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The Stability of Linear Continuous Time Delay Systems Over the Finite Time Interval: An Overview

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This paper gives a detailed overview of the work and the results of many authors in the area of Non-Lyapunov (finite time stability, technical stability, practical stability, final stability) of particular class of linear continuous time delays systems.
This survey covers the period since 1995 up to nowadays and has strong intention to present the main concepts and contributions that have been derived during the mentioned period through the whole world, published in the respectable international journals or presented at workshops or prestigious conferences.

Key words: **continuous system, linear system, system stability, Non-Lyapunov stability, time delay system, system over a finite time interval.**

Introduction

THE problem of investigation of time delay system has been exploited over many years. Delay is very often encountered in different technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time delay, regardless if it is present in the control or/and state, may cause undesirable system transient response, or generally, even an instability. Consequently, the problem of stability analysis of this class of systems has been one of the main interests of many researchers. In general, the introduction of time lag factors makes the analysis much more complicated. In the existing stability criteria, mainly two ways of approach have been adopted. Namely, one direction is to contrive the stability condition which does not include information on the delay, and the other is the method which takes it into account. The former case is often called the delay-independent criteria and generally provides nice algebraic conditions. Numerous reports have been published on this matter, with a particular emphasis on the application of Lyapunov's second method, or on using the idea of the matrix measure *Lee, Diant* (1981), *Mori* (1985), *Mori et al*. (1981), *Hmamed* (1986), *Lee et al*. (1986)).

In practice one is not only interested in system stability (e.g. in the sense of Lyapunov), but also in bounds of system trajectories. A system could be stable but still completely useless because it possesses undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain subsets of state-space which are defined *a priori* in a given problem.

Besides that, it is of particular significance to consider the behavior of dynamical systems only over a finite time interval.

These boundedness properties of system responses, i.e. the solution of system models, are very important from the engineering point of view. Therefore, numerous definitions of the so-called technical and practical stability were introduced. Roughly speaking, these definitions are essentially based on the predefined boundaries for the perturbation of initial conditions and allowable perturbation of system response. In the engineering applications of control systems, this fact becomes very important and sometimes crucial, for the purpose of characterizing in advance, in quantitative manner, possible deviations of the system response. Thus, the analysis of these particular boundedness properties of solutions is an important step, which precedes the design of control signals, when finite time or practical stability control is considered.

Motivated by "brief discussion" on practical stability in the monograph *La Salle*, *Lefchet*, (1961), *Weiss* and *Infante* (1965, 1967) have introduced various notations of stability over the finite time interval for continuous-time systems and constant set trajectory bounds. Further development of these results was due to many other authors *Michel* (1970), *Grujic* (1971), *Lashirer, Story* (1972)). Practical stability of simple and interconnected systems with respect to time-varying subsets was considered in *Michel* (1970) and *Grujic* (1975). A more general type of stability ("practical stability with settling time", practical exponential stability, etc.) which includes many previous definitions of finite stability was introduced and considered in *Grujic* (1971, 1975.a, 1975.b).

A concept of finite-time stability, called "final stability", was introduced in *Lashirer*, *Story* (1972) and further development of these results was due to *Lam* and *Weiss* (1974).

In the context of practical stability for linear generalized state-space systems, various results were first obtained in

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Debeljkovic, *Owens* (1985) and *Owens*, *Debeljkovic* (1986).

An analysis of nonlinear singular and implicit dynamic systems in terms of the generic qualitative and quantitative concepts, which contain technical and practical stability types as special cases, has been introduced and studied in *Bajic* (1988, 1992).

In this short overview, the results in the area of finite and practical stability were only concerned for continuous linear control systems.

Here we present the problem of sufficient conditions that enable system trajectories to stay within the *a priori* given sets for the particular class of time-delay systems. To the best knowledge of authors, these problems, using this approach, are not yet analyzed for the time-delay systems by the other authors.

Notations and preliminaries

A linear, multivariable time-delay system can be represented by a differential equation:

$$\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + B_0\mathbf{u}(t) + B_1\mathbf{u}(t-\tau), \quad (1)$$

and with the associated function of the initial state:

$$\mathbf{x}(t) = \boldsymbol{\psi}_x(t), \qquad \mathbf{u}(t) = \boldsymbol{\psi}_u(t), \quad -\tau \le t \le 0. \quad (2)$$

Equation (1) is referred to as *nonhomogenous* or forced state equation, $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ control vector, A_0, A_1, B_0 and B_1 are constant system matrices of appropriate dimensions, and τ is the pure time delay, $\tau = const.$ ($\tau > 0$).

A dynamical behavior system (1) with initial functions (2) is defined over time interval $J = \{t_0,\ t_0 + T\}$, where the quantity T may be either a positive real number or symbol $+\infty$, so finite time stability and practical stability can be treated simultaneously.

It is obvious that $J \in \mathbb{R}$.

Time invariant sets, used as bounds of system trajectories, are assumed to be open, connected and bounded.

Let the index β stands for the set of all allowable states of the system and the index α for the set of all initial states of the system, such that the set $S_\alpha \subseteq S_\beta$.

In general, one may write:

$$S_\rho = \left\{\mathbf{x}(t):\ \|\mathbf{x}(t)\|_Q^2 < \rho\right\}, \quad (3)$$

where Q will be assumed to be a symmetric, positive-definite, real matrix.

S_ε denotes the set of all allowable control actions.

Let $\|\mathbf{x}(\cdot)\|_{(\cdot)}$ be any vector norm (e.g., $\cdot$ = 1, 2, ∞) and $\|(\cdot)\|$ the matrix norm induced by this vector.

Here, we use $\|\mathbf{x}(t)\|_2 \overset{\Delta}{=} \left(\mathbf{x}^T(t)\mathbf{x}(t)\right)^{1/2}$ and $\|(\cdot)\|_2 = \lambda_{\max}^{1/2}(A^*A)$.

The upper indices * and T denote the transpose conjugate and the transpose, respectively.

Matrix measure has been widely used in the literature when dealing with the stability of time delay systems.

The matrix measure $\mu(\cdot)$ for any matrix $A \in \mathbb{C}^{n\times n}$ is defined as follows

$$\mu(A) \overset{\Delta}{=} \lim_{\varepsilon\to 0} \frac{\|1+\varepsilon A\| - 1}{\varepsilon}. \quad (4)$$

The matrix measure defined in (4) can be subdefined in three different ways, depending on the norm utilized in its definitions, *Coppel* (1965), *Desoer*, *Vidysagar* (1975):

$$\mu_1(A) = \max_k \left(\mathrm{Re}(a_{kk}) + \sum_{\substack{i=1\\ i\neq k}}^{n} |a_{ik}|\right), \quad (5.a)$$

$$\mu_2(A) = \frac{1}{2}\max_i \lambda_i(A^* + A), \quad (5.b)$$

and

$$\mu_\infty(A) = \max_i \left(\mathrm{Re}(a_{ii}) + \sum_{\substack{k=1\\ k\neq i}}^{n} |a_{ki}|\right), \quad (5.c)$$

From *Mori* (1988), the following inequality holds:

$$-\|F\|_2 \le -\mu(-F) \le \mu(F) \le \|F\|_2. \quad (5.d)$$

Basic notations

$\mathbb{R}$	- Real vector space
$\mathbb{C}$	- Complex vector space
$F = (f_{ij}) \in \mathbb{R}^{n\times n}$	- real matrix
F^T	- Transpose of the matrix F
$F > 0$	- Positive definite matrix
$F \ge 0$	- Positive semi definite matrix
$\lambda(F)$	- Eigenvalue of the matrix F
$\sigma_{(\ )}(F)$	- Singular values of the matrix F
$\sigma\{F\}$	- Spectrum of the matrix F
$\|F\|$	- Euclidean matrix norm $\|F\| = \sqrt{\lambda_{\max}(A^T A)}$
$\Rightarrow$	- Follows
$\mapsto$	- Such that

Time invariant time delay systems stability definitions

In the context of finite or practical stability for a particular class of *nonlinear singularly perturbed multiple time delay systems* various results were, *for the first time*, obtained in *Feng*, *Hunsarg* (1996). It seems that theirs definitions are very similar to those in *Weiss*, *Infante* (1965, 1967), clearly addopted to time delay systems.

It should be noticed that those definitions are significantly different from the definition presented by the autor of this paper.

Definition 1. A system is *stable* with respect to the set $\{\alpha,\ \beta,\ -\tau,\ T,\ \|\mathbf{x}\|\}$, $\alpha \le \beta$ if for any trajectory $\mathbf{x}(t)$ the condition $\|\mathbf{x}_0\| < \alpha$ implies $\|\mathbf{x}(t)\| < \beta \quad \forall t \in [-\Delta,\ T]$, $\Delta = \tau_{\max}$, *Feng*, *Hunsarg* (1996).

Definition 2. A system is *contractively stable* with respect to the set $\{\alpha,\ \beta,\ -\tau,\ T,\ \|\mathbf{x}\|\}$, $\gamma < \alpha < \beta$, if for any trajectory $\mathbf{x}(t)$ the condition $\|\mathbf{x}_0\| < \alpha$ implies

(i) Stabilty w.r.t. $\{\alpha,\ \beta,\ -\tau,\ T,\ \|\mathbf{x}\|\}$,

(ii) There exists $t^{*} \in \left]0,\ T\right[$ such that $\|\mathbf{x}(t)\| < \gamma$ for all $\forall t \in \left]t^{*},\ T\right[$ *Feng, Hunsarg* (1996).

Definition 3. Autonomous system (**1**) satisfying initial condition (2) is finite time stable w.r.t. $\{\zeta(t),\ \beta,\ J\}$ if and only if

$$\boldsymbol{\psi}_x(t) < \zeta(t),$$

implies:

$$\|\mathbf{x}(t)\|_2 < \beta, \quad t \in J,$$

$\zeta(t)$ being the scalar function with the property $0 < \zeta(t) \le \alpha,\ \ -\tau \le t \le 0, -\ \tau \le t \le 0$, where α is a real positive number and $\beta \in \mathbb{R}$ and $\beta > \alpha$, *Debeljkovic et al.* (1997.a, 1997.b, 1997.c, 1997.d), *Nenadic et al.* (1997).

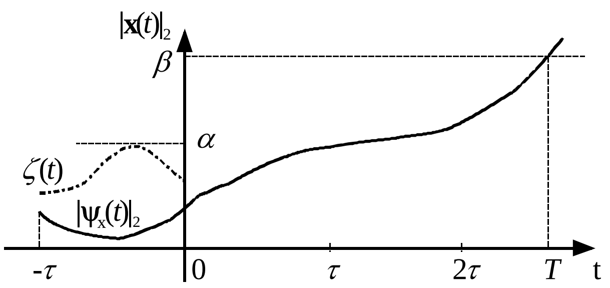

Figure 1. Illustration of the preceding definition

Definition 4. System (1), with $\mathbf{u}(t-\tau) \equiv 0$, $\forall t$, satisfying initial condition (2) is finite time stable w.r.t. $\{\zeta(t),\ \beta,\ \varepsilon,\ \tau,\ J,\ \mu(A_0 \neq 0)\}$ if and only if

$$\boldsymbol{\Psi}_x(t) \in S_\alpha, \qquad \forall t \in [-\tau,\ 0],$$

and

$$\mathbf{u}(t) \in S_\varepsilon, \qquad \forall t \in J,$$

imply

$$\mathbf{x}(t_0,\ t,\ \mathbf{x}_0) \in S_\beta, \quad \forall t \in [0,\ T],$$

Debeljkovic et al. (1997.b, 1997.c)

Definition 5. System (**1**) satisfying initial condition (2) is finite time stable w.r.t. $\{\alpha, \beta, \varepsilon_\Psi, \varepsilon, \tau, J, \mu_2(A_0) \neq 0\}$ if and only if

$$\Psi_x(t) \in S_\alpha, \quad \Psi_u(t) \in S_{\varepsilon\Psi}, \quad \forall t \in [-\tau,\ \ 0], \quad (13)$$

$$\mathbf{u}(t) \in S_\varepsilon, \quad \forall t \in J,$$

imply:

$$\mathbf{x}(t, t_0, \mathbf{x}_0, \mathbf{u}(t)) \in S_\beta, \qquad \forall t \in J,$$

Debeljkovic et al. (1997.b, 1997.c).

Stability theorems

Theorem 1. Autonomous system (1) with initial function (2) is finite time stable with respect to $\{\alpha,\ \beta,\ \tau,\ J\}$ if the following condition is satisfied

$$\|\Phi(t)\|_2 < \frac{\sqrt{\beta/\alpha}}{1+\tau\|A_1\|_2}, \qquad \forall t \in [0,\ T], \tag{6}$$

where $\|(\cdot)\|$ is the Euclidean norm and $\Phi(t)$ is the fundamental matrix of system (1), *Nenadic et al.* (1997), *Debeljkovic et al*. (1997.a).

Proof. The solution of (**1**) with initial function (2) can be expressed in terms of the fundamental matrix as

$$\mathbf{x}(t) = \Phi(t)\boldsymbol{\psi}_x(0) + \int_{-\tau}^{0} \Phi(t-\theta-\tau)A_1\boldsymbol{\psi}_x(\theta)d\theta, \tag{7}$$

Using the above equation, one can get

$$\begin{aligned}
&\mathbf{x}^T(t)\mathbf{x}(t) = \boldsymbol{\psi}_x^T(0)\Phi^T(t)\Phi(t)\boldsymbol{\psi}_x(0) \\
&+\boldsymbol{\psi}_x^T(0)\Phi^T(t)\int_{-\tau}^{0}\Phi(t-\eta-\tau)A_1\boldsymbol{\psi}_x(\eta)d\eta \\
&+\left(\int_{-\tau}^{0}\boldsymbol{\psi}^T(\theta)A_1^T\Phi^T(t-\theta-\tau)d\right)\theta\,\Phi(t)\boldsymbol{\psi}_x(0) \\
&+\int_{-\tau}^{0}\boldsymbol{\psi}_x^T(\theta)A_1^T\Phi^T(t-\theta-\tau)d\theta\int_{-\tau}^{0}\Phi(t-\eta-\tau)A_1\boldsymbol{\psi}_x\eta d\theta
\end{aligned} \tag{8}$$

Using the abbreviations

$$\Phi(t)\boldsymbol{\psi}_x(0) = \mathbf{a}(t) \in \mathbb{R}^{n\times 1}, \tag{9}$$

$$\Phi(t-\theta-\tau)A_1\boldsymbol{\psi}_x(\theta) = \mathbf{b}(t,\theta) \in \mathbb{R}^{n\times 1}, \tag{10}$$

it is obvious that, if one introduces

$$\int_{-\tau}^{0}\mathbf{b}(t,\theta)d\theta = \mathbf{c}(t) \in \mathbb{R}^{n\times 1} \tag{11}$$

then (8) becomes

$$\begin{aligned}\mathbf{x}^T(t)\mathbf{x}(t) &= \boldsymbol{\psi}_x^T(0)\Phi^T(t)\Phi(t)\boldsymbol{\psi}_x(0) \\ &+\mathbf{a}^T(t)\mathbf{c}(t)+\mathbf{c}^T(t)\mathbf{a}(t)+\mathbf{c}^T(t)\mathbf{c}(t)\end{aligned}. \tag{12}$$

A well-known result from the theory of quadratic forms gives

$$\boldsymbol{\psi}^T(0)\left(\Phi^T(t)\Phi(t)\right)\boldsymbol{\psi}_x(0) \le \lambda_M(t)\boldsymbol{\psi}_x^T(0)\boldsymbol{\psi}_x(0), \tag{13}$$

where

$$\lambda_M(t) = \max\sigma\left\{\Phi^T(t)\Phi(t)\right\}, \tag{14}$$

and where $\sigma\{F\}$ denotes the spectrum of the matrix F.

Also, it is easy to see that $\mathbf{a}^T(t)\mathbf{c}(t) = \mathbf{f}(t) \in \mathbb{R}^1$ and $\mathbf{c}^T(t)\mathbf{a}(t) = f(t) \in \mathbb{R}^1$, so it follows that $\mathbf{a}^T(t)\mathbf{c}(t) = \mathbf{c}^T(t)\mathbf{a}(t)$.

Now one can write

$$\begin{aligned}\mathbf{x}^T(t)\mathbf{x}(t) &\le \lambda_M(t)\boldsymbol{\psi}_x^T(0)\boldsymbol{\psi}_x(0) \\ &+2\mathbf{a}^T(t)\mathbf{c}(t)+\|\mathbf{c}(t)\|^2\end{aligned}. \tag{15}$$

Moreover

$$f(t) \le |f(t)|, \qquad \left|\int_a^b \varphi(t)\,dt\right| \le \int_a^b |\varphi(t)|\,dt\,, \tag{16}$$

so it follows that

$$\mathbf{a}^T(t)\mathbf{c}(t) \le \left\|\boldsymbol{\psi}_x^T(0)\right\| \cdot \left\|\Phi^T(t)\right\| \cdot \int_{-\tau}^{0} \|\mathbf{b}(t,\theta)\|\, d\theta \tag{17}$$

Now, equation (15) can be rewritten as

$$\begin{aligned}&\mathbf{x}^T(t)\mathbf{x}(t) \le \lambda_M(t)\boldsymbol{\psi}_x^T(0)\boldsymbol{\psi}_x(0)\\ &+2\left\|\boldsymbol{\psi}_x^T(0)\right\|\cdot\left\|\Phi^T(t)\right\|\cdot\|A_1\|\int_{-\tau}^{0}\|\Phi(t-\theta-\tau)\|\cdot\|\boldsymbol{\psi}_x(\theta)\|\,d\theta \\ &+\|A_1\|^2\left(\int_{-\tau}^{0}\|\Phi(t-\theta-\tau)\|\cdot\|\boldsymbol{\psi}_x(\theta)\|\,d\theta\right)^2\end{aligned} \tag{18}$$

However, if

$$m(t) \le \|\Phi(t-\theta-\tau)\|\cdot\|\boldsymbol{\psi}_x(\theta)\| \le \wp(t), \quad \forall\theta\in[-\tau,\ 0] \tag{19}$$

then

$$\tau\cdot m(t) \le \int_{-\tau}^{0}\|\Phi(t-\theta-\tau)\|\cdot\|\boldsymbol{\psi}_x(\theta)\|d\theta \le \tau\cdot\wp(t) \tag{20}$$

It is easy to show that

$$\|\Phi(t-\theta-\tau)\|_{\theta\in[-\tau,\,0]} \le \|\Phi(t)\|, \quad \|\boldsymbol{\psi}_x(\theta)\|_{\theta\in[-\tau,\,0]} < \sqrt{\alpha} \tag{21}$$

and

$$\|\boldsymbol{\psi}_x(0)\| < \sqrt{\alpha}, \qquad \left\|\Phi^T(t)\right\| = \|\Phi(t)\|, \tag{22}$$

so equation (18) is now

$$\begin{aligned}\mathbf{x}^T(t)\mathbf{x}(t) &\le \lambda_M(t)\boldsymbol{\psi}_x^T(0)\boldsymbol{\psi}_x(0)\\ &+2\left\|\boldsymbol{\psi}_x^T(0)\right\|\cdot\left\|\Phi^T(t)\right\|^2\cdot\|A_1\|\cdot\tau\cdot\sqrt{\alpha}\,.\\ &+\left\|\Phi^T(t)\right\|^2\cdot\|A_1\|^2\cdot\tau^2\cdot\alpha\end{aligned} \tag{23}$$

Also, from the simple fact that

$$\lambda_M(t) \le \left\|\Phi^T(t)\Phi(t)\right\| \le \left\|\Phi^T(t)\right\|\cdot\|\Phi(t)\| = \|\Phi(t)\|^2 \tag{24}$$

it follows that

$$\begin{aligned}\mathbf{x}^T(t)\mathbf{x}(t) &\le \|\Phi(t)\|^2\boldsymbol{\psi}_x^T(0)\boldsymbol{\psi}_x(0)\\ &+2\left\|\boldsymbol{\psi}_x^T(0)\right\|\cdot\left\|\Phi^T(t)\right\|^2\cdot\|A_1\|\cdot\tau\cdot\sqrt{\alpha}\,.\\ &+\left\|\Phi^T(t)\right\|^2\cdot\|A_1\|^2\cdot\tau^2\cdot\alpha\end{aligned} \tag{25}$$

If one chooses

$$\boldsymbol{\psi}_x^T(0)\boldsymbol{\psi}_x(0) < \alpha\,, \tag{26}$$

then, it immediately follows that

$$\begin{aligned}\mathbf{x}^T(t)\mathbf{x}(t) &\le \left\|\Phi^T(t)\right\|^2\alpha + 2\|\Phi(t)\|^2\cdot\|A_1\|\cdot\tau\cdot\alpha\\ &+\|\Phi(t)\|^2\cdot\|A_1\|^2\cdot\tau^2\cdot\alpha\\ &=\|\Phi(t)\|^2\alpha(1+\tau\|A_1\|)^2\end{aligned} \tag{27}$$

Applying the basic condition of the *Theorem*, i.e. (6), to the preceding inequality, one can get

$$\mathbf{x}^T(t)\mathbf{x}(t) < \left(\frac{\sqrt{\beta/\alpha}}{1+\tau\|A_1\|}\right)^2\alpha\cdot(1+\tau\|A_1\|)^2 < \beta\,, \tag{28}$$

which was to be proved, *Nenadic et al*. (1997), *Debeljkovic et al*. (1997.a), Q.E.D.

When $\tau = 0$ or $\|A_1\| = 0$, the problem is reduced to the case of the ordinary linear systems, *Angelo* (1970).

Theorem 2: Autonomous system (1) with initial function (2) is finite time stable w.r.t. $\{\alpha,\ \beta,\ \tau,\ T\}$ if the following condition is satisfied

$$e^{\mu_2(A_0)t} < \frac{\sqrt{\beta/\alpha}}{1+\tau\,\|A_1\|_2}\,, \qquad \forall t\in[0,\ T], \tag{29}$$

where $\|(\cdot)\|$ denotes the Euclidean norm, *Debeljkovic et al*. (1997.b).

Theorem 3. Autonomous system (1) with initial function (2) is finite time stable with respect to $\{\alpha,\ \beta,\ \tau,\ T,\ \mu_2(A_0)\neq 0\}$ if the following condition is satisfied

$$e^{\mu_2(A_0)t} < \frac{\beta/\alpha}{1+\mu_2^{-1}(A_0)\cdot\|A_1\|_2\cdot\left(1-e^{-\mu_2(A_0)\tau}\right)}\,, \quad \forall t\in[0,\ T] \tag{30}$$

Debeljkovic et al. (1997.c, 1997.d).

Theorem 4. System (**1**), with initial function (2) is finite time stable w.r.t. $\{\zeta(t),\ \beta,\ \varepsilon,\ \tau,\ J,\ \mu_2(A_0)\neq 0,\ B_1=0\}$ if the following condition is satisfied

$$e^{\mu_2(A_0)t} < \frac{\beta/\alpha}{\phi}\,, \tag{31}$$

$$\begin{aligned}\phi &= \mu^{-1}(A_0)\left(\mu(A_0)+\|A_1\|_2\left(1-e^{-\mu_2(A_0)\tau}\right)\right)\\ &+\mu^{-1}(A_0)\gamma\,\|B_0\|_2\left(1-e^{-\mu_2(A_0)t}\right), \quad \forall t\in J.\end{aligned} \tag{32}$$

where,

$$\gamma = \frac{\varepsilon}{\alpha}, \qquad \mu_2(A_0) = \frac{1}{2}\lambda_{\max}\left(A_0 + A_0^T\right), \tag{33}$$

Debeljkovic et al. (1997.c).

Theorem 5. System (1), with $\mathbf{u}(t-\tau) \equiv 0$, $\forall t$, satisfying initial condition (2) is finite time stable w.r.t. $\{\zeta(t),\ \beta,\ \varepsilon,\ \tau,T,\ \mu_2(A_0)\neq 0,\ B_1=0\}$, if the following condition is satisfied

$$(1+\tau\,\|A_1\|_2)+\gamma\,\|B_0\|_2\,t < \frac{\beta}{\alpha}, \quad \forall t\in J\,, \tag{34}$$

where γ is given with (33), *Debeljkovic et al*. (1997.c).

Theorem 6. Autonomous system (1) with initial function (2) is finite time stable with respect to $\{\sqrt{\alpha},\ \sqrt{\beta},\ \tau,\ T,\ \mu(A_0)=0\}$ if the following condition is satisfied,

$$1+\tau\,\|A_1\|_2 < \sqrt{\beta/\alpha}\,, \ \forall t\in[0,\ T], \tag{35}$$

Debeljkovic et al. (1997.d):

Theorem 7. A system given by (1), with initial function (2) is finite time stable w.r.t. $\{\alpha,\beta,\varepsilon_\Psi,\varepsilon,\tau,J,\mu_2(A_0)\neq 0\}$ if the following condition is satisfied

$$\frac{e^{\mu_2(A_0)t}}{\mu_2(A_0)} < \beta/\alpha\cdot\delta,\quad \forall t\in J\ , \tag{36}$$

where

$$\delta=\mu_2(A_0)+a_1\left(\pi_1\left(1-e^{-\mu_2(A_0)\tau}\right)+\pi_2\left(1-e^{-\mu_2(A_0)t}\right)\right) \tag{36.a}$$

$$\pi_1=1+b_1(\gamma+\gamma_\psi),\qquad \pi_2=\gamma(b_0+b_1), \tag{36.b}$$

$$a_1=\|A_1\|,\ \ b_1=\frac{\|B_0\|}{a_1},\ \ b_0=\frac{\|B_0\|}{a_1}, \tag{36.c}$$

$$\gamma=\frac{\varepsilon}{\alpha},\qquad \gamma_\psi=\frac{\varepsilon_\psi}{\alpha} \tag{36.d}$$

Debeljkovic et al. (1998.a, 1988.b, 1998.d).

The results that will be presented in the sequel enable to check finite time stability of the autonomous system to be considered, namely the system given by (1) and (2), without finding the fundamental matrix or corresponding matrix measure.

Equation (2) can be rewritten in its general form as:

$$\begin{aligned}&\mathbf{x}(t_0+\theta)=\boldsymbol{\psi}_x(\theta)\ -\tau\le\theta\le 0\\&\boldsymbol{\psi}_x(\theta)\in\mathcal{C}[-\tau,\ \ 0]\end{aligned}\ , \tag{37}$$

where t_0 is the initial time of observation of the system (1) and $\mathcal{C}[-\tau,\ 0]$ is a *Banach space* of continuous functions over a time interval of the length τ, mapping the interval $[(t-\tau),\ \ t]$ into $\mathbb{R}^n$ with the norm defined in the following manner:

$$\|\boldsymbol{\psi}\|_{\mathcal{C}}=\max_{-\tau\le\theta\le 0}\|\boldsymbol{\psi}(\theta)\|. \tag{38}$$

It is assumed that the usual smoothness conditions are present so that there is no difficulty with questions of existence, uniqueness, and continuity of solutions with respect to the initial data.

Moreover, one can write:

$$\mathbf{x}(t_0+\theta)=\boldsymbol{\psi}_x(\theta)\,, \tag{39}$$

as well as:

$$\dot{\mathbf{x}}(t_0)=\mathbf{f}(t_0,\boldsymbol{\psi}_x(\theta))\,. \tag{40}$$

Theorem 8. The autonomous system given by (1) with initial function (2) is finite time stable w.r.t. $\{t_0,J,\alpha,\beta\}$ if the following condition is satisfied

$$\left(1+(t-t_0)\sigma_{\max}\right)^2 e^{2(t-t_0)\sigma_{\max}}<\frac{\beta}{\alpha},\qquad \forall t\in J\ , \tag{41}$$

$\sigma_{\max}(\cdot)$ being the largest singular value of the matrix $(\cdot)$, namely

$$\sigma_{\max}=\sigma_{\max}(A_0)+\sigma_{\max}(A_1)\,. \tag{42}$$

Debeljkovic et al. (1998.c).

Proof. In accordance with the known property of norm, one can immediately write

$$\begin{aligned}\|\dot{\mathbf{x}}(t)\|&=\|A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau)\|\\&\le\|A_0\mathbf{x}(t)\|+\|A_1\mathbf{x}(t-\tau)\|\\&\le\|A_0\|\cdot\|\mathbf{x}(t)\|+\|A_1\|\cdot\|\mathbf{x}(t-\tau)\|\end{aligned} \tag{43}$$

where $\|(\cdot)\|$ denotes the induced matrix norm, as well as:

$$\|\mathbf{x}(t-\tau)\|\le\sup_{t-\tau\le t^*\le t}\|\mathbf{x}(t^*)\| \tag{44}$$

Employing the previous inequality, (43) may be written in the following form

$$\begin{aligned}\|\dot{\mathbf{x}}(t)\|&\le\sigma_{\max}(A_0)\|\mathbf{x}(t)\|+\sigma_{\max}(A_1)\sup_{t-\tau\le t^*\le t}\|\mathbf{x}(t^*)\|\\&\le\sigma_{\max}\sup_{t-\tau\le t^*\le t}\|\mathbf{x}(t^*)\|,\qquad t>t_0+\tau\end{aligned} \tag{45}$$

or

$$\|\dot{\mathbf{x}}(t)\|\le\sigma_{\max}\sup_{t-\tau\le t^*\le t}\|\mathbf{x}(t^*)\|,\qquad t>t_{0^+} \tag{46}$$

However,

$$\begin{aligned}\|\dot{\mathbf{x}}(t_{0^+})\|&=\|A_0\boldsymbol{\psi}_x(0)+A_1\boldsymbol{\psi}_x(-\tau)\|\\&\le\sigma_{\max}\|\boldsymbol{\psi}_x\|_{\mathcal{C}}\end{aligned}\ . \tag{47}$$

so, combining (46) and (47), it yields

$$\|\dot{\mathbf{x}}(t_0)\|\le\left(\sup_{t-\tau\le t^*\le t}\|\mathbf{x}(t^*)\|+\|\boldsymbol{\psi}_x\|_{\mathcal{C}}\right)\cdot\sigma_{\max}\,. \tag{48}$$

To obtain the final result one has to integrate (1), so that

$$\begin{aligned}\int_{t_0}^{t}\dot{\mathbf{x}}(t)dt&=\int_{t_{0^+}}^{t}\mathbf{f}(v,\mathbf{x}(v),\mathbf{x}(v-\tau))dv\\&=\int_{t_{0^+}}^{t}(A_0\mathbf{x}(v)+A_1\mathbf{x}(v-\tau))dv\end{aligned}\ . \tag{49}$$

It is obvious

$$\begin{aligned}\|\mathbf{x}(t)\|&\le\|\mathbf{x}(t_0)\|+\left\|\int_{t_0}^{t}(A_0\mathbf{x}(v)+A_1\mathbf{x}(v-\tau))dv\right\|\\&\le\|\mathbf{x}(t_0)\|+\int_{t_0}^{t}\|(A_0\mathbf{x}(v)+A_1\mathbf{x}(v-\tau))dv\|\\&\le\|\mathbf{x}(t_0)\|+\int_{t_0}^{t}\left(\sup_{v-\tau\le t^*\le v}\|\mathbf{x}(t^*)\|+\|\boldsymbol{\psi}_x\|_{\mathcal{C}}\right)\sigma_{\max}\,dv\\&\le\|\mathbf{x}(t_0)\|+\\&\quad+\sigma_{\max}\left((t-t_0)\|\boldsymbol{\psi}_x\|_{\mathcal{C}}+\int_{t_0}^{t}\sup_{v-\tau\le t^*\le v}\|\mathbf{x}(t^*)\|dv\right)\end{aligned} \tag{50}$$

and since:

$$\|\mathbf{x}(t_0)\|=\|\boldsymbol{\psi}_x(0)\|\le\|\boldsymbol{\psi}_x\|_{\mathcal{C}}\,, \tag{51}$$

it is possible to write:

$$\|\mathbf{x}(t)\| \le \|\boldsymbol{\psi}_x\|_{\mathcal{C}} \cdot \left(1+(t-t_0)\sigma_{\max}\right) + \sigma_{\max}\int_{t_0}^{t} \sup_{v-\tau \le t^* \le v} \left\|\mathbf{x}\left(t^*\right)\right\| dv \,. \tag{52}$$

It is clear that:

$$\varphi(t) = \left(1+(t-t_0)\sigma_{\max}\right)\cdot\|\boldsymbol{\psi}_x\|_{\mathcal{C}}\,, \tag{53}$$

is a nondecreasing function, so that:

$$\sup_{t-\tau \le t^* \le t} \left\|\mathbf{x}\left(t^*\right)\right\| \le \varphi(t) + \int_{t_0}^{t} \sup_{v-\tau \le t^* \le v} \left\|\mathbf{x}\left(t^*\right)\right\| dv \tag{54}$$

If a very well known *Bellman-Gronwall* lemma, *Hale* (1971), is applied, one can get:

$$\|\mathbf{x}(t)\| \le \sup_{t-\tau \le t^* \le t} \left\|\mathbf{x}\left(t^*\right)\right\| \le \left(1+(t-t_0)\sigma_{\max}\right)\cdot\|\boldsymbol{\psi}_x\|_{\mathcal{C}}\, e^{(t-t_0)\sigma_{\max}}\,, \tag{55}$$

or equivalently

$$\|\mathbf{x}(t)\|^2 \le \left(1+(t-t_0)\sigma_{\max}\right)^2\cdot\|\boldsymbol{\psi}_x\|_{\mathcal{C}}^2\, e^{2(t-t_0)\sigma_{\max}}\,. \tag{56}$$

According to (39), one can get

$$\|\mathbf{x}(t)\|^2 \le \alpha\left(1+(t-t_0)\sigma_{\max}\right)^2 e^{2(t-t_0)\sigma_{\max}}\,, \tag{57}$$

and finally, applying the basic condition of the *Theorem* 8, namely (41), it is obvious that:

$$\|\mathbf{x}(t)\|^2 < \beta, \qquad \forall t \in J \tag{58}$$

that had to be proved, Q.E.D, *Debeljkovic et al*. (1998.c).

Theorem 9. The autonomous system given by (**1**) with initial function (2) is finite time stable w.r.t. $\{t_0, J, \alpha, \beta\}$ if the following condition is satisfied

$$e^{2(t-t_0)\cdot\sigma_{\max}} < \frac{\beta}{\alpha}, \qquad \forall t \in J\,, \tag{59}$$

where $\sigma_{\max}(\cdot)$ is defined in (42), *Debeljkovic et al*. (1998.c).

Note 1. In the case when in the *Theorem* 9

$$A_1 = 0\,, \tag{60}$$

e.g. A_1 is the null matrix, we have the result similar to that presented in *Angelo* (1974).

Fractional order time delay systems

Itroduction

Recently, there have been some advances in the control theory of fractional differential systems for stability questions, *Matignon* (1994).

A fractional order means that the delay differential equation order is non-integer. However, for fractional order dynamic systems, it is difficult to evaluate the stability by simply examining its characteristic equation either by finding its dominant roots or by using other algebraic methods.

At the moment, a direct check of the stability of fractional order systems using polynomial criteria (e.g.,Routh's or Jury's type) is not possible, because the characteristic equation of the system is, in general, not a polynomial but a pseudopolynomial function of fractional powers of the complex variable *s*.

Thus there remain only geometrical methods of the complex analysis based on the so-called argument principle (e. g. Nyquist type) which can be used for the stability check in the BIBO sense (bounded- input bounded-output).

Also, for linear fractional differential systems of finite dimensions in a state-space form, both internal and external stabilities are investigated by *Matignon* (1996,1998).

An analytical approach was suggested by *Chen*, *Moore* (2002.a) who considered the analytical stability bound using *Lambert function* W for a class of second-order ordinary delay differential equations (DDE) and case of the linear fractional–order (DDE) *Chen*, *Moore* (2002.b).

On the other side, the approach which will be presented in the sequel does not demand any solving of the delay differential equation (DDE) but it is based on forming the corresponding criteria (criterion of practical stability and finite time stability) in which the basis matrices of the system A_0, A_1 exclusively appear, where the basis matrices may contain tuning parameters which inflence the stability of the system more obviously than in both papers *Chen*, *Moore* (2002.a, 2002.b). For the first time, a finite time stability test procedure is proposed for linear and nonlinear autonomous time-invariant delay fractional order systems.

Here, we examine the problem of sufficient conditions that enable system trajectories to stay within the *a priori* given sets for the particular class of linear and perturbed nonlinear autonomous fractional order time-delay systems.

Preliminaries on fractional differential systems

Although the fractional order calculus is a 300-years-old topic, the theory of fractional-order derivative was developed mainly in the 19th century.

Even though the idea of fractional order operators is as old as the idea of the integer order ones is, it has been in last decades when the use of fractional order operators and operations has become more and more popular among many research areas.

The theoretical and practical interest of these operators is nowadays well established, and its applicability to science and engineering can be considered as emerging new topics.

Fractional differential equations (FDEs) have been the focus of many studies due to their appearance in various fields such as physics, chemistry, and engineering *Torvik*, *Bagley* (1984), *Mainardi* (1996) and Podlubny (1999).

Moreover, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes.

The mathematical modelling and simulation of systems and processes, based on the description of their properties in terms of fractional derivatives, naturally leads to differential equations of fractional order and to necessity to analyse and solve such equations.

Also, due mainly to the works *Oldham* and his co-authors *Oldham* (1972), *Oldham*, *Spanier* (1974), electrochemistry is one of those fields in which fractional-order integrals and derivatives have a strong position and bring practical results where the idea of using a half-order fractional integral of the current ${}_0D_t^{-1/2}i(t)$ was proposed *Oldham* (1972).

Moreover, *Matignon* (1994) gives the model of the pressure wave transmission through an air-filled tube with viscothermic perturbation and discusses the stability of the transfer function *as example of a fractional delay system*.

Recently, in their paper, *Hotzel*, Fliess (1998) considered the BIBO stability and control of finite-dimensional fractional time delay systems.

The use of fractional-order derivatives and integrals as boundary controls, i.e.the *Crone* controller design of a complex order of an active suspension systems have been recently studied by *Lannuse et al.* (2003).

These systems give rise to the same type modules as linear fractional time delay systems.

Also, there are new results in colloid and interface science where the general fractional order model of liquid-liquid interfaces is considered.

A new theory of electroviscoelasticity describes the behavior of electrified liquid-liquid interfaces in fine dispersed systems, and is based on a new constitutive model of liquids, *Spasic et al.*(2005).

Also, taking into account a small transport time-delay τ, the electromagnetic oscillation of the "continuum" particle can be obtained by the linear time delay fractional order of differential equation, *Spasic*, *Lazarevic*, (2004).

Linear autonomous fractional order time delay systems

Also, it is shown that the fractional-order time delay state space model of PD^{α} control of the *Newcastle robot* can be presented by the linear time delay fractional order of differential equation in the state space form *Lazarevic* (2006)

$$\frac{d^{\alpha}\mathbf{x}(t)}{dt^{\alpha}} = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau)$$
$$ {}_0D_t^{\alpha}\left(f(t)\right) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f^{(1)}(\tau)}{(t-\tau)^{\alpha}}\,d\tau, \quad 0<\alpha<1 \tag{61.a}$$

and with the associated function of the initial state

$$\mathbf{x}(t) = \boldsymbol{\psi}_x(t), \quad -\tau \le t \le 0, \tag{61.b}$$

for $0<\alpha<1$ and where $\Gamma(\cdot)$ is the well known *Euler`s gamma function*.

Also, for a case of multiply time delays in state fractional order systems the time delay system can be presented as

$$\frac{d^{\alpha}\mathbf{x}(t)}{dt^{\alpha}} = A_0\mathbf{x}(t) + \sum_{i=1}^{n} A_i\mathbf{x}(t-\tau_i), \quad 0 \le \tau_1 < \tau_2 < \tau_3 < \ldots < \tau_i < \ldots < \tau_n = \Delta \tag{62.a}$$

and with the associated function of the initial state:

$$\mathbf{x}(t) = \boldsymbol{\psi}_x(t), \quad -\Delta \le t \le 0. \tag{62.b}$$

Stabilty definitions

Definition 6. System (**61.a**) satisfying initial condition (61.b) is finite stable w.r.t $\{t_0, J, \delta, \varepsilon, \tau\}$, $\delta<\varepsilon$ if and only if

$$\left\|\boldsymbol{\psi}_x(t)\right\|_{\mathcal{C}} < \delta, \quad -\tau \le t \le 0,$$

implies

$$\left\|\mathbf{x}(t)\right\| < \varepsilon, \ \forall t \in J,$$

where δ is a real positive number and $\varepsilon \in \mathbb{R}$, $\delta<\varepsilon$, *Lazarevic*, *Debeljkovic* (2003).

Definition 7. System (**62.a**) satisfying initial condition (62.b) is finite stable w.r.t. $\{t_0, J, \delta, \varepsilon, \Delta\}$, $\delta<\varepsilon$ if and only if

$$\left\|\boldsymbol{\psi}_x(t)\right\|_{\mathcal{C}} < \delta, \quad \forall t \in J_{\Delta}, \quad J_{\Delta} = [-\Delta, \ 0] \in \mathbb{R}$$

implies:

$$\left\|\mathbf{x}(t)\right\| < \varepsilon, \quad \forall t \in J,$$

where δ is a real positive number and $\varepsilon \in \mathbb{R}$, $\delta<\varepsilon$, *Lazarevic*, *Debeljkovic* (2003).

Stabilty theorems

Theorem 10. System (61.a) satisfying initial condition (61.b) is finite time stable w.r.t. $\{t_0, J, \delta, \varepsilon, \tau\}$, $\delta<\varepsilon$, if the following condition is satisfied

$$\left(1+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sigma_{\max}^{A}(t-t_0)\right)\cdot e^{\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sigma_{\max}^{A}(t-t_0)} \le \frac{\varepsilon}{\delta}, \quad \forall t \in J = [t_0, \ (t_0+T)] \tag{63}$$

where $\sigma_{\max}(\cdot)$ being the largest singular value of the matrix $(\cdot)$, namely

$$\sigma_{\max}^{A} = \sigma_{\max}(A_0) + \sigma_{\max}(A_1), \tag{64}$$

and $\Gamma(\cdot)$ is the Euler's gamma function, *Lazarevic*, *Debeljkovic* (2003).

Theorem 11. The system given by (62.a) satisfying initial condition (62.b) is finite time stable w.r.t. $\{\delta, \varepsilon, \Delta, t_0, J\}$, $\delta<\varepsilon$, if the following condition is satisfied

$$\left(1+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sigma_{\max}^{A}(t-t_0)\right)\cdot e^{\frac{T^{\alpha}}{\Gamma(\alpha+1)}\sigma_{\max}^{A}(t-t_0)} \le \frac{\varepsilon}{\delta}, \quad \forall t \in J = [t_0, \ (t_0+T)]$$

where $\sigma_{\max}(\cdot)$ being the largest singular value of the matrix A_i, $i = 0,1,2\ldots,n$, namely

$$\sigma_{\max}^{A} = \sigma_{\max}(A_0) + \sigma_{\max}(A_1) + \ldots + \sigma_{\max}(A_n), \tag{65}$$

Lazarevic, *Debeljkovic* (2003).

Nonlinear autonomous fractional order time delay systems

Now we consider a class of fractional non-linear autonomous system with time delay described by the state space equation:

$$\frac{d^{\alpha}\mathbf{x}(t)}{dt^{\alpha}} = \left(A_0 + \Delta A_0\right)\mathbf{x}(t) + \left(A_1 + \Delta A_1\right)\mathbf{x}(t-\tau) + f_N(\mathbf{x}(t)), \tag{66.a}$$

with the associated function of the initial state:

$$\mathbf{x}(t) = \boldsymbol{\psi}_X(t), \quad -\tau \le t \le 0, \tag{66.b}$$

and the vector functions $f_N(\mathbf{x}(t))$ satisfied

$$\| f_N(\mathbf{x}(t)) \| \le c_0 \| \mathbf{x}(t) \|, \quad t \in [0, \infty[, \tag{67}$$

where $c_0 \in \mathbb{R}^+$ is a known real positive number, *Lazarevic, Debeljkovic* (2007).

Theorem 12 Nonlinear perturbed system (66.a) satisfying initial condition (66.b) and the vector functions $f_N(\mathbf{x}(t))$ satisfied (67) is finite time stable w.r.t. $\{\delta, \varepsilon, t_0, J\}$, $\delta < \varepsilon$, if the following condition is satisfied:

$$\left(1 + \frac{\mu_p (t-t_0)^\alpha}{\Gamma(\alpha+1)}\right) e^{\frac{\mu_p (t-t_0)^\alpha}{\Gamma(\alpha+1)}} \le \varepsilon / \delta, \quad \forall t \in J, \tag{68}$$

where

$$\begin{aligned} \mu_{Aoco} &= \sigma_{Ao} + \gamma_{\Delta A_0} + c_0 \\ \sigma_{A1\Delta} &= \sigma_{A1} + \gamma_{\Delta A_1} \\ \mu_p &= \mu_{Aoco} + \sigma_{A1\Delta} \\ \gamma_{\Delta A_0} &= \|\Delta A_0\|, \quad \gamma_{\Delta A_1} = \|\Delta A_1\| \end{aligned}, \tag{69}$$

and $\Gamma(\cdot)$ the Euler's gamma function, *Lazarevic, Debeljkovic* (2007).

Conclusion

In the circumstances when it is possible to establish a suitable connection between the fundamental matrices of a linear time delay system and a non-delay system, some of these results enable an efficient procedure for testing finite time stability of a particular class of linear time delay systems.

The matrix measure has been widely used in the literature dealing with stability and asymptotic stability of time-delay systems. This approach has been used here to develop some results which have an evident advantage over those derived earlier. In that sense, delay dependent criteria expressed by simple inequalities have been derived yielding sufficient conditions of the non-Lyapunov stability of the system considered. Moreover, a new theorem has been presented, enabling the application of a very well-known *Bellman-Gronwall* lemma for time delay systems.

The same idea has been used to establish analogous results for fractional order time delay systems.

APPENDIX A

Some additional result

Lemma 1. Let $Q(t)$ be an $n \times n$ characteristic matrix for autonomous system (1) with initial function (2), also continuous and differentiable in $[0, \tau]$ and zero elsewhere.

Define the following vector:

$$\mathbf{y}(t) = \mathbf{x}(t) + \int_0^\tau Q(t)\mathbf{x}(t-\theta)\, d\theta, \tag{A.1}$$

where the matrix $Q(t)$ satisfy the following matrix equation:

$$\dot{Q}(\theta) = (A_0 + Q(0)) \cdot Q(\theta) Q, \quad \theta \in [0, \tau], \tag{A.2}$$

with the boundary value:

$$Q(\tau) = A_1, \tag{A.3}$$

Lee, *Diant* (1981).

If

$$V(\mathbf{y}(t)) = \mathbf{y}^T(t)\mathbf{y}(t), \tag{A.4}$$

is the aggregation function for system (1), then

$$\dot{V}(\mathbf{y}(t)) = \mathbf{y}^T(t)(-R)\mathbf{y}(t), \tag{A.5}$$

where:

$$-R = (A_0 + Q(0))^T + (A_0 + Q(0)), \tag{A.6}$$

The proof is omitted, for the sake of brevity and can be found in *Lee* and *Diant* (1981).

Theorem A.1 If λ_M is the maximal eigenvalue of the matrix $(-R)$ being defined by (A.6), then

$$\int_0^\tau \| Q(\theta)\mathbf{x}(t-\theta) \| \, d\theta \le \| Q(0) \| \int_0^\tau e^{\frac{\lambda_M}{2}\theta} \| \mathbf{x}(t-\theta) \| \, d\theta \tag{A.7}$$

Debeljkovic et al. (2001).

APPENDIX B

Basic mathematical tools for fractional calculus

The fractional integro-differential operators (fractional calculus) represent a generalization of integration and derivation to non-integer order (fractional) operators. The idea of fractional calculus has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695.

The theoretical and practical interest of these operators is nowadays well established, and its applicability to science and engineering can be considered as emerging new topics. Even if they can be thought of as somehow ideal, they are, in fact, useful tools for both the description of a more complex reality, and the enlargement of the practical applicability of the common integer order operators.

At first, one can generalize the differential and integral operators into one fundamental D_t^α operator t which is known as fractional calculus *Podlubny* (1999), *Oldham, Spanier* (1974)

$${}_aD_t^\alpha = \begin{cases} \dfrac{d^\alpha}{dt^\alpha} & \operatorname{Re}\alpha > 0, \\ 1 & \operatorname{Re}\alpha = 0 \\ \int_a^t (d\tau)^{-\alpha} & \operatorname{Re}\alpha < 0. \end{cases}, \tag{B.1}$$

The two definitions generally used for the fractional differintegral are the Grunwald definition and the Riemann-Liouville (RL) definition.

The Grunwald definition is given here:

$${}_aD_t^\alpha f(t) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{j=0}^{[(t-a)/h]} (-1)^j \binom{\alpha}{j} f(t-jh) \tag{B.2}$$

where a, t are the limits of the operator and [x] means the integer part of $\mathbf{x}$.

The RL definition of the fractional derivative is given by the expression:

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \qquad \text{(B.3)} $$

for $(n-1<\alpha<n)$ and where $\Gamma(.)$ is the well-known Euler's gamma function as follows:

$$ \Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt, \quad z = x+iy, \quad \Gamma(z+1) = z\Gamma(z), \text{(B.4)} $$

Fractional differentiation is a linear operator

$$ D^{\alpha}\left(\lambda f(t) + \mu g(t)\right) = \lambda D^{\alpha} f(t) + \mu D^{\alpha} g(t), \quad \text{(B.5)} $$

where D^{α} denotes any mutation of the fractional differentiation.

Closely related to fractional-order differentiation is fractional order integration i.e the Riemann-Liouville fractional integral is defined as

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad \alpha > 0 \qquad \text{(B.6)} $$

The property of the Riemann-Liouville fractional derivative is that for $\alpha > 0$ and $t > a$

$$ {}_aD_t^{\alpha}\left({}_aD_t^{-\alpha} f(t)\right) = f(t), \qquad \text{(B.7)} $$

which generalizes an analogous property of integer derivative and integrals.

For convenience, the Laplace domain is usually used to describe the fractional integro-differential operation for solving engineering problems.

The formula for the Laplace transform of the RL fractional derivative has the form

$$ \int_0^{\infty} e^{-st}\, {}_0D_t^{\alpha} f(t)\, dt = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^k\, {}_0D_t^{\alpha-k-1} f(t)_{|t=0} \quad \text{(B.8)} $$

Also, there is another definition of the fractional differintegral introduced by *Caputo* (1967).

Caputo's definition can be written as

$$ {}_a^CD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad n-1<\alpha<n, \text{(B.9)} $$

and its Laplace transform

$$ \int_0^{\infty} e^{-st}\, {}_0D_t^{\alpha} f\ (t)\, dt = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \qquad \text{(B.10)} $$
$$ n-1<\alpha<n $$

which implies that all the initial values of the considered equation are presented by a set of only classical integer-order derivatives (evaluated at the initial time) with a well-known physical meaning.

In view of our objective to provide a suitable mathematical treatment of fractional derivative phenomena, the following notation is introduced:

$$ \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \begin{cases} f^{(n)}(t) & if \quad \alpha = n \in \mathbb{N} \\ {}_a^CD_t^{\alpha} f(t) & if \quad n-1<\alpha<n. \end{cases} \qquad \text{(B.11)} $$

We also note that in the limit cases the definition (B.9) yields ${}_a^CD_t^n f(t) = f^{(n)}(t)$ (using delta distribution), Mainardi (1996).

The relation between the two fractional derivatives, Riemann-Liouville and Caputo, is:

$$ {}_a^CD_t^{\alpha} f(t) = {}_a^{RL}D_t^{\alpha}\left(f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{t^k}{k!} \right). \quad \text{(B.12)} $$

The Caputo and Riemann-Liouville formulation coincide when the initial conditions are zero.

Also, *Lorenzo*, *Hartley* (1998) considered variable prehistories of $\mathbf{x}(t)$ in $t<0$, and its effects were taken into account for the fractional derivative in terms of the initialization function.

Moreover, using the short memory principle *Podlubny* (1999) and taking into account (2) one can obtain a correct initial function where it is assumed that there is no difficulty with questions of continuity of solutions with respect to the initial data (function).

References

[1] ANGELO,H.: *Linear Time Varying Systems*, Allyn and Bacon, Boston, 1974.

[2] BAJIĆ,V.B.: *Generic Stability and Boundedness of Semistate Systems*, IMA Journal of Mathematical Control and Information, 5 (2), (1988)103–115.

[3] BAJIĆ,V.B.: *Generic Concepts of System Behavior and the Subsidiary Parametric Function Method*, SACAN, Link Hills, RSA, 1992.

[4] CAPUTO,M,: *Linear models of dissipation whose Q is almost frequency independent*, Geophys. J. Royal Astronom .Soc., 13, (1967) 529-539.

[5] CHEN,J., XU,D., SHAFAI,B.: *On Sufficient Conditions for Stability Independent of Delay*, IEEE Trans. Automat. Control AC-40 (9), (1995) 1675-1680.

[6] CHEN,Y.Q., MOORE,K.L.: *Analytical stability bound for delayed second order systems with repeating poles using Lambert function W*, Automatica, 38 (5), (2002.a) 891-895.

[7] CHEN,Y.Q., MOORE,K.L.: *Analytical Stability Bound for a Class of Delayed Fractional-Order Dynamic Systems*, Nonlinear Dynamics, Vol.29, (2002.b)191-200.

[8] COPPEL,W.A.: S*tability and Asymptotic Behavior of Differential Equations*, Boston: D.C. Heath, 1965.

[9] DEBELJKOVIĆ,D.LJ., OWENS,D.H.: *On Practical Stability*, Proc. MELECON Conference, Madrid (Spain), (1985) 103–105.

[10] DEBELJKOVIĆ,D.LJ., NENADIĆ,Z.LJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B.: *On Practical and Finite-Time Stability of Time-Delay Systems*, Proc. ECC 97, Brussels (Belgium), (1997.a) 307–311.

[11] DEBELJKOVIĆ,D.LJ., NENADIĆ,Z.LJ., KORUGA,DJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B.: *On Practical Stability of Time-Delay Systems: New Results*, Proc. 2nd ASCC 97, 22 – 25 July, Seoul (Korea) (1997.b) 543–546.

[12] DEBELJKOVIĆ,D.LJ., LAZAREVIĆ,M.P., KORUGA,DJ., TOMAŠEVIĆ,S.: *On Practical Stability of Time Delay System Under Perturbing Forces*, Proc. AMSE 97, Melbourne (Australia), (1997.c) 442–446.

[13] DEBELJKOVIĆ,D.LJ., NENADIĆ,Z.LJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B.: *On the Stability of Linear Systems with Delayed State Defined over Finite Time Interval*, Proc. CDC 97, San Diego, California (USA), (1997.d).2771–2772.

[14] DEBELJKOVIĆ,D.LJ., LAZAREVIĆ,M.P., KORUGA,DJ.: *Finite Time Stability for the Metal Strips Cold Rolling*, Proc. ASI (International Workshop on Automation in Steel Industry), Kyongju (Korea), July 16 – 18, (1997.e) 233 – 238.

[15] DEBELJKOVIĆ, D. LJ., KORUGA, DJ., MILINKOVIĆ, S. A., JOVANOVIĆ,M.B., JACIĆ, LJ. A.: *Further Results on Non-Lyapunov Stability of Time Delay Systems*, Proc. MELECON 98, Tel-Aviv (Israel), May 18-20, Vol.1, (1998.a) 509 – 512.

[16] DEBELJKOVIĆ, D. LJ., KORUGA, DJ., MILINKOVIĆ, S. A., JOVANOVIĆ,M.B.: *Non - Lyapunov stability analysis of linear time delay systems*, Preprints DYCOPS 5, 5th IFAC Symposium on Dynamics and Process Systems, Corfu (Greece), June 8 - 10, (1998.b) 549 - 553.

[17] DEBELJKOVIĆ, D. LJ., LAZAREVIĆ, M. P., MILINKOVIĆ, S. A., JOVANOVIĆ,M.B.: *Finite time stability analysis of linear time delay systems: Bellman - Gronwall approach*, Preprints IFAC Workshop on Linear Time Delay Systems, Grenoble (France), July 6-7, (1998.c) 107-112.

[18] DEBELJKOVIĆ, D. LJ., MILINKOVIĆ, S. A., JOVANOVIĆ, M. B., JACIĆ,LJ.A., KORUGA,DJ.: *Further results on Non - Lyapunov stability of time delay systems*, Preprints 5th IFAC Symposium on Low Cost Automation, Shenyang (China), September 8 - 10 (1998.d), TS13 6 - 10.

[19] DEBELJKOVIĆ,D.LJ., KORUGA,DJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B., JACIĆ,LJ.A.: *Further results on Non - Lyapunov stability of linear systems with delayed state*, Proc. XII CBA - Brazilian Automatic Control Conference, Uberlandia (Brazil), September 14 - 18 Vol. IV, (1998.e) 1229 – 1233.

[20] DEBELJKOVIĆ,D.LJ., LAZAREVIĆ,M.P., KORUGA,DJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B.: *Further results on the stability of linear nonautonomous systems with delayed state defined over finite time interval*, Proc. ACC 2000, Chicago Illinois (USA), June 28 - 30, (2000.a) 1450 - 1451.

[21] DEBELJKOVIĆ,D.LJ., LAZAREVIĆ,M.P., KORUGA,DJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B.: *Further results on the stability of linear nonautonomous systems with delayed state defined over finite time interval*, Proc. APCCM (The 4th Asia - Pacific Conference on Control and Measurements), 9 - 12 July Guilin (China) (2000.b) D.9.

[22] DEBELJKOVIĆ,D.LJ., LAZAREVIĆ,M.P., KORUGA, DJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B.: *Further results on the stability of linear nonautonomous systems with delayed state defined over finite time interval and application to different chemical processes*, CHISA 2000, 27 - 31 Avgust, Praha (Czech Republic), (2000.c) CD – Rom

[23] DEBELJKOVIĆ,D.LJ., LAZAREVIĆ,M.P., KORUGA, DJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B., JACIĆ,LJ.A.: *Further Results on Non – Laypunov Stabilty of the Linear Nonautonomous Systems with Delayed State*, Facta Universitatis, (YU), MACR, 3, No.11, (2001) 231 - 243.

[24] DESOER,C.A., VIDYSAGAR,M.: *Feedback Systems: Input - Output Properties*, Academic Press, New York 1975.

[25] FANG,H.H., HUNGSARD,H.J.: *Stabilization of Nonlinear Singularly Perturbed Multiple Time Delay Systems by Dither*, Journal of Dynamic Systems, Measurements and Control, Vol. 118, March (1996) 177 – 181.

[26] GRUJIĆ,LJ.T.: *On Practical Stability*, 5th Asilomar Conf. on Circuits and Systems, (1971) 174–178.

[27] GRUJIĆ,LJ.T.: *Non-Lyapunov Stability Analysis of Large-Scale Systems on Time-Varying Sets*, Int. J. Control, 21 (3), (1975.a) 401–415.

[28] GRUJIĆ,LJ.T.: *Practical Stability with Settling Time on Composite Systems*, Automatika (Yu), T. P. 9, (1975.b) 1–11.

[29] GRUJIĆ,LJ.T.: *Uniform Practical and Finite-Time Stability of Large-Scale Systems*, Int. J. System Science, 6 (2), (1975.c) 181–195.

[30] HALE,J.K.: *Functional Differential Equations*, Springer, New York, 1971.

[31] HMAMED,A.: *On the Stability of Time Delay Systems: New Results*, Int. J. Control 43 (1), (1986) 321–324.

[32] HOTZEL,R., FLIESS, M.: *On Linear Systems with a Fractional Derivation: Introductory Theory and Examples*, Mathematics and Computers in Simulations, 45, (1998) 385-395.

[33] LAM, L., WEISS, L.: *Finite Time Stability with Respect to Time-Varying Sets*, J. Franklin Inst., 9, (1974) 415–421.

[34] LANUSSE, P., POINOT, T., COIS, O., OUSTALOUP, A., TRIGEASSOU, J.: *Tuning of an Active Suspension System using a Fractional Controller and a Closed-Loop Tuning*, 11th Proc. of ICAR 2003, Portugal (2003)258-263.

[35] LA SALLE, LEFSCHET, S.: *Stability by Lyapunov's Direct Method*, Academic Press, New York, 1961.

[36] LASHIRER, A. M., STORY, C.: *Final-Stability with Applications*, J. Inst. Math. Appl., 9, (1972) 379–410.

[37] LAZAREVIĆ, P. M.: *Finite Time Stability Analysis of PD^{α} Fractional Control of Robotic Time Delay Systems*, Journal of Mechanics Research Communications, 33, Iss.2, (2006) 269-279.

[38] LAZAREVIĆ, P. M., DEBELJKOVIĆ,D. LJ., NENADIĆ, Z. LJ., MILINKOVIĆ,S.A.: *Finite time stability of time delay systems*, IMA Journal of Mathematical Control and Information, Vol.17, No.3, (2000) 101 - 109.

[39] LAZAREVIĆ, P. M., DEBELJKOVIĆ,D. LJ.: *Finite time stability analysis of linear autonomous fractional order systems with delayed state*, Preprints of IFAC Workshop on Time delay systems, INRIA, Rocquencourt, Paris, (France), September 8 – 10., (2003)., CD-Rom.

[40] LAZAREVIĆ, P. M., DEBELJKOVIĆ, D. LJ.: *Robust Finite Time Stability of Perturbed Nonlinear Autonomous Fractional Order Time Delay Systems*, Preprints of IFAC – The 5-th IFAC DECOM-TT Workshop Cesme - Izmir, Republic of Turkey, May 17-20, (2007), 375-381

[41] LAZAREVIĆ, P. M., DEBELJKOVIĆ, D. LJ.: *Finite Time Stability Analysis of Linear Autonomous Fractional Order Systems with Delayed State*, Asian Journal of Control, Vol.7, No.4, (2005) 440–447.

[42] LEE, T. N., DIANT, S.: *Stability of Time Delay Systems*, IEEE Trans. Automat. Control AC 31 (3) (1981) 951-953.

[43] LEE, E. B., LU, W. S., WU, N. E.: *A Lyapunov Theory for Linear Time Delay Systems*, IEEE Trans. Automat. Control AC-31 (3), (1986) 259–262.

[44] LEE, T. N., DIANT, S.: *Stability of Time Delay Systems*, IEEE Trans. Automat. Control AC-26 (4), (1981) 951–953.

[45] LORENZO, C., HARTLEY,T.: *Initialization, conceptualization and application*, NASA TP -1998-208415, December 1998.

[46] MAINARDI,F.: *Fractional Relaxation-Oscillation and Fractional Diffusion-Wave Phenomena*, Chaos,Solitons & Fractals Vol.7 No.9, (1996) 1461-1477.

[47] MATIGNON,D.: *Representations en variables d`eta de modeles de guides d`ondes avec derivation fractionnaire*, These de doctorat, Universite Paris-Sud,Orsay, 1994.

[48] MATIGNON,D.: *Stability result on fractional differential equations with applications to control processing*, In IMACS - SMC Proceeding, July, Lille, France, (1996)963- 968.

[49] MATIGNON,D.: *Stability properties for generalized fractional differential systems*, ESAIM: Proceedings, 5 (1998) 145 – 158.

[50] MICHEL,A.N.: *Stability, Transient Behavior and Trajectory Bounds of Interconnected Systems*, Int. J. Control, 11 (4), (1970) 703–715.

[51] MORI,T.: *Criteria for Asymptotic Stability of Linear Time Delay Systems*, IEEE Trans. Automat. Control, AC-30, (1985) 158–161.

[52] MORI,T., FUKUMA,N., KUWAHARA,M.: *Simple Stability Criteria for Single and Composite Linear Systems with Time Delays*, Int. J. Control 34 (6), (1981) 1175–1184.

[53] MOVLJANKULOV,H., FILATOV,A.: *Ob odnom približennom metode postroenija rešenii integral`nyh uravneii*, Tr.In-ta kibern. s VC AN UZSSR. Voprosy vyčislitel`noi i prikladnoi matematiki, Vyp. 12, Taškent (1972).

[54] NENADIĆ, Z. LJ., DEBELJKOVIĆ,D. LJ., MILINKOVIĆ, S. A.: *On Practical Stability of Time Delay Systems*, Proc. ACC97, Albuquerque, New Mexico (USA), (1997) 3235–3236.

[55] OLDHAM, K. B.: *A Signal – Independent Electro Analytical Method*, Anal. Chem., Vol 44, No 1, (1972) 196-198.

[56] OLDHAM, K. B., SPANIER, J.: *The Fractional Calculus*, Academic Press, New York, 1974.

[57] OWENS, D. H., DEBELJKOVIĆ, D. LJ.: *On Non-Lyapunov Stability of Discrete Descriptor Systems*, Proc. 25th Conference on Decision and Control, Athens (Greece), (1986) 2138–2139.

[58] PODLUBNY, I.: *Fractional Differential Equations*, Academic Press, San Diego 1999.

[59] SPASIĆ, A. M., LAZAREVIĆ, M. P.: *Electro visco-elasticity of Liquid-Liquid Interfaces: fractional-order model (New constitutive models of liquids)*, Lectures in Rheology, Department of Mechanics, Faculty of Mathematics, University of Belgrade, (2004).

[60] SPASIĆ,A. M., LAZAREVIĆ, M. P., KRSTIĆ, D.: *Chapter: Theory of electroviscoelasticity*, 371-394 in Finely Dispersed Particles: Micro-Nano -and Atto-Engineering, Dekker - CRC Press - Taylor & Francis, Florida, (2005) 950.

[61] TORVIK,P.J., BAGLEY,R.L.: *On the appearance of the fractional derivatives in the behaviour of real materials*, J. Appl. Mech. (Trans

SME), 51, (1984) 294-298.

[62] WEISS,L., INFANTE, E. F.: *On the Stability of Systems Defined over Finite Time Interval*, Proc. National Acad. Science, 54 (1), (1965) 44–48.

[63] WEISS,L., INFANTE,E. F.: *Finite Time Stability under Perturbing Forces on Product Spaces*, IEEE Trans. Automat. Cont., AC-12 (1), (1967) 54–59.

[64] WILLIAMS, J., OTTO, E.: *A Generalized Chemical Processing Model for the Investigation of Computer Model*", AIEE Transaction Commun. Electron, Vol.79 (1960) 458-473.

[65] ZAVAREI, M., JAMSHIDI, M.: *Time - Delay Systems: Analysis*, Optimization and Applications, North-Holland, Amsterdam 1987.

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Stabilnost linearnih kontinualnih sistema sa čistim vremenskim kašnjenjem na konačnom vremenskom intervalu: Pregled rezultata

Ovaj rad daje detaljan pregled radova i rezultata mnogih autora na polju Neljapunovske stabilnosti (stabilnost na konačnom vremenskom intervalu, tehnička stabilnost, praktična stabilnost, krajnja stabilnost) posebne klase linearnih kontinualnih sistema sa čistim vremenskim kašnjenjem u stanju.
Ovaj pregled obuhvata period posle 1995.god., pa sve do današnjih dana i ima snažnu nameru the predstavi glavne koncepte i doprinose koji su stvoreni u pomenutom periodu u celom svetu a koji su publikovani u respektabilnim međunarodnim časopisima ili prezentovani na prestižnim međunarodnim konferencijama.

Ključne reči: **kontinualni sistem, linearni sistem, stabilnost sistema, Neljapunovska stabilnost, sistem sa kašnjenjem, sistem na konačnom vremenskom intervalu.**

Устойчивость линейных непрерывных систем со чистым временным запаздыванием на конечном временном интервале: Обзор результатов

Настоящая работа даёт подробный обзор работ и результатов многих авторов в области исследования неляпуновой устойчивости (устойчивость на конечном временном интервале, техническая устойчивость, практическая устойчивость, конечная устойчивость) особого класса линейных непрерывных систем со чистым временным запаздыванием в состоянии.
Этот обзор результатов охватывают период после 1995-ого года до сих пор и у него выразительное намерение представить основные концепции и вклады в этой области созданные в целом мире в упомянутом периоде и опубликованные в передовых международных журналах или показаны и представлены на выдающихся международных конференциях.

Ключевые слова: непрерывная система, линейная система, устойчивость системы, неляпуновая устойчивость, система со запаздыванием, система на конечном временном интервале.

Stabilité des systèmes linéaires continus à délai temporel pur chez l'intervalle temporelle finie: présentation des résultats

Ce travail donne un compte-rendu des travaux et des résultats de nombreux auteurs dans le domaine de la stabilité de non Lyapunov (stabilité chez l'intervalle temporelle finie, stabilité technique, stabilité pratique, stabilité finale) de classe particulière des systèmes linéaires continus à délai temporel pur. Ce bulletin recouvre la période depuis 1995 jusqu'à nos jours et a l'intention de présenter les concepts principaux et les contributions réalisés au cours de la période citée dans le monde entier et publiés dans des revues internationales renommées ou bien présentés lors des conférences internationales de prestige.

Mots clés: **système continu, système linéaire, stabilité du système, stabilité non Lyapunov, système à délai, système chez l'intervalle temporelle finie.**