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Comparative Analysis of Different Methods in Mathematical Modelling of the Recuperative Heat Exchangers

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The heat exchangers are frequently used as constructive elements in various plants and their dynamics is very important. Heat exchangers are used in military aircraft of all sizes, flying different missions, in many different aircraft systems Their operation is usually controlled by manipulating inlet fluid temperatures or mass flow rates. On the basis of the accepted and critically clarified assumptions, a linearized mathematical model of the cross-flow heat exchanger has been derived, taking into account the wall dynamics. The model is based on the fundamental law of energy conservation, covers all heat accumulation storages in the process, and leads to the set of partial differential equations (PDE), the solution of which is not possible in a closed form. In order to overcome the solution difficulties, this paper analyses different methods for modeling the heat exchanger: the approach based on the Laplace transform, the approximation of partial differential equations based on the finite difference method, the method of physical discretization and the transport approach. Specifying the input temperatures and output variables, under constant initial conditions, the step transient responses have been simulated and presented in a graphic form in order to compare the results for the four characteristic methods considered in this paper and to analyze their practical significance.

Key words: **heat exchanger, mathematical model, comparative analysis, Laplace transform, finite difference method, discretization method.**

Introduction

THE heat exchangers are designed to achieve certain requirements in the steady state which implies that the transient response of the heat exchanger must be known to define a correct control strategy.

The heat exchangers are frequently used in military and civil aircraft as oil coolers and fuel heaters with heat exchangers placed inside fuel tanks. In large transport aircraft the cabin pressurization systems cool the hot engine bleed air to a temperature suitable for use within the cabin with a heat exchanger. The air conditioning system is also based on a heat exchanger, known as an air-cycle machine, in order to provide air at the desired temperature to the cabin and flight deck.

The mathematical model of the heat exchanger must be known in order to determine the transient response.

Many researchers have been working on this problem within the last decade such as *Čermak et al.* (1968), *Roetzel, Huan* (1992), *Romie* (1984), *Spiga, Spiga* (1987), *Tan, Spinner* (1991).

Solving of this problem in the mentioned papers is based on two approaches:

1. numerical solving of PDE which pulls drawbacks such as convergence problems, stability, stiffness, etc.
2. Laplace transform which is complicated in this case and demands a numerical inversion to the original time domain.

These problems and the similar ones have motivated the research to invent new approaches for modeling these processes in order to obtain wider practical use and not to lose their veracity as well.

In this paper the four characteristic methods for modelling heat exchangers are considered:

1. Analytical approach using the Laplace transform.
2. Approximation of PDE based on finite differences.
3. The method of physical discretization.
4. Transport approach.

The comparative analysis of these methods and their practical significance is carried out on the model of a cross flow heat exchanger.

Finally, the step responses of the derived mathematical models are presented in a graphic form in order to compare these results with different methods.

Analytical approach based on the laplace transform

In this paper, the recuperative cross-flow heat exchanger is observed, shown in Fig.1, as a process with distributed parameters. Among many kinds of water-to-air heat exchangers, the cross-flow geometry is very common. The geometry of cross-flow heat exchangers can be complicated, but in this paper we observe the case with the simplest geometry that can be easily computed. This heat exchanger consists of a single tube with the fluid (water or other fluid) flow inside and the cross flow of hot air outside.

The mathematical model of the cross flow heat exchanger, shown in Fig.1, is carried out on the basis of the following assumptions:

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- Fluid in the tube is incompresible and viscous.
- Temperature field fluid in the tube is one-dimensional.
- Temperature field of the tube wall is one-dimensional.
- Fluid enthalpy can be expressed by means of a corresponding temperature.
- Inlet air temperature is a function of time.
- Specific heat of fluid, wall and air has constant values.
- There is convective heat transfer between the cross flow air and the tube wall, conduction along the tube wall, and convection between the tube wall and the fluid in the tube wall.
- Newton's law of heat convection determines exactly enough of the amount of heat exchange in the steady state and the transient state.
- Heat transfer coefficients on both sides of the tubewall have constant values.
- The fluid flow in the tube is one-phased.

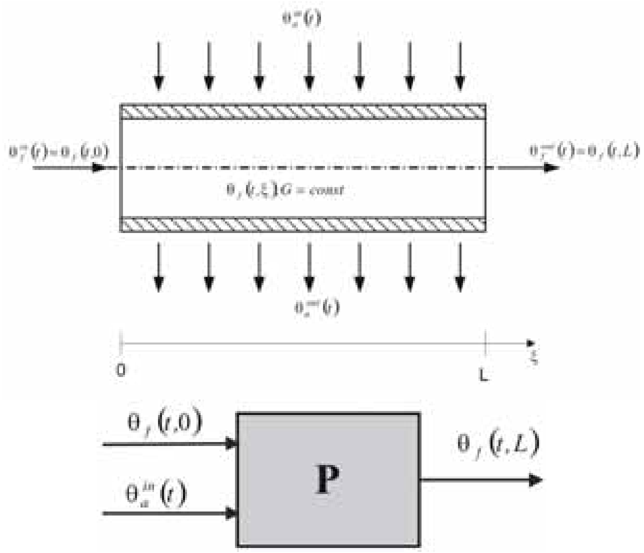

Figure 1. Symbolic-functional scheme and the diagram of the process of the cross-flow heat exchanger

On the basis of these assumptions, the mathematical model of the cross- flow heat exchanger after some simple mathematical transformations can be written in the following form.

- On the outside of the tube:

$$G_a c_a \left(\theta_a^{in}(t) - \theta_a^{out}(t)\right) = 2\alpha_{aw}\pi r_o L\left(\theta_a(t) - \theta_w(t,\xi)\right) \quad (1)$$

where L is the length of the tube, G_a is the mass-flow rate of air, c_a is the specific heat of air, θ_a^{in} and θ_a^{out} are incoming and outgoing air temperatures, α_{aw} is the heat transfer coefficient beween the hot air and the tube wall, r_0 is the outer radius of the tube, θ_a is the air temperature sorrounding the tube, and θ_w is the tube wall temperature. For convenience, the air temperature can be assumed to be approximately:

$$\theta_a(t) = \frac{\theta_a^{in}(t) + \theta_a^{out}(t)}{2} \quad (2)$$

This can be substituted in eq. (1).

- In the water:

$$\rho_f c_f r_i^2 \pi \frac{\partial \theta_f(t,\xi)}{\partial t} = \alpha_{wf} 2\pi r_i \left(\theta_w(t,\xi) - \theta_f(t,\xi)\right) - -G_f c_f \frac{\partial \theta_f(t,\xi)}{\partial \xi} \quad (3)$$

where ρ_f is the fluid (water) density, c_f is the specific heat of the fluid in the tube, r_i is the inner radius, θ_f is the fluid temperature, α_{wf} is the heat transfer coefficient between the wall and the fluid, and G_f is the mass-flow rate of the fluid.

- Finally, in the wall of the tube:

$$\rho_w c_w \pi \left(r_0^2 - r_i^2\right)\frac{\partial \theta_w(t,\xi)}{\partial t} = \lambda_w \pi \left(r_0^2 - r_i^2\right)\frac{\partial^2 \theta_w(t,\xi)}{\partial \xi^2} + +2\pi r_0 \alpha_{aw}\left(\theta_a(t) - \theta_w(t,\xi)\right) - 2\pi r_i \alpha_{wf}\left(\theta_w(t,\xi) - \theta_f(t,\xi)\right) \quad (4)$$

where ρ_w is the wall density, c_w is the specific heat of the wall, λ_w is the wall termal conductivity.

The boundary and initial conditions are:

$$\theta_f(t,0) = \theta_f^{in}(t),\ \theta_f(t,L) = \theta_f^{out}(t), \quad (5)$$

$$\theta_f(0,\xi) = 0,\ \theta_w(0,\xi) = 0$$

Introducing relative deviations in this form:

$$\overline{\Delta\theta}_f(t,\xi) = \frac{\theta_f(t,\xi) - \theta_{fN}(\xi)}{\theta_{fN}(\xi_N)}$$

$$\overline{\Delta\theta}_w(t,\xi) = \frac{\theta_w(t,\xi) - \theta_{wN}(\xi)}{\theta_{wN}(\xi_N)} \quad (6)$$

$$\overline{\Delta\theta}_a^{in}(t) = \frac{\theta_a^{in}(t) - \theta_{aN}^{in}}{\theta_{aN}^{in}}$$

where θ_{fN}, θ_{wN}, θ_{aN}^{in} are nominal values.

From eq. (2), (3), (4), (5) the mathematical model of the cross-flow heat exchanger is obtained in the following form:

$$\frac{\partial \overline{\Delta\theta}_f(t,\xi)}{\partial t} = a_1 \overline{\Delta\theta}_w(t,\xi) - a_2 \overline{\Delta\theta}_f(t,\xi) - a_3 \frac{\partial \overline{\Delta\theta}_f(t,\xi)}{\partial \xi} \quad (7)$$

$$\frac{\partial \overline{\Delta\theta}_w(t,\xi)}{\partial t} = b_1 \frac{\partial^2 \overline{\Delta\theta}_w(t,\xi)}{\partial \xi^2} - b_2 \overline{\Delta\theta}_w(t,\xi) + + b_3 \overline{\Delta\theta}_f(t,\xi) + b_4 \overline{\Delta\theta}_a^{in}(t) \quad (8)$$

where:

$$a_1 = \frac{2\alpha_{wf}}{\rho_f c_f r_i}\frac{\theta_{wN}}{\theta_{fN}},$$

$$a_2 = \frac{2\alpha_{wf}}{\rho_f c_f r_i},$$

$$a_3 = \frac{G_f}{\rho_f r_i^2 \pi},$$

$$b_1 = \frac{\lambda_w}{\rho_w c_w}, \quad (9)$$

$$b_2 = \frac{2G_a c_a\left(r_0\alpha_{aw} + r_i\alpha_{wf}\right) + 2\alpha_{aw}\alpha_{wf} r_0 r_i \pi L}{\rho_w c_w \left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{wa}\pi r_0 L\right)},$$

$$b_3 = \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_0^2 - r_i^2\right)}\frac{\theta_{fN}}{\theta_{wN}},$$

$$b_4 = \frac{2r_0\alpha_{aw}G_a c_a}{\rho_w c_w\left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 L\right)}\frac{\theta_{aN}^{in}}{\theta_{wN}}.$$

Taking the Laplace transform of (6), (7) first with respect to *t*, and then with respect to the spatial coordinate ξ, $L_{\xi\to\sigma}$, with initial conditions (5) we find:

$$\left(s + a_2 + a_3\sigma\right)\overline{\Delta\theta_f}\left(s,\sigma\right) = a_1\overline{\Delta\theta_w}\left(s,\sigma\right) + a_3\overline{\Delta\theta_f}\left(s,0\right) \quad (10)$$

$$\begin{aligned}&\left(s - b_1\sigma^2 + b_2\right)\overline{\Delta\theta_w}\left(s,\sigma\right) = \\ &= b_3\overline{\Delta\theta_f}\left(s,\sigma\right) - b_1\sigma\overline{\Delta\theta_w}\left(s,0\right) + b_4\overline{\Delta\theta_a^{in}}\left(s\right)\end{aligned} \quad (11)$$

Sollution of the system (10), (11) for $\theta_f\left(s,\sigma\right)$ has the following form:

$$\begin{aligned}\theta_f\left(s,\sigma\right) &= \frac{a_3b_1\sigma^2 - M\left(s\right)}{a_3b_1\sigma^3 + P\left(s\right)\sigma^2 - Q\left(s\right)\sigma - R\left(s\right)}\overline{\Delta\theta_f}\left(s,0\right) - \\ &- \frac{a_1b_4}{a_3b_1\sigma^3 + P\left(s\right)\sigma^2 - Q\left(s\right)\sigma - R\left(s\right)}\overline{\Delta\theta_a^{in}}\left(s\right) + \\ &+ \frac{a_1b_1\sigma}{a_3b_1\sigma^3 + P\left(s\right)\sigma^2 - Q\left(s\right)\sigma - R\left(s\right)}\overline{\Delta\theta_w}\left(s,0\right)\end{aligned} \quad (12)$$

where

$$M\left(s\right) = a_3s + a_3b_2,\ P\left(s\right) = b_1s + a_2b_1,$$

$$Q\left(s\right) = a_3s + a_3b_2, \quad (13)$$

$$R\left(s\right) = s^2 + \left(a_2 + b_2\right)s + a_2b_2 - a_1b_3$$

Taking the inverse Laplace transform of (12) with respect to the spatial coordinate, $L_{\sigma\to\xi}^{-1}$, and switching $\xi = L$, appropriate transfer functions are obtained in the following form:

$$\begin{aligned}W_1\left(s,L\right) &= \frac{\overline{\Delta\theta_f}\left(s,L\right)}{\overline{\Delta\theta_f}\left(s,0\right)} = F\left(\sigma_1,s,L\right) + \\ &+ F\left(\sigma_2,s,L\right) + F\left(\sigma_3,s,L\right)\end{aligned}, \quad (14)$$

$$\begin{aligned}W_2\left(s,L\right) &= \frac{\overline{\Delta\theta_f}\left(s,L\right)}{\overline{\Delta\theta_a^{in}}\left(s\right)} = G\left(\sigma_1,s,L\right) + \\ &+ G\left(\sigma_2,s,L\right) + G\left(\sigma_3,s,L\right)\end{aligned} \quad (15)$$

where

$$F\left(\sigma_n,s,L\right) = \frac{a_3b_1\sigma_n^2 - U\left(s\right)}{3a_3b_1\sigma_n^2 + V\left(s\right)\sigma_n - U\left(s\right)}e^{-\sigma_n L}, \quad (16)$$

$$G\left(\sigma_n,s,L\right) = \frac{-b_4a_1}{3a_3b_1\sigma_n^2 + V\left(s\right)\sigma_n - U\left(s\right)}e^{-\sigma_n L}$$

and

$$U\left(s\right) = a_3s + a_3b_2,\ V\left(s\right) = 2\left(b_1s + a_2b_1\right) \quad (17)$$

$\sigma_n, n = 1,2,3$ are the roots of the equation

$$\begin{aligned}&a_3b_1\sigma^3 + \left(b_1s + a_2b_1\right)\sigma^2 - \left(a_3s + a_3b_2\right)\sigma - \\ &-s^2 - \left(a_2 + b_2\right)s + a_1b_3 - a_2b_2 = 0\end{aligned} \quad (18)$$

In order to obtain the step response for the mathematical model of the cross-flow heat exchanger it is necessary to take another inverse Laplace transform with respect to time, $L_{s\to t}^{-1}$. Considering the complexity of the right side of Eqs. (14) and (15) this can be done using the numerical methods for the inverse Laplace transform, *Abate, Choudhury* (1999), *Abate, Valko* (2004).

Applying the *Gaver-Stehfest* algorithm for the inverse Laplace transform of (14),(15) we find:

$$\frac{\overline{\Delta\theta_f}\left(t,L\right)}{\overline{\Delta\theta_f}\left(t,0\right)} = f(t,M) = \frac{\ln(2)}{t}\sum_{k=1}^{2M}\varsigma_k F\left(\frac{k\ln(2)}{t}\right) \quad (19)$$

$$\frac{\overline{\Delta\theta_f}\left(t,L\right)}{\overline{\Delta\theta_a^{in}}\left(t\right)} = g(t,M) = \frac{\ln(2)}{t}\sum_{k=1}^{2M}\varsigma_k G\left(\frac{k\ln(2)}{t}\right) \quad (20)$$

where

$$\varsigma_k = \left(-1\right)^{M+k}\sum_{j=[(k+1)/2]}^{k\wedge M}\frac{j^{M+1}}{M!}\binom{M}{j}\binom{2j}{j}\binom{j}{k-j} \quad (21)$$

with $\left[(k+1)/2\right]$ being the greatest integer lower than or equal to $\left(k+1\right)/2$, $k \wedge M \equiv \min\left\{k,M\right\}$ and the positive integer *M*.

Differential-discrete mathematical model of the cross-flow heat exchanger

The system of PDE is very complex to be solved analytically as it was considered in the analytical aproach for modeling the cross-flow heat exchanger. A numerical method based on the finite differences is developed to approximate infinite-dimensional equations by finite-dimensional ones. The spatial coordinate of the cross-flow heat exchanger is discretized by means of the finite difference method. In this manner the system of PDE is transformed to the system of ODE. Discretization of the spatial variable is carried out by means of substituting partial derivation with respect to the spatial coordinate in certain numbers of points. In the physical sense, the observed heat exchanger is divided into equal *p*-cells with the spatial coordinate discretization.

The mathematical model of the cross-flow heat exchanger is derived in the previous section, described with Eqs. (1), (3), and (4). The finite difference method has a different form for the first, *k*-th (*k* = 2,3,..., *p*-1) and the last cell. This method has a lot of different forms. Regarding a one-dimensional problem, the finite difference method has the following form:

$$\Delta\xi = \xi_{k+1} - \xi_k = l,\ \theta_{f,k}\left(t\right) = \theta_f\left(t,k\right),\ k=1,2,...,p$$

$$\frac{\partial\theta_f\left(t,\xi\right)}{\partial\xi}\Big|_k \cong \frac{1}{2l}\left(\theta_{f,k+1}\left(t\right) - \theta_{f,k-1}\left(t\right)\right), \quad (22)$$

$$\frac{\partial^2\theta_f\left(t,\xi\right)}{\partial\xi^2}\Big|_k \cong \frac{1}{l^2}\left(\theta_{f,k+1}\left(t\right) - 2\theta_{f,k}\left(t\right) + \theta_{f,k-1}\left(t\right)\right).$$

Substituting (22) into (3), (4) the differential-discrete model of the cross-flow heat exchanger is obtained in the following form:

- for the first cell:

$$\frac{d\theta_{f,1}(t)}{dt} = \frac{2\alpha_{wf}}{\rho_f c_f r_u}\theta_{w,1}(t) - \frac{2\alpha_{wf}}{\rho_f c_f r_u}\theta_{f1}(t) - \frac{G_f}{\rho_f r_u^2 \pi}\left(\frac{\theta_{f,2}(t) - \theta_f^{in}(t)}{2l}\right) \quad (23)$$

$$\frac{d\theta_{w,1}(t)}{dt} = \frac{\lambda_w}{\rho_w c_w}\left(\frac{\theta_{w,2}(t) - 2\theta_{w,1}(t)}{l^2}\right) + \frac{2r_s\alpha_{aw}G_a c_a}{\rho_w c_w\left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 L\right)}\theta_a^{in}(t) + \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_0^2 - r_i^2\right)}\theta_{f,1}(t) - \frac{2G_a c_a\left(r_0\alpha_{aw} + r_i\alpha_{wf}\right) + 2\alpha_{aw}\alpha_{wf}r_0 r_i \pi L}{\rho_w c_w\left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 L\right)}\theta_{w,1}(t) \quad (24)$$

k-th cell:

$$\frac{d\theta_{f,k}(t)}{dt} = \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{w,k}(t) - \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{f,k}(t) - \frac{G_f}{\rho_f r_i^2 \pi}\left(\frac{\theta_{f,k+1}(t) - \theta_{f,k-1}(t)}{2l}\right) \quad (25)$$

$$\frac{d\theta_{w,k}(t)}{dt} = \frac{\lambda_w}{\rho_w c_w}\left(\frac{\theta_{w,k+1}(t) - 2\theta_{w,k}(t) + \theta_{w,k-1}(t)}{l^2}\right) + \frac{2r_0\alpha_{aw}G_a c_a}{\rho_w c_w\left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 L\right)}\theta_a^{in}(t) + \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_0^2 - r_i^2\right)}\theta_{f,k}(t) - \frac{2G_a c_a\left(r_0\alpha_{aw} + r_i\alpha_{wf}\right) + 2\alpha_{aw}\alpha_{wf}r_0 r_i \pi L}{\rho_w c_w\left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 L\right)}\theta_{w,k}(t) \quad (27)$$

- last cell:

$$\frac{d\theta_{f,p}(t)}{dt} = \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{w,p}(t) - \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{f,p}(t) + \frac{G_f}{\rho_f r_i^2 \pi}\frac{\theta_{f,p-1}(t)}{2l}$$

$$\frac{d\theta_{w,p}(t)}{dt} = -\frac{\lambda_w}{\rho_w c_w}\frac{2\theta_{w,p}(t) - \theta_{w,p-1}(t)}{l^2} + \frac{2r_0\alpha_{aw}G_a c_a}{\rho_w c_w\left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 L\right)}\theta_a^{in}(t) + \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_0^2 - r_i^2\right)}\theta_{f,p}(t) - \frac{2G_a c_a\left(r_0\alpha_{aw} + r_i\alpha_{wf}\right) + 2\alpha_{aw}\alpha_{wf}r_0 r_i \pi L}{\rho_w c_w\left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 L\right)}\theta_{w,p}(t)$$

where

$$\theta_f(t,L) = \theta_{f,p}(t),\ \theta_{w,0}(t) = 0, \quad (29)$$

$$\theta_{f,p+1}(t) = 0,\ \theta_{f,0}(t) = \theta_f^{in}(t).$$

Introduce relative deviations and define the state variables in this form:

$$\overline{\Delta\theta}_{f,k}(t) = \frac{\theta_{f,k}(t) - \theta_{fN,k}}{\theta_{fN,k}} = x_k(t),$$

$$\overline{\Delta\theta}_{w,k}(t) = \frac{\theta_{w,k}(t) - \theta_{wN,k}}{\theta_{wN,k}} = x_k^*(t),$$

$$\overline{\Delta\theta}_f^{in}(t) = \frac{\theta_f^{in}(t) - \theta_{fN}^{in}}{\theta_{fN}^{in}} = u_1(t), \quad (30)$$

$$\overline{\Delta\theta}_a^{in}(t) = \frac{\theta_a^{in}(t) - \theta_{aN}^{in}}{\theta_{aN}^{in}} = u_2(t),$$

$$\mathbf{x}_k(t) = \begin{bmatrix} x_k(t) \\ x_k^*(t) \end{bmatrix},\ \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

The mathematical model of the cross-flow heat exchanger is obtained in the state space:

- for the k-th cell:

$$\dot{\mathbf{x}}_k(t) = A_k^1\mathbf{x}_{k-1}(t) + A_k^2\mathbf{x}_k(t) + A_k^3\mathbf{x}_{k+1}(t) + B_k\mathbf{u}(t) \quad (31)$$

where

$$A_k^1 = \begin{pmatrix} a_{k3} & 0 \\ 0 & b_{k3} \end{pmatrix},\ A_k^2 = \begin{pmatrix} -a_{k1} & a_{k4} \\ b_{k4} & -b_{k1} \end{pmatrix},$$

$$A_k^3 = \begin{pmatrix} -a_{k2} & 0 \\ 0 & b_{k2} \end{pmatrix},\ B_k = \begin{pmatrix} 0 & 0 \\ 0 & b_{k5} \end{pmatrix},$$

$$K = 2,3,\ldots,p\text{-}1 \quad (32)$$

and

$$a_{k1} = \frac{2\alpha_{wf}}{\rho_f c_f r_i},\ a_{k2} = \frac{G_f}{\rho_f r_i^2 \pi 2l}\frac{\theta_{fN,k+1}}{\theta_{fN,k}},$$

$$a_{k3} = \frac{G_f}{\rho_f r_i^2 \pi 2l}\frac{\theta_{fN,k-1}}{\theta_{fN,k}},\ a_{k4} = \frac{2\alpha_{wf}}{\rho_f c_f r_i}\frac{\theta_{wN,k}}{\theta_{fN,k}},$$

$$b_{k1} = \left(\frac{2\lambda_w}{\rho_w c_w l^2} + \frac{2G_a c_a\left(r_0\alpha_{aw} + r_i\alpha_{wf}\right) + 2\alpha_{aw}\alpha_{wf}r_0 r_i \pi L}{\rho_w c_w\left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 L\right)}\right)$$

$$b_{k2} = \frac{\lambda_w}{\rho_w c_w l^2}\frac{\theta_{wN,k+1}}{\theta_{wN,k}},\ b_{k3} = \frac{\lambda_w}{\rho_w c_w l^2}\frac{\theta_{wN,k-1}}{\theta_{wN,k}},$$

$$b_{k4} = \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_0^2 - r_i^2\right)}\frac{\theta_{fN,k}}{\theta_{wN,k}},\ k=1,2,\ldots,p \quad (33)$$

$$b_{k5} = \frac{2r_0\alpha_{aw}G_a c_a}{\rho_w c_w\left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 L\right)}\frac{\theta_{aN}^{in}}{\theta_{wN,k}}.$$

The mathematical model of the first and the last cell in the state space will be slightly modified, because these two cells, through control boundary, have contact with enviroment.

The mathematical model of the first cell has the following form:

$$\dot{\mathbf{x}}_1(t) = A_1^2 \mathbf{x}_1(t) + A_1^3 \mathbf{x}_2(t) + B_1 \mathbf{u}(t) \tag{34}$$

where

$$A_1^2 = \begin{pmatrix} -a_{11} & a_{14} \\ b_{14} & -b_{11} \end{pmatrix},\ A_1^3 = \begin{pmatrix} -a_{12} & 0 \\ 0 & b_{12} \end{pmatrix}, \tag{35}$$

$$B_1 = \begin{pmatrix} a_{15} & 0 \\ 0 & b_{15} \end{pmatrix},$$

while this model in the state space for the last cell can be written as follows:

$$\dot{\mathbf{x}}_p(t) = A_p^1 \mathbf{x}_{p-1}(t) + A_p^2 \mathbf{x}_p(t) + B_p \mathbf{u}(t) \tag{36}$$

where

$$A_p^1 = \begin{pmatrix} a_{p3} & 0 \\ 0 & b_{p3} \end{pmatrix},\ A_p^2 = \begin{pmatrix} -a_{p1} & a_{p4} \\ b_{p4} & -b_{p1} \end{pmatrix}, \tag{37}$$

$$B_p = \begin{pmatrix} 0 & 0 \\ 0 & b_{p5} \end{pmatrix}.$$

The state equation and the output of the cross-flow heat exchanger are given with:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{38}$$

$$x_i(t) = \mathbf{c}^T \mathbf{x}(t),$$

$$A = \begin{pmatrix} A_1^2 & A_1^3 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ A_2^1 & A_2^2 & A_2^3 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & A_3^1 & A_3^2 & A_3^3 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & A_{p-1}^1 & A_{p-1}^2 & A_{p-1}^3 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & A_p^1 & A_p^2 \end{pmatrix}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_{p-1} \\ B_p \end{bmatrix},\ \mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ \dots \\ \vdots \\ \dots \\ 1 \\ 0 \end{bmatrix},\ \mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_p(t) \end{bmatrix},$$

$$x_i(t) = \overline{\Delta\theta}_{f,p}(t).$$

The method of physical discretization

To avoid partial differential equations, the method of physical discretization suggested by *Grujić* (1978) is used. By this method the observed heat exchanger is divided into the same p cells with the spatial coordinate discretization. These cells are assumed to have homogenous fields of specific physical values. The qualitative description of the process with distributed parameters can be obtained taking into consideration a big enough number of cells.

The advantage of such an approach is avoiding complex PDE to be set and solved. Ballance differential equations are set for an arbitrary cell, taking especially into consideration the first an the last cell which through control boundary have contact with environment. The flowing index indicates the position of the cell in the heat exchanger.

In the physical sense, the observed cross-flow heat exchanger is divided into equal p cells with the spatial coordinate discretization. Assuming that the number of cells is big enough, each of these cells can be considered as a process with certain parameters. The fundamental law of energy conservation is derived for every cell, so the mathematical model of the cross-flow heat exchanger can be reperesented as a system of ODE.

Considering the k-th cell, the ballance equation can be written in the following form:

$$\frac{d\theta_{f,k}(t)}{dt} = \frac{G_f}{\rho_f r_i^2 \pi l}\left(\theta_{f,k-1}(t) - \theta_{f,k}(t)\right) + \\ + \frac{2\alpha_{wf}}{\rho_f c_f r_i}\left(\theta_{w,k}(t) - \theta_{f,k}(t)\right) \tag{39}$$

$$\frac{d\theta_{w,k}(t)}{dt} = \frac{\lambda_w}{\rho_w c_w l}\left(\theta_{w,k-1}(t) - \theta_{w,k}(t)\right) + \\ + \frac{2r_0\alpha_{aw}}{\rho_w c_w\left(r_0^2 - r_i^2\right)}\left(\theta_a(t) - \theta_{w,k}(t)\right) - \\ - \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_0^2 - r_u^2\right)}\left(\theta_{w,k}(t) - \theta_{f,k}(t)\right) \tag{40}$$

$$\left(\theta_a^{in}(t) - \theta_a^{out}(t)\right) = \frac{2\alpha_{aw}\pi r_0 l}{G_a c_a}\left(\theta_a(t) - \theta_{w,k}(t)\right) \tag{41}$$

where

$$\theta_a(t) = \frac{\theta_a^{in}(t) + \theta_a^{out}(t)}{2} \tag{42}$$

Using Eqs. (39) - (42) the mathematical model of the cross flow heat exchanger can be obtained in the following form:

- k-th cell:

$$\frac{d\theta_{f,k}(t)}{dt} = -\left(\frac{2\alpha_{wf}}{\rho_f c_f r_i} + \frac{G_f}{\rho_f r_i^2 \pi l}\right)\theta_{f,k}(t) + \\ + \frac{G_f}{\rho_f r_i^2 \pi l}\theta_{f,k-1}(t) + \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{w,k}(t) \tag{43}$$

$$\frac{d\theta_{w,k}(t)}{dt} = \frac{4\alpha_{aw}^2 r_0^2 \pi l}{\rho_w c_w\left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 l\right)}\theta_{w,k}(t) - \\ - \frac{\lambda_w}{\rho_w c_w l}\theta_{w,k}(t) - \frac{2r_0\alpha_{aw} + 2r_i\alpha_{wf}}{\rho_w c_w\left(r_0^2 - r_i^2\right)}\theta_{w,k}(t) + \\ + \frac{2r_i\alpha_{wf}}{\rho_w c_w\left(r_0^2 - r_i^2\right)}\theta_{f,k}(t) + \frac{\lambda_w}{\rho_w c_w l}\theta_{w,k-1}(t) + \\ + \frac{2r_0\alpha_{aw}\left(G_a c_a - \alpha_{aw}\pi r_0 l\right)}{\rho_w c_w\left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 l\right)}\theta_a^{in}(t) \tag{44}$$

- for the first cell:

$$\frac{d\theta_{f,1}(t)}{dt} = -\left(\frac{2\alpha_{wf}}{\rho_f c_f r_i} + \frac{G_f}{\rho_f r_i^2 \pi l}\right)\theta_{f,1}(t) + \\ + \frac{G_f}{\rho_f r_i^2 \pi l}\theta_f^{in}(t) + \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{w,1}(t) \tag{45}$$

$$\begin{aligned}\frac{d\theta_{w,1}(t)}{dt} &= \frac{4\alpha_{aw}^2 r_0^2 \pi l}{\rho_w c_w \left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 l\right)}\theta_{w,1}(t) - \\ &- \frac{\lambda_w}{\rho_w c_w l}\theta_{w,1}(t) - \frac{2r_0\alpha_{aw} + 2r_i\alpha_{wf}}{\rho_w c_w \left(r_0^2 - r_i^2\right)}\theta_{w,1}(t) + \\ &+ \frac{2r_i\alpha_{wf}}{\rho_w c_w \left(r_0^2 - r_i^2\right)}\theta_{f,1}(t) + \\ &+ \frac{2r_0\alpha_{aw}\left(G_a c_a - \alpha_{aw}\pi r_0 l\right)}{\rho_w c_w \left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 l\right)}\theta_a^{in}(t)\end{aligned} \tag{46}$$

- for the last cell:

$$\begin{aligned}\frac{d\theta_{f,p}(t)}{dt} &= -\left(\frac{2\alpha_{wf}}{\rho_f c_f r_u} + \frac{G_f}{\rho_f r_i^2 \pi l}\right)\theta_{f,p}(t) \\ &+ \frac{G_f}{\rho_f r_i^2 \pi l}\theta_{f,p-1}(t) + \frac{2\alpha_{wf}}{\rho_f c_f r_i}\theta_{w,p}(t)\end{aligned} \tag{46}$$

$$\begin{aligned}\frac{d\theta_{w,p}(t)}{dt} &= \frac{4\alpha_{aw}^2 r_o^2 \pi l}{\rho_w c_w \left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 l\right)}\theta_{w,p}(t) - \\ &- \frac{\lambda_w}{\rho_w c_w l}\theta_{w,p}(t) - \frac{2r_0\alpha_{aw} + 2r_i\alpha_{wf}}{\rho_w c_w \left(r_0^2 - r_i^2\right)}\theta_{w,p}(t) + \\ &+ \frac{2r_i\alpha_{wf}}{\rho_w c_w \left(r_0^2 - r_i^2\right)}\theta_{f,p}(t) + \frac{\lambda_w}{\rho_w c_w l}\theta_{w,p-1}(t) + \\ &+ \frac{2r_0\alpha_{aw}\left(G_a c_a - \alpha_{aw}\pi r_0 l\right)}{\rho_w c_w \left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 l\right)}\theta_a^{in}(t)\end{aligned} \tag{48}$$

Introducing relative deviations and defining the state variables in this form:

$$\overline{\Delta\theta}_{f,k}(t) = \frac{\theta_{f,k}(t) - \theta_{fN,k}}{\theta_{fN,k}} = x_k(t),$$

$$\overline{\Delta\theta}_{w,k}(t) = \frac{\theta_{w,k}(t) - \theta_{wN,k}}{\theta_{wN,k}} = x_k^*(t), \tag{49}$$

$$\overline{\Delta\theta}_f^{in}(t) = \frac{\theta_f^{in}(t) - \theta_{fN}^{in}}{\theta_{fN}^{in}} = u_1(t),$$

$$\overline{\Delta\theta}_a^{in}(t) = \frac{\theta_a^{in}(t) - \theta_{aN}^{in}}{\theta_{aN}^{in}} = u_2(t).$$

The mathematical model for the k-th cell is obtained in the following form:

$$\frac{dx_k(t)}{dt} = -a_{k1}x_k(t) + a_{k2}x_{k-1}(t) + a_{k3}x_k^*(t) \tag{50}$$

$$\begin{aligned}\frac{dx_k^*(t)}{dt} &= b_{k1}x_k^*(t) + b_{k2}x_{k-1}^*(t) + \\ &+ b_{k3}x_k(t) + b_{k4}u_2(t)\end{aligned} \tag{51}$$

where $k = 1,2,\dots,p$ and

$$a_{k1} = \frac{2\alpha_{wf}}{\rho_f c_f r_i} + \frac{G_f}{\rho_f r_i^2 \pi l},\ a_{k2} = \frac{G_f}{\rho_f r_u^2 \pi l}\frac{\theta_{fN,k-1}}{\theta_{fN,k}},$$

$$a_{k3} = \frac{2\alpha_{wf}}{\rho_f c_f r_i}\frac{\theta_{wN,k}}{\theta_{fN,k}},$$

$$\begin{aligned}b_{k1} &= \frac{4\alpha_{aw}^2 r_0^2 \pi l}{\rho_w c_w \left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 l\right)} - \\ &- \frac{\lambda_w}{\rho_w c_w l} - \frac{2r_0\alpha_{aw} + 2r_i\alpha_{wf}}{\rho_w c_w \left(r_0^2 - r_i^2\right)}\end{aligned}, \tag{52}$$

$$b_{k2} = \frac{\lambda_w}{\rho_w c_w\ l}\frac{\theta_{wN,k-1}}{\theta_{wN,k}},\ b_{k3} = \frac{2r_i\alpha_{wf}}{\rho_w c_w \left(r_0^2 - r_i^2\right)}\frac{\theta_{fN,k}}{\theta_{wN,k}}$$

$$b_{k4} = \frac{2r_0\alpha_{aw}\left(G_a c_a - \alpha_{aw}\pi r_0 l\right)}{\rho_w c_w \left(r_0^2 - r_i^2\right)\left(G_a c_a + \alpha_{aw}\pi r_0 l\right)}\frac{\theta_{gN}^u}{\theta_{zN,k}}.$$

The mathematical model for the k-th cell of the cross-flow heat exchanger is obtained in the state space:

$$\frac{d\mathbf{x}_k(t)}{dt} = A_k^1\mathbf{x}_{k-1}(t) + A_k^2\mathbf{x}_k(t) + B_k\mathbf{u}(t) \tag{53}$$

where

$$A_k^1 = \begin{pmatrix} a_{k2} & 0 \\ 0 & b_{k2} \end{pmatrix},\ A_k^2 = \begin{pmatrix} -a_{k1} & a_{k3} \\ b_{k3} & b_{k1} \end{pmatrix},$$

$$B_k = \begin{pmatrix} 0 & 0 \\ 0 & b_{k4} \end{pmatrix}, \tag{54}$$

$$\mathbf{x}_k(t) = \left[x_k(t)\ \ x_k^*(t)\right]^T,\ \mathbf{u}(t) = \left[u_1(t)\ \ u_2(t)\right]^T$$

For the first cell of the cross-flow heat exchanger the state space equation has the following form:

$$\dot{\mathbf{x}}_1(t) = A_1^2\mathbf{x}_1(t) + B_1\mathbf{u}(t) \tag{55}$$

where

$$A_1^2 = \begin{pmatrix} -a_{11} & a_{13} \\ b_{13} & b_{11} \end{pmatrix},\ B_1 = \begin{pmatrix} a_{12} & 0 \\ 0 & b_{14} \end{pmatrix} \tag{56}$$

For the last cell the state space equation can be written as follows:

$$\dot{\mathbf{x}}_p(t) = A_p^1\mathbf{x}_{p-1}(t) + A_p^2\mathbf{x}_p(t) + B_p\mathbf{u}(t) \tag{57}$$

$$A_p^1 = \begin{pmatrix} a_{p2} & 0 \\ 0 & b_{p2} \end{pmatrix},\ A_p^2 = \begin{pmatrix} -a_{p1} & a_{p3} \\ b_{p3} & b_{p1} \end{pmatrix}, \tag{58}$$

$$B_p = \begin{pmatrix} 0 & 0 \\ 0 & b_{p4} \end{pmatrix}.$$

The state equation and the output of the cross-flow heat exchanger are given with:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \tag{59}$$

$$\mathbf{x}_i = \mathbf{c}^T\mathbf{x}(t)$$

$$A=\begin{pmatrix} A_1^2 & 0 & 0 & 0 & \dots & 0 & 0 \\ A_2^1 & A_2^2 & 0 & 0 & \dots & 0 & 0 \\ 0 & A_3^1 & A_3^2 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & A_p^1 & A_p^2 \end{pmatrix}$$

$$B=\begin{bmatrix} B_1 \\ B_2 \\ B_3 \\ \vdots \\ B_p \end{bmatrix},\ \mathbf{c}=\begin{bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ 0 \\ \dots \\ \vdots \\ \dots \\ 1 \\ 0 \end{bmatrix},\ \mathbf{x}(t)=\begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \\ \vdots \\ \mathbf{x}_p(t) \end{bmatrix},$$

$$x_i(t)=\theta_{f,p}(t)=x_p(t).$$

The transport approach

The transport approach represents the special metohod for mathematical modelling of heat exchangers. This approach is based on dividing the observed heat exchanger into the same p cells.

The fundamental law of energy conservation is derived by means of the finite differences for every cell with respect to the finite time interval Δt. In this manner the system of partial differential equations is transformed to the system of algebric equations which are easy to solve by numeric, itterative methods, *Ilić* (2002).

Let us observe the k-th cell, k=1,2,..., p, of the cross-flow heat exchanger. Considering the heat ballance within the observed cell, the heat input and output by fluid flow through the boundary of the cell can be identified as well as the heat exchange between the fluid in the cell and the wall.

Consider the changes of the specific physical values in a particular interval defined with t and $t+\Delta t$. The amount of heat exchange into the k-th cell for the time interval Δt can be written in the following form:

$$\overline{Q}(\Delta t)=\frac{Q(t)+Q(t+\Delta t)}{2}\Delta t,$$

$$\overline{Q}_{in}(\Delta t)=\frac{Q_{in}(t)+Q_{in}(t+\Delta t)}{2}\Delta t, \tag{60}$$

$$\overline{Q}_{out}(\Delta t)=\frac{Q_{out}(t)+Q_{out}(t+\Delta t)}{2}\Delta t,$$

$$\overline{Q}_{wf}(\Delta t)=\frac{Q_{wf}(t)+Q_{wf}(t+\Delta t)}{2}\Delta t.$$

where $\overline{Q}_{in}(\Delta t)$ is the heat change at the cell inlet for the time interval Δt, $\overline{Q}_{out}(\Delta t)$ is the heat change at the cell outlet and $\overline{Q}_{wf}(\Delta t)$ the heat exchange between the wall and the fluid in the cell.

The heat flow can be presented as:

$$Q_{in}(t)=c_f G_f \theta_f^{in}(t),\ Q_{out}(t)=c_f G_f \theta_f^{out}(t), \tag{61}$$

$$Q_{wf}(t)=\alpha_{wf} A_{wf}\left(\theta_w(t)-\theta_f(t)\right).$$

Assuming that fluid temperature in the cell can be expressed in the folowing form:

$$\theta_{fsr}(t)=\frac{\theta_f^{in}(t)+\theta_f^{out}(t)}{2}. \tag{62}$$

The heat content in the fluid for the observed cell can be expressed in the following form:

$$\overline{Q}(t)=m_f c_f \theta_{fsr}(t)=m_f c_f \frac{\theta_f^{in}(t)+\theta_f^{out}(t)}{2}, \tag{63}$$

The total amount of heat accumulated in the observed cell for the time interval $((t+\Delta t)-t)=\Delta t$ can be written as follows:

$$\begin{aligned}\Delta\overline{Q}(\Delta t)&=\overline{Q}(t+\Delta t)-\overline{Q}(t)=\\&=\overline{Q}_{in}(\Delta t)-\overline{Q}_{out}(\Delta t)+\overline{Q}_{wf}(\Delta t)\end{aligned} \tag{64}$$

where

$$\overline{Q}_{in}(\Delta t)=c_f G_f \frac{\theta_f^{in}(t)+\theta_f^{in}(t+\Delta t)}{2}\Delta t,$$

$$\overline{Q}_{out}(\Delta t)=c_f G_f \frac{\theta_f^{out}(t)+\theta_f^{out}(t+\Delta t)}{2}\Delta t, \tag{65}$$

$$\begin{aligned}\overline{Q}_{wf}(\Delta t)&=\frac{\alpha_{wf}A_{wf}\Delta t}{2}\left(\theta_w(t)-\frac{\theta_f^{in}(t)+\theta_f^{out}(t)}{2}\right)+\\&+\frac{\alpha_{wf}A_{wf}\Delta t}{2}\left(\theta_w(t+\Delta t)-\frac{\theta_f^{in}(t+\Delta t)+\theta_f^{out}(t+\Delta t)}{2}\right)\end{aligned}$$

The ballance equation for the observed cell is obtained in the following form:

- for the fluid in the tube:

$$\begin{aligned}&m_f c_f \theta_{fsr,k}(t+\Delta t)-m_f c_f \theta_{fsr,k}(t)=\\&\frac{c_f G_f \Delta t}{2}\left(\theta_{f,k}^{in}(t)-\theta_{f,k}^{in}(t+\Delta t)\right)-\\&-\frac{c_f G_f \Delta t}{2}\left(\theta_{f,k}^{out}(t)-\theta_{f,k}^{out}(t+\Delta t)\right)+\\&+\frac{\alpha_{wf}A_{wf}\Delta t}{2}\left(\theta_{w,k}(t)-\frac{\theta_{f,k}^{in}(t)+\theta_{f,k}^{out}(t)}{2}\right)+\\&+\frac{\alpha_{wf}A_{wf}\Delta t}{2}\left(\theta_{w,k}(t+\Delta t)-\frac{\theta_{f,k}^{in}(t+\Delta t)+\theta_{f,k}^{out}(t+\Delta t)}{2}\right)\end{aligned} \tag{66}$$

- for the wall:

$$\begin{aligned}&m_w c_w\left(\theta_{w,k}(t+\Delta t)-\theta_{w,k}(t)\right)=\frac{\alpha_{aw}A_{aw}\Delta t}{2}\left(\theta_a(t)-\theta_{w,k}(t)\right)+\\&+\frac{\alpha_{aw}A_{aw}\Delta t}{2}\left(\theta_a(t+\Delta t)-\theta_{w,k}(t+\Delta t)\right)-\\&-\frac{\alpha_{wf}A_{wf}\Delta t}{2}\left(\theta_{w,k}(t)-\frac{\theta_{f,k}^{in}(t)+\theta_{f,k}^{out}(t)}{2}\right)-\\&-\frac{\alpha_{wf}A_{wf}\Delta t}{2}\left(\theta_{w,k}(t+\Delta t)-\frac{\theta_{f,k}^{in}(t+\Delta t)+\theta_{f,k}^{out}(t+\Delta t)}{2}\right)\end{aligned} \tag{67}$$

- for the air:

$$G_a c_a\left(\theta_a^{in}(t)-\theta_a^{out}(t)\right)=\alpha_{aw}A_{aw}\left(\theta_a(t)-\theta_{w,k}(t)\right) \tag{68}$$

The mathematical model for the observed cell after a simple mathematical transformation can be obtained in the following form:

$$\theta^{out}_{f,k}(t+\Delta t) = a_1\theta^{out}_{f,k}(t) + a_2\theta^{in}_{f,k}(t) + \\ + a_3\theta_{w,k}(t) + a_4\theta^{in}_{f,k}(t+\Delta t) + a_3\theta_{w,k}(t+\Delta t) \tag{69}$$

$$\theta_{w,k}(t+\Delta t) = b_1\theta_{w,k}(t) + b_2\theta^{in}_{a}(t) + \\ + b_2\theta^{in}_{a}(t+\Delta t) + b_3\theta^{in}_{f,k}(t) + b_3\theta^{in}_{f,k}(t+\Delta t) + \\ + b_3\theta^{out}_{f,k}(t) + b_3\theta^{out}_{f,k}(t+\Delta t) \tag{70}$$

where

$$a_1 = \frac{2m_f c_f - 2c_f G_f \Delta t - \alpha_{wf} A_{wf} \Delta t}{2m_f c_f + 2c_f G_f \Delta t + \alpha_{wf} A_{wf} \Delta t},$$

$$a_2 = \frac{2m_f c_f + 2c_f G_f \Delta t - \alpha_{wf} A_{wf} \Delta t}{2m_f c_f + 2c_f G_f \Delta t + \alpha_{wf} A_{wf} \Delta t},$$

$$a_3 = \frac{2\alpha_{wf} A_{wf} \Delta t}{2m_f c_f + 2c_f G_f \Delta t + \alpha_{wf} A_{wf} \Delta t}, \tag{71}$$

$$b_1 = \frac{\left(2m_w c_w - \alpha_{aw} A_{aw}\Delta t - \alpha_{wf} A_{wf}\Delta t\right)\left(2G_a c_a + \alpha_{aw} A_{aw}\right) + \alpha_{aw}^2 A_{aw}^2 \Delta t}{\left(2m_w c_w + \alpha_{aw} A_{aw}\Delta t + \alpha_{wf} A_{wf}\Delta t\right)\left(2G_a c_a + \alpha_{aw} A_{aw}\right) - \alpha_{aw}^2 A_{aw}^2 \Delta t}$$

$$a_4 = \frac{2c_f G_f \Delta t - 2m_f c_f - \alpha_{wf} A_{wf} \Delta t}{2m_f c_f + 2c_f G_f \Delta t + \alpha_{wf} A_{wf} \Delta t},$$

$$b_2 = \frac{2\alpha_{aw} A_{aw}\Delta t G_a c_a}{\left(2m_w c_w + \alpha_{aw} A_{aw}\Delta t + \alpha_{wf} A_{wf}\Delta t\right)\left(2G_a c_a + \alpha_{aw} A_{aw}\right) - \alpha_{aw}^2 A_{aw}^2 \Delta t}$$

$$b_3 = \frac{\alpha_{wf} A_{wf}\Delta t\left(2G_a c_a + \alpha_{aw} A_{aw}\right)}{2\left(2m_w c_w + \alpha_{aw} A_{aw}\Delta t + \alpha_{wf} A_{wf}\Delta t\right)\left(2G_a c_a + \alpha_{aw} A_{aw}\right) - 2\alpha_{aw}^2 A_{aw}^2 \Delta t}$$

Eliminating $\theta_{w,k}(t+\Delta t)$ from Eqs. (69) and (70) the outlet temperature from the k-th cell can be expressed in the following form:

$$\theta^{out}_{f,k}(t+\Delta t) = c_1\theta^{out}_{f,k}(t) + c_2\theta^{in}_{f,k}(t) + c_3\theta_{w,k}(t) + \\ + c_4\theta^{in}_{a}(t) + c_5\theta^{in}_{f,k}(t+\Delta t) + c_4\theta^{in}_{a}(t+\Delta t) \tag{72}$$

where

$$c_1 = \frac{a_1 + a_3 b_3}{1 - a_3 b_3},\ c_2 = \frac{a_2 + a_3 b_3}{1 - a_3 b_3},\ c_3 = \frac{a_3 + a_3 b_1}{1 - a_3 b_3}, \tag{73}$$

$$c_4 = \frac{a_3 b_2}{1 - a_3 b_3},\ c_5 = \frac{a_4 + a_3 b_3}{1 - a_3 b_3}.$$

Eq. (72) is used for the simulation of the process in the cross-flow heat exchanger.

Assuming, in the first step, $\theta^{in}_{f,k}(t+\Delta t) = \theta^{in}_{f,k}(t)$ and $\theta^{in}_{a}(t+\Delta t) = \theta^{in}_{a}(t)$ where Δt is small enough, the outlet temperature is determinated for the k-th cell, k=1,2,..., p. Repeating this procedure for different values of Δt the fluid temperature of cross-flow heat exchanger at the cells outlet is obtained as a function of time.

Simulation and comparative analysis

This section presents the analysis of the mathematical model of the cross-flow heat exchanger with respect to different methods considered in the previous sections. The characteristic of a real cross-flow heat exchanger are given in Table 1.

The observed heat exchanger is divided into five cells. If the results are unsatifactory the number of cells must be increased.

Table 1.

	Parameter	Dimension	Value
1.	c_w	kJ/kgK	0.53
2.	ρ_w	Kg/m^3	$7.85\cdot10^3$
3.	λ_w	W/mK	$0.04\cdot10^3$
4.	r_0	m	0.016
5.	r_i	m	0.012
6.	w_f	m/s	0.5
7.	G_f	kg/s	0.12
8.	G_a	kg/s	1.8
9.	ρ_f	Kg/m^3	952.38
10.	c_f	kJ/kgK	4.233
11.	α_{aw}	W/(m^{2}K)	220
12.	ρ_a	Kg/m^3	0.748
13	α_{wf}	W/(m^{2}K)	3000
14.	c_a	kJ/kgK	1.097
15.	L	m	10
16.	θ^{in}_{fN}	$^\circ C$	110
17.	θ^{in}_{aN}	$^\circ C$	220

Results of the simulation for the analytical approach based on Laplace transformation

Fig.2 and 3 show the step response of the outlet temperature of the vfluid in the tube for the step change of the inlet fluid temperature and the inlet air temperature. Since the temperature function obtained using the approach explained in Section 2 is complex valued, the amplitude of this function is used for the simulation.

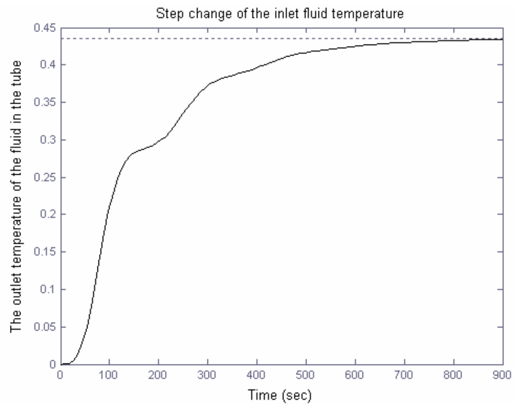

Figure 2. Step response of the outlet temperature of the fluid in the tube for the step change of the inlet fluid temperature

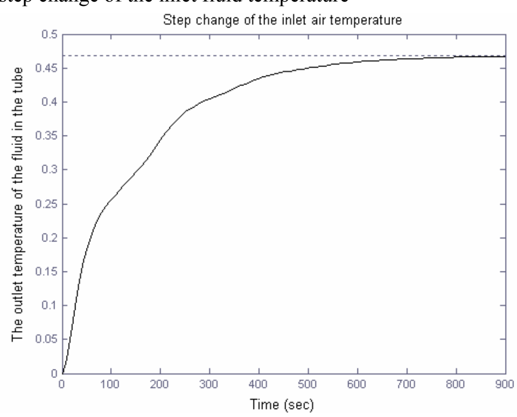

Figure 3. Step response of the outlet temperature of the fluid in the tube for the step change of the inlet air temperature

Results of the simulation for the differential discrete mathematical model based on the finite difference method

Figures 4, 5, 6 and 7 show the step response in the fluid outlet temperature and the temperature profile in the outlet of each cell for the step change in the fluid inlet temperature and the step change in the air inlet temperature.

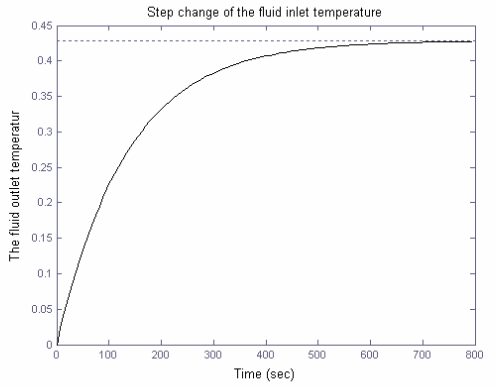

Figure 4. Step response of the outlet temperature of the fluid in the tube for the step change of the inlet fluid temperature

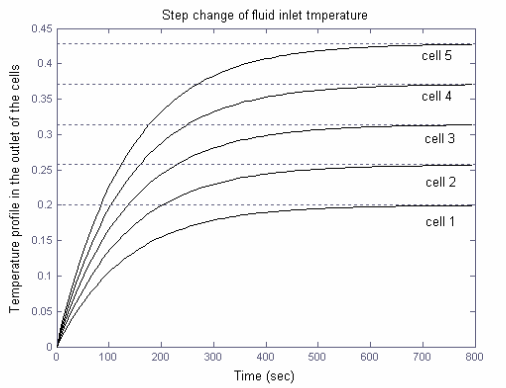

Figure 5. Temperature profile in the outlet of each cell for the step change in the fluid inlet temperature

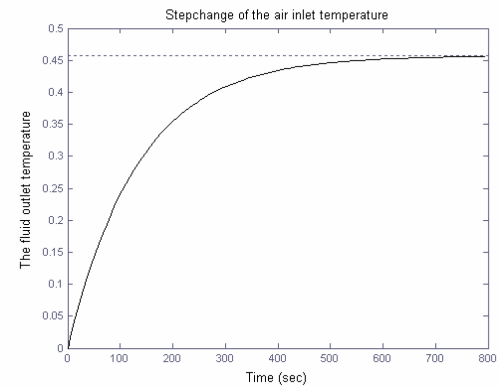

Figure 6. Step response of the outlet temperature of the fluid in the tube for the step change of the inlet air temperature

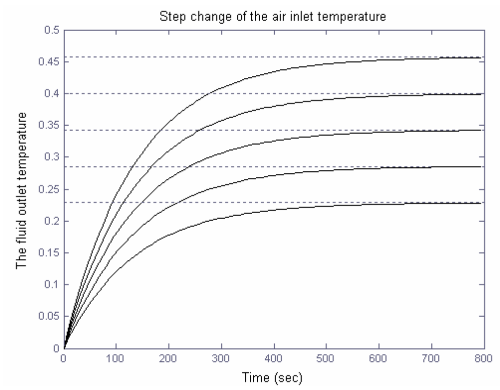

Figure 7. Temperature profile in the outlet of each cell for the step change in the air inlet temperature

Results of the simulation for the mathematical model based on the physical discretization

Fig. 8, 9, 10 and 11 show the step response in the fluid outlet temperature and the temperature profile in the outlet of each cell for the step change in the fluid inlet temperature and the step change in the air inlet temperature.

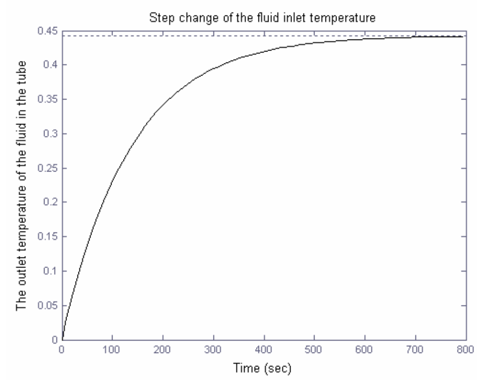

Figure 8. Step response of the outlet temperature of the fluid in the tube for the step change of the inlet fluid temperature

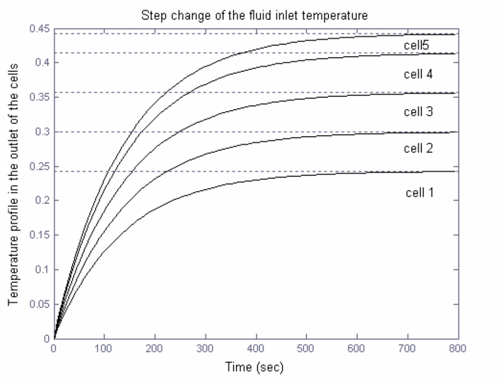

Figure 9. Temperature profile in the outlet of each cell for the step change in the fluid inlet temperature

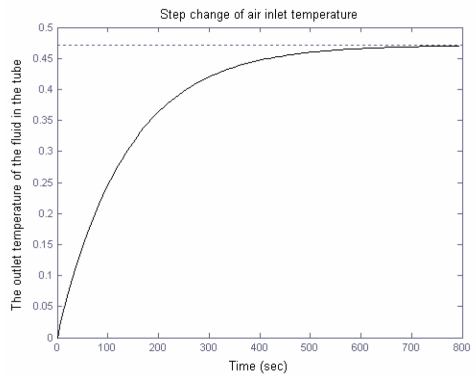

Figure 10. Step response of the outlet temperature of the fluid in the tube for the step change of the inlet air temperature

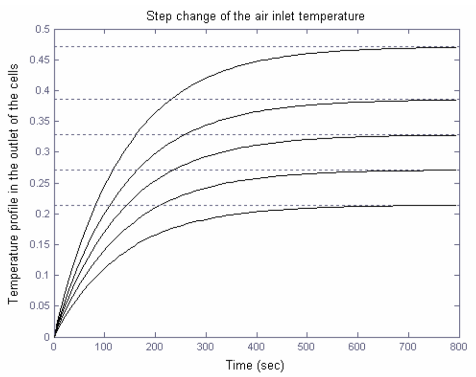

Figure 11. Temperature profile in the outlet of each cell for the step change in the air inlet temperature

Results of the simulation for the mathematical model based on the transport approach

Fig. 12 and 13 show the step response in the fluid outlet temperature and the temperature profile in the outlet of each cell for the step change (from 220ºC to 257ºC) in the air inlet temperature.

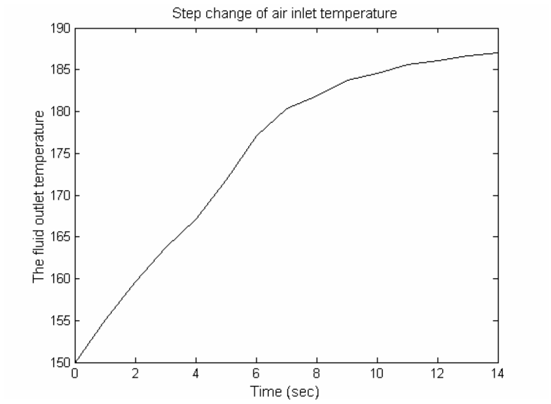

Figure 12. Step response of the outlet temperature of the fluid in the tube for the step change of the inlet air temperature

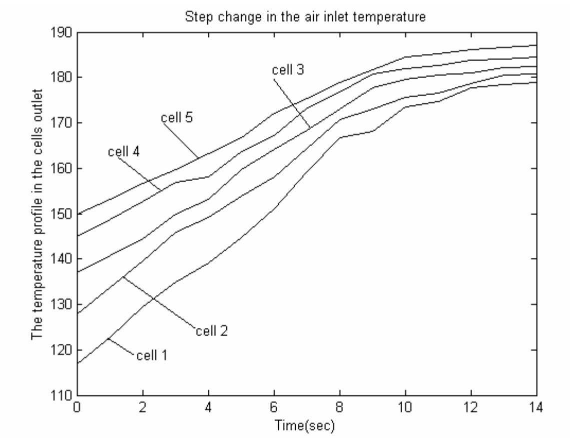

Figure 13. Temperature profile in the outlet of each cell for the step change in the air inlet temperature

Fig. 14 and 15 show the step response in the fluid outlet temperature and the temperature profile in the outlet of each cell for the step change (from 110ºC to 127ºC) in the fluid inlet temperature.

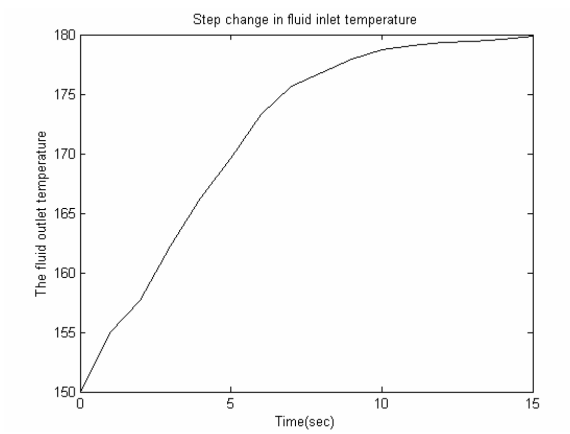

Figure 14. Step response of the outlet temperature of the fluid in the tube for the step change of the inlet fluid temperature

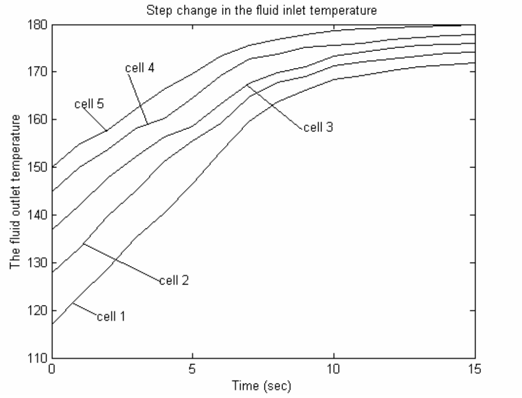

Figure 15. Temperature profile in the outlet of each cell for the step change in the fluid inlet temperature

Finally, it should be pointed out that the temperature changes obtained by the aplication of the transport approach are in total coordinates while the step responses obtained by the application of previous methods are represented by the relative deviations.

Conclusion

In this paper some characteristic methods in mathematical modelling of the heat exchangers are considered.

Analyzing the analytical approach using the Laplace transform it can be concluded that the system of PDE is very complex to be solved analytically and this model has only academic significance while its practical usage is limited.

In order to overcome the solution difficulties, the procedure of differential discrete modelling is applied, leading to the set of ordinary differential equations of a rather high order. This procedure is based on the discretization of spatial coordinates using the finite difference method.

One of possible methods to avoid partial differential equations is the method of physical discretization which implies dividing the observed heat exchanger into an appropriate number of cells. If the number of cells is big enough it can be assumed that the process within one cell is a process with certain parameters. This procedure results in ordinary differential equations of a high order that are still simple enough to be solved.

The transport approach is based both on the spatial and time discretization which transforms the system of partial differential equations to the system of the algebric equations that can be solved using numerical, iterative procedures. This procedure should be carried out taking into account the conditions for the convergence of numerical procedures.

The graphical results of simulations of the different mathematical models of cross-flow heat exchangers are presented in this section.

Comparing the step responses obtained by the application of different methods for mathematical modelling of the heat exchanger it can be concluded that results agree very well with each other.

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Komparativna analiza različitih metoda matematičkog modeliranja rekuperativnih razmenjivača toplote

Rekuperativni razmenjivači toplote se veoma često koriste kao konstruktivni elementi različitih postrojenja tako da je poznavanje njihove dinamike veoma važno. Razmenjivači toplote se takođe koriste i u vojnim letilicama svih veličina, različitih namena, kao delovi različitih sistema ovih letilica. Njihovo funkcionisanje je najčešće kontrolisano regulacijom ulaznih temperatura radnih fluida ili veličine njegovog masenog protoka. Na bazi realno usvojenih pretpostavki izveden je linearizovani matematički model razmenjivača toplote sa unakrsnim strujanjem radnih fluida, uzimajući u obzir i dinamiku zida razmenjivača. Model je zasnovan na fundamentalnom zakonu konzervacije energije, uzimajući u obzir sve akumulatore toplote ovog procesa i predstavljen je pomoću sistema parcijalnih diferencijalnih jednačina (PDE), čije rešenje nije, u opštem slučaju, moguće u zatvorenom obliku. Kao jedan od mogućih načina da se izbegne rešavanje sistema parcijalnih diferencijalnih jednačina, u ovom radu se razmatraju različite metode matematičkog modeliranja razmenjivača toplote: prilaz zasnovan na primeni Laplasove transformacije, aproksimacija parcijalnih diferencijalnih jednačina pomoću metode konačnih razlika, metoda fizičke diskretizacije i transportni prilaz. Za konkretno usvojene vrednosti parametara razmenjivača toplote izvršena je simulacija njegovog rada i dati su grafički prikazi odskočnih odziva za sve razmatrane metode i analiziran je njihov praktični značaj.

Ključne reči: **razmenjivač toplote, matematički model, uporedna analiza, Laplasova transformacija, metoda konačnih razlika, metoda diskretizacije.**

Сравннительный анализ различных методов математического моделирования рекуперативных теплообменников

Рекуперативные теплообменники очень часто использованы в роли конструктивных элементов различных оборудований, из-за чего очень важным является познание их динамики. Теплообменники тоже употребляются во военных летательных аппаратах всех размеров, с различными назначениями и в роли составных частей различных систем этих летательных аппаратов. Их функционирование в большинстве случаев контролировано регулированием входящих температур рабочих жидкостей или значением их протока массы. На основании действительно принятых предположений выведена линейная математическая модель теплообменника с перекрёстным потоком рабочих жидкостей, учитывая и динамику стены теплообменника. Модель обоснована на фундаментальном законе консервации энергии, учитывая все аккумуляторы теплоты этого процесса и представлена путём систем частичных дифференциальных уравнений (ЧДУ), чьё решение в общем случае возможно только в закрытой форме. В роли одного из возможных способов во избежание решений систем частичных дифференциальных уравнений, в настоящей работе рассматриваются различные методы математического моделирования теплообменников: подход обоснован на трансформации Лапласа, приближение (апроксимация) частичных дифференциальных

уравнений при помощи метода конечных разниц, метода физической дискредитации и транспортный подход. Для конкретно приемлемых значений параметров теплообменника проведена симуляция его работы и даны графики отскакивающих отзывов для всех рассматриванных методов и анализировано их практическое значение.

Kly~evwe slova: **теплообменник, математическая модель, сравннительный анализ, трансформация Лапласа, метод конечных элементов, метод дискретизации.**

Analyse comparative des différentes méthodes de modélisation mathématique des échangeurs récupérables de chaleur

Les échangeurs récupérables de chaleur sont très souvent utilisés comme les éléments constructifs pour les différentes installations et la connaissance de leur dynamique est très importante. On utilise les échangeurs de chaleur dans les avions militaires de toutes tailles, de diverses missions, comme les parties de différents systèmes de ces avions. Leur fonctionnement est contrôlé le plus souvent par le réglage des températures d'entrée des fluides moteurs ou par la grandeur de leur écoulement de masse. A la base des hypothèses réellement adoptées on a dérivé le modèle mathématique linéarisé de l'échangeur de chaleur à l'écoulement croisé de fluides moteurs considérant la dynamique du paroi de l'échangeur. Le modèle est conçu sur la loi fondamentale de la conservation d'énergie tenant compte de tous les accumulateurs de chaleur de ce processus et il est représenté par le système des équations partielles différentielles (EPD) dont la solution , généralement, n'est pas possible en forme fermée. Comme un des moyens d'éviter la solution du système des équations partielles différentielles, dans ce travail on considère les différentes méthodes de la modélisation mathématique pour les échangeurs de chaleur : approche basée sur l'application de la transformation de Laplace, approximation des équations partielles différentielles par la méthode des différences finies, méthode de discrétisation physique et approche de transport. Pour les valeurs adoptées des paramètres des échangeurs de chaleur, on a effectué la simulation de leur fonctionnement et on a donné les formes graphiques des réponses pour toutes les méthodes étudiées en analysant leur importance pratique.

***Mots clés*: échangeur de chaleur, modèle mathématique, analyse comparative, transformation de Laplace, méthode des différences finies, méthode de discrétisation.**