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Analogy Between a New Formulation of the Euler-Bernoulli Equation and the Algorithm for Forming Mathematical Models of Robot Motion

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With new knowledge collected through generations, the intensive development of new technical areas such as robotics especially strengthened by the development of the data computing process demanded and enabled that elastic deformation was considered as a real dynamic value depending on system parameters. The elastic deformation amplitude and its frequency are dynamic values which depend on the total dynamics of the robot system movements (forces) and also on the mechanism configuration, weight, length of the segments of the reference trajectory choice, dynamic characteristics of the motor movements, etc. We define a general form of the equation of the flexible line of a complex robotic system of arbitrary configuration, using the Euler-Bernoulli equation. The relation between the Euler-Bernoulli equation and the equation of motion at the point of elastic line tip is explained. A mathematical model of the actuators also comprises coupling between elasticity forces. The analogy between the Euler-Bernoulli equation solutions, defined by Daniel Bernoulli in the original form, and the procedure of the „direct kinematics“ solutions in the robotics, is presented.

Key words: **robotic, kinematics, motion dynamics, Euler-Bernoulli equations, process modeling, elastic deformation, coupling, stiffness matrix, motion simulation, programmed trajectory.**

Designations

DOF	– degree of freedom
$t(s)$	– time
$dt = 0.000107$ (s)	– sample time
$T = 4\,(\mathrm{s})$	– whole period time
$p_s = [x \quad y \quad z \quad \psi \quad \wp \quad \varphi]^T$	– Cartesian (external) coordinates
$\varphi = [p_{1,1} \quad p_{1,2} \quad p_{1,3} \quad p_{,4} \quad \dots \quad p_{1,n}]^T$	– vector of internal coordinates
$x_{i,j}, y_{i,j}, z_{i,j}$	– local coordinate frame, which is set in the base of considered mode
x_j, y_j, z_j	– local coordinate frame, which is set in the base of the considered link
x, y, z	– basic coordinate frame, which is set in the root of the considered robotic system
$j = 1, 2, 3, \dots, n_i$	– serial number of the mode of the considered link
$i = 1, 2, 3, \dots, m$	– ordinal number of the link
$k = n_1 + n_2 + \dots + n_m$	– whole number of the modes in the considered robotics configuration
$M_{i,j} \in R^1\,(\mathrm{Nm})$	– load moment for the mode tip
$\varepsilon_{i,j} \in R^1\,(\mathrm{Nm})$	– bending moment for the mode tip
$\varepsilon_j \in R^n\,(\mathrm{Nm})$	– bending moment vector for each mode tip of the considered link
$\varepsilon = [\varepsilon_{1,1}\ \varepsilon_{1,2} \dots \varepsilon_{1,n_1}\ \varepsilon_{2,1}\ \varepsilon_{2,2} \dots \varepsilon_{2,n_2} \dots \varepsilon_{m,n_m}]^T$	– vector of bending moments
$\varepsilon_m = [\varepsilon_{1,1}\ \varepsilon_{2,1}\ \varepsilon_{3,1} \dots \varepsilon_{m,1}]^T$	
$\varsigma \in R^1\,(\mathrm{Nm})$	– elasticity moment of the gear
$\hat{\#}_{i,j}$	– quantities that are related to an arbitrary point of the elastic line of the mode, for example: $M_{i,j}, x_{i,j}, \varepsilon_{i,j}$
$\#_{i,j}$	– quantities that are not designated by " ^" are defined for the mode tip, for example: $M_{i,j}, x_{i,j}, \varepsilon_{i,j}$
$\#_j$	– quantities which characterize link
$\#^{\circ}$	– quantities that define a desired value
$\bar{\theta} \in R^1\,(\mathrm{rad})$	– rotation angle of the motor shaft after the reducer
$\vartheta \in R^1\,(\mathrm{rad})$	– bending angle of the considered mode
$\omega \in R^1\,(\mathrm{rad})$	– rotation angle of the considered mode tip (see [21])
$\xi \in R^1\,(\mathrm{rad})$	– deflection angle of the gear
$\beta_{i,j} \in R^1\,(\mathrm{Nm}^2)$	– flexural rigidity
$\eta_{i,j} \in R^1\,(\mathrm{s})$	– factor which characterizes a part of damping in all flexural characteristics
$H \in R^{kxk}$	– inertial matrix
$h \in R^k$	– centrifugal, gravitational, Coriolis vector
J_e	– Jacobian matrix mapping the effect of the external contact force
$T_{sti,j} \in R^1\,(\mathrm{m})$	– stationary part of flexible deformation caused by stationary moments that vary continuously over time
$T_{toi,j} \in R^1\,(\mathrm{m})$	– oscillatory part of flexible deformation
$a_j \in R^1\,(\mathrm{m})$	– usually normal distance between j-th and $j+1$-th joints

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$\alpha_j \in R^1(^\circ)$ – angle between the axes z_{j-1} and z_j about the axis x_j.

$d_j \in R^1(\mathrm{m})$ – distance between normal l_{j-1} and l_j along the axis of j-th joint

$R = 0.272(\Omega)$ – rotor circuit resistance

$u(V)$ – voltage

$i(A)$ – rotor current

$C_E = 6.1(V/(\mathrm{rad/s}))$ – proportionality constants of the electromotive force

$C_M = 6.1(\mathrm{Nm/A})$ – proportionality constants of the moment

$B = 0(\mathrm{Nm/(rad/s)})$ – coefficient of viscous friction

$I = 4.52(\mathrm{kgm^2})$ – inertia moments of the rotor and reducer

$S = 0.0446$ – expression defining the reducer geometry

$\Diamond \in R^{kxk}$ – matrix characterizing the mutual influence of the bending moments modes of all the links

$\Diamond_m \in R^{mxm}$ – characterizes the influence of the bending moment of each mode on the motor dynamics

$\Theta \in R^{kxk}$ – matrix characterizing the robot configuration

$l = 0.3(\mathrm{m})$ – length of mode

$D = 0.0169(\mathrm{m})$ – outside diameter

$\xi = 0.7$ – ratio between the inside and the outside diameter of the link cylinder

$f \in R^1(\mathrm{m})$ – flexure

$m_b = 1(\mathrm{kg})$, $J_b = 0.00125(\mathrm{kgm^2})$ – mass in the link base

$m = 2(\mathrm{kg})$, $j = 0.0025(\mathrm{kgm^2})$ – mass in the link tip

$I_{mom} = 0.3042 * 10^{-8}(\mathrm{m^4})$ – inertia moments of the cross-section of mode

$E_k(\mathrm{Nm})$ – kinetic energy

$E_p(\mathrm{Nm})$ – potential energy

$\Phi(\mathrm{Nm/s})$ – dissipative energy

ϕ – generalized coordinate

$g(\mathrm{m/s^2})$ – gravity acceleration

$u \in R^1(\mathrm{V})$ – control signal

$C_s = 7.0278 \cdot 10^3(\mathrm{kg/s^2})$ – characteristics of the stiffness of the mode considered link

$B_s = 50(\mathrm{kg/s})$ – characteristics of the damping of the mode considered link

$C_\xi = 1.8143 \cdot 10^3(\mathrm{Nm/rad})$ – characteristics of the stiffness of the gear

$B_\xi = 20(\mathrm{Nm/(rad/s)})$ – characteristics of the damping of the gear

$I_0 = 0.7854 \cdot 10^{-9}(\mathrm{m^4})$ – polar moment of inertia, which we obtain depending on the diameter and the thickness of cross-section joints

$a_{uv} = 0.03(\mathrm{m})$ – length at which deflection joints occurred

$E_I = 69.3 \cdot 10^9(\mathrm{N/m^2})$ – module of elasticity for aluminium

$\delta\bar{\theta}(t_0) = 0(\mathrm{rad})$, $\delta\dot{\bar{\theta}}(t_0) = 0(\mathrm{rad/s})$ – initial exceptions of the angle turning powers

$K_{lp} = 40000$, $K_{lv} = 400$ – position, velocity control gains for movement stabilization

Introduction

MODELING of elastic robotic systems has been a challenge to researchers in the last four decades. In paper [18] the authors extend the integral manifold approach to the control of flexible joint robot manipulators from the known parameter case to the adaptive case. Paper [22] presents the derivation of the equations of motion for the application of mechanical manipulators with flexible links. In [23] the equations are derived using Hamilton's principle and they are nonlinear integro-differential equations. There are methods of variables separation and Galerkin's approach suggested in [24] for the boundary-value problem with the time-dependent boundary condition. The first detailed presentation of the procedure for creating reference trajectory was given in [1].

A mathematical model of a mechanism with one degree of freedom (DOF), with one elastic gear, was defined by Spong, 1987, in [29]. Based on the same principle, the elasticity of gears is introduced in the mathematical model in this paper, as well in [12-17].

However, when the introduction of link flexibility in the mathematical model is concerned, it is necessary to point out to some essential problems in this domain.

In our paper we do not use the "assumed modes technique", proposed by Meirovitch in [28] (and used by all authors until today in [5-8], [19-21], [25-27], [30] etc.). We disagree with him.

The LMA ("Lumped-mass approach") is a method which defines motion equation at any point of a considered mechanism. If any link of the mechanism is elastic then we can also define the motion equation at any point of the presented link. We do not know exactly when this approach was stated. It defines the dynamic equation at any point of a mechanism during movement. The LMA in [2-4] gives the possibility to analyze the motion of any point of each mode. Papers with this research topic (approach) were rare in robotics journals in the last two decades.

The EBA ("Euler-Bernoulli approach") assumes the use of the Euler-Bernoulli equations which appeared in 1750. The EBA gives the possibility to analyze a flexible line form of each mode in the course of task realization. The EBA is an approach that is still in the focus of researchers' interest and it was analyzed most often in the last decades.

In the pertinent literature no relationship has been established between the LMA and the EBA.

We consider that the EBA and the LMA are two comparative methods addressing the same problem but from different aspects, see [12], [13], [15-17].

Using the EBA, we obtain the equations of the flexible line model of each mode and by setting boundary conditions we obtain model equations of motion at the point of the tip (or any other point) of each mode, which is in fact the LMA. As the equation of motion for the mode tip point is essentially an LMA, it follows directly from the equation of the flexible line obtained via the EBA for the preset boundary conditions.

In the meantime, from 1750 when the Euler Bernoulli equation was published until today, our knowledge, especially in the robotics, the oscillation theory and the elasticity theory, has progressed significantly. As a consequence, this paper points out the necessity of the extension of the Euler Bernoulli equation from many aspects.

In the previous [5-8], [19-21], [25-27], [30] etc., the general solution of the motion of an elastic robotic system has been obtained by considering flexural deformations as transversal oscillations that can be determined by the method of particular integrals of D. Bernoulli.

We consider that any elastic deformation can be presented by superimposing D. Bernoulli's particular solutions of the oscillatory character and the stationary solution of the forced character. The elastic deformation is a dynamic value which depends on the total dynamics of the robot system movements, see [12], [13], [15-17].

The reference trajectory is calculated from the overall dynamic model when the robot tip is tracking a desired trajectory in a reference regime in the absence of disturbances.

Elastic deformation (of flexible links and elastic gears) is a quantity which is, at least, partly encompassed by the reference trajectory. It is assumed that all elasticity characteristics in the system (both of stiffness and damping) are "known" at least partly and at that level they can be included into the process of defining the reference motion. Thus defined reference trajectory allows the possibility of applying very simple control laws via PD local feedback loops which ensures reliable tracking of the robotic tip considered in the space of the Cartesian coordinates to the level of known elasticity parameters.

The "Assumed modes technique" from [28] was used by all authors in the last 40 years to form the Euler Bernoulli beam equation. In our paper we form the Euler Bernoulli equation but we do not use the "assumed modes technique" in contrast to our contemporaries.

We think that the "assumed modes technique" was and still can be useful in some other research areas but it is used in a wrong way in robotics, theory of oscillations and theory of elasticity.

Let us emphasize once again that in this paper we propose a mathematical model solution that includes in its root the possibility for simultaneous analyzing both present phenomena – the elasticity of gears and the flexibility of links. The idea originated from [4], but is based on new principles.

The area which we deal with, the robotics, is very important, because the modeling of the robot system movement dynamics with both rigid and elastic elements comes from it directly. The robotics is the area that can offer a solution and it represents the foundation of the further research in many other areas. The reason for that is quite simple: the robotics progressed significantly in the last 40 years. It is important to emphasize the importance of the further research but now based on new principles which will be set in this paper.

Our future work should be directed towards the implementation of gears elasticity and links flexibility on any model of a rigid mechanism and also on a model of reconfigurable rigid robot as given in [10], [11] or any other type of mechanism. The mechanism should be modeled to contain elastic elements and to generate vibrations which are used for conveying particulate and granular materials in [9].

The procedure of defining the dynamic model with all elements of coupling is presented completely as well as with dynamic effects of the present forces defined in Section 2. We presented the kinematic model and the analogy between the Euler-Bernoulli equation solutions defined by Daniel Bernoulli and the procedure of the „direct kinematics" and „inverse kinematics" solutions in the Robotics in Section 3. Section 4 analyzes simulation example for movement dynamic of an elastic robotic pair with an elastic gear and a flexible link in the presence of only one mode. Section 5 gives some concluding remarks.

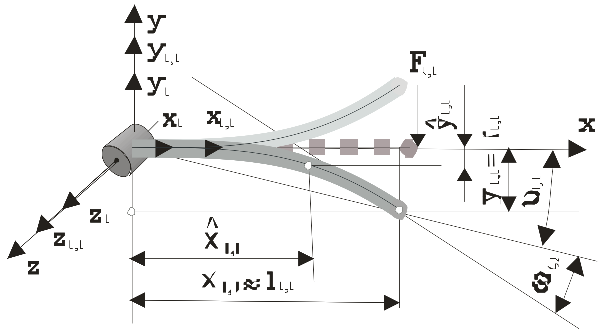
Dynamics of elastic robotic systems

The Euler-Bernoulli equation was written in 1750. They did not even dream about the robotics and the knowledge we have now at our disposal. However, although it was conceived more than 250 years ago, the Euler-Bernoulli equation is still valid and it can be connected logically with the contemporary knowledge from the robotics. The "Source form of the Euler-Bernoulli equation" of the elastic line of beam bending has the following form:

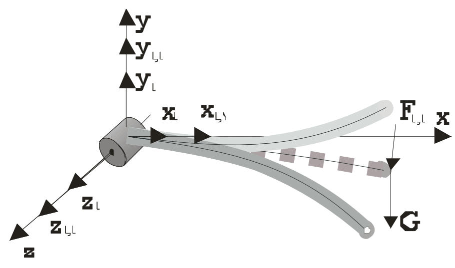
$$\hat{M}_{1,1} + \hat{\varepsilon}_{1,1} = 0, \qquad \hat{\varepsilon}_{1,1} = \beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^{2}} \tag{1}$$

Equation (1) was defined under the assumption that the elasticity force is opposed only by the inertial force proper. Besides, it is supposed by the definition that the motion in (1) is caused by an external force $F_{1,1}$, suddenly added and then removed. Bernoulli presumed the horizontal position of the observed body as its stationary state (in this case it matches the position x - axis, see Fig. 1a). At such presumption, the oscillations happen just around the x - axis.

If Bernoulli, at any case, had included the gravity force G in (1), the situation would have been more real. Then the stationary body position would not have matched the x - axis position, but the body position would have been little lower and the oscillations would have happened around the new stationary position as presented in Fig. 1b).



a) idealized motion according to D.Bernoulli



b) motion in case the presence of a gravity force

Figure 1. Motion of an elastic body

The Euler-Bernoulli equation (1) should be expanded as in [12], [13], [15-17] from several aspects in order to be applicable in a broader analysis of elasticity of robot mechanisms. By supplementing these equations with the expressions that come out directly from the motion dynamics of elastic bodies, they become more complex.

The motion of the considered robotic system mode is far more complex than the motion of the body presented in Fig.1a). This means that the equations that describe the robotic system (its modes) must also be more complex than (1) formulated by Euler and Bernoulli. Hence, we should especially emphasize the necessity of expanding the source equations in the following way:

- Based on the known laws of dynamics, (1) is to be supplemented by all the forces that participate in the formation of the bending moment of the considered mode.
- Damping is an omnipresent flexibility characteristic of real systems, so that it is naturally included in the Euler-Bernoulli equation.

$$\hat{M}_{1,1} + \beta_{1,1}\frac{\partial^2\left(\hat{y}_{1,1} + \eta_{1,1}\dot{\hat{y}}_{1,1}\right)}{\partial\hat{x}_{1,1}^2} = 0 \tag{2}$$

$\eta_{1,1}$ is a factor characterizing the share of damping in the total flexibility characteristic.

The load moment is composed of all forces acting on the first mode of the link and these are inertial forces (own and coupled forces), centrifugal, gravitational, Coriolis forces (own and coupled), forces due to relative motion of one mode with respect to the other, coupled elasticity forces of the other modes, as well as the force of the environment dynamics, which is via the Jacobian matrix transferred to the motion of the first mode that come out directly from the motion dynamics of elastic bodies. In that case, the model of the elastic line of the elastic link's first mode has the "New form of the Euler-Bernoulli equation":

$$\begin{aligned}&\hat{H}_{1,j}\frac{d^2\hat{y}_{1,j}}{dt^2} + \hat{h}_{1,1} + j_{1,1}^T F_{uk} + \\ &+\Diamond_{1,j}\varepsilon_1 + \beta_{1,1}\frac{\partial^2\left(\hat{y}_{1,1} + \eta_{1,1}\dot{\hat{y}}_{1,1}\right)}{\partial\hat{x}_{1,1}^2} = 0\end{aligned} \tag{3}$$

Let us consider a robotic system with m links, whereby the first link contains n_1 modes. $i = 1,\ldots m$. The model of the elastic line of this complex elastic robotic system is given in the matrix form by the following "New form of the Euler-Bernoulli equation":

$$\hat{H}\cdot\frac{d^2\hat{y}}{dt^2} + \hat{h} + j_e^T\cdot F_{uk} + \Diamond\cdot\Theta\cdot\varepsilon_1 + \hat{\varepsilon} = 0 \tag{4}$$

Robotics researchers are especially interested in the motion of the first mode tip.

The equation of motion of the forces involved at any point of the elastic line of first mode, including the point of the first mode tip, can be defined from the Euler-Bernoulli eq. (3). The equation of motion of all forces at the first mode tip for the given boundary conditions can be defined by:

$$\begin{aligned}&\hat{H}_{1,j}\frac{d^2\hat{y}_{1,j}}{dt^2} + h_{1,1} + j_{e1,1}{}^T\cdot F_{uk} + \\ &+\Diamond_{1,j}\cdot\varepsilon_1 + \beta_{1,1} = 0\left|\begin{array}{l}\Sigma F = 0\left(\Sigma M = 0\right)\\ \text{at the point of}\\ \text{first mode tip}\end{array}\right.\end{aligned} \tag{5}$$

Eq.(5) is interesting because it allows one to calculate the position of the first mode tip. If we know the position of each mode tip, we can always calculate the position of the link tip too and eventually the position of the robot tip.

The equation of motion of all the forces at the point of each mode tip of any link can be defined from the Euler-Bernoulli eq.(4) by setting the boundary conditions.:

$$H\frac{d^2y}{dt^2} + h + j_e^T\cdot F_{uk} + \Diamond\cdot\Theta\cdot\varepsilon + \varepsilon = 0\left|\begin{array}{l}\Sigma F = 0\left(\Sigma M = 0\right)\\ \text{at the tip of}\\ \text{any mode of}\\ \text{link considered}\end{array}\right. \tag{6}$$

The mathematical model of all m motors can be written in a vector form as:

$$\begin{aligned}&u = R\cdot i + C_E\cdot\dot{\bar{\theta}},\\ &C_M\cdot i = I\cdot\ddot{\bar{\theta}} + B\cdot\dot{\bar{\theta}} - S\cdot\left(z_m\cdot\varepsilon + \varepsilon_m\right)\left|\begin{array}{l}\Sigma M = 0\\ \text{about the rotation axis}\\ \text{of the each motors}\end{array}\right.\end{aligned} \tag{7}$$

Example: Dynamic model of a typical robotic pair

We shall observe a dynamic pair with an elastic gear and an elastic link. See Fig.2. Behind the engine we have the Harmonic-drive reducer, represented with the spring stiffness $C_\xi\,(\mathrm{Nm/rad})$ and the damping $B_\xi\,(\mathrm{Nm/(rad/s)})$. Behind the reducer there is an elastic link which divides the whole mass into two parts, m(kg) and m_b(kg) $\delta \approx 0$.

The link is observed as an elastic stick of the fixed length l(m) and a constant cross-section. The stiffness of the link is marked with $C_s\,(\mathrm{kg/s^2})$, and the damping of the link is marked with $B_s\,(\mathrm{kg/s})$.

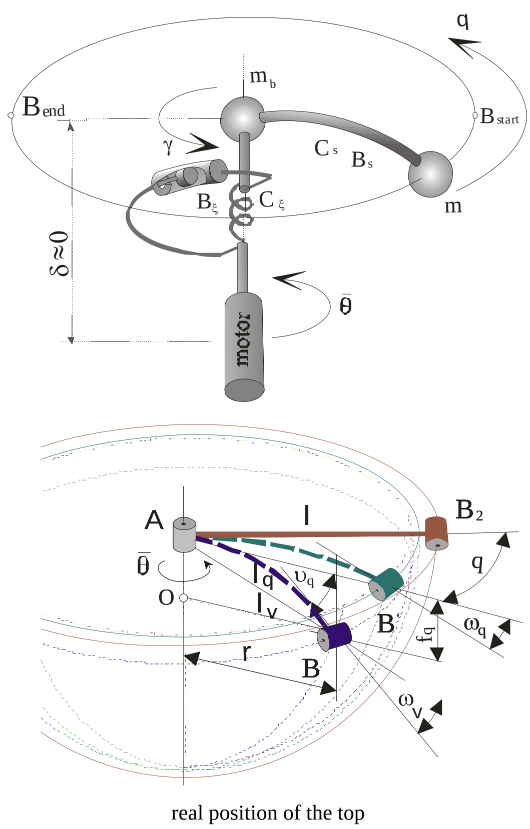


Figure 2. Typical robotic pair in the "horizontal" plane

The chosen robot pair is shown in Fig.2. The elasticity coordinate of this "kinematic" pair is q and it is formed by: $\bar{\theta}$ - the turning angle of the motor shaft behind the gear, ξ - the deflection angle of the gear and ϑ_r - the bending angle of the link (one mode) in the horizontal plane.

$$q = \bar{\theta} + \xi + \vartheta_r\,,\ \gamma = \bar{\theta} + \xi\,,\ q = \gamma + \vartheta_r \tag{8}$$

The angles $\bar{\theta}$ and ϑ_r are in the same plane for the rotational robotic pair in space. The angle ϑ_q is the angle of the link bending around the z_2 -axis, see Figures 2 and 3.

$$l_q = l \cdot \cos\vartheta_q\,,\ l_v = l_q \cdot \cos\vartheta_v \tag{9}$$

We will introduce one simplification that $\cos\vartheta_q \approx 1$, for $\vartheta_q \approx 0$, $\cos\vartheta_v \approx 1$, $\vartheta_v \approx 0$.

We accepted that the top of the mode has been moving continuously on the surface of the ball, the radius of which is l without shortening for the considered mode, see Fig.3.

$$l_q \approx l, \quad l_v \approx l \tag{10}$$

$$tg\vartheta_q = \frac{f_q}{l},\ tg\vartheta_r = \frac{f_r}{l}\,.$$

For small bending angles we adopt that $\vartheta_q \approx tg\vartheta_q$, $\vartheta_r \approx tg\vartheta_r$:

$$\vartheta_q = \frac{f_q}{l}, \quad \vartheta_r = \frac{f_r}{l} \tag{11}$$

$$r = l \cdot \cos\vartheta_q \tag{12}$$

The dynamic model of final equations for the motion of the considered robotic pair was obtained by applying Lagrange's equations.

According to Figures 2 and 3, we defined four generalized coordinates q, γ, ϑ_q and $\bar{\theta}$.

We shall not set the terms for kinematic, potential and dissipative energy of the rotation shaft of the engine because all these expressions were already realized long time ago and we shall only transcribe them in a corresponding way to the adapted elastic robot pair. We shall put the expression for kinetic, potential and dissipative energy moving of mass m_b, such as the mass m around the observed joint, and the expressions which appear as a consequence of the elastic deformations of the link and the gear.

We express the angles ϑ_r and ξ with the generalized coordinates (see Fig.3), respectively:

$$\vartheta_r = q - \gamma, \quad \xi = \gamma - \bar{\theta} \tag{13}$$

The relations between the angle of the bend link $\vartheta_r\left(\vartheta_q\right)$ and the angle of the turning top link $\omega_r\left(\omega_q\right)$, see [31], are:

$$\omega_r = \frac{1}{2}\vartheta_r, \quad \omega_q = \frac{1}{2}\vartheta_q \tag{14}$$

The kinetic and potential energy of the mechanism presented in Fig.2 are denoted as $\hat{\hat{E}}_k$ and $\hat{\hat{E}}_p$. All the angles in the expression for kinetic, potential and dissipative energy characterizing flexibility of the links should also be expressed via generalized coordinates.

Thus the total potential $\hat{\hat{E}}_p$ and the dissipative energy Φ are:

$$\hat{\hat{E}}_p = E_{p0} + E_{pels} + E_{pel\xi} \tag{15}$$

$$\Phi = \Phi_{els} + \Phi_{el\xi} \tag{16}$$

The potential and dissipative energy as a result of the elasticity of the links are, respectively

$$E_{pels} = \frac{1}{2}\cdot C_s \cdot f_q^2 + \frac{1}{2}\cdot C_s \cdot f_r^2\,,$$

$$\Phi_{els} = \frac{1}{2}\cdot B_s \cdot f_q^2 + \frac{1}{2}\cdot B_s \cdot f_r^2\,.$$

We introduce the multiplication and the division of the same expression with l^2,

$$E_{pels} = \frac{1}{2}\cdot C_s \cdot \frac{f_q^2}{l^2}\cdot l^2 + \frac{1}{2}\cdot C_s \cdot \frac{f_r^2}{l^2}\cdot l^2\,,$$

$$\Phi_{pels} = \frac{1}{2}\cdot B_s \cdot \frac{f_q^2}{l^2}\cdot l^2 + \frac{1}{2}\cdot B_s \cdot \frac{f_r^2}{l^2}\cdot l^2\ .$$

Since (11) and (13) are valid, then it follows that:

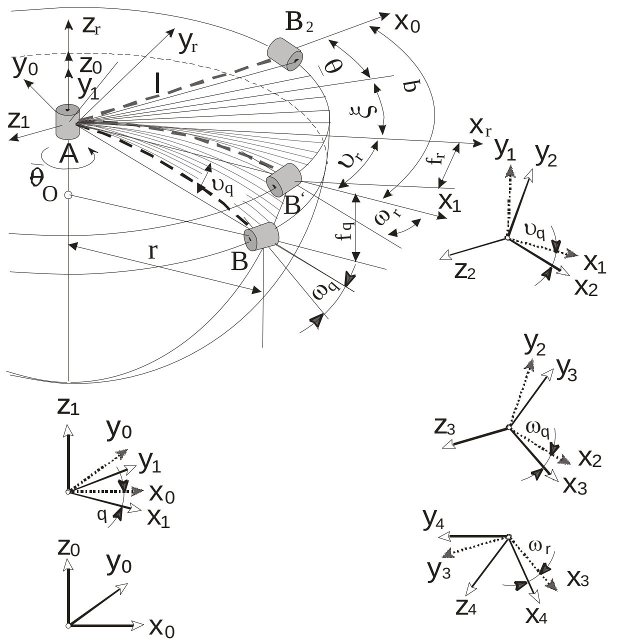


Figure 3. Position of the top after introducing simplification

$$E_{pels} = \frac{1}{2}\cdot C_s \cdot \vartheta_q^2 \cdot l^2 + \frac{1}{2}\cdot C_s \cdot \left(q-\gamma\right)^2 \cdot l^2 \tag{17}$$

$$\Phi_{els} = \frac{1}{2}\cdot B_s \cdot \vartheta_q^2 \cdot l^2 + \frac{1}{2}\cdot B_s \cdot \left(\dot{q}-\dot{\gamma}\right)^2 \cdot l^2 \tag{18}$$

Since (15) the potential $E_{pel\xi} = \frac{1}{2}\cdot C_\xi \cdot \xi^2$ energy and the dissipative $\Phi_{el\xi} = \frac{1}{2}\cdot B_\xi \cdot \xi^2$ energy are:

$$E_{pel\xi} = \frac{1}{2}\cdot C_\xi \cdot \left(\gamma - \bar{\theta}\right)^2 \tag{19}$$

$$\Phi_{el\xi} = \frac{1}{2}\cdot B_\xi \cdot \left(\dot{\gamma} - \dot{\bar{\theta}}\right)^2 \tag{20}$$

Let us define the equation of the flexible line of the link mode in the horizontal plane. The expressions $\hat{\hat{E}}_k$ and $\hat{\hat{E}}_p$ should be defined for any point of the first link (one mode) $\hat{l} = \int_0^{\hat{l}} d\hat{x}_\gamma$.

By applying Lagrange's equation with respect to the first generalized coordinate q using the expressions $\hat{E}_{kq}$, $\hat{E}_{pq}$, E_{pels} , $E_{pel\xi}$, Φ_{els} , $\Phi_{el\xi}$ we obtain the load moment $\hat{M}_q$ which is opposed by the flexibility force $\hat{\varepsilon}_r = \beta \cdot \frac{\partial^2\left(\hat{y}_r + \eta \cdot \dot{\hat{y}}_r\right)}{\partial \hat{x}_r^2}$ in the plane $x_r - y_r$

This is just the procedure for obtaining the „New form of the Euler-Bernoulli equation" by which the motion of any point on the flexible line of the link (one mode) in the horizontal plane:

$$\hat{H}_{1,1} \cdot \ddot{q} + \hat{H}_{1,2} \cdot \ddot{\gamma} + \hat{h}_1 + \beta \cdot \frac{\partial^2\left(\hat{y}_r + \eta \cdot \dot{\hat{y}}_r\right)}{\partial \hat{x}_r^2} = 0 \tag{21}$$

$$\hat{H}_{1,1} = m \cdot \hat{l}^2 \cdot \cos^2\left(\vartheta_q\right) + \frac{9}{4}\hat{J}_{zz} \cdots$$

Following the same analogy, we define the equation of the flexible line of the same link (one mode) in the vertical plane.

The expressions $\hat{\hat{E}}_{km}$ and $\hat{\hat{E}}_p$ should be defined for any point of the link (one mode) $\hat{l} = \int_0^{\hat{l}} d\hat{x}_1$.

Thus we obtain the expressions $\hat{E}_{k\upsilon_q}$ and $\hat{E}_{p\upsilon_q}$. By applying Lagrange's equation with respect to the third generalized coordinate υ_q using the expressions $\hat{E}_{k\upsilon_q}$, $\hat{E}_{p\upsilon_q}$, E_{pels} , $E_{pel\xi}$, Φ_{els} , $\Phi_{el\xi}$ we obtain the load moment $\hat{M}_{\upsilon_q}$ which is opposed by the flexibility force $\hat{\varepsilon}_{\upsilon_q} = \beta \cdot \frac{\partial^2\left(\hat{y}_1 + \eta \cdot \dot{\hat{y}}_1\right)}{\partial \hat{x}_1^2}$ in the plane $x_1 - y_1$.

This is just the procedure for obtaining the „New form of the Euler-Bernoulli equation" by which the motion of any point on the flexible line of the same segment in the vertical plane (one mode):

$$\hat{H}_{3,3} \cdot \ddot{\vartheta}_q + \hat{h}_3 + \beta \cdot \frac{\partial^2\left(\hat{y}_1 + \eta \cdot \dot{\hat{y}}_1\right)}{\partial \hat{x}_1^2} = 0 \tag{22}$$

The expressions $\hat{\hat{E}}_k$ and $\hat{\hat{E}}_p$ should be defined for the full length of the link $l = x_r$. Thus, for the given boundary conditions, by applying Lagrange's equation with respect to the first generalized coordinate q , we can obtain the equation of motion of the tip point of the considered elastic line link.

$$H_{1,1} \cdot \ddot{q} + H_{1,2} \cdot \ddot{\gamma} + h_1 + B_s \cdot l \cdot \dot{f}_r \cdot C_s \cdot l \cdot f_r = 0 \tag{23}$$

$$H_{1,1} = m \cdot l^2 \cdot \cos^2\left(\vartheta_q\right) + \frac{9}{4} J_{zzz} \cdots$$

The expressions $\hat{\hat{E}}_k$ and $\hat{\hat{E}}_p$ should be defined for the full length of the link *l*. Following the same procedure and by applying Lagrange's equation with respect to the second generalized coordinate γ, we define the equation of motion.

$$H_{2,1} \cdot \ddot{q} + H_{2,2} \cdot \ddot{\gamma} - B_s \cdot l \cdot \dot{f}_r - C_s \cdot l \cdot f_r \cdot B_\xi \cdot \xi \cdot C_\xi \cdot \xi = 0 \tag{24}$$

The expressions $\hat{\hat{E}}_k$ and $\hat{\hat{E}}_p$ should be defined for the full length of the link *l*. Following the same procedure and by applying Lagrange's equation with respect to the third generalized coordinates ϑ_q , we obtain the equation of motion.

$$H_{3,3} \cdot \ddot{\vartheta}_q + h_3 + B_s \cdot l \cdot \dot{f}_q + C_s \cdot l \cdot f_q = 0 \tag{25}$$

By applying Lagrange's equation with respect to the fourth coordinate $\overline{\theta}$, we obtain the motion equation of the motors:

$$\begin{aligned} u &= R \cdot i + C_E \cdot \dot{\overline{\theta}}, \\ C_M \cdot i &= I \cdot \ddot{\overline{\theta}} + B \cdot \dot{\overline{\theta}} - S \cdot \left(B_\xi \cdot \dot{\xi} + C_\xi \cdot \xi\right) \end{aligned} \tag{26}$$

In this paper, we choose the positional control law with local feedback for the realization of simulations:

$$u = K_{lp} \cdot \left(\overline{\theta}^\circ - \overline{\theta}\right) + K_{lv} \cdot \left(\dot{\overline{\theta}}^\circ - \dot{\overline{\theta}}\right) \tag{27}$$

Note: Eq.(21) cannot be equated to (23) because they are equations of different types. Eq. (21) is an equation of flexible lines (Euler-Bernoulli equation (EBA)) of the link in the horizontal plane, while (23) is an equation of motion (LMA) at the point of the tip of the same link in the horizontal plane. Eq. (22) is the Euler-Bernoulli equation of the link in the vertical plane, while (25) is an equation of motion at the point of the tip of the same link in the vertical plane.

All examples which we have analyzed in our papers up to now have two modes, see [12], [13], [15-17]. They have been manually developed, which limits their complexity. The example which we analyze in this paper has 4 DOFs. The segment has only one mode unlike the example in [12], [13], [15-17]. Since we are now at the stage of the pioneer research steps in this field, we believe that these equations should be first analyzed on the simplest examples in order to make public experts accept the idea of the new interpretation of the Euler-Bernoulli equation and a root of the equation. For these reasons, everything is shown on a little "simpler" example, step by step. It was developed in the period of the first research steps, but it does not make it less important. Considering the fact that it has not been published and that it gives the possibility of explaining all analyzed phenomena, we find it appropriate because it helps understanding this field.

Kinematics of elastic robotic systems

First we can analyse the solution of Euler-Bernoulli eq.(1). The general solution of motion, i.e. the form of transversal oscillations of flexible beams can be found in the method of particular integrals of D. Bernoulli, that is, see Fig.1a):

$$\hat{y}_{t01}\left(\hat{x}_{1,j}, t\right) = \hat{X}_{1,1}\left(\hat{x}_{1,1}\right) \cdot \hat{T}_{t01,1}\left(t\right) \tag{28}$$

By superimposing the particular solutions (28), any transversal oscillation can be presented in the following form:

$$\hat{y}_{t01}\left(\hat{x}_{1,j},t\right)=\sum_{j=1}^{\infty}\hat{X}_{1,j}\left(\hat{x}_{1,j}\right)\cdot\hat{T}_{t01,j}\left(t\right) \tag{29}$$

As already mentioned, equations (1), (28), (29) are defined under the assumption that the elasticity force is opposed only by the inertial force proper. The solution (28), (29) of D. Bernoulli satisfies these assumptions.

The Bernoulli solution (28), (29) describes only partially the nature of motion of real elastic beams. More precisely, it is only one component of motion. This fact is overlooked, and the original equations (28), (29) are widely used in the literature to describe the robotic system motion. This is very inadequate because valuable pieces of information about the complexity of the elastic robotic system motion are thus lost.

The Daniel Bernoulli solution (28), (29) should be expanded.

By superposing the particular solution of oscillatory nature, and the stationary solution of forced nature, see Fig. 1b), any flexible deformation of a considered mode may be presented in the following general form:

$$\hat{y}_{1,1}=\hat{X}_{1,1}\left(\hat{x}_{1,1}\right)\cdot\left(\hat{T}_{st1,1}\left(t\right)+\hat{T}_{t01,1}\left(t\right)\right) \tag{30}$$

$\hat{T}_{st1,1}$ is the stationary part of flexible deformation caused by stationary forces that vary continuously over time. $\hat{T}_{t01,1}$ is the oscillatory part of flexible deformation as in (28).

By superposing solutions (30), any flexible deformations of a flexible link with an infinite number of degrees of freedom (modes) may be presented in the following form:

$$\hat{y}_{1}\left(\hat{x}_{1,j},t\right)=\sum_{j=1}^{\infty}\hat{X}_{1,j}\left(\hat{x}_{1,j}\right)\cdot\left(\hat{T}_{st1,j}\left(t\right)+\hat{T}_{t01,j}\left(t\right)\right) \tag{31}$$

The solution of system (4) and dynamic motor motion, i.e. the form of its elastic line, can be obtained in the presence of the dynamics (angle) of rotation of each motor, as well as by taking into account the robotic configuration.

$$\begin{aligned}
\hat{y}&=\hat{a}\left(\hat{x}_{i,j},\hat{T}_{sti,j},T_{t0i,j},\overline{\theta},\alpha,t\right)\\
\hat{x}&=\hat{b}\left(\hat{x}_{i,j},\hat{T}_{sti,j},T_{t0i,j},\overline{\theta},\alpha,t\right)\\
\hat{z}&=\hat{c}\left(\hat{x}_{i,j},\hat{T}_{sti,j},T_{t0i,j},\overline{\theta},\alpha,t\right)\\
\hat{\psi}&=\hat{d}\left(\hat{x}_{i,j},\hat{T}_{sti,j},T_{t0i,j},\overline{\theta},\alpha,t\right)\\
\hat{\xi}&=\hat{e}\left(\hat{x}_{i,j},\hat{T}_{sti,j},T_{t0i,j},\overline{\theta},\alpha,t\right)\\
\hat{\varphi}&=f\left(\hat{x}_{i,j},\hat{T}_{sti,j},T_{t0i,j},\overline{\theta},\alpha,t\right)
\end{aligned} \tag{32}$$

Thus we defined the position and orientation of each point of the elastic line in the space of Cartesian coordinates. It should be pointed out that the form of elastic line comes out directly from the dynamics of the system motion.

The robot tip motion is defined by the sum of the stationary and oscillatory motion of each mode tip plus the dynamics of motion of the motor powering each link, as well by the included robot configuration (solution of (6) and (7)):

$$\begin{aligned}
y&=a\left(x_{i,j},T_{sti,j},T_{t0i,j},\overline{\theta},\alpha,t\right)\\
x&=b\left(x_{i,j},T_{sti,j},T_{t0i,j},\overline{\theta},\alpha,t\right)\\
z&=c\left(x_{i,j},T_{sti,j},T_{t0i,j},\overline{\theta},\alpha,t\right)\\
\psi&=d\left(x_{i,j},T_{sti,j},T_{t0i,j},\overline{\theta},\alpha,t\right)\\
\xi&=e\left(x_{i,j},T_{sti,j},T_{t0i,j},\overline{\theta},\alpha,t\right)\\
\varphi&=f\left(x_{i,j},T_{sti,j},T_{t0i,j},\overline{\theta},\alpha,t\right)
\end{aligned} \tag{33}$$

From (33) we can calculate the motion of each mode tip and link, and finally, of the robot tip motion.

Example: Kinematic model of a typical robotic pair

In order to define the shape and position of the elastic line of the mode link from Fig.2, during the realization of the robot task in the space of Cartesian coordinates, it is necessary to find solutions (21), (24), (22) and (26). The general solution of the dynamics movement of the observed model is given with (32).

Then, (23), (24), (25) and (26) will be valid and according to these conditions, all generalized coordinates $q,\gamma,\vartheta_q,\overline{\theta}$ could be calculated. According to (13), (14), other values of the system ξ, ϑ_r, ω_r, ω_q could be defined. According to the analogy with robot systems, these values can be named ‘internal coordinates’.

A geometric link between these characteristics (internal coordinates) and the space of Cartesian coordinates (external coordinates) was defined by the use of the transformation matrix, or so-called “direct kinematics” in the robotics. In order to describe space coordinates of the moving of this link top we shall need four matrices of rotation (see Fig.3):

* rotation around z_1 axis in the point “A” for the angle q, $a_1=0$, $d_1=0$, $\alpha_1=0$. The transformation matrix is T_{e1}^{0}.

* turning of the link top around the z_2 axis in the point “A” for the angle ϑ_q, $a_2=l$, $d_2=0$, $\alpha_2=90^\circ$. The transformation matrix is T_{e2}^{1}.

* turning of the link top around the z_3 axis in the point “B`” in the same plane for the angle ω_q, $a_3=0$, $d_3=0$, $\alpha_3=0^\circ$. The transformation matrix is T_{e3}^{2}.

* turning of the link top around the z_4 axis in the point “B” for the angle ω_r, $a_4=0$, $d_4=0$, $\alpha_4=90^\circ$. The transformation matrix is T_{e4}^{3}.

$$T_{e1}^{0}=\begin{bmatrix}\cos q & -\sin q & 0 & 0\\ \sin q & \cos q & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{34}$$

$$T_{e2}^{1}=\begin{bmatrix}\cos\vartheta_q & 0 & \sin\vartheta & l\cdot\cos\vartheta_q\\ \sin\vartheta_q & 0 & -\cos\vartheta_q & l\cdot\sin\vartheta_q\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{35}$$

$$T_{e3}^{2}=\begin{bmatrix}\cos\omega_q & -\sin\omega_q & 0 & 0\\ \sin\omega_q & \cos\omega_q & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{36}$$

$$T_{e4}^{3} = \begin{bmatrix} \cos\omega_r & 0 & \sin\omega_r & 0 \\ \sin\omega_r & 0 & -\cos\omega_r & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (37)$$

The total transformation matrix is calculated as a product of the transformation matrices of these four rotations.

$$T_{e4}^{0} = T_{e1}^{0} \cdot T_{e2}^{1} \cdot T_{e3}^{2} \cdot T_{e4}^{3} \quad (38)$$

In accordance with the analyzed example, it is evident that the rotational robotic pair from the elastic effects can compensate only for the stick bending in the direction of those forces that act in the direction of the motor shaft turning. The bending angles ϑ_q, ω_q, ω_r, which are not in the direction of the motor shaft turning, cannot be compensated for and they form a continual mistake in the tracking of reference trajectory.

As we have defined the transformation matrix for the same manipulator T_{e4}^{0}, given by (38), now it is easy to form the Jacobian matrix J_e.

The Jacobian matrix for a manipulator with elastic joints and links maps the velocity vector of the external coordinates $\dot{p}_s$ into the velocity vector of the internal coordinates $\dot{\phi}$:

$$\dot{p}_s = J(\phi) \cdot \dot{\phi}. \quad (39)$$

Where $\dot{p}_s = [\dot{x}\ \dot{y}\ \dot{z}\ \dot{\psi}\ \dot{\wp}\ \dot{\varphi}]^T$ defines the velocity of a given point of the robotic system in the Cartesian coordinates, whereas $\dot{\phi} = [\dot{\rho}_{1,1}\ \dot{\rho}_{1,2}\ \dot{\rho}_{1,3}\ \dot{\rho}_{1,4}\ \ldots\ \dot{\rho}_{1,n}]^T$ defines the velocity vector of the internal coordinates. In this example see Fig.2, eq.(39) has a form:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -l \cdot \sin\vartheta_q \cdot \cos q & -l \cdot \cos\vartheta_q \cdot \sin q \\ -l \cdot \sin\vartheta_q \cdot \sin q & +l \cdot \cos\vartheta_q \cdot \cos q \\ +l \cdot \cos\vartheta_q & 0 \end{bmatrix} \cdot \begin{bmatrix} \dot{\vartheta}_q \\ \dot{q} \end{bmatrix} \quad (40)$$

The elements of the Jacobian are only functions of the elements of the homogenous transformation matrix T_{e4}^{0}.

Here we have one more innovation concerning the known considerations. In robotics the reference trajectory is defined in a purely kinematic way i.e. geometric and now in the presence of the elasticity elements we can include also the elastic deformation values at the reference level i.e. at the level of knowing the elasticity characteristics during the reference trajectory defining.

There are two aspects in defining the reference trajectory of the motor angle (see [12-17]):

1. Elastic deformation is considered as a quantity which is not encompassed by the reference trajectory.
2. Elastic deformation is a quantity which is at least partly encompassed by the reference trajectory.

It is clear now that (33), generally, and in our example (38) serve for the calculation of the robot tip movement during the robot task realization and that based on the motor rotating angles, elastic deformation values and all other kinematic and dynamic robot mechanism characteristics (such as its geometry, configuration, weight disposal, motor characteristics, reference trajectory choice as well as many other important characteristics that influence the robot system movement dynamics). We, in the robotics, call this procedure the solution of „direct kinematics“.

In this way we presented the analogy:

between		
the Euler-Bernoulli equation solutions which were defined by Daniel Bernoulli by (28) (or (29)) in the original form (i.e. the form of elastic line solutions (32) in the extended form or the form of equation of the motion solutions of tip (33))	and	the procedure of the „direct kinematics“ (38) in the robotics.

The analogy between the Euler-Bernoulli equation and its solution and modern knowledge from the Robotics is presented in this way.

Simulation Example

We are analysing the behavior of a robotic pair with an elastic gear and an elastic link as in Fig.2. The tip of the robot started from the position “B_{start}” $\left(\overset{\circ}{q}_{start} = 0(\text{rad})\right)$ and moves directly to the point “B_{end}” $\left(\overset{\circ}{q}_{end} = \pi(\text{rad})\right)$ in a predicted time of $T = 4(\text{s})$.

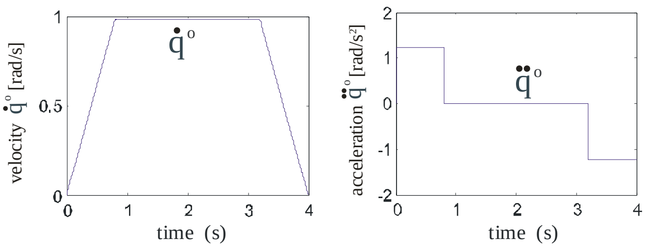


Figure 4. Reference velocity and acceleration in the space of internal coordinates

The trapezoidal profile of velocity together with the time of acceleration and deceleration from $0.2 \cdot T$ is adopted. See Fig.4.

The links have a tube form. The outside diameter is $D(\text{m})$, and the ratio between the inside and the outside diameter of the cylinder is ξ.

The characteristics of stiffness and damping of the gear in the real and reference regimes are not the same and neither are the stiffness and damping characteristics of the link. $C_\xi = 0.99 \cdot C_\xi^0$, $B_\xi = 0.99 \cdot B_\xi^0$, $C_s = 0.99 \cdot C_s^0$, $B_s = 0.99 \cdot B_s^0$. The only disturbance in the system is the ignorance of the rigidity characteristics and damping.

The defined parameters of the mechanism with their values can significantly influence the stability of a robot system. Fig.5 gives the change of the generalised coordinates $q, \gamma, \bar{\theta}$ during the realization of the robot task. Since we control directly the position of turning of the angle of the motors behind the reducer $\bar{\theta}$, it is obvious that we have almost an ideal tracking comparing to the reference.

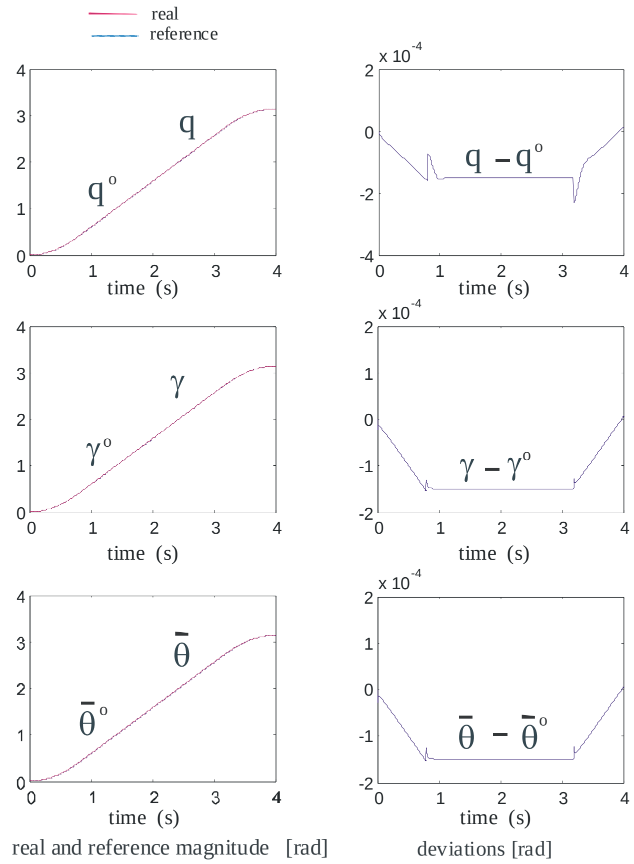


Figure 5. Position and deviation from the references in the space of internal coordinates

In Fig.6 we can see that during the free motion from the point "B_{start}" to the point "B_{end}", in the space of external coordinates, there is a good tracking of the reference trajectory.

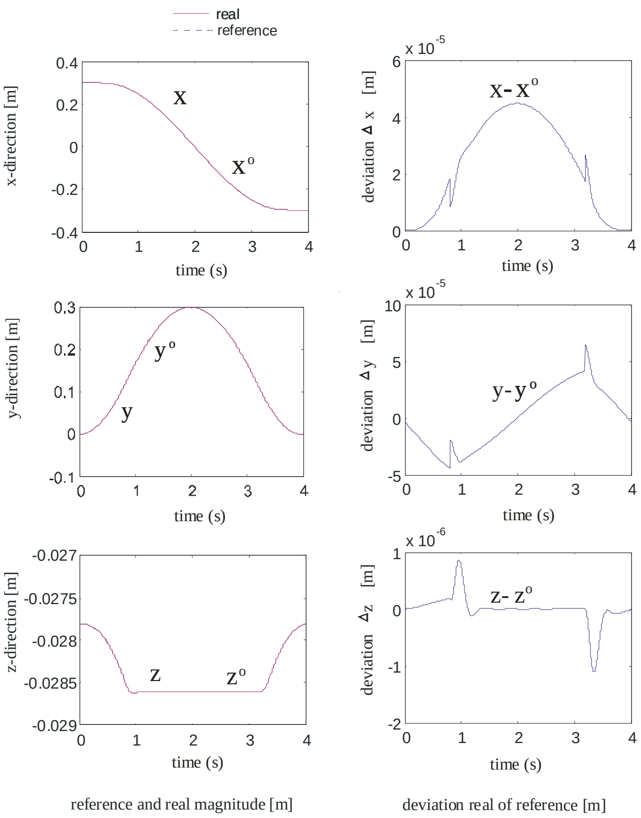


Figure 6. Position and deviation from the references in the space of external coordinates

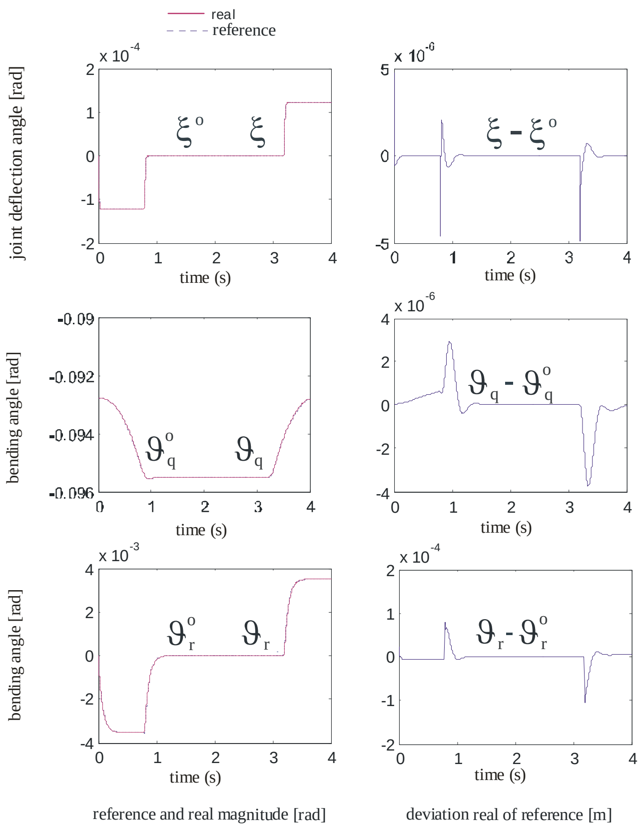


Figure 7. Elastic deformations and deviation from the reference

Fig.7 shows the level of elastic deformations, the angle of the gear deflection ξ and the angle of the link bend ϑ_q as well as ϑ_r during the realization of the robot task. In the same figure, the deviation of these values from the referent ones can be seen as well.

Fig.8 depicts the signal control along the realization of the robot task.

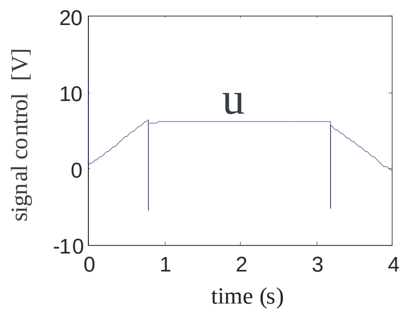


Figure 8. Signal control

Conclusion

It is pointed out that the elastic deformation is the consequence of the total mechanism dynamics which is essentially different from a widely used method that implies the adaptation of the "assumed modes technique".

The analogy between the solution of the Euler-Bernoulli equation which Daniel Bernoulli defined in the original form and „the direct kinematics solution" was defined and the basic solution for further analysis of elastic robotic systems was presented.

With fundamental approach to the analysis of flexibility of complex mechanisms, a wide field of working on analyzing and modeling complex mechanical constructions as well as on the implementation of different controls of laws was opened. All this was presented for a relatively "simple" mechanism that offered the possibility of analyzing the phenomena involved. Through the analysis and modeling of an elastic mechanism we attempted to give a contribution to the development of this area.

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Aanalogija između novog oblika Ojler-Bernulijeve jednačine i algoritma za definisanje matematičkog modela kretanja robota

Sa novim, generacijski skupljenim znanjima, intenzivnim razvojem novih tehničkih oblasti kao što je robotika, posebno osnažena razvojem kompjuterskih tehnologija, podstiče i omogućuje da elastična deformacija bude razmatrana realno kao dinamička veličina koja zavisi od parametara sistema. Amplituda elastične deformacije kao i frekvencija su dinamičke veličine koje zavise od ukupne dinamike kretanja robotskog sistema (sila) i takođe od konfiguracije mehanizma, težina, dužina segmenata, odabrane referentne trajektorije, dinamičkih karakteristika

kretanja motora i td. Mi definišemo opštu formu jednačine elastične linije kompleksnog robotskog sistema određene konfiguracije, koristeći Ojler-Bernulijeve jednačinu. Prikazana je veza između Ojler-Bernulijeve jednačine i jednačine ravnoteže u tački vrha elastične linije. Matematički model motora takođe obuhvata sprezanje između sila elastičnosti. Prikazana je analogija između rešenja Ojler-Bernulijeve jednačine, koje je definisao Daniel Bernuli u originalnoj formi, i procedure rešenja „direktne kinematike" u robotici.

Ključne reči: **robotika, kinematika, dinamika kretanja, Ojler-Bernulijeve jednačine, modelovanje procesa, elastična deformacija, sprezanje, matrica krutosti, simulacija kretanja, programirana trajektorija.**

Analogi} me`du novwmi formami $jler-Bernulli уravneni} i algorifma dl} opredeleni} matemati~eskoj modeli dvi`eni} robota

С новыми, на протяжении многих поколений накапливающимися знаниями, с интенсивным развитием новых технических областей, а в том числе и робототехники, особенно усиленной развитием вычислительного процесса, побуждается и обеспечивается чтобы эlasti~na} deformaci} реально была рассматривана как dinami~eskая veli~ina, kotora} zavisit от параметров системы. Амплитуды эlasti~nой deformacии, а в том числе и частота, являются dinami~eskими veli~inами зависящими ot sovokupnoj dinamiki dvi`eni} robototehni~eskoj sistemw (сил) и тоже от konfiguraciи механизма, весов, длины сегментов, отобранной относительной траектории, от динамических характеристик движения двигателя и так далее. Мы определяем общую форму уравнения &lasti~noj linii сложной robototehni~eskой sistemы определённой konfiguracii, пользуясь уravneniем $jler-Bernulli. Здесь показана связь между уravneniем $jler-Bernulli и уravneniем равновесия в точке вершины &lasti~noj linii. Matemati~eska} modelx dvigatel} to`e ohvatwvaet sv}zwvanie me`du silami &lasti~nosti. Здесь показана аналогия между решениями uravneniя $jler-Bernulli, kotorwe opredelil Daniel Bernulli в подлинной форме, и процедуры решения чпрямой кинематикич в робототехнике.

Kly~evwe slova: robototehnika, **кинематика**, dinamika dvi`eni}, uravneni} $jler-Bernulli, modelirovanie processa, &lasti~na} deformaci}, sv}zwvanie, matrica `ëstkosti, imitaci} dvi`eni}, programmirovana} traektori}.

Analogie entre de la nouvelle forme de l'équation Euler-Bernoulli et l'algorithme pour la définition du modéle mathématique du mouvement chez robot

Avec les nouvelles connaissances accumulées par des générations et par le développement intensif de nouveaux domaines techniques tel que la robotique, renforcée en particulier par le développement des technologies numériques, la déformation élastique peur être considérée réellement comme la valeur dynamique qui dépend des paramètres du système. L'amplitude de la déformation élastique ainsi que la fréquence sont les valeurs dynamiques qui dépendent de la dynamique totale du mouvement du système robotique (forces) et aussi de la configuration du mécanisme, poids, longueur des segments, trajectoire référentielle choisie, caractéristiques dynamiques du mouvement de moteur etc. Nous définissons la forme générale de l'équation de la ligne élastique du système robotique complexe de la configuration déterminée au moyen de l'équation Euler-Bernoulli et l'équation de l'équilibre au point du sommet de la ligne élastique. Le modèle mathématique du moteur comprend aussi le couplage entre les forces d'élasticité. On a présenté l'analogie entre la solution de l'équation Euler-Bernoulli et le processus de la solution pour « cinématique directe » en robotique.

Mots clés: **robotique, cinématique, dynamique de mouvement, équations Euler-Bernoulli, modélisation du processus, déformation élastique, couplage, matrice de rigidité, simulation du mouvement, trajectoire programmée.**