\lambda}_{\max}() \mathbf{x}^{T}(k-1) A_{0}^{T} P A_{0} \mathbf{x}(k-1),$$
(61)

where:

$$\overline{\lambda}_{\max}(\) = \max \begin{cases} \mathbf{x}^{T}(k)A_{1}^{T}PA_{1}\mathbf{x}(k):\\ \left(2A_{0}^{T}PA_{0}-P\right) = -Q,\\ \mathbf{x}^{T}(k)A_{0}^{T}PA_{0}\mathbf{x}(k) = 1 \end{cases}$$
(62)

Since this manipulation is independent of k, it can be written:

$$\mathbf{x}^{T}(k+1)A_{0}^{T}PA_{0}\mathbf{x}(k+1) < < \frac{1}{2}\overline{\lambda}_{\max}(\mathbf{x}(k)\mathbf{x}^{T}(k)A_{0}^{T}PA_{0}\mathbf{x}(k),$$
(63)

or:

³ $2\mathbf{u}^{T}(t)\mathbf{v}(t) \leq \mathbf{u}^{T}(t)\Gamma^{-1}\mathbf{u}(t) + \mathbf{v}^{T}(t)\Gamma\mathbf{v}(t), \Gamma > 0$

$$\ln \mathbf{x}^{T} (k+1) A_{0}^{T} P A_{0} \mathbf{x} (k+1) < < \ln \frac{1}{2} \overline{\lambda}_{\max} () \mathbf{x}^{T} (k) A_{0}^{T} P A_{0} \mathbf{x} (k) < \ln \frac{1}{2} \overline{\lambda}_{\max} () + \ln \mathbf{x}^{T} (k) A_{0}^{T} P A_{0} \mathbf{x} (k),$$
(64)

and:

$$\ln \mathbf{x}^{T} (k+1) A_{0}^{T} P A_{0} \mathbf{x} (k+1) - -\ln \mathbf{x}^{T} (k) A_{0}^{T} P A_{0} \mathbf{x} (k) < \ln \overline{\lambda}_{\max}^{\frac{1}{2}} (),$$
(65)

If we apply the summing $\sum_{j=k_0}^{k_0+k-1}$ on both sides of (65) for $\forall k \in K_N$, one can obtain:

$$\sum_{j=k_{0}}^{k_{0}+k-1} \ln \mathbf{x}^{T} (k+1) A_{0}^{T} P A_{0} \mathbf{x} (k+1) - \\ -\ln \mathbf{x}^{T} (k) A_{0}^{T} P A_{0} \mathbf{x} (k) \leq \sum_{j=k_{0}}^{k_{0}+k-1} \ln \lambda_{\max}^{\frac{1}{2}} () \leq (66)$$
$$\leq \ln \prod_{j=k_{0}}^{k_{0}+k-1} \lambda_{\max}^{\frac{1}{2}} ().$$

It can be shown:

so that, for (66), it seems to be:

$$\ln \mathbf{x}^{T} \left(k_{0}+k\right) A_{0}^{T} P A_{0} \mathbf{x} \left(k_{0}+k\right) - \\ -\ln \mathbf{x}^{T} \left(k_{0}\right) A_{0}^{T} P A_{0} \mathbf{x} \left(k_{0}\right) < \ln \prod_{j=k_{0}}^{k_{0}+k-1} \overline{\lambda}_{\max}^{\frac{1}{2}} \left(\right) \qquad (68)$$
$$< \ln \overline{\lambda}_{\max}^{\frac{1}{2}k} \left(\right), \quad \forall k \in \mathcal{K}_{N},$$

as well as:

$$\ln \mathbf{x}^{T} \left(k_{0}+k\right) A_{0}^{T} P A_{0} \mathbf{x} \left(k_{0}+k\right) \leq \ln \prod_{j=k_{0}}^{k_{0}+k-1} \overline{\lambda}_{\max}^{\frac{1}{2}} \left(\right)$$

$$\leq \ln \overline{\lambda}_{\max}^{\frac{1}{2}k} \left(\right) \quad \forall k \in \mathcal{K}_{N} + \ln \mathbf{x}^{T} \left(k_{0}\right) A_{0}^{T} P A_{0} \mathbf{x} \left(k_{0}\right).$$
(69)

Taking into account the fact that $\|\mathbf{x}_0\|_{A_0^T P A_0}^2 < \alpha$ and condition of *Theorem* 8, eq. (44), one can get:

$$\ln \mathbf{x}^{T} (k_{0} + k) A_{0}^{T} P A_{0} \mathbf{x} (k_{0} + k) <$$

$$< \ln \overline{\lambda}_{\max}^{\frac{1}{2}k} () + \ln \mathbf{x}^{T} (k_{0}) A_{0}^{T} P A_{0} \mathbf{x} (k_{0})$$

$$< \ln \alpha \cdot \overline{\lambda}_{\max}^{\frac{1}{2}k} () < \ln \alpha \cdot \frac{\beta}{\alpha} < \ln \beta, \quad \forall k \in \mathcal{K}_{N}.$$
(70)

Q.E.D.

Remark 2. The assumption det $A_1 \neq 0$ does not reduce the generality of this result, since this condition is not crucial when discrete time systems are considered.

Remark 3. Lyapunov asymptotic stability and Finite time stability are independent concepts: a system that is Finite time stable may not be Lyapunov asymptotically stable; conversely, a Lyapunov asymptotically stable system could not be Finite time stable if, during the transients, its motion exceeds the prespecified bounds (β) .

Attractivity property is guaranteed by (44.c), e.g. by the Lyapunov equation and system motion within the prespecified boundaries is well provided by (44.a).

Remark 4. For the numerical treatment of this problem, λ_{max} () can be calculated in the following way:

$$\overline{\lambda}_{\max}\left(\begin{array}{c}\right) = \max_{\mathbf{x}}\left\{\right\} = \overline{\lambda}_{\max}\left(A_{1}^{T}PA_{1}\left(A_{0}^{T}PA_{0}\right)^{-1}\right), \quad (71)$$

Kalman, Bertram (1960.b).

Remark 5. These results are in some sense analogous to those given in Amato et al. (2003), although the results presented there are derived for continuous time varying systems.

Now we proceed to develop delay independent criteria, for finite time stability of a system under consideration, not to be necessarily asymptotic stable, e.g. so we reduce the previous demand that the basic system matrix A_0 should be a discrete stable matrix.

Theorem 9. Suppose the matrix $(I - A_1^T A_1) > 0$.

A system, given by (26), is *finite time stable* with respect to $\{k_0, \mathcal{K}_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if the following condition is satisfied:

$$\lambda_{\max}^{k}(\) < \frac{\beta}{\alpha}, \quad \forall k \in \mathcal{K}_{N},$$
 (72.a)

where:

$$\lambda_{\max}\left(\right) = \lambda_{\max}\left(A_0^T \left(I - A_1^T A_1\right) A_0 + \beta I\right)$$
(72.b)

Proof. Now we consider again a system given by (26). Define:

$$V(\mathbf{x}(k)) = \mathbf{x}^{T}(k)\mathbf{x}(k) + \mathbf{x}^{T}(k-1)\mathbf{x}(k-1), \quad (73)$$

as a tentative Lyapunov-like function for the system, given (26).

Then, the $\Delta V(\mathbf{x}(k))$ along the trajectory is obtained as:

$$\Delta V(\mathbf{x}(k)) = V(\mathbf{x}(k+1)) - V(\mathbf{x}(k))$$

= $\mathbf{x}^{T}(k+1)\mathbf{x}(k+1) - \mathbf{x}^{T}(k-1)\mathbf{x}(k-1)$
= $\mathbf{x}^{T}(k)A_{0}^{T}A_{0}\mathbf{x}(k)$
+ $2\mathbf{x}^{T}(k)A_{0}^{T}A_{1}\mathbf{x}(k-1)$
+ $\mathbf{x}^{T}(k-1)A_{1}^{T}A_{1}\mathbf{x}(k-1)$
- $\mathbf{x}^{T}(k-1)\mathbf{x}(k-1)$ (74)

From (74), one can get:

$$\mathbf{x}^{T}(k+1)\mathbf{x}(k+1) = \mathbf{x}^{T}(k)A_{0}^{T}A_{0}\mathbf{x}(k) + 2\mathbf{x}^{T}(k)A_{0}^{T}A_{1}\mathbf{x}(k-1) + \mathbf{x}^{T}(k-1)A_{1}^{T}A_{1}\mathbf{x}(k-1).$$
(75)

Using the very well known inequality⁴, with a particular choice:

$$\Gamma = \left(I - A_1^T A_1\right) > 0, \qquad (76)$$

I being the identity matrix, it can be obtained:

$$\mathbf{x}^{T} (k+1) \mathbf{x} (k+1) \leq \mathbf{x}^{T} (k) A_{0}^{T} A_{0} \mathbf{x} (k)$$

+
$$\mathbf{x}^{T} (k) A_{1} (I - A_{1}^{T} A_{1})^{-1} A_{1}^{T} \mathbf{x} (k)$$
(77)
+
$$\mathbf{x}^{T} (k-1) \mathbf{x} (k-1),$$

and the fact that it is more than obvious, that one can adopt

$$\left\| \mathbf{x} (k-1) \right\|^{2} < \beta \left\| \mathbf{x} (k) \right\|^{2}, \ \forall \mathbf{x} (k) \in \mathcal{S}_{\beta},$$
(78)

it is clear that (77) reduces to:

$$\mathbf{x}^{T}(k+1)\mathbf{x}(k+1) < < \mathbf{x}^{T}(k)A_{0}^{T}\left(\left(I - A_{1}A_{1}^{T}\right)^{-1} + \beta I\right)A_{0}\mathbf{x}(k)$$
(79)
 < $\lambda_{\max}(A_{0}, A_{1}, \beta)\mathbf{x}^{T}(k)\mathbf{x}(k),$

where:

$$\lambda_{\max}\left(A_0, A_1, \beta\right) = \lambda_{\max}\left(A_0^T \left(I - A_1 A_1^T\right)^{-1} A_0 + \beta I\right), \quad (80)$$

with obvious property, that gives the natural sence to this problem:

$$\lambda_{\max}\left(A_0, A_1, \beta\right) = \lambda_{\max}\left(A_0^T \left(I - A_1 A_1^T\right)^{-1} A_0 + \beta I\right) \ge 0, (81)$$

when.

$$\left(I - A_1 A_1^T\right) \ge 0. \tag{82}$$

Folloving the procedure from the previous section, it can be written:

$$\ln \mathbf{x}^{T} (k+1) \mathbf{x} (k+1) - \ln \mathbf{x}^{T} (k) \mathbf{x} (k) < \ln \lambda_{\max} ()$$
(83)

If we apply the summing $\sum_{i=k_0}^{k_0+k-1}$ on both sides of (82)

for $\forall k \in K_N$, one can obtain:

$$\ln \mathbf{x}^{T} (k_{0} + k) \mathbf{x} (k_{0} + k)$$

$$\leq \ln \prod_{j=k_{0}}^{k_{0}+k-1} \lambda_{\max} () \leq \ln \lambda_{\max}^{k} ()$$

$$+ \ln \mathbf{x}^{T} (k_{0}) \mathbf{x} (k_{0}), \quad \forall k \in K_{N}.$$
(84)

Taking into account the fact that $\|\mathbf{x}_0\|^2 < \alpha$ and condition of Theorem 9, eq. (72.a), one can get:

⁴ $2\mathbf{u}^{T}(t)\mathbf{v}(t-\tau) \leq \mathbf{u}^{T}(t)\Gamma^{-1}\mathbf{u}(t) + \mathbf{v}^{T}(t-\tau)\Gamma\mathbf{v}(t-\tau), \Gamma > 0$

$$\ln \mathbf{x}^{T} (k_{0} + k) \mathbf{x} (k_{0} + k) <$$

< $\ln \lambda_{\max}^{k} (A_{0}, A_{1}, \beta) + \ln \mathbf{x}^{T} (k_{0}) \mathbf{x} (k_{0})$
< $\ln \alpha \cdot \lambda_{\max}^{k} (A_{0}, A_{1})$
< $\ln \alpha \cdot \frac{\beta}{\alpha} < \ln \beta, \quad \forall k \in K_{N}.$

Q.E.D.

Remark 6. In the case when A_1 is the null matrix, the result given by (85) reduces to the one given in *Debeljković* (2001), developed for ordinary discrete time systems.

Remark 7. Different final sufficient conditions can be derived with particular choices of the matrix Γ in (76).

Theorem 10. Suppose the matrix $(I - A_1^T A_1) > 0$.

A system, given by (26), is *practically unstable* with respect to $\{k_0, \mathcal{K}_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if there exists a real, positive number δ , $\delta \in]0, \alpha[$ and a time instant $k, k = k^* : \exists ! (k^* > k_0) \in \mathcal{K}_N$ for which the next condition is fulfilled:

$$\lambda_{\min}^{k^*} > \frac{\beta}{\delta}, \quad k^* \in K_N.$$
(86)

Proof. Let:

$$V(\mathbf{x}(k)) = \mathbf{x}^{T}(k)\mathbf{x}(k) + \mathbf{x}^{T}(k-1)\mathbf{x}(k-1).$$
(87)

Then following the identical procedure as in the previous *Theorem*, one can get:

$$\ln \mathbf{x}^{T}(k+1)\mathbf{x}(k+1) - \ln \mathbf{x}^{T}(k)\mathbf{x}(k) > \ln \lambda_{\min}(), \quad (88)$$

where:

$$\lambda_{\max}\left(A_0, A_1, \beta\right) = \lambda_{\max}\left(A_0^T \left(I - A_1 A_1^T\right)^{-1} A_0 + \beta I\right) (89)$$

If we apply the summing $\sum_{j=k_0}^{k_0+k-1}$ on both sides of (87)

for $\forall k \in K_N$, one can obtain:

$$\ln \mathbf{x}^{T} (k_{0} + k) \mathbf{x} (k_{0} + k) \geq \ln \prod_{j=k_{0}}^{k_{0}+k-1} \lambda_{\max} ()$$

$$\geq \ln \lambda_{\max}^{k} () + \ln \mathbf{x}^{T} (k_{0}) \mathbf{x} (k_{0}), \quad \forall k \in K_{N}.$$
(90)

It is clear that for any \mathbf{x}_0 follows: $\delta < \|\mathbf{x}_0\|^2 < \alpha$ and for some $k^* \in K_N$ and taking into account the basic condition of *Theorem* 10, (86), one can get:

$$\ln \mathbf{x}^{T} (k_{0} + k^{*}) \mathbf{x} (k_{0} + k^{*})$$

$$> \ln \lambda_{\max}^{k^{*}} (A_{0}, A_{1}, \beta) + \ln \mathbf{x}^{T} (k_{0}) \mathbf{x} (k_{0})$$

$$> \ln \delta \cdot \lambda_{\max}^{k^{*}} (A_{0}, A_{1}, \beta) > \ln \delta \cdot \frac{\beta}{\delta} > \ln \beta,$$
(91)

for some $k^* \in K_N$.

Q.E.D.

Conclussion

The concept of practical (finite time) stability is of particular importance in engineering since it expresses realistically the strong demands which are imposed on dynamical behavior of real automatic control systems.

Definitions and theorems were established and proved for a few classes of autonomous time-discrete and discrete time delay systems, which guarantee attractive practical and only practical stability within the prespecified timeinvariant sets in state space.

Moreover, based on classical definitions, some new theorems were derived for the so-called *finite time stability* as well as the corresponding results *concerning instability problems*.

The developed results represent sufficient conditions for this type of non–Lyapunov stability. A discrete version of a very well-known Bellman–Gronwall Lemma was also mentioned and can be used for practical proofs in concept of practical instability of forced linear discrete–time systems.

APENDIX A

Notation

[]	closed interval
] [open interval
\wedge	and
\vee	or
\vee	exclusive or
\rightarrow	maps
\Rightarrow	follows
\Leftrightarrow	if and only if
\forall	for every
Э	exist
∃!	exist at least one
Ξ	do not exist
:	with property
Э	so that
	so that
∈	belongs
∉	do not belong
{ }	set, sequence
\cup	union of sets
\cap	intersection of sets
\subset	subset
\	set difference
Δ	set symmetric difference
~	equivalent sets
S	open set
∂S	boundary of set S
\overline{S}	closure of set S
\mathcal{S}^{c}	complement of set S
int S	interior of set S
Ø	empty or null set
<u> </u>	upon definition
Δ	finite backward difference
∇	particular meaning, symbol
•	dot
×	multiplication
Σ	summation
П	product

$\ (\cdot)\ $	norm
grad	gradient
det	determinant
exp	eksponent
inf	infinum
max	maksimum
min	minimum
sup	supremum
\mathbb{R}	all real numbers
\mathbb{R}_+	all the non-negative real numbers
\mathbb{R}^{n}	n dimensional real vector space
degree	degree of polynomial
det()	determinant of matrix ()
$diag\{ \}$	diagonal matrix { }
Ind()	index of matrix ()
rang()	rank of matrix ()
<i>tr</i> ()	trace of matrix ()
$\mathcal{L}\{ \}$	Laplace transform

Q. E. D. end of the proof

APENDIX B

Some necessary mathematics

Define the total difference of $V(k, \mathbf{x}(k))$ along the trajectory of the systems given by (1-3), *Michel, Wu* (1969):

$$\Delta V(k, \mathbf{x}(k)) =$$

$$= V(k+1, \mathbf{x}(k+1)) - V(k, \mathbf{x}(k))$$

$$= (\nabla V(k, \mathbf{x}(k)))^T \cdot \Delta \mathbf{x}(k)$$

$$+ V(k+1, \mathbf{x}(k)) - V(k, \mathbf{x}(k)),$$
(B.1)

where:

$$\nabla V(k, \mathbf{x}(k)) = \frac{\nabla V(k, \mathbf{x}(k)) = \frac{V(x_1(k+1), \dots (k+1)) - V(x_1(k), \dots (k+1))}{x_1(k+1) - x_1(k)}}{\frac{V(\cdot x_2(k+1), \dots (k+1)) - V(\cdot x_2(k), \dots (k+1))}{x_2(k+1) - x_2(k)}}$$

$$\frac{V(\dots, x_n(k+1), (k+1)) - V(\dots, x_n(k), (k+1))}{x_n(k+1) - x_n(k)}$$
(B.2)

In (B.1) " \cdot " denotes the dot product of two vectors and:

$$\Delta \mathbf{x}(k) = \mathbf{x}(k+1) - \mathbf{x}(k), \qquad (B.3)$$

is the finite difference.

Definition B.1 Function $V(k, \mathbf{x}(k))$ is said to possess the property Γ if the vector $\nabla V(k, \mathbf{x}(k))$ is unique regardless of the particular path taken when going from one specific point to another in state space \mathbb{R}^n , *Michel, Wu* (1969). Next, let:

$$\Delta V(k, \mathbf{x}(k)) = \Delta V_{\mathbf{f}}(k, \mathbf{x}(k)) + + (\nabla V(k, \mathbf{x}(k)))^{T} \cdot \mathbf{f}(k, \mathbf{x}(k))$$
(B.4)

where:

$$\Delta V_{\mathbf{f}}(k, \mathbf{x}(k)) \equiv \Delta V(k, \mathbf{x}(k))_{\mathbf{f}=0}$$
(B.5)

with the function f(k, x(k)) in the linear combination presented in (3).

Besides that, we use the following notation:

$$V_{M}^{\alpha}(k) = \max_{\|\mathbf{x}(k)\| < \alpha} V(k, \mathbf{x}(k))$$

$$V_{m}^{\alpha}(k) = \min_{\|\mathbf{x}(k)\| < \alpha} V(k, \mathbf{x}(k))$$

$$V_{M}^{\beta\alpha}(k) = \max_{\alpha \le \|\mathbf{x}(k)\| < \beta} V(k, \mathbf{x}(k))$$

$$V_{m}^{\beta\alpha}(k) = \min_{\alpha \le \|\mathbf{x}(k)\| < \beta} V(k, \mathbf{x}(k))$$

$$V_{m}^{(\alpha+\rho),\alpha}(k) = \max_{\alpha \le \|\mathbf{x}(k)\| < (\alpha+\rho)} V(k, \mathbf{x}(k))$$

$$V_{m}^{(\alpha+\rho),\alpha}(k) = \max_{\alpha \le \|\mathbf{x}(k)\| < \alpha} V(k, \mathbf{x}(k))$$

$$V_{m}^{\alpha,(\alpha-\rho)}(k) = \min_{(\alpha-\rho) \le \|\mathbf{x}(k)\| < \alpha} V(k, \mathbf{x}(k))$$

$$V_{m}^{\alpha,(\alpha-\rho)}(k) = \min_{(\alpha-\rho) \le \|\mathbf{x}(k)\| < \alpha} V(k, \mathbf{x}(k))$$

$$V_{m}^{\alpha,(\alpha-\rho)}(k) = \max_{\mathbf{x} \in S_{(\cdot)}} V(k, \mathbf{x}(k))$$

$$V_{\overline{M}(\cdot)}(k) = \max_{\mathbf{x}(k) \in S_{(\cdot)}} V(k, \mathbf{x}(k))$$

$$V_{\overline{M}(\cdot)}(k) = \max_{\mathbf{x}(k) \in S_{(\cdot)}} V(k, \mathbf{x}(k))$$

$$V_{m(\cdot)}(k) = \min_{\mathbf{x}(k) \in \delta_{(\cdot)}} V(k, \mathbf{x}(k))$$

$$V_{m(\cdot)}(k) = \max_{\mathbf{x}(k) \in \delta_{(\cdot)}} V(k, \mathbf{x}(k))$$

$$V_{\overline{M}(\cdot)}(k) = \max_{\mathbf{x}(k) \in \delta_{(\cdot)}} V(k, \mathbf{x}(k))$$

$$V_{\overline{M}(\cdot)}(k) = \max_{\mathbf{x}(k) \in \delta_{(\cdot)}} V(k, \mathbf{x}(k))$$

$$V_{\underline{M}(\cdot)}(k) = \max_{\mathbf{x}(k) \in \delta_{(\cdot)}} V(k, \mathbf{x}(k))$$

$$V_{\underline{M}(\cdot)}(k) = \min_{\mathbf{x}(k) \in \delta_{(\cdot)}} V(k, \mathbf{x}(k))$$

Instead of general sets, let the sets be defined as:

$$S_{\kappa\xi} = \left\{ \mathbf{x}(k) \in \mathbb{R}^n : \| \mathbf{x}(k) \| < \xi \right\}$$
(B.8)

$$\overline{\mathcal{S}}_{\kappa\xi} = \left\{ \mathbf{x}(k) \in \mathbb{R}^n : \| \mathbf{x}(k) \| \le \xi \right\}$$
(B.9)

$$\partial S_{\kappa\xi} = \overline{S}_{\kappa\xi} \setminus S_{\kappa\xi} = \left\{ \mathbf{x}(k) \in \mathbb{R}^n : \| \mathbf{x}(k) \| = \xi \right\}$$
(B.10)

The consequences are as follows:

$$V_{M\xi}(k) = \max_{\|\mathbf{x}(k)\| < \xi} V(k, \mathbf{x}(k))$$

$$V_{\overline{M}\xi}(k) = \max_{\|\mathbf{x}(k)\| \le \xi} V(k, \mathbf{x}(k))$$

$$V_{M\xi}^{\partial}(k) = \max_{\|\mathbf{x}(k)\| \ge \xi} V(k, \mathbf{x}(k))$$

$$V_{\underline{M}\xi}(k) = \max_{\|\mathbf{x}(k)\| \le \xi} V(k, \mathbf{x}(k))$$

$$V_{m\xi}(k) = \min_{\|\mathbf{x}(k)\| \le \xi} V(k, \mathbf{x}(k))$$

$$V_{\overline{m}\xi}(k) = \min_{\|\mathbf{x}(k)\| \le \xi} V(k, \mathbf{x}(k))$$

$$(B.11)$$

$$(B.12)$$

$$V_{\underline{m}\xi}(k) = \min_{\|\mathbf{x}(k)\| > \xi} V(k, \mathbf{x}(k))$$

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Received: 25.02.2010.

Dalji rezultati u proučavanju stabilnosti linearnih diskretnih sistema sa čistim vremenskim kašnjenjem na konačnom vremenskom intervalu: Sasvim drugačiji prilaz

U ovom radu su izvedeni dovoljni uslovi praktične stabilnosti i stabilnosti na konačnom vremenskom intervalu posebne klase linearnih diskretnih sistema sa čistim vremenskim kašnjenjem tipa $\mathbf{x}(k+1) = A_0 \mathbf{x}(k) + A_1 \mathbf{x}(k-1)$.

Kada je bio razmatran koncept stabilnosti na konačnom vremenskom intervalu ovi novi uslovi, koji ne uzimaju u obzir iznos čisto vremenskog kašnjenja, bili su izvedeni korišćenjem prilaza koji počiva na korišćenju tzv. kvazi Ljapunovljevih funkcija.

Kada se pak razmatrala praktična stabilnost i atraktivna praktična stabilnost prethodno pomenuti prilaz bio je kombinovan sa klasičnom ljapunovskom tehnikom, a sve sa ciljem da se garantuju osobine privlačnja kretanja razmatranog sistema.

Ključne reči: linearni sistem, disketni sistem, sistem sa kašnjenjem, sisem na konačnom vremenskom intervalu, stabilnost sistema, neljapunovska stabilnost, asimptotska stabilnost.

Дальнейшие результаты в исследовании ystoj ~ivostu linejnwh neprerwvnwh sistem so ~istwm vremennwm zapazdwvaniem na kone~nom vremennom intervale: Совсем иной подход

В нasto} ей rabote выведены удовлетворительные условия prakti~eskoй ustoj~ivostи и ustoj~ivostи na kone~nom vremennom intervale osobogo klassa linejnwh neprerwvnwh sistem so ~istwm vremennwm zapazdwvaniem типа $\mathbf{x}(k+1) = A_0 \mathbf{x}(k) + A_1 \mathbf{x}(k-1)$.

Когда был рассматриван черновик **ustoj~ivost**и **na kone~nom vremennom intervale**, эти новые условия, которые не учитывают размер **so ~istwm vremennwm zapazdwvaniem**, были выведены с использованием подхода обоснованного на использовании так называемых ложных ляпуновых функций.

А когда была рассматривана **prakti~eska} ustoj~ivostx** и привлекательная **prakti~eska} ustoj~ivostx**, предварительно упомянутый подход был комбинирован с классической ляпуновой техникой, а всё это с целью гарантии особенностей притягивания движения рассматриваемой системы.

Kly~evwe slova: ліпејпа) sistema, нергегична) sistema, sistema so zapazdwvaniem, система na kone~nom vremennom intervale, ustoj~ivostx sistemw, nel)punovaя ustoj~ivostь, асимптотическая ustoj~ivostx.

Nouveaux résultats dans les recherches sur la stabilité des systèmes linéaires discrets à délai temporel pur chez l'intervalle temporelle finie: une approche toute différente

Ce papier donne les conditions suffisantes de la stabilité pratique ainsi que la stabilité pour l'intervalle temporelle finie de clase particulière des systèmes linéaires discrets à délai temporel pur du type $\mathbf{x}(k+1) = A_0 \mathbf{x}(k) + A_1 \mathbf{x}(k-1)$.

Quand on a considéré le concept de la stabilité pour l'intervalle temporelle finie, ces nouvelles conditions, qui ne prennent pas en considération la totalité du délai temporel pur, ont été réalisées via l'approche basée sur l'emploi des quasi équations de Lyapunov. Lorsqu'on a considéré la stabilité pratique et la stabilité pratique attractive, déjà citées, l'approche était combinée avec la technique classique de Lyapunov, dans le but de garantir les caractéristiques attrayantes du comportement du système observé.

Mots clés: système linéaire, système discret, système à délai, système sur l'intervalle temporelle finie, stabilité du système, stabilité de non Lyapunov, stabilité asymptotique.