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Models of Technological Processes on the Basis of Vibro-impact Dynamics

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The paper presents the series of the models of technological processes on the basis of vibro-impact dynamics for typical technical machines and systems. The analysis of the vibro-impact motions for the operation of some machines is shown for some specific examples, for which dynamic models are made. The given examples for the abstraction into characteristic vibro-impact models of real systems are: vibro-impact hammer, hand strike - rotary hammer, printer and vibro-rammer (mechanical vibrator). For each of the listed models of characteristic vibro-impact systems there are differential equations derived, representing the vibro-impact process dynamics followed by the appropriate initial and impact conditions and the conditions of elongation restrictions as well. The specifics vibro-impact dynamics for different properties of standard light elements (linear, nonlinear elasticity, high-elasticity, hereditary and creeping properties) included in the basic dynamic oscillator model is investigated. The basics of the energy transfer between the elements of the vibro-impact system analysis are presented. For some typical cases, the phase portraits and the energy curves are shown, representing the original results of the authors.

Key words: **impact, impact effect, vibrations, vibro-impact dynamics, process dynamics, differential equations.**

Introduction

INVESTIGATION of the Vibro-Impact System dynamics and nonlinear phenomena, taking into account influence of some discontinuity, is the aim of numerous researchers from all over the world. The aims of the international conferences on vibro-impact systems, the next one in China (ICoVIS 2010), as well as the previous one in Russia (DyVIS 2009), are to bring together academic and industrial experts to present new knowledge in the area of vibro-impact dynamics in real technological processes as well as in advances in theory of nonlinear dynamics in the abstractions of the real vibro-impact systems to the theoretical models and corresponding analytical, numerical solutions and comparison with corresponding experimental data.

These international conferences are very important for interactions on new advances in theory and realizations of technological processes on the basis of vibro-impact dynamical processes. This is possible only with scientific and professional interactions between academic and industrial experts from the fields of non-linear and structural dynamics, continuum mechanics, materials science, physics, applied mathematics, and mechanical, aerospace, civil and systems engineering.

The areas covered by research of the vibro-impact dynamics include: excitation, synchronisation and stabilisation of vibro-impact processes; dynamics of vibro-impact machines and technological processes; vibration protection of operators and structures in severe environment; synergistic effects of repeated impacts on solids and granular media; non-linear fluid-solid interactions; analytical, experimental and numerical methods for the analysis of vibro-impact systems and processes; synthesis and optimization of vibro-impact systems; and measurements and applications of vibro-impact processes.

In reference [48] vibro-impact systems are presented as systems of particular interest for studying. This is due to the fact that even when the behaviour of the system between impacts is linear, the impacts make the system highly nonlinear. Vibroimpact systems with deterministic excitation have been well studied and described in detail in a number of references and books [1-4], [6], [8-11] and [34]. Systems with elastic impacts can be studied by means of the method which transforms an impact system to a system without impacts. However, this method is effective only in the cases where the motion between impacts is linear or has a polynomial nonlinearity. In the case of inelastic impacts, the impact condition is usually expressed by the Dirac delta function on the right-hand side of the equation of motion. The coefficient of the delta function expresses the value of the impact impulse. In this case, the problem can be solved by the method of averaging. Stochastic systems with linear behavior between impacts and a one-sided limiter located at the equilibrium position have been thoroughly studied. The exact expressions for the probability density and spectral density in such systems have been obtained by means of the same transformation. The main purpose of the presented study [48] is to obtain analytical estimates for the average energy of stochastic vibro-impact systems with inelastic impacts. This paper [48] presents calculations that enable the extension of the application area of the energy balance method to stochastic vibroimpact systems with interference, clearance, and two-

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sided impact. In addition, the results are compared with the approximate results obtained by means of the quasi-conservative averaging and the numerical simulation results. The results of the numerical simulation of a vibro-impact system with two degrees of freedom are given.

In reference [8], a time-optimal feedback control driving the particle from an arbitrary initial state to a prescribed terminal position is constructed for a simple vibro-impact system consisting of a particle moving on a line segment between two rigid stops. The control variable is the force applied to the particle, the magnitude of this force being constrained.

In reference [4] a vibro-impact interaction between a rotor and a floating sealing ring is studied. The two-mass model of a “high-speed rotor-sealing ring” is considered. The model includes an unbalanced flexible rotor with elastic bearings. The rotor rotates inside the floating sealing ring. The ring is able to contact with the casing. The hydrodynamic forces in the clearance between the rotor and the ring as well as the dry friction between the ring and the casing are taken into account. The investigation of flow-coupled vibro-impact oscillations of the rotor and the ring are presented. For these regimes, the analytical solution is obtained as well as the numerical results. The main dynamic features of these behaviour and stability domains are discussed.

In reference [9], the response analyses of vibro-impact systems to random excitation are greatly facilitated by using certain piecewise-linear transformations of state variables which reduce the impact-type nonlinearities (with velocity jumps) to nonlinearities of the "common" type-without velocity jumps. This reduction resulted in some exact and approximate asymptotic solutions for stationary probability densities of the response for random vibrations with white-noise excitation. Moreover, if a linear system with a single barrier has its static equilibrium position exactly at the barrier, then the transformed equation of free vibration is found to be perfectly linear in case of the elastic impact. The transformed excitation term contains a signature-type nonlinearity, which is found to be of no importance in the case of a white-noise random excitation. Thus, an exact solution for the response spectral density was obtained previously for such a vibroimpact system, which may be called ”pseudolinear”, for the case of a white-noise excitation. This paper presents the analysis of a lightly damped pseudolinear single degree of freedom vibro-impact system under a non-white random excitation. The solution is based on the Fourier series expansion of a signum function for a narrow-band response. The formulae for the mean square response are obtained for a resonant case, where the (narrow-band) response occurs predominantly with frequencies, close to the system's natural frequency; and for a non-resonant case, where frequencies of the narrow-band excitation dominate the response. The results obtained may be applied directly for studying the response of moored bodies to ocean wave loading, and may also be used for establishing and verifying procedures for the approximate analysis of general vibro-impact systems.

Bifurcation phenomena of an electro-vibroimpact system have been investigated by means of the numerical analysis and presented in reference [31]. It has been shown that the system undergoes transition from a chaotic motion to a periodic motion as the control frequency of the solid state relay (one of the system parameters) varies. A close co-relationship with an experimental bifurcation diagram has been observed. A periodic motion has been identified to yield better system performance over a chaotic motion. The foundation of implementing an optimal feedback control strategy is established. Studies on vibro-impact systems have revealed very rich system dynamics due to the presence of nonlinearities in the system characteristics (Hinrichs *et al*. [30], 1997; Pavlovskaia and Wiercigroch [39-40], 2003; Peterka [41], [42], [43], 1996). The construction of Poincar`e maps, bifurcation diagrams and basins of attraction are useful to understand the qualitative dynamics of the system. The considered electro-vibro-impact system is a discontinuous system, both from a mathematical and a physical point of view. A detailed approach to describe and solve dynamical systems with motion dependent discontinuities was undertaken by Wiercigroch (2000). An important result from that piece of work was the clarification of accurate mathematical modelling of such systems and the numerical realisation of the analytical solution.

A large variety of dynamic responses is known to exist for nonlinear discontinuous systems. For example, systems exhibiting dry friction are known to behave in a chaotic manner, as demonstrated by Stefanski *et al*. [49] (2003). For detailed presentation of the current advances in the field of the vibroimpact systems see previous cited reference [31] written by Jee-Hou Ho and Ko-Choong Woo.

In Serbia there are a few researchers specialist in area of the vibro-impact dynamics. Some of these are among the VTI researchers. Three research projects supported by the Ministry of Sciences of Republic Serbia in basic sciences (in the period of 1990-2010), with project leader Hedrih (Stevanović) K., resulted in a number of titles in the field of vibro-impact dynamics. One of these is a doctoral thesis entitled *Stability of the deterministic and stochastic processes of the vibro-impact systems* [33] written by Mitic Sl. defended in 1994, and a corresponding monograph based on research results was published (in 2005). A series of the published papers by Mitić Sl.and Hedrih K. [34-38] in the period from 1992 to 1997 and by Hedrih (Stevanović) Katicam and Jović Srdjan [27-28] and Hedrih (Stevanović) Katica, Raičević Vladimir and Jović Srdjan [29] in 2009 is also a result of the cited projects. An MSc thesis entitled *Energy analysis of the vibro-impact systems*, written by Jović Srdjan in 2009, has been submitted as well.

All vibro-impact systems with single or multi degree of freedom can be divided into the systems with one-sided constraint and the systems with two-sided constraints along each degree of freedom. In the case of one-sided constraint, the coordinate of the limiter, measured from the equilibrium position of the system can be negative (interference), zero, or positive (clearance), while in the case of two-sided constraint, the limiters can be arranged symmetrically or asymmetrically relative to the equilibrium position.

A series of the vbro-impact systems represents the piecewise-conservative vbro-impact systems. The energy loss in such systems occurs at certain discrete time instants between impacts. These instants are unknown beforehand and depend only on the position and/or velocity of the system. Vibro-impact systems with impact-induced dominant losses are characteristic representatives of the class of piecewise-conservative systems. The idea of the method as applied to vibro-impact systems implies the consideration of the behaviour of the energy of the system between impacts and the balance of the energy before and after an impact. It is important that this method does not require the change in the system energy per period to be small, as is the case for the quasi-conservative averaging method.

Due to the fact that even when the behaviour of the system between impacts is linear, the impacts make the system highly nonlinear by discontinuity, we made a choice of the vibro-impact models with one degree of freedom and one side impact limiter of the elongations representing abstractions of the real technological process vibro-impact dynamics. The characteristic vibro-impact dynamics for different properties of the standard light elements, as linear, nonlinear elasticity, high-elasticity, hereditary and creeping properties included in the structure of the basic dynamic oscillator model, is investigated. The basics of energy transfer between the elements of the vibro-impact system analysis are presented. For some typical cases, the phase portraits and the energy curves are shown as the original results of the authors.

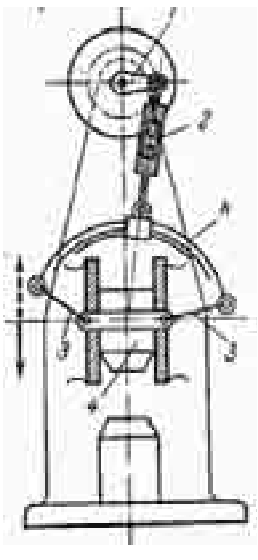
Vibroimpact hammer

This model was studied through the adjusting method by Кобринский А.Е., Кобринский А.А. [32]. A vibro-impact hammer (Fig.1) represents a machine for which the vibro-impact regime is the basis of the technological – operation process. The basis of the operation of this machine lies in the vibro-impact ramrod - striking effect.

Here is a brief description of the machine operation process: driving force leads to reversion of the crank mechanism (1) and is transferred through the connecting rod (2) and the spring element (c) to the rings (3). They are linked with the ramrod - striker (4). While the crank mechanism spins, the ramrod strikes the anvil (5), where frequency and intensity of the impact are regulated by the spinning speed of the crank mechanism. The initial position of the hammer is regulated by changing the length of the connecting rod. In most of operation processes the hammer moves periodically with sufficient accuracy.

Studies of the vibro-impact hammer dynamics (Fig.1.a*) are reduced and abstracted to the analysis of the dynamical vibro-impact models shown in Fig.1.b*.

Vibro-impact hammer, abstracted into a dynamical model with a single degree of freedom is shown in the dynamic model in Fig.1.a* as a vibro-impact system on the basis of a free oscillations harmonic oscillator with one side limited elongations and a spring with linear stress-strain constitutive relation.



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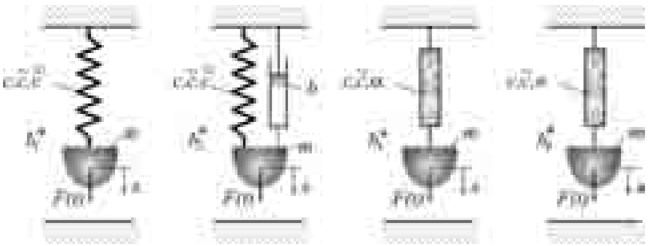


b* c* d* e*

Figure 1. a* Vibroimpact Hammer – machine with corresponding abstraction into different dynamical models of vibro-impact systems with one side impact limit elongations: a* oscillator with an ideal elastic linear/nonlinear spring with one side impact limit elongations; b*oscillator with an ideal elastic spring and a damper with one side impact limit elongations, d* hereditary oscillator with one side impact limit elongations; e* fractional order oscillator with one side impact limit elongations.

Taking into account that the model is with one degree of freedom, we take x as a generalized coordinate. The form of the ordinary differential equation of the vibro-impact motion is:

$$\ddot{x}+\omega^2 x = 0 \qquad (1)$$

where: $\frac{c}{m}=\omega^2$, the square of the eigen circular frequency depending of the spring rigidity c and the mass m. The ordinary differential equation is accompanied by the initial conditions:

$$x(0)=x_0 \quad \text{and} \quad \dot{x}(0)=\dot{x}_0 \qquad (2)$$

and by the impact conditions determined by one side impact limit of the elongation caused by the collision with the limiter**:**

$$x_{ul_i}=x(t_{ul_i -})=\delta\,, \quad i=1,2,3,...,n \qquad (3)$$

The impact (collision) conditions are:

a* for an ideally elastic impact

$$\dot{x}_{odl_i}=\dot{x}(t_{ul_i +})=-\dot{x}(t_{ul_i -})\,, \qquad (4)$$

b* for an arbitrary case between the ideally elastic and the ideally plastic impacts:

$$\dot{x}_{odl_i}=\dot{x}(t_{ul_i +})=-k\dot{x}(t_{ul_i -}) \ (5)$$

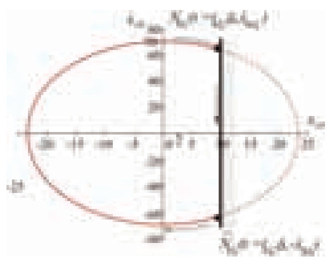
The solution of the ordinary linear differential equation (1) for the no-impact system is:

$$x(t)=A_1\cos\omega t+B_1\sin\omega t \qquad (6)$$

And for given initial copnditions (2) the particular solution is:

$$x(t)=x_0\cos\omega t+\frac{\dot{x}_0}{\omega}\sin\omega t \qquad (7)$$

The integral constants A_1 and B_1 are derived from the initial conditions (2).



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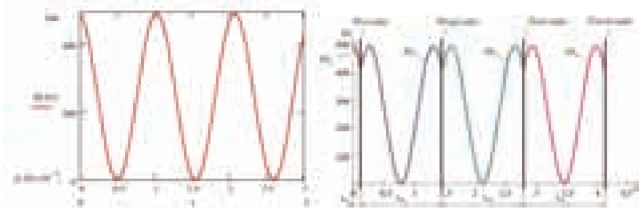

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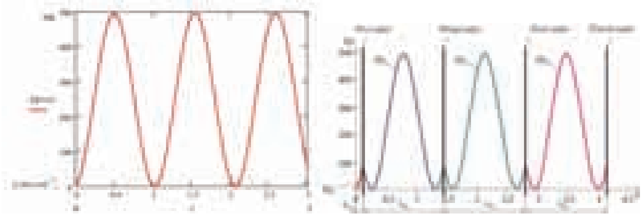

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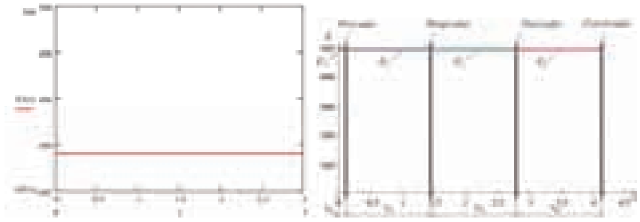
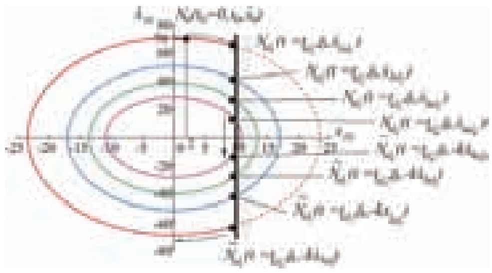


d*

Figure 2. $k=1$: a* Phase portrait piecewise linear vibro-impact oscillator; b* the graphics of kinetic energy $Ek=f(t)$; c* the graphics of potential energy $Ep=f(t)$; d* the graphics of total mechanical energy $E=f(t)$.

The solution of the primary no-impact system dynamics is in the basis of which the solution of a corresponding impact system dynamics is founded. In the considered example the model of vibro-impact dynamics is structured by one side impact limiter of the elongations and solution is structured for each interval between two followed impacts, determined by the corresponding values of integral constants according to the impact conditions (4) or (5).,



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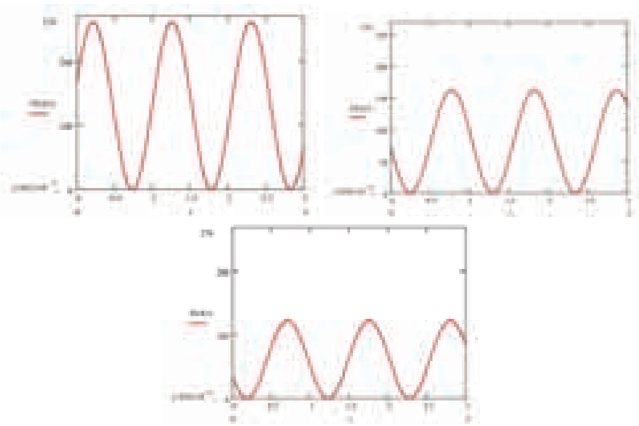

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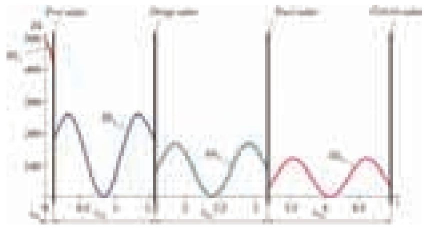

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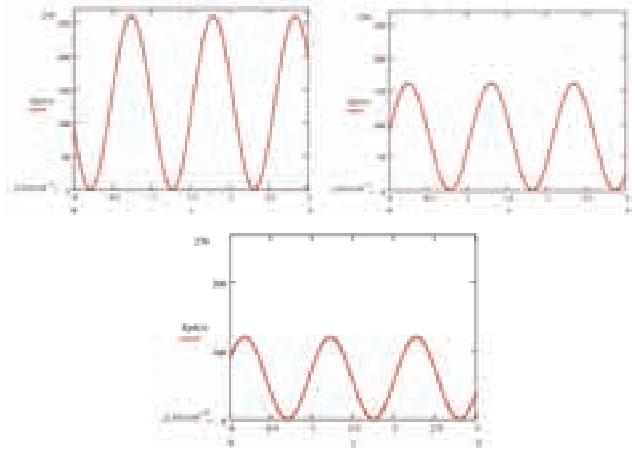

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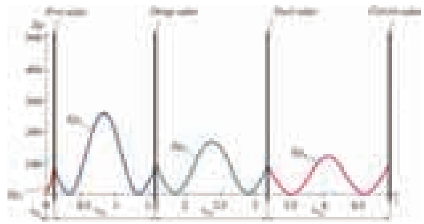

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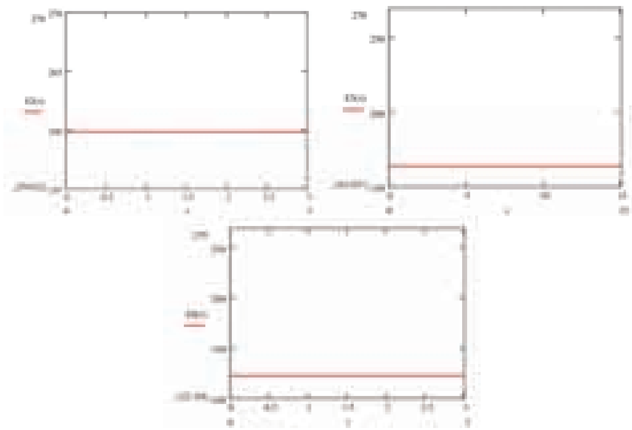

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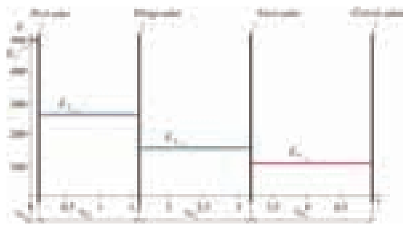


g*

Figure 3. $k=0,65$ a* *Phase portrait; b*and c* the graphics of kinetic energy* $Ek=f(t)$ for the followed intervals between impacts; d* *and e* the graphics of potential energy* $Ep=f(t)$ for followed intervals between impact; f* *and g* the graphics of the vibro-impact system total mechanical energy* $E=f(t)$ for the followed intervals between impacts.

and the coefficient of the restitution, or the conditions of the one side impact limit of the elongations (3).

The solution of differential equation (1) for the observed vibro-impact system in the period after the first impact is in the form:

$$x_i(t) = \delta\cos\omega t - \frac{v_{ul_{i-1}}}{\omega}\sin\omega t \quad i>1 \tag{8}$$

where $v_{ul_{i-1}}$ is the velocity before the i-th impact, as well as the velocity of the i-th impact.

The energy analysis for this case of the considered model of the vibro-impact system is shown in Figures 2 and 3, through a phase portrait containing one phase trajectory $(k=1)$ or a series of phase trajectories (k =0.65) and the curves of kinetic energy change, potential energy and vibro-impact system total mechanical energy as a time function in the following intervals of motions between each two impacts and for the case when the coefficient of the restitution takes the following values: k = 1 (see Fig.2) and k =0.65 (see Fig.3).

Vibroimpact hammer **is** presented through the dynamical model in Fig.1.b* as a vibro-impact system based on the forced harmonic oscillator with one side limited elongations.

The differential equation has the form:

$$\ddot{x} + \omega^2 x = h\cos\Omega t \tag{9}$$

where: $\frac{c}{m} = \omega^2$, $h = \frac{F_0}{m}$, and F_0 amplitude and Ω circular frequency of the external single frequency excitation. The ordinary differential equation (9) is accompanied by initial conditions in the form (2) and by impact conditions determined by one side impact limit of the elongation caused by the colision with the limiter in form (3). The impact conditions are in form (4) or (5).

The solution of differential equation (9) of a no-impact system is:

$$x(t) = A_1\cos\omega t + B_1\sin\omega t + \frac{h}{\omega^2-\Omega^2}\cos\Omega t \tag{10}$$

and for the given initial conditions in form (2) it is:

$$\begin{aligned} x(t) = {} & (x_0 - \frac{h}{\omega^2-\Omega^2})\cos\omega t + \\ & + \frac{\dot{x}_0}{\omega}\sin\omega t + \frac{h}{\omega^2-\Omega^2}\cos\Omega t \end{aligned} \tag{11}$$

The integrating constants A_1 and B_1 are derived from initial conditions (2) for the no- resonant case $\Omega \neq \omega$.

The solutions for the primary no-impact system on the basis of which a vibro-impact system is structured are used for finding solutions of the vibro-impact system where each period between impacts is determined by the integral constant according to the impact conditions or the coefficient of the restitution (4) or (5) and the conditions of elongations restrictions (3) (time after each collision runs from zero to the next collision).

The solutions of differential equation (9) for the observed vibro-impact system for the followed intervals between two impacts are in the following forms:

1. in the second interval after the first impact is in the following form:

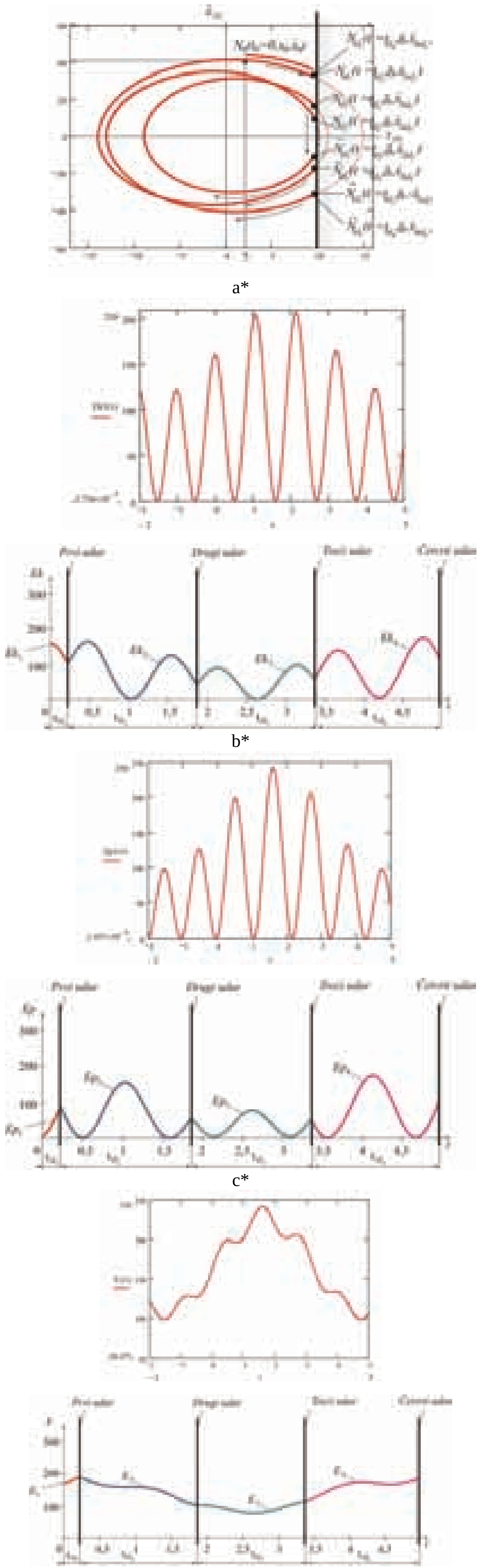


Figure 4. k=1 : a* Phase portrait piecewise linear vibro-impact oscillator for the forced no resonance vibro-impact regimes; b* the graphics of kinetic energy $Ek=f(t)$; c* the graphics of potential energy $Ep=f(t)$; d* the graphics of total mechanical energy $E=f(t)$

$$x_2(t)=\left[\delta-\frac{h}{\omega^2-\Omega^2}\cos\Omega\left(t_{ul_1}\right)\right]\cos\omega t+ +\frac{1}{\omega}\left[-v_{ul_1}+\frac{h}{\omega^2-\Omega^2}\Omega\sin\left(\Omega t_{ul_1}\right)\right]\sin\omega t+ +\frac{h}{\omega^2-\Omega^2}\cos\left\langle\Omega\left(t+t_{ul_1}\right)\right\rangle \tag{12}$$

2. in the third interval after the second impact is in the form:

$$x_3(t)=\left[\delta-\frac{h}{\omega^2-\Omega^2}\cos\Omega\left(t_{ul_{12}}\right)\right]\cos\omega t+ +\frac{1}{\omega}\left[-v_{ul_2}+\frac{h}{\omega^2-\Omega^2}\Omega\sin\left(\Omega t_{ul_{12}}\right)\right]\sin\omega t+ +\frac{h}{\omega^2-\Omega^2}\cos\left\langle\Omega\left(t+t_{ul_{12}}\right)\right\rangle \tag{13}$$

where: $t_{ul_{12}}=t_{ul_1}+t_{ul_2}$ etc.

The energy analysis is presented, in Fig.4, through the phase portrait and the graphics of kinetic energy changes, potential energy and vibro-impact system total mechanical energy as a time function in the following intervals of vibrations between two followed impacts.

Vibroimpact hammer is shown in Fig.1.c* by the dynamic model as a vibro-impact system based on free oscillations of the oscillator with a damping force, which is linearly proportional to the system velocity and with one side impact limiter of elongation (limited elongations).

The differential equation has the form:

$$\ddot{x}+2\zeta\omega\dot{x}+\omega^2x=0 \tag{14}$$

where:

$$\frac{c}{m}=\omega^2 \text{ and } \zeta=\frac{b}{2\sqrt{\dfrac{c}{m}}}$$

damping coefficient.

The ordinary differential equation (14) is accompanied by the initial conditions in the form (2) and by the impact conditions determined by one side impact limit of the elongation caused by the colision with the limiter in form (3). The impact conditions are in form (4) or (5).

The solution of the differential equation (14) of the no-impact system (for the damping coefficient $\zeta<1$) is in the form:

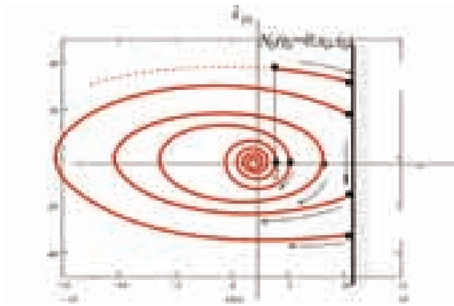
$$x(t)=e^{-\zeta\omega t}\left[A_1\cos\left(\omega t\sqrt{1-\zeta^2}\right)+B_1\sin\left(\omega t\sqrt{1-\zeta^2}\right)\right]. \tag{15}$$

and for the given initial conditions in the form (2) is:

$$x(t)=e^{-\zeta\omega t}\left[x_0\cos\left(\omega t\sqrt{1-\zeta^2}\right)+\frac{\dot{x}_0+\zeta\omega x_0}{\omega\sqrt{1-\zeta^2}}\sin\left(\omega t\sqrt{1-\zeta^2}\right)\right] \tag{16}$$

The solutions of the primary no-impact system on the basis of which a vibro-impact impact system is structured are used for finding solutions of the systems where each period between two followed impacts is determined by corresponding integral constants based on the collision conditions (3) and (4) or (5).

The solution of differential equation (14) of the vibro-impact system for the interval after the first and the i -th impact is in the following form:



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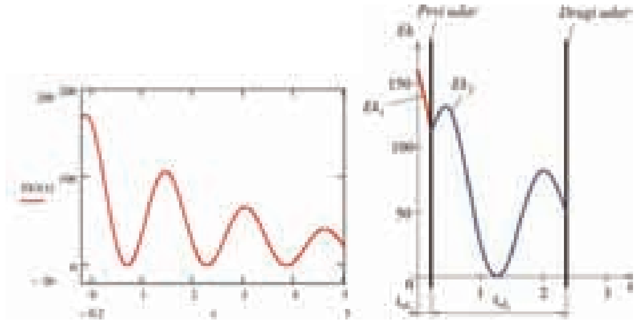

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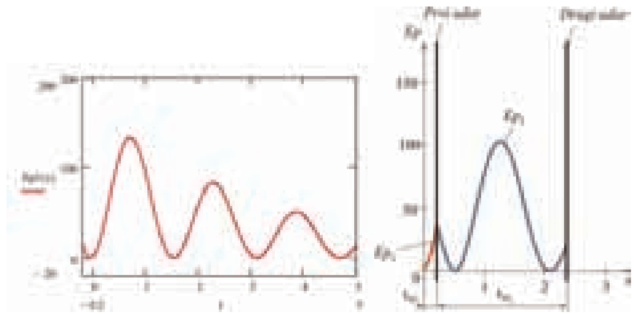

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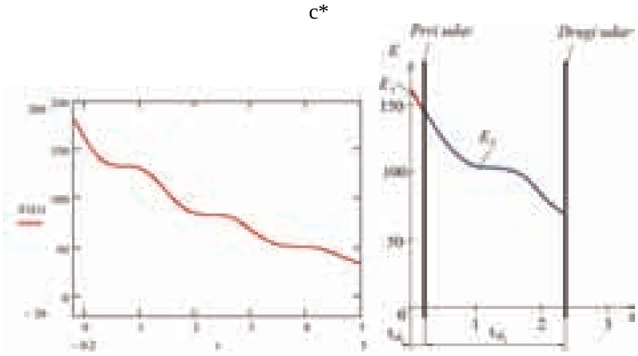

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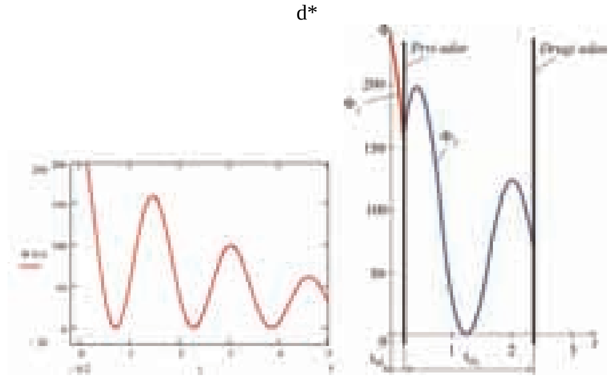


e*

Figure 5. $k=1$: a* Phase portrait piecewise linear vibro-impact oscillator for the linear damped regimes; b* the graphics of kinetic energy $Ek=f(t)$; c* the graphics of potential energy $Ep=f(t)$; d* the graphics of total mechanical energy $E=f(t)$; e* Rayleigh's function of the system energy dissipation.

$$x_i(t) = e^{-\zeta\omega t}\left[\delta\cos\left(\omega t\sqrt{1-\zeta^2}\right) + \frac{-v_{ul_{i-1}} + \zeta\omega\delta}{\omega\sqrt{1-\zeta^2}}\sin\left(\omega t\sqrt{1-\zeta^2}\right)\right] \quad (17)$$

$$i > 1$$

The energy analysis is presented in Fig.5 through the phase portrait and the curves of kinetic energy changes, the potential energy, the vibro-impact system total mechanical energy and Rayleigh's function of the system energy dissipation as time functions in the next intervals of the motion between two followed impacts.

Vibroimpact hammer is shown in Fig.1.b* by the dynamic model as a vibro-impact system based on forced linear oscillations of the damped linear oscillator, in which damping force is linearly proportional to the system velocity and with one side impact limiter of the elongation (limited elongations).

The differential equation has the form:

$$\ddot{x} + 2\zeta\omega\dot{x} + \omega^2 x = h\cos\Omega t \quad (18)$$

where: $\frac{c}{m} = \omega^2$, $\zeta = \frac{b}{2\sqrt{\frac{c}{m}}}$, $h = \frac{F_0}{m}$ and F_0 is the amplitude and Ω circular frequency of the external single frequency excitation. The ordinary differential equation (18) is accompanied by initial conditions in form (2) and by impact conditions determined by one side impact limit of the elongation caused by the collision with the limiter in form (3). The impact conditions are in form (4) or (5).

The solution of differential equation (18) of the no-impact system (damping coefficient $\zeta < 1$) is in the form (see reference [44-47] by Račković):

$$x(t) = e^{-\zeta\omega t}\left[A_1\cos\left(\omega t\sqrt{1-\zeta^2}\right) + B_1\sin\left(\omega t\sqrt{1-\zeta^2}\right)\right] + N\cos(\Omega t - \gamma_0) \quad (19)$$

where:

$$N = \frac{h}{\sqrt{\left(\omega^2 - \Omega^2\right) + 4\zeta^2\omega^2\Omega^2}} \quad (20)$$

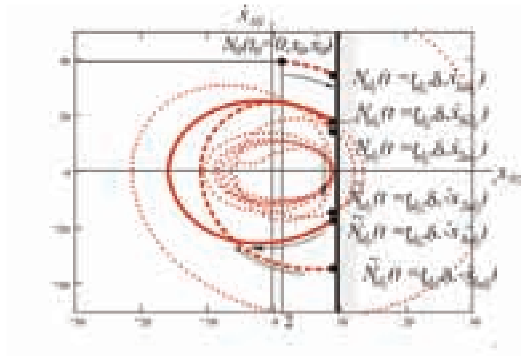
and

$$tg\gamma_0 = \frac{2\zeta\omega\Omega}{\omega^2 - \Omega^2}. \quad (21)$$

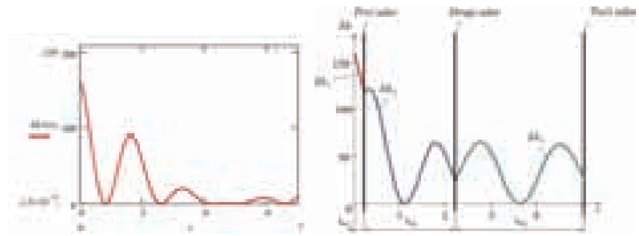
and for the given initial conditions in form (2) the particular solution is:

$$x(t) = e^{-\zeta\omega t}\left\{(x_0 - N\cos\gamma_0)\cos\left(\omega t\sqrt{1-\zeta^2}\right) + \left[\left(\frac{\dot{x}_0 + \zeta\omega(x_0 - N\cos\gamma_0)}{\omega\sqrt{1-\zeta^2}} - \frac{\Omega N\sin\gamma_0}{\omega\sqrt{1-\zeta^2}}\right)\sin\left(\omega t\sqrt{1-\zeta^2}\right)\right]\right\} + N\cos(\Omega t - \gamma_0) \quad (22)$$

As in previous cases, the solutions of the primary no-impact system on the basis of which a vibro-impact impact system is structured are used for finding solutions of the systems where each period between two followed impacts is determined by the corresponding integral constants based on the collision conditions (3) and (4) or (5).



a*



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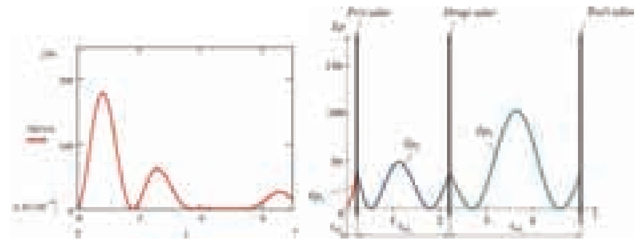

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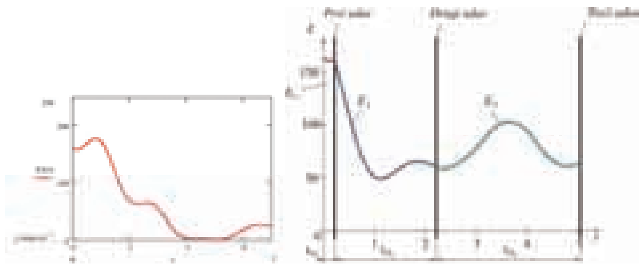

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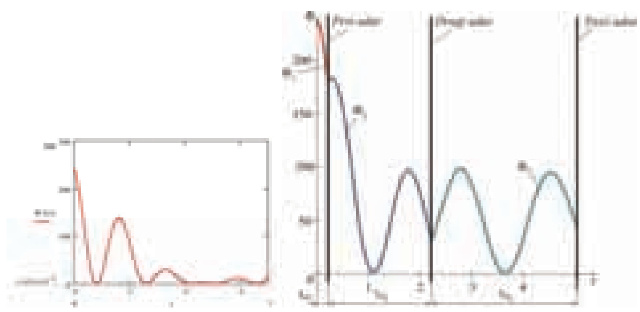


e*

Figure 6. $k = 1$: a* Phase portrait piecewise linear vibro-impact oscillator for the linear damped and forced regimes; b* the graphics of kinetic energy $Ek = f(t)$; c* the graphics of potential energy $Ep = f(t)$; d* the graphics of total mechanical energy $E = f(t)$; e* Rayleigh's function of the system energy dissipation.

The solutions of differential equation (18) for the observed vibro-impact system for the followed intervals between two impacts are in the following forms:

1* in the second interval after the first impact and between the first and the second impact, is in the following form:

$$x_2(t)=e^{-\zeta\omega t}\left\{\left[\delta-N\cos\left(\Omega t_{ul_1}-\gamma_0\right)\right]\cos\left(\omega t\sqrt{1-\zeta^2}\right)+ +\frac{-v_{ul_1}+\zeta\omega\left(\delta-N\cos\left(\Omega t_{ul_1}-\gamma_0\right)\right)}{\omega\sqrt{1-\zeta^2}}+ +\frac{\Omega N\sin\left(\Omega t_{ul_1}-\gamma_0\right)}{\omega\sqrt{1-\zeta^2}}\sin\left(\omega t\sqrt{1-\zeta^2}\right)\right\} + N\cos\left\langle\Omega\left(t+t_{ul_1}\right)-\gamma_0\right\rangle \tag{23}$$

2* in the third interval after the second impact, for the interval between the second and the third impact, is in the form:

$$x_3(t)=e^{-\zeta\omega t}\left\{\left[\delta-N\cos\left(\Omega t_{ul12}-\gamma_0\right)\right]\cos\left(\omega t\sqrt{1-\zeta^2}\right)+ +\frac{-v_{ul_2}+\zeta\omega\left(\delta-N\cos\left(\Omega t_{ul12}-\gamma_0\right)\right)}{\omega\sqrt{1-\zeta^2}}+ +\frac{\Omega N\sin\left(\Omega t_{ul12}-\gamma_0\right)}{\omega\sqrt{1-\zeta^2}}\sin\left(\omega t\sqrt{1-\zeta^2}\right)\right\}+ + N\cos\left\langle\Omega\left(t+t_{ul12}\right)-\gamma_0\right\rangle \tag{24}$$

where: $t_{ul12}=t_{ul_1}+t_{ul_2}$ etc.

The energy analysis is presented in Fig.6 through the phase portrait and the graphics of kinetic energy changes, potential energy and vibro-impact system total mechanical energy as a time function in the next intervals of vibrations between two followed impacts.

Vibroimpact hammer is shown in Fig.1. b* by the dynamic model as a vibro-impact system based on the forced nonlinear moving oscillator with a damping force which is linearly proportional to the system velocity, and the vibro-impact oscillator with one side impact limiter of the elongations.

The differential equation for the defined case has the following form for the general case:

$$\ddot{x}+\omega^2x+\varepsilon f(x)+2\zeta\omega\dot{x}=h\cos\Omega t \tag{25}$$

where: $\frac{c}{m}=\omega^2$, $\frac{\tilde{c}}{m}=\tilde{\omega}^2$, $\frac{\tilde{\tilde{c}}}{m}=\tilde{\tilde{\omega}}^2$, $\zeta=\frac{b}{2\sqrt{\frac{c}{m}}}$, $h=\frac{F_0}{m}$, ε small parametre.

The property of the spring is defined by a nonlinear stress-strain constitutive relation in the form of the force-spring elongation relation F_c and x expressed by:

$$F_c=-cx-\tilde{\varepsilon}f(x) \tag{26}$$

The ordinary differential equation (26) is accompanied by initial conditions in form (2) and by impact conditions determined by one side impact limit of the elongation caused by the collision with the limiter in form (3). The impact conditions are in form (4) or (5).

For **free oscillations** of this oscillator without the damping force and the spring nonlinear low restitution force, the form of the differential equation is:

$$\ddot{x}+\omega^2x+\varepsilon f(x)=0 \tag{27}$$

The first integral of the previous equation (26) is in the form:

$$\dot{x}^2=C_i-\omega^2x^2-2\varepsilon\int f(x)dx \tag{28}$$

where the integral constant is in the following form

$$C_i=\dot{x}_{oi}^2+\omega^2x_{oi}^2+\left[2\varepsilon\int f(x)dx\right]_{x=x_{oi}} \tag{29}$$

The solution of differential equation (27) in the first asymptotic approximation of the nonlinear oscillator is shown in the following form:

$$x(t)=a(t)\cos\psi(t) \tag{30}$$

where the amplitude $a(t)$, the phase $\varphi(t)$ or the full phase $\psi(t)$ are the time functions determined through the system of the first order of differential equations by the amplitude $a(t)$ and the ful phase $\psi(t)$ in the folowing form:

$$\frac{da}{dt}=\varepsilon A_1(a)$$

$$\frac{d\psi}{dt}=\omega+\varepsilon B_1(a). \tag{31}$$

From the system of differential equations of the first approximations for the amplitude $a(t)$ and the ful phase $\psi(t)$ (31) corresponding to the first approximation of the solution of system dynamics, we conclude that the system oscillations are with a constant amplitude equal to the amplitude at the initial moment of the motion while the first asymptotic approximation of the solution is no isochronous, and the system oscillates with a frequency and a period depending on the initial amplitude and phase

$$a(t)=a_0=const. \tag{32}$$

$$\omega_{nonlinear}\left(a_0\right)=\omega+\varepsilon\frac{1}{2a_0\pi\omega}\int_0^{2\pi}f\left(a_0\cos\psi\right)\cos\psi d\psi \tag{33}$$

while the period of the nonlinear no-impact oscillation is equal to

$$T_{nonlinear}\left(a_0\right)=\frac{2\pi}{\omega+\varepsilon\frac{1}{2a_0\pi\omega}\int_0^{2\pi}f\left(a_0\cos\psi\right)\cos\psi d\psi} \tag{34}$$

The full phase of the nonlinear no-impact oscillation in the first approximation is

$$\psi(t)=\omega_{nonlinear}\left(a_0\right)t+\psi_0$$

$$\psi(t)=\left[\omega+\varepsilon\frac{1}{2a_0\pi\omega}\int_0^{2\pi}f\left(a_0\cos\psi\right)\cos\psi d\psi\right]t+\psi_0 \tag{35}$$

The solution of the nonlinear differential equation describing free nonlinear oscillations in the first asymptotic approximation of non-linear oscillator in no-impact regime is in the form:

$$x(t) = a_0 \cos\psi(t) \tag{36}$$

For the special example of the spring nonlinear constitutive relation describing the spring force in the nonlinear form by the spring deformation in the form:

$$F_c = -cx - \tilde{c}x^3 - \tilde{\tilde{c}}x^5 \tag{37}$$

the equation of the phase trajectory describing free nonlinear oscillations of the defined nonlinear no-impact oscillator is in the form:

$$\dot{x}^2 = \tilde{C}_i - \omega^2 x^2 - \frac{1}{2}\tilde{\omega}^2 x^4 - \frac{1}{3}\tilde{\tilde{\omega}}^2 x^6 \tag{38}$$

where:

$$\tilde{C}_i = \dot{x}_{oi}^2 + \omega^2 x_{oi}^2 + \frac{1}{2}\tilde{\omega}_{oi}^4 x_{oi}^2 + \frac{1}{3}\tilde{\tilde{\omega}}^2 x_{oi}^6 \tag{39}$$

The solution of the nonlinear differential equation describing a chosen example of the free nonlinear oscillations in the first asymptotic approximation of non-linear oscillator in no-impact regime is in form (36) where the full phase $\psi(t)$ is in the following form:

$$\psi(t) = \left[\omega + \frac{3}{8}\frac{\tilde{\omega}^2}{\omega}a_0^2 + \frac{5}{16}\frac{\tilde{\tilde{\omega}}^2}{\omega}a_0^4\right]t + \psi_0 \tag{40}$$

The eigen circular frequency of the free nonlinear no-impact oscillations in the first approximation is:

$$\omega_{nonlinear}(a_0) = \omega + \frac{3}{8}\frac{\tilde{\omega}^2}{\omega}a_0^2 + \frac{5}{16}\frac{\tilde{\tilde{\omega}}^2}{\omega}a_0^4 \tag{41}$$

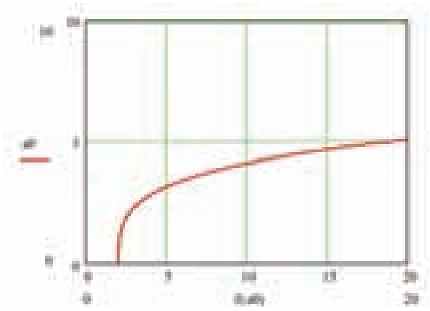


Figure 7. Amplitude-frequency graph $\omega_{nonlinear}(a_0)$ or the skeleton amplitude frequency characteristic

It depends on the initial amplitude and is no-isochronous, while the corresponding period of free nonlinear no-impact oscillations is equal to:

$$T_{nonlinear}(a_0) = \frac{2\pi}{\omega + \frac{3}{8}\frac{\tilde{\omega}^2}{\omega}a_0^2 + \frac{5}{16}\frac{\tilde{\tilde{\omega}}^2}{\omega}a_0^2} \tag{42}$$

In Fig.7 the **a**mplitude frequency graph $\omega_{nonlinear}(a_0)$ or the skeleton amplitude frequency characteristic is presented.

We can notice that the considered free nonlinear oscillations are not isochronous, and the period of oscillations and the circular frequency of the free nonlinear no-impact oscillator depend on the initial conditions, initial amplitude and the full phase of oscillating.

For the case of the vibro-impact nonlinear free vibrations of the vibro-impact system with one side impact limit of the elongations, the vibrations are no isochronous vibrations, and for ideal elastic impacts, the system dynamics is conservative with a constant period of vibro-impact oscillations, depending on initial conditions, initial kinetic and potential energies, and with a shorter period in comparison with the corresponding no impact dynamics. This is caused by discontinuity and jumps of the alternations of the velocity directions during each time interval between two subsequent impacts of numerous impacts in very, very short time periods approaching to zero.

The energy analysis is presented in Fig.8 through the phase portrait and the history graphs of kinetic energy changes, potential energy and vibro-impact system total mechanical energy as a time function in the following intervals of motion.

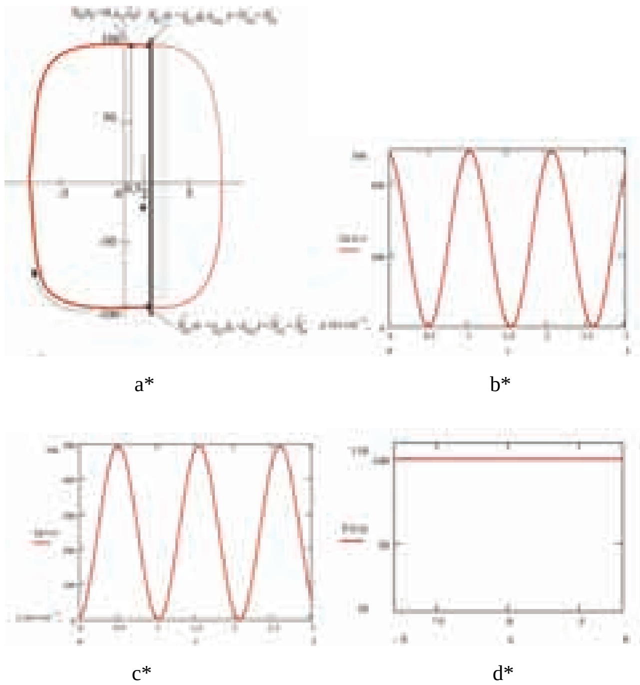


Figure 8. $k = 1$: a* Phase portrait of the nomlinear vibro-impact oscillator; b* the graphics of kinetic energy $Ek = f(t)$; c* the graphics of potential energy $Ep = f(t)$; d* the graphics of total mechanical energy $E = f(t)$.

For **forced oscillations** of the observed nonlinear oscillator with a damping force which is linearly proportional to the system velocity and with one side impact limiter of the elongations, the differential equation has the form (25):

$$\ddot{x} + \omega^2 x + \varepsilon f(x) + 2\zeta\omega\dot{x} = h\cos\vartheta \tag{25*}$$

where $\theta = \Omega t + \theta_0$, and θ_0 is the inital phase of the single frequency external excitation. The spring is nonlinear elastic decribed by the stress-strain constitutive relation in the form of the relation between force and spring elongation, F_c and x, expressed by (37). The ordinary differential equation (25*) is accompanied by the initial conditions in form (2) and by the impact conditions determined by the one side impact limit of the elongation caused by the collision with the limiter in form (3). The impact conditions are in form (4) or (5).

By using the asymptotic method Krilov-Bogoljubov-Mitropolyski for the case of the principal resonant range state $\Omega \approx \omega$, the first asimptotic approxiamtion of the solution is in the form:

$$x = a(t)\cos\psi(t) \tag{43}$$

where: the amplitude $a(t)$, the phase $\varphi(t)$ or the full phase $\psi(t) = \theta(t) + \varphi$ are time functions determined by the system of the first order differential equations in the first asymptotic aproximation. This system is expressed by the amplitude $a(t)$ and the phase $\varphi(t)$ in the following form of

$$\frac{da}{dt} = \varepsilon A_1(a,\varphi)$$

$$\frac{d\psi}{dt} = \omega - \Omega + \varepsilon B_1(a,\varphi). \tag{44}$$

For determining the functions $A_1(a,\varphi),\ B_1(a,\varphi)...$, we differentiate the assumed solution (43) and submit it in the nonlinear differential equation (25*) taking into consideration the assumed derivations of the amplitude $a(t)$ and the phase $\varphi(t)$ in the appropriate approximation by (44) and we obtain:

$$\frac{da}{dt} = -\zeta\omega a - \frac{h}{\omega + \Omega(t)}\cos\varphi$$

$$\frac{d\varphi}{dt} = \omega - \Omega(t) + \varepsilon\frac{1}{2a\pi\omega}\int_0^{2\pi} f\left(a\cos\psi\right)\cos\psi\, d\psi + \tag{45}$$

$$+\frac{h}{(\omega + \Omega(t))a}\sin\varphi$$

For a special considered example of nonlinear oscillations, for case (37), the differential equation of the no impact system dynamics is in the following form

$$\ddot{x} + \omega^2 x + \tilde{\omega}^2 x^3 + \tilde{\tilde{\omega}}^2 x^5 + 2\zeta\omega\dot{x} = h_0\cos\vartheta \tag{46}$$

where we assumed that the spring force of nonlinear elasticity has the form (37).

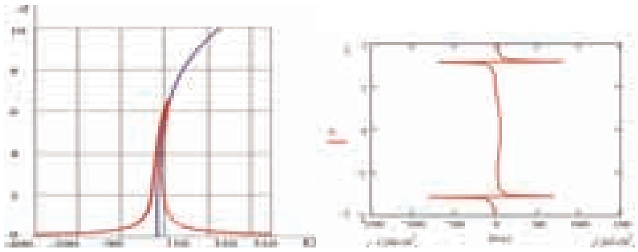
In the case of the principal resonant range state $\Omega \approx \omega$, the first asimptotic approximation of the solution is in the form (43), where the amplitude $a(t)$ and the phase $\varphi(t)$ or the full phase $\psi(t) = \vartheta + \varphi$ are the time functions determined by the system of the first order differential equations in the first approximation are in the form:

$$\frac{da}{dt} = -\zeta\omega a - \frac{h_0}{\omega + \Omega(t)}\cos\varphi$$

$$\frac{d\varphi}{dt} = \omega - \Omega(t) + \frac{3}{8}\frac{\tilde{\omega}^2}{\omega}a^2 + \frac{5}{16}\frac{\tilde{\tilde{\omega}}^2}{\omega}a^4 + \tag{47}$$

$$+\frac{h_0}{(\omega + \Omega(t))a}\sin\varphi$$

In Fig.9 the a* amplitude frequency characteristics of the nonlinear oscillator for stationary resonant regimes with selection curve are presented. The dotted line presents no stable amplitudes, and two resonant jumps of the amplitude as well as of the phase appear at boundaries of the frequency interval of the unstable amplitude branch. In Fig.9 the b* phase frequency characteristic of the nonlinear oscillator for stationary resonant regimes is presented.



a* b*

Figure 9. a*Amplitude frequency characteristics of the nonlinear oscillator for stationary resonant regimes with selection curve; b* phase frequency characteristics of the nonlinear oscillator for stationary resonant regimes.

For the investigation of the vibro-impact system dynamics with one side impact limiter of the system elongation, it is necessary to use the first asymptotic approximation $x = a(t)\cos\psi(t)$ according to the system (47) of the first order differential equations in the first approximation of the amplitude $a(t)$ and phase $\varphi(t)$ or the full phase $\psi(t) = \vartheta + \varphi$ and by use numerical integration and and taking into account initial conditions as well a impact conditions to fined time of the first impact, and corresponding velocity of the first impact. Then it is necessary to solve the following task:

Using one side impact limit of the elongation in the form:

$$x_{ul_i} = x(t_{ul_i -}) = \delta\,,\ i = 1,2,3,...,n \tag{48}$$

for each impact it is necessary to solve the following task: to determine the time of the corresponding impact: $t_{ul_i -} = ?$ and other corresponding kinetic parameters.

For the first impact time $t_{ul_1 -} = ?$, it is necessary to solve numerically the following task:

$$x_{ul_1} = a(t_{ul_1 -})\cos\psi\left(t_{ul_1 -}\right) = \delta \Rightarrow t_{ul_1 -} = ? \tag{49}$$

simultaneously using the following first order differential equations according to the unknown amplitude $a(t)$ and phase $\varphi(t)$ or the full phase $\psi(t) = \vartheta + \varphi$ and starting with the initial conditions (2), $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$:

$$\frac{da}{dt} = -\zeta\omega a - \frac{h_0}{\omega + \Omega(t)}\cos\varphi$$

$$\frac{d\varphi}{dt} = \omega - \Omega(t) + \frac{3}{8}\frac{\tilde{\omega}^2}{\omega}a^2 + \frac{5}{16}\frac{\tilde{\tilde{\omega}}^2}{\omega}a^4 + \tag{50}$$

$$+\frac{h_0}{(\omega + \Omega(t))a}\sin\varphi$$

for determining the amplitude $a(t_{ul_1 -})$ and the phase $\varphi(t_{ul_1 -})$ or the full phase $\psi(t_{ul_1 -}) = \vartheta\left(t_{ul_1 -}\right) + \varphi\left(t_{ul_1 -}\right)$ at the moment of the first impact. After determining the time of the first impact $t_{ul_1 -}$ numerically, it is necessary to calculate the velocity of the nonlinear system at the moment before and after the first impact in the following form:

The impact velocity $\dot{x}(t_{ul_1 -})$ before the first impact at the start of the first impact:

$$\dot{x}_{ul_1} = \dot{a}(t_{ul_1 -})\cos\psi\left(t_{ul_1 -}\right) - a(t_{ul_1 -})\dot{\psi}\left(t_{ul_1 -}\right)\sin\psi\left(t_{ul_1 -}\right) \tag{51}$$

The velocity $\dot{x}(t_{ul_i+})$ at the moment after the first impact, at the moment of the finish of the first impact:

$$\dot{x}_{odl_1} = \dot{x}(t_{ul1+})$$
$$\dot{x}_{odl_1} = -\dot{x}_{ul_1} = -\dot{a}(t_{ul1-})\cos\psi\left(t_{ul1-}\right) + \qquad (52)$$
$$+a(t_{ul1-})\dot{\psi}\left(t_{ul1-}\right)\sin\psi\left(t_{ul1-}\right)$$

For the period between the first and the second impact, the initial conditions are:

$$x_{odl_1} = a(t_{ul1+})\cos\psi\left(t_{ul1+}\right) = \delta \text{ and } \dot{x}_{odl_i} = \dot{x}(t_{ul_i+}) \, . \qquad (53)$$

For finding the second impact time $t_{ul2-} = ?$, it is necessary to solve numerically the following task:

$$x_{ul_2} = a(t_{ul_2-})\cos\psi\left(t_{ul_2-}\right) = \delta \Rightarrow t_{ul_2-} = ? \qquad (54)$$

simultaneously by using the following first order differential equatins according to the unknown amplitude $a(t)$ and phase $\varphi(t)$ or the full phase $\psi(t) = \vartheta + \varphi$ and starting with the initial conditions (53):

$$\frac{da}{dt} = -\zeta\omega a - \frac{h_0}{\omega + \Omega(t)}\cos\varphi$$
$$\frac{d\varphi}{dt} = \omega - \Omega(t) + \frac{3}{8}\frac{\tilde{\omega}^2}{\omega}a^2 + \frac{5}{16}\frac{\tilde{\tilde{\omega}}^2}{\omega}a^4 + \qquad (50^*)$$
$$+\frac{h_0}{\left(\omega + \Omega(t)\right)a}\sin\varphi$$

for determining the amplitude $a(t_{ul2-})$ and the phase $\varphi(t_{ul21-})$ or the full phase $\psi(t_{ul2-}) = \vartheta\left(t_{ul2-}\right) + \varphi\left(t_{ul2-}\right)$ at the moment of the second impact. After determining the time $t_{ul2-} = ?$ of the second impact t_{ul2-} numerically, it is necessary to calculate the velocity of the nonlinear system at the moment before and after the second impact in the following form:

The impact velocity $\dot{x}(t_{ul2-})$ before the first impact at the start of the first impact:

$$\dot{x}_{ul_2} = \dot{a}(t_{ul2-})\cos\psi\left(t_{ul2-}\right) -$$
$$-a(t_{ul2-})\dot{\psi}\left(t_{ul2-}\right)\sin\psi\left(t_{ul2-}\right) \qquad (55)$$

The velocity $\dot{x}(t_{ul2+})$ at the moment after the first impact, at the moment of the finish of the first impact:

$$\dot{x}_{odl_2} = \dot{x}(t_{ul2+})$$
$$\dot{x}_{odl_2} = -\dot{x}_{ul_2} = -\dot{a}(t_{ul2-})\cos\psi\left(t_{ul2-}\right) + \qquad (56)$$
$$+a(t_{ul2-})\dot{\psi}\left(t_{ul2-}\right)\sin\psi\left(t_{ul2-}\right)$$

For the period between the first and second impacts, the initial conditions are:

$$x_{odl_2} = a(t_{ul2+})\cos\psi\left(t_{ul2+}\right) = \delta \text{ and } \dot{x}_{odl_2} = \dot{x}(t_{ul2+}) \qquad (57)$$

It is not dificult to use analogy for the next impacts to determine times and corresponding impact kinetic parameters.

The impact velocity $\dot{x}(t_{ul_i-})$ before the i-th impact at the start of the f i-th impact:

$$\dot{x}_{ul_i} = \dot{a}(t_{ul_i-})\cos\psi\left(t_{ul_i-}\right) - a(t_{ul_i-})\dot{\psi}\left(t_{ul_i-}\right)\sin\psi\left(t_{ul_i-}\right) \qquad (58)$$

The velocity $\dot{x}(t_{ul_i+})$ at the moment after the i-th impact, at the moment of the finish of the i-th impact:

$$\dot{x}_{odl_i} = \dot{x}(t_{ul_i+})$$
$$\dot{x}_{odl_i} = -\dot{x}_{ul_i} = -\dot{a}(t_{ul_i-})\cos\psi\left(t_{ul_i-}\right) + \qquad (59)$$
$$+a(t_{ul_i-})\dot{\psi}\left(t_{ul_i-}\right)\sin\psi\left(t_{ul_i-}\right)$$

For the period between the i-th and $i+1$-th impacts, the initial conditions are:

$$x_{odl_i} = a(t_{ul_i+})\cos\psi\left(t_{ul_i+}\right) = \delta \text{ and } \dot{x}_{odl_i} = \dot{x}(t_{ul_i+}) \, . \qquad (60)$$

The basic problem is that there are no analytical solutions for the nonlinear vibro-impact system, so it is necessary to find numerical solutions.

It is also possible to use the phase plane method and the cross sections of the phase trajectory for stationary resonant regimes and to determine some necessary kinetic parameters of the vibro-impact system oscillations.

Vibroimpact hammer is shown in Figure 1. c* by the dynamic model as a vibro-impact system based on the standard hereditary oscillator.

A standard light hereditary element is formed on the basis of the serial, parallel or combined coupling of rheological basic elements and springs and also with possible modifications. One set of possible standard light elements massless is presented in Fig.10 (see References [11-30]).

The basic rheological elements are:
- a* ideally elastic element represented by the spring;
- b* viscous element which is schematically presented by a damper.

The integro - differential equation of hereditary oscillator dynamics, containing mass m and standard light hereditary element has the following form:

$$m\ddot{x} + cx - c\int_0^t \Re(t-\tau)x(\tau)d\tau = F(t) \qquad (61)$$

where:

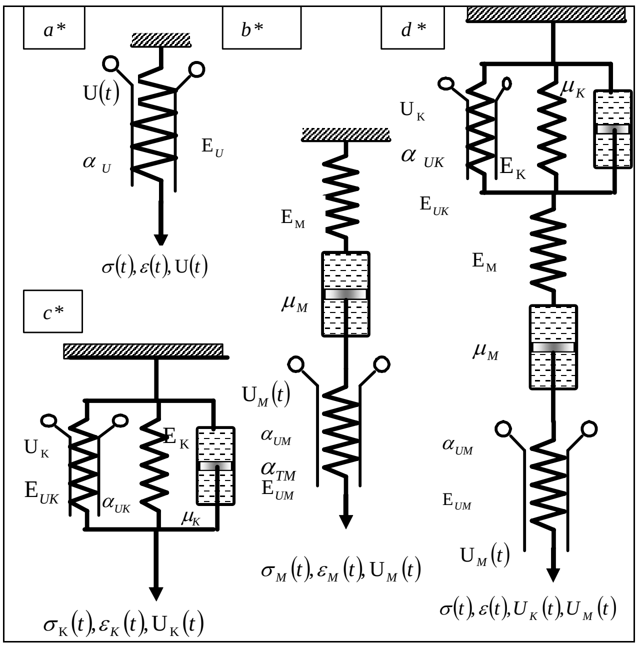


Figure 10. a* Schematic view of the piezoelastic element; b* Schematic view of the piezomodified Maxvell's elasticviscose hereditary element; c* Schematic view of the piezomodified Kelvin-Foight's viscoelastic hereditory element; d* Schematic view of piezomodified Burgers's hereditory element

$$\Re(t-\tau)=\frac{c-\tilde{c}_1}{nc}e^{\frac{t-\tau}{nc}} \quad (62)$$

is the resolvent or the core of relaxation, $\beta=\frac{1}{n}$ is the relaxation coefficient, $c,\tilde{c}_1$ are the coefficients of the rigidity of the light heredity element for the case of momentaneous as well as long-term loading, and n is the time of the element relaxation.

The integro-differential equation (61) in the differential form

$$nm\dddot{x}(t)+m\ddot{x}(t)+nc\dot{x}(t)+\tilde{c}y(t)=n\dot{F}(t)+F(t) \quad (63)$$

is accompanied by three initial conditions (see Refs. [11-16]):

$$x(0)=x_0,\quad \dot{x}(0)=\dot{x}_0,\quad \ddot{x}(0)=\ddot{x}_0=g \quad (64)$$

and by the impact conditions determined by one side impact limit of the elongation caused by the colision with the limiter**:**

$$x_{ul_i}=x(t_{ul_i-})=\delta,\ i=1,2,3,...,n \quad (65)$$

The impact (collision) conditions are:

- a* for an ideally elastic impact

$$\dot{x}_{odl_i}=\dot{x}(t_{ul_i+})=-\dot{x}(t_{ul_i-}), \quad (66)$$

- b* for an arbitrary case between the ideally elastic and the ideally plastic impacts:

$\dot{x}_{odl_i}=\dot{x}(t_{ul_i+})=-k\dot{x}(t_{ul_i-})$ (66) General ssolution of the ordinary linear differential equation (63) or integro-differential equation (61) for no impact system dynamics is in the form:

$$x(t)=Ae^{\lambda_1 t}+Be^{\lambda_2 t}+De^{\lambda_3 t}, \quad (67)$$

where $\lambda_i,\ i=1,2,3$ are eigen characteristic numbers and the roots of the characteristic equations

$$nm\lambda^3+m\lambda^2+nc\lambda+\tilde{c}_1=0 \quad (68)$$

Let's present the roots of the previous characteristic equation in the complex form

$$\lambda_0=-\delta_0,\quad \lambda_{1,2}=-\delta\pm i\omega \quad (69)$$

and after their introduction in the characteristic eq. (69) we obtain:

$$(\lambda+\delta_0)(\lambda+\delta+i\omega)(\lambda+\delta-i\omega)=0 \quad (70)$$

or in the form

$$\lambda^3+(\delta_0+2\delta)\lambda^2+\left(\omega^2+\delta^2+2\delta\delta_0\right)\lambda+ +\delta_0\left(\omega^2+\delta^2\right)=0 \quad (71)$$

After comparing the coefficients of equations (68) and (71) of the corresponding exponents, we obtain the relations between the kinetic parameters of the hereditary oscillator in the following forms:

$$\frac{\delta_0\left(\omega^2+\delta^2\right)}{\left(\omega^2+\delta^2+2\delta\delta_0\right)}=\frac{\tilde{c}}{nc},\ \delta_0+2\delta=\frac{1}{n},\ \left(\omega^2+\delta^2+2\delta\delta_0\right)=\frac{c}{m} \quad (72)$$

from which it follows:

$$\delta_0=\frac{\tilde{c}}{nc}\left(1+\frac{2\delta\delta_0}{\omega^2+\delta_0^2}\right)$$

$$\delta=\frac{1}{2n}-\frac{\delta_0}{2}=\frac{c-\tilde{c}}{2nc}-\frac{\tilde{c}}{nc}\frac{\delta\delta_0}{\omega^2+\delta_0^2} \quad (73)$$

$$\omega^2=\frac{c}{m}\left[1-\frac{\delta(\delta+2\delta_0)}{\omega^2}\right]$$

In the first approximation, taking into account that ratio $\left(\frac{\delta}{\omega}\right)^2$ is small, the kinetic parameters $\delta_0,\delta,\ \omega$ of the hereditary oscillator in the first approximation are obtained in the forms:

$$\delta_0=\frac{\tilde{c}}{nc},\ \ \delta=\frac{c-\tilde{c}}{2nc},\ \ \omega^2=\frac{c}{m} \quad (74)$$

By using expressions (74) of the first approximation and putting them in the expressions (73), the kinetic parameters $\delta_0,\delta,\ \omega$ of the hereditary oscillator in the second approximation are obtained in the forms (see Refs. [11-16]):

$$\delta_0=\frac{\tilde{c}}{nc}\left[1+\frac{(c-\tilde{c})\tilde{c}}{c^2}\frac{1}{n^2\omega^2}\right]$$

$$\delta=\frac{c-\tilde{c}}{2nc}\left[1-\left(\frac{\tilde{c}}{c}\right)^2\frac{1}{n^2\omega^2}\right] \quad (75)$$

$$\omega^2=\frac{c}{m}\left[1-\frac{c-\tilde{c}}{c}\frac{c+3\tilde{c}}{4c}\frac{1}{n^2\omega^2}\right]$$

For many visco-elastic hereditary materials, the time of the hereditary element relaxation is $n\sim 50[\sec]$. For the frequency of the hereditary oscillator $f=1[hertz]$ or $\omega=2\pi f=6,28\left[\sec^{-1}\right]$ the dimensionless ratio has the following value $\frac{1}{n^2\omega^2}\approx 4\cdot 10^{-5}$. The experimental results were obtained by O.A. Goroshko and presented in the Monograph Goroshko O.A. and Hedrih (Stevanović) K. [11] published in 2001. In this way, the values of the hereditary oscillator coefficients δ_0,δ and the circular frequency ω are defined by expressions (75) with a high degree of precision.

By using the previous considerations and the approximation of the standard hereditary oscillator coefficients δ_0,δ and the circular frequency ω defined by expressions (78), the solution of the equation (61) or (63), for the standard hereditary no impact oscillator, can be written in the following form (see Ref. [11]):

$$(t)=mg\left[\frac{1}{\tilde{c}}+\left(\frac{1}{c}-\frac{1}{\tilde{c}}\right)e^{-\delta_0 t}-\frac{e^{-\delta t}}{c}\cos\omega t- -\frac{3}{2}\frac{c-\tilde{c}}{c^2}\frac{1}{n^2\omega^2}\sin\omega t\right] \quad (76)$$

For the initial conditions $y(0)=0$, $\dot{y}(0)=0$, $\ddot{y}(0)+P(0)=mg$, where $P(0)=cy(0)$, corresponding to applied heavy material particle with the weight mg and with the zero initial velocity of the hereditary oscillator, the material particle corresponds to the unstressed and undeformed natural state of the hereditary element in the hereditary oscillator.

The motion of this considered hereditary oscillator in the defined initial conditions represents damped oscillations in accordance with the curve of rheology.

An estimation of precision expressions of values of the hereditary oscillator with weak singular rheological elements for the coefficients: δ_0 of rheology, δ for decrement and ω for circular frequency expressed by Γ - Euler function leads us to the conclusion, presented in reference [4], that these expressions are with a high step of precision.

For the vibro-impact dynamics it is necessary to use impact conditions of the impact listed elongations and to find the time of the first impacts and the corresponding system velocity at the moment of impacts. The procedure and the algorithm are same as in the previous examples, but hereditary vibro-impact needs new investigations according the hereditary element memory of the history of loading and according this determinations of the initial conditions of each next interval between two subsequent impacts. This is a serious task for new investigations, since up to now there is not enough knowledge in scientific literature.

Vibroimpact hammer shown in Fig.1 e* through the dynamic model as a vibroimpac system based on the fraction order oscillator.

The basic elements of the fractional order oscillator are:

* *Material particle* with mass m, having one degree of motion freedom, defined by the following coordinate x;

* *Standard light fractional order coupling element* of negligible mass in the form of axially stressed rod without bending, with the ability to resist deformation under static and dynamic conditions. *Standard light creep constraint element* for which the stress-strain relation for the restitution force as the function of element elongation is given by fractional order derivatives in the form:

$$P(t) = -\left\{c_0 x(t) + c_\alpha \, \mathrm{D}_t^\alpha \left[x(t)\right]\right\} \qquad (77)$$

where $\mathrm{D}_t^\alpha[\bullet]$ is operator of the α^{th} derivative with respect to the time t in the following form:

$$\mathrm{D}_t^\alpha\left[x(t)\right] = \frac{d^\alpha x(t)}{dt^\alpha} = x^{(\alpha)}(t) = = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{x(\tau)}{(t-\tau)^\alpha} d\tau \qquad (78)$$

where c, c_α are the rigidity coefficients– momentary and prolonged one, and α a rational number between 0 and 1, $0<\alpha<1$. Here $\Gamma(1-\alpha)$ - is the Euler's gama-function.

The differential equation of the oscillation of the basic friction order oscillator on the basis of which this vibroimpact system is constructed is

$$m\ddot{x} + cx + c_\alpha \, \mathrm{D}_\alpha^t\left[x(t)\right] = F(t) \qquad (79)$$

The ordinary fractional order differential equation (77) is accompanied by initial conditions in the form (2) and by impact conditions determined by one side impact limit of the elongation caused by the colision with the limiter in the form (3). The impact conditions are in the form (4) or (5).

The solution of this fractional order differential equation (79) is determined by the Laplace transformation, by developing into the series (expansion) of Laplace transformation and searching the inverse Laplace transform.

It is possible to starts with the fractional order differential equation (79) in the following form:

$$\ddot{x}(t) \pm \omega_\alpha^2 x^{(\alpha)}(t) + \omega_0^2 x(t) = 0 \qquad (80)$$

This fractional order differential equation (89) for the observed vibro-impact system on the basis of the fraction order oscillator for the generalized coordinate $x(t)$ could be solved by using the Laplace transform. Thus we obtain (for detail see References by Gorenflo, R., Mainardi, F., [10], by Bačlić, B. S., Atanacković, T. M., [5] or by Hedrih (Stevanović) K. and Filipovski A. [26])

$$x(p) = \mathrm{D}\left[x(t)\right] = \frac{p x(0) + \dot{x}(0)}{p^2 + \omega_0^2\left[1 \pm \frac{\omega_\alpha^2}{\omega_0^2} \mathbf{R}(p)\right]} \qquad (81)$$

where $\mathsf{L}\left[\mathsf{D}_t^\alpha\left[x(t)\right]\right] = \mathbf{R}(p)\mathsf{L}\left[x(\mathrm{t})\right]$ is the Laplace transform of the friction order $\frac{d^\alpha x(t)}{dt^\alpha}$ for $0 \le \alpha \le 1$.For creep rheological material these Laplace transforms of the initial conditions are in the form:

$$\mathrm{L}\left[\mathrm{D}_t^\alpha\left[x(t)\right]\right] = \mathbf{R}(p)\mathrm{L}\left[x(t)\right] - \frac{d^{\alpha-1}}{dt^{\alpha-1}} x(0) = = p^\alpha \, \mathrm{L}\left[x(t)\right] - \frac{d^{\alpha-1}}{dt^{\alpha-1}} x(0) \qquad (82)$$

where the initial value is equal to

$$\left.\frac{d^{\alpha-1} x(t)}{dt^{\alpha-1}}\right|_{t=0} = 0 \qquad (83)$$

so, in that case the Laplace transform of time-function is given by the following expression:

$$\mathrm{L}\left\{T(t)\right\} = \frac{p x_0 + \dot{x}_0}{\left[p^2 \pm \omega_\alpha^{\ 2} p^\alpha + \omega_0^{\ 2}\right]} \qquad (84)$$

For the boundary cases, when material parameters α have the following values: $\alpha = 0$ i $\alpha = 1$, we have two special simple cases with known corresponding fractional-differential equations and solutions. In these cases the fractional-differential equations are:

1* $\ddot{x}(t) \pm \tilde{\omega}_{0\alpha}^2 x^{(0)}(t) + \omega_0^2 x(t) = 0$ for $\alpha = 0$ (85)

where: $x^{(0)}(t) = x(t)$, and

2* $\ddot{x}(t) \pm \omega_{1\alpha}^2 x^{(1)}(t) + \omega_0^2 x(t) = 0$ for $\alpha = 1$ (86)

where: $x^{(1)}(t) = \dot{x}(t)$.

The solutions of the differential equations (85) and (86) are:

2.1.*
$$x(t) = x_0 \cos t\sqrt{\omega_0^2 \pm \tilde{\omega}_{0\alpha}^2} + \frac{\dot{x}_0}{\sqrt{\omega_0^2 \pm \tilde{\omega}_{0\alpha}^2}} \sin t\sqrt{\omega_0^2 \pm \tilde{\omega}_{0\alpha}^2}$$

for $\alpha = 0$. (87)

2.1.* $$x(t)=e^{\mp\frac{\omega_1^2}{2}t}\left\{x_0\cos t\sqrt{\omega_0^2-\frac{\omega_{1\alpha}^4}{4}}+\frac{\dot{x}_0}{\sqrt{\omega_0^2-\frac{\omega_{1\alpha}^4}{4}}}\sin t\sqrt{\omega_0^2-\frac{\omega_{1\alpha}^4}{4}}\right\}$$

for $\alpha=1$ and for $\omega_0>\frac{1}{2}\omega_{1\alpha}^2$ (for soft creep properties) (88)

or for strong creep.

2.2.* $$x(t)=e^{\mp\frac{\omega_{1\alpha}^2}{2}t}\left\{x_0 Ch\, t\sqrt{\frac{\omega_{1\alpha}^4}{4}-\omega_0^2}+\frac{\dot{x}_0}{\sqrt{\frac{\omega_{1\alpha}^4}{4}-\omega_0^2}}Sh\, t\sqrt{\frac{\omega_{1\alpha}^4}{4}-\omega_0^2}\right\}$$

for $\alpha=1$ and for $\omega_0<\frac{1}{2}\omega_{1\alpha}^2$. (89)

For critical cases:

2.3.* $$x(t)=e^{\mp\frac{\omega_{1\alpha}^2}{2}t}\left\{x_0+\frac{2\dot{x}_0}{\omega_{1\alpha}^2}t\right\} \qquad (90)$$

for $\alpha=1$ and for $\omega_0=\frac{1}{2}\omega_{1\alpha}^2$.

The fractional-differential equation (80) for the general case, when α is a real number from the interval $0<\alpha<1$, can be solved by the Laplace transform. Using this, we obtain:

$$\mathrm{L}\left\{\frac{d^\alpha x(t)}{dt^\alpha}\right\}=p^\alpha\,\mathrm{L}\{x(t)\}-\left.\frac{d^{\alpha-1}x(t)}{dt^{\alpha-1}}\right|_{t=0}=p^\alpha\,\mathrm{L}\{x(t)\} \qquad (91)$$

and by introducing for initial conditions of fractional derivatives in the form (91), and after taking the Laplace transform of the equation (80), we obtain the corresponding equation according Laplace transform $\mathrm{L}\{x(t)\}$. By anlysing the previous Laplace transform of equation (90) of the solution, we can conclude that we can consider two cases.

For the case when $\omega_0^2\neq 0$, the Laplace transform solution can be developed into series by the following way:

$$\mathrm{L}\{x(t)\}=\frac{px_0+\dot{x}_0}{p^2\left[1+\frac{\omega_\alpha^2}{p^2}\left(\pm p^\alpha+\frac{\omega_0^2}{\omega_\alpha^2}\right)\right]}=\left(x_0+\frac{\dot{x}_0}{p}\right)\frac{1}{p}\frac{1}{1+\frac{\omega_\alpha^2}{p^2}\left(\pm p^\alpha+\frac{\omega_0^2}{\omega_\alpha^2}\right)} \qquad (92)$$

$$\mathrm{L}\{x(t)\}=\left(x_0+\frac{\dot{x}_0}{p}\right)\frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k\omega_\alpha^{2k}}{p^{2k}}\left(\pm p^\alpha+\frac{\omega_0^2}{\omega_\alpha^2}\right)^k \qquad (93)$$

$$\mathrm{L}\{x(t)\}=\left(x_0+\frac{\dot{x}_0}{p}\right)\frac{1}{p}\sum_{k=0}^{\infty}\frac{(-1)^k\omega_\alpha^{2k}}{p^{2k}}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j p^{\alpha j}\omega_\alpha^{2(j-k)}}{\omega_o^{2j}} \qquad (94)$$

In (94) it is assumed that the expansion leads to the convergent series (for detail see References by Gorenflo, R., Mainardi, F.,[10], by Bačlić, B. S., Atanacković, T. M., [5] or by Hedrih (Stevanović) K. and Filipovski A. [26]). The inverse Laplace transform $\mathrm{L}\{x(t)\}$ of the previous Laplace transform of solution (94) in term-by-term steps is based on the known theorem, and yields the following solution of differential equation (80) of the time function in the following form of time series:

$$x(t)=\mathrm{L}^{-1}\,\mathrm{L}\{x(t)\}=$$
$$=x_0\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j\omega_\alpha^{2j}t^{-\alpha j}}{\omega_o^{2j}\Gamma(2k+1-\alpha j)}+\dot{x}_0\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j\omega_\alpha^{-2j}t^{-\alpha j}}{\omega_o^{2j}\Gamma(2k+2-\alpha j)} \qquad (95)$$

Using the initial conditions of the motion and impact conditions we can get the analysis of the vibro-impact system on the basis of the fraction order oscillator.

In the previous expression (95), there are two constants determined by the initial conditions, the initial coordinate x_0, and the initial velocity $\dot{x}_0$. For each impact it is necessary to determine time $t_{ul_i-}=?$ of the i-th impact into the elongation limiter and to solve numerically the following equation:

$$x_{ul_i}=x(t_{ul_i-})=\delta,\quad i=1,2,3,...,n \qquad (96)$$

The velocity $\dot{x}(t_{ul_i+})$ at the moment after the i-th impact, i.e. at the moment of the finish of the i-th impact, is:

$$\dot{x}_{odl_i}=\dot{x}(t_{ul_i+})=\dot{x}_{odl_i}=-\dot{x}_{ul_i} \qquad (97)$$

Or in the developed form:

$$x_0\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j\omega_\alpha^{2j}t_{ul_i-}^{-\alpha j}}{\omega_o^{2j}\Gamma(2k+1-\alpha j)}+\dot{x}_0\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}t_{ul_i-}^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j\omega_\alpha^{-2j}t_{ul_i-}^{-\alpha j}}{\omega_o^{2j}\Gamma(2k+2-\alpha j)}=\delta \qquad (98)$$

For the period between the i-th and the $i+1$-th impacts, the initial conditions are:

$$x_{odl_i}=x(t_{ul_i+})=\delta \text{ and } \dot{x}_{odl_i}=\dot{x}(t_{ul_i+})=-\dot{x}_{ul_i}. \qquad (99)$$

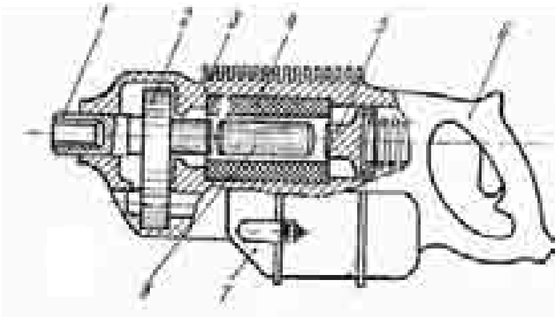
The basic problem is that, for the fractional order, the vibro-impact system analytical solutions are presented by time series expansion, but it is necessary to solve numerical roots of the expansion equal to the constant.

$$x_{i+1}(t)=\mathrm{L}^{-1}\,\mathrm{L}\{x_{i+1}(t)\}=$$
$$=\delta\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}t^{2k}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j\omega_\alpha^{2j}t^{-\alpha j}}{\omega_o^{2j}\Gamma(2k+1-\alpha j)}-\dot{x}_{il_i-}\sum_{k=0}^{\infty}(-1)^k\omega_\alpha^{2k}t^{2k+1}\sum_{j=0}^{k}\binom{k}{j}\frac{(\mp 1)^j\omega_\alpha^{-2j}t^{-\alpha j}}{\omega_o^{2j}\Gamma(2k+2-\alpha j)} \qquad (100)$$

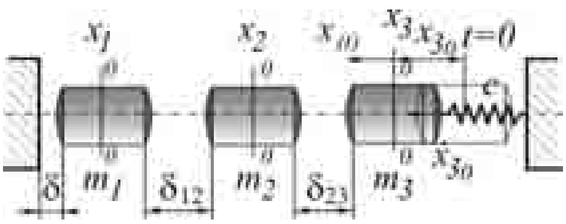
Manual stroke - rotary hammer

The manual stroke-rotary effect machine based on the vibro-impact model with three degrees of freedom, noted by the coordinate x_i, $i=1,2,3$, is used for drilling channels in rocks, concrete, asphalt, etc. The construction scheme is shown in Fig.11.a *, and a model in 1.b*.

The principle of operation of this machine consists of the following:



a*



b*

Figure 11. a* Manual stroke-rotary hammer, and b* the scheme of the dynamic model [32]

The device patron (1) receives a drive through the redactor (2) of the electro-starter (7). The operating device is located in the patron. The cabinet of the machine is connected to the electric motor (4) where the anchorless is located (8) and has the role of the striker. When the electromagnets are plugged, the striker starts the oscillatory- transactional motion. During the direct motion, the striker strikes the repulsion mass - an element which is used to alleviate the effect of the stroke (5), which is connected to the cabinet through a spring. The repulsion interface is set to reduce the effect of the strokes on the handle (6) of the machine. The dynamic model of this machine is shown in Fig.11.b *.

The dynamic model consists of three material particles with the following masses m_1, m_2 and m_3. The mass m_3 is connected by the spring with the rigidity c, and connected to the real support, while the mass m_1 performs strike action on the processed element.

The link between the dynamic model and the manual machine with a strike - rotary effect can be shown in the following way: the moving restrictor imitates the mutual effect of the operating device of the mass m_1 and external environment (the processed element). The mass m_2 is activated due to the oscillatory motion. The repulsion mass m_3 and the hardness spring c form an oscillatory system.

This model is based on the presented facts represents a one-dimensional multimass vibroimpact system of the type "chain", which, in a general case, may have different masses, and these masses form stroke pairs with different restitution coefficients during collisions (impacts). Analyses of this type of vibro-impact systems are undertaken by Кобринский А.Е., Кобринский А.А [32], through the adjustment method, by the statistical method of Бабицкий В.И. [1-3] and others.

The observed dynamic model on the basis of vibro-impact action-dynamics has three degrees of freedom. For the generalized coordinates we use: x_1, x_2 and x_3. The kinetic energy of the system is

$$Ek=\frac{1}{2}m_1x_1^2+\frac{1}{2}m_2x_2^2+\frac{1}{2}m_3x_3^2 \qquad (101)$$

Potential energy of the system is

$$Ep=\frac{1}{2}cx_3^2 \qquad (102)$$

By using Lagrange's second order differential equations or the material system dynamics, we get to the differential equations of the observed vibro-impact system dynamics in the following forms:

$$m_3\ddot{x}_3+cx_3=0$$

or

$$\ddot{x}_3+\omega_3^2x_3=0 \qquad (103)$$

$$m_2\ddot{x}_2=0 \qquad (104)$$

$$m_1\ddot{x}_1=0 \qquad (105)$$

where: $\frac{c}{m_3}=\omega_3^2$.

We join the initial conditions and corresponding impact (collision-clearance) conditions to these differential equations.

The initial conditions are:

$$x_1(0)=0, x_2(0)=0, x_3(0)=-x_{30}$$

$$\dot{x}_1(0)=0, \dot{x}_2(0)=0, \dot{x}_3(0)=\dot{x}_{30} \qquad (106)$$

First we determine under what conditions the material particle of the mass m_3 would strike the material particle of the mass m_2.

The general solution of the first differential equation (103) is

$$x_3(t)=A_3\cos\omega_3t+B_3\sin\omega_3t \qquad (107)$$

By using the initial conditions (106) we determine the integrating constants A_3 and B_3, so the particular law of motion of the material particle of the mass m_3 in the motion interval until the first collision (impact or clearance) is as follows:

$$x_3(t)=-x_{30}\cos\omega_3t+\frac{\dot{x}_{30}}{\omega_3}\sin\omega_3t \qquad (107^*)$$

In order to determine the velocity of the impact between the subsequent material particle of the system, it is necessary to determine the time period in which the material particle of the mass m_3 would strike the material particle of the mass m_2. That time point before the first impact contact between the masses is determined from the condition

$$x_3\left(t_{ul_1}\right)=\delta_{23}=-x_{30}\cos\omega_3 t_{ul_1}+\frac{\dot{x}_{30}}{\omega_3}\sin\omega_3 t_{ul_1} \quad (108)$$

The time t_{ul_1} is determined by the numerical procedure or by using the existing programs for solving equations for root determination.

In MatchCad we can require for zero functions

$$f_3\left(t_{ul_1}\right)=\delta_{23}+x_{30}\cos\omega_3 t_{ul_1}-\frac{\dot{x}_{30}}{\omega_3}\sin\omega_3 t_{ul_1}=0 \quad (109)$$

After determing t_{ul_1} (numeric value is determined from the first section of the zero line with the function graphic) we get the value of the velocity of the material particle just before the imapact (stroke):

$$v_{ul_1}=\dot{x}_3\left(t_{ul_1-}\right)=x_{30}\omega_3\sin\omega_3 t_{ul_1}+\dot{x}_{30}\cos\omega_3 t_{ul_1} \quad (110)$$

Under the assumption that the impact contact (stroke) is completely elastic $(k=1)$ and that the velocity just before the impact contact (stroke) is the same as the velocity immediately after the impact contact (stroke) with the opposite sign direction, we obtain

$$v_{odl_1}=v_{ul_1}. \quad (111)$$

The second system motion interval (after the impact contact (stroke, collision) of the material particle of the mass m_3 with the material particle of the mass m_2 is formed under the initial conditions $x_2(t_{ul_1+})=\delta_{23}$ i $\dot{x}_2(t_{ul_1+})=v_{odl_1}=v_{ul_1}$, where the time after each impact contact (stroke) runs from zero to the next impact contact (stroke).

By solving the second differential equation (105), we obtain the general law of the system motion in another time interval

$$x_2(t)=C_2 t+B_2$$

By using the initial conditions and the impact contact (stroke) conditions, we get the law of motion of the material particle of m_2 in the second movement interval.

$$x_2(t)=v_{ul_1}t \quad (112)$$

The time when the material particle of the mass m_2 contacts (strikes) the material particle of the mass m_1 can be calculated from the following conditions:

$$x_2(t_{ul_2})=\delta_{12}=v_{ul_1}t_{ul_2} \quad (113)$$

where

$$t_{ul_2}=\frac{\delta_{12}}{v_{ul_1}} \quad (114)$$

After the determination of the time interval when the material particle of the mass m_2 contacts (strikes) the material particle of the mass m_1, we can also determine the velocity of the impact contact (stroke) in this time interval.

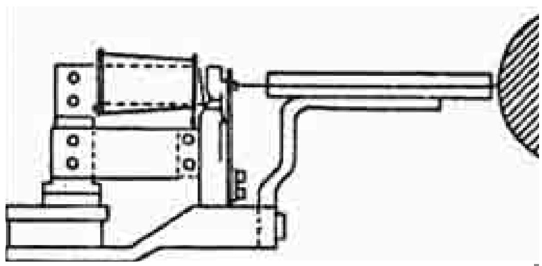
$$v_{odl_2}=v_{ul_2}=v_{odl_1} \quad (115)$$

According to this observation, we can notice that the velocity of the material particle of the mass m_1 which strikes the processed element or in our case so-called elongation restrictor is equal to the outgoing speed after the impact contact (stroke) of the material particle of the mass m_2 with the particle of the mass m_1. In this idealized case when the impact contact (stroke) is ideally elastic $(k=1)$ and when there is no notion friction, the incoming velocity during the first impact contact (stroke) is completely transferred to the processed element. After the impact contact (stroke) with the processed element the material particle of the mass m_1 is moving in the opposite direction and now it strikes the material particle of the mass m_2. A further analysis of this multi-mass vibro-impact system can be presented in some particular cases.

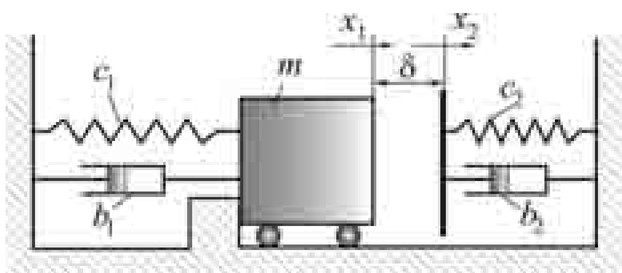
Printer

An application of the vibro-impact action is characteristic for the printer on the basis of accumulated energy (Fig.12 a*). The armature of the strike writing hammer is represented as a heavy mass connected to the left fixed wall by a spring and a damper. The spring is previously loaded and has a linear characteristic. The negative loss is characterized by the effect of the permanent magnet (for details see reference by Tung P.C., Shaw S. W.: [50] presented as a method for the improvement of the impact printer performance.

The position of the heavy material particle is limited with a simulated beam on the printer armature. The periodic sinus pulsing moving force removes the hammer so that it strikes the tape and the paper on the typing machine, which are related to the linear elastic spring and the linear damping force. A corresponding simple one-dimensional single-mass vibro-impact system is shown on the dynamic model (Fig.12 b*). A moving limiter (as a one side restrictor of elongation) is characteristic for this model. This model was studied by Tung P.C. and Shaw S.W. [50].



a*



b*

Figure 12. Printer, a* sheme [50], b* the dynamc model

This vibro-impact system is specific because it has one (out of two) degree of freedom from the moment when the material particle of the mass m_1 (the hammer) strikes the paper of negligible mass but of non-negligible high elasticity. Until that moment there are two coordinates of the movement taken: x_1 elongation of the hammer and x_2 elongation of the paper.

The kinetic energy of the system

$$Ek = \frac{1}{2} m\dot{x}_1^2 + \frac{1}{2} m_p \dot{x}_2^2 \tag{116}$$

where $m_p = 0$.

The potential energy of the system

$$Ep = \frac{1}{2} c_1^2 + \frac{1}{2} c_2^2 \tag{117}$$

The Rayleigh function of the system energy dissipation

$$\Phi = \frac{1}{2} b_1 x_1^2 + \frac{1}{2} b_2 x_2^2 \tag{118}$$

By using the Lagrange's equation of the second kind, we can write the differential equations of the system dynamics in the following form:

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + c_1 x_1 = 0 \tag{119}$$

$$b_2\dot{x}_2 + c_2 x_2 = 0 \tag{120}$$

This system of differential equations can be applied until the system dynamics when $x_1 < x_2$

In the moment when $x_1 = x_2$, the observed vibro-impact system has one degree of freedom of motion and the appropriate differential equation has the form:

$$m_1\ddot{x}_3 + (b_1 + b_2)\dot{x}_3 + (c_1 + c_2)x_3 = 0 \tag{121}$$

where x_3 is the coordinate from the moment when the hammer and the paper are in coupled contact.

In order to reach the velocity immediately before the impact contact (stroke) of the material particle of the mass *m* (the hammer) with the paper, it is necessary to solve the differential equations (119) and (120).

The general law of the system dynamics of the hammer before the coupled impact contact (stroke) by solving the differential equation (119) has the form

$$x_1(t) = e^{-\kappa t}\left(A_1 \cos\omega_p t + B_1 \sin\omega_p t\right) \tag{122}$$

where: $\kappa = \frac{b_1}{2m}$, $\omega_1^2 = \frac{c_1}{m}$, $\omega_{p1}^2 = \omega_1^2 - \kappa^2$

By using the initial conditions of the motion of the hammer $x_1(0) = 0$ and $\dot{x}_1(0) = \dot{x}_{10}$, we determine the integral constants $A_1 = 0, B_1 = \frac{\dot{x}_{10}}{\omega_{p1}}$ on the basis of which we get the law of the motion of the hammer before the coupled impact contact (stroke) on the paper.

$$x_1(t) = \frac{\dot{x}_{10}}{\omega_{p1}} e^{-\delta t} \sin\omega_{p1} t \tag{123}$$

By solving the differential equation (121) we get the expression of the law of the motion of the paper just before the impact contact (stroke).

$$x_2(t) = e^{-\frac{c_2}{b_2}t} \tag{124}$$

We consider that at the initial moment of time the paper did not move which means the coordinate is equal to zero, $x_2 = 0$.

After determinating the expression for the law of motions of the material particle of the mass *m* (the hammer) and the paper and according to the impact contact (stroke) conditions, we can determine the time of the impact contact (stroke) and the impact contact (stroke) incoming velocity.

The impact contact (stroke) conditions are $x_1 + \delta = x_2$. Since the paper does not move, $x_2 = 0$ and the time that passes before the first impact contact (stroke) is derived from the relation

$$x_1(t_{ul_1}) = \delta = \frac{\dot{x}_{10}}{\omega_{p1}} e^{-\kappa t_{ul_1}} \sin\omega_{p1} t_{ul_1} \tag{125}$$

Time t_{ul_1} is determined by a numeric procedure or by using the existing programs for equations solving in order to obtain corresponding roots.

In MatchCad we require the zero roots of the functions

$$f_1(t_{ul_1}) = \delta - \frac{\dot{x}_{10}}{\omega_{p1}} e^{-\kappa t_{ul_1}} \sin\omega_{p1} t_{ul_1} = 0 \tag{126}$$

After determining t_{ul_1} (numeric value is determined from the first section of the zero line with the function graphic), we get the velocity of the material body immediately before the stroke

$$v_{ul_1} = \dot{x}_1(t_{ul_1 -}) = -\frac{\kappa \dot{x}_{10}}{\omega_{p1}} e^{-kt_{ul_1}} \sin\omega_{p1} t_{ul_1} + \\ + \dot{x}_{10} e^{-\kappa t_{ul_1}} \cos\omega_{p1} t_{ul_1} \tag{127}$$

The expression determining time of the law of the motion of the observed vibroimpact system dynamics in a time interval after the coupling impact contact (stroke) when the hammer and the paper continue moving together until the moment when the mutual moving velocity is equal to zero $\dot{x}_3 = 0$ can be defined by solving the differential equation (121):

$$x_3(t) = e^{-\kappa_1 t}\left(A_3 \cos\omega_{p3} t + B_3 \sin\omega_{p3} t\right) \tag{128}$$

where:

$$\kappa_1 = \frac{b_1 + b_2}{2m}, \ \omega_3^2 = \frac{c_1 + c_2}{m}, \ \omega_{p3}^2 = \omega_3^2 - \kappa_1^2 \tag{129}$$

According to the impact contact (stroke) conditions $x_{3(0)} = \delta, \ \dot{x}_{3(0)} = v_{ul_1} = \dot{x}_1(t_{ul_1 -})$ we determine the integrational constants in the form:

$$A_3 = \delta, \ B_3 = \frac{\dot{x}_1(t_{ul_1}) + \kappa_1 \delta}{\omega_{p3}} \tag{130}$$

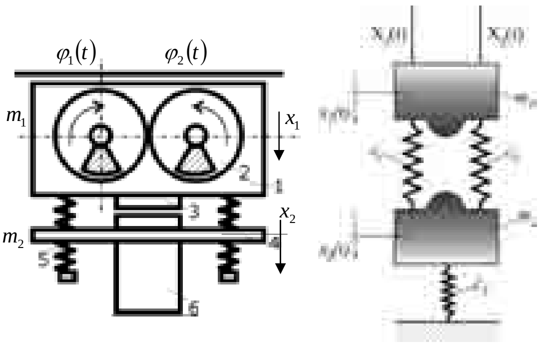
and the expression for the functional time depenent coordinate of the motion of the observed one-dimensional single-mass vibro-impact system equals to

$$x_3(t) = e^{-\kappa_1 t}\left(\delta \cos\omega_{p3} t + \frac{\dot{x}_1(t_{ul_1}) + \kappa_1 \delta}{\omega_{p3}} \sin\omega_{p3} t\right) \tag{131}$$

This expression for the functional time dependent coordinate of the motion of the observed one-dimensional single-mass vibro-impact system can be apllied until the velocity is $\dot{x}_3(t) = 0$. After that moment the further motion and dynamics of the hammer and the paper are questionable. Furter analyses can be given only for special cases.

Vibro-rammer (mechanical vibrators)

Mechanical vibratorsa re used for strong induction of vibratory and vibroimpact effects machines. Fig.13 a* gives a kinematics scheme of the vibrohammer for implementing pillars and other building elements.



a* b*

Figure 13. Vibro-rammer, a* scheme, b* dynamical model [44]

The principle of the vibro-rammer operation is as follows. The vibrohammer consists of two parts: vibromass (1) with vibrators (2) and the ramrod (3), and the lower plate which stands in hard connection with the column/pillar that is being rammed. The vibromass is connected to the plates by springs (5). Depending on the hardness and the initial regulation of the springs, the rammer and the anvil can be placed with a certain gap. Due to the spinning of the non-balance – vibrator (2), a linear oscillatory motion of the vibromass is obtained, which is followed by the periodical compact contacts of the ramrod/rammer (3) and the anvil (6).

A rough approximation of the vibro-rammer suitable for testing the vibroimpact process is a dynamical model shown in Fig.13.b* which represents a double-mass vibroimpact system with two striking pairs. This issue was studied by Регльскене В.Л. [44].

A double-mass vibroimpact system on the basis of forced oscillations has two degrees of movement freedom. The generalized coordinates are x_1 i x_2.

The kinetic energy of the vibroimpact system is

$$Ek = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \qquad (132)$$

The potential energy of the vibroimpact system has the form:

$$Ep = \frac{1}{2}2c_1\left(x_2 - x_1\right)^2 + \frac{1}{2}c_2x_2^2 \qquad (133)$$

Periodic two frequency force that initiates the impact contact of the rammer and the anvil is determined through the excitation:

$$F(t) = \left[X_1(t) + X_2(t)\right] = F_{01}\cos\Omega_1 t + F_{02}\cos\Omega_2 t \qquad (134)$$

By using Lagrange's equation of the second kind for the case of forced undamped oscillations of the double-mass vibro-impact system, we get the differential equations of motion:

$$m_1\ddot{x}_1 + 2c_1\left(x_1 - x_2\right) = F_{01}\cos\Omega_1 t + F_{02}\cos\Omega_2 t$$

$$m_2\ddot{x}_2 - 2c_2\left(x_1 - x_2\right) + c_2x_2 = 0 \qquad (135)$$

F**ree oscillations** of the double-mass vibroimpact action are solved first.

The solution of the differential equations system is derived through the first and second particular integrals, e.g.

$$x_1 = x_1^{(1)} + x_1^{(2)} = A_1^{(1)}\cos\left(\omega_1 t - \alpha_1\right) + A_1^{(2)}\cos\left(\omega_2 t - \alpha_2\right)$$

$$x_2 = x_2^{(1)} + x_2^{(2)} = = A_2^{(1)}\cos\left(\omega_1 t - \alpha_1\right) + A_2^{(2)}\cos\left(\omega_2 t - \alpha_2\right) \qquad (136)$$

The frequency equation of differential equations system (135) has the form

$$f\left(\omega^2\right) = \begin{vmatrix} \left(2c_1 - m_1\omega^2\right) & -2c_1 \\ -2c_1 & \left(2c_1 + c_2 - m_2\omega^2\right) \end{vmatrix} = 0 \qquad (137)$$

or by developing the determinant we get

$$m_1m_2\omega^4 - \left[m_1\left(2c_1 + c_2\right) + 2m_2c_1\right]\omega^2 - 2c_1c_2 = 0 \qquad (137^*)$$

The roots of the previous frequency equation are the square of the eigen frequency of free vibrations:

$$\omega_{1,2}^2 = = \frac{m_1(2c_1 + c_2) + 2c_1m_2 \pm \sqrt{\left[m_1(2c_1 + c_2) + 2c_1m_2\right]^2 + 8m_1m_2c_1c_2}}{2m_1m_2} \qquad (138)$$

with the note that $\omega_1 < \omega_2$. In order to determine the eigen amplitudes $A_1^{(1)}$ and $A_1^{(2)}$ we have to find the coefficients $\eta_{21}^{(1)}$ and $\eta_{21}^{(2)}$.

$$\eta_{21}^{(1)} = 1 - \frac{m_1}{2c_1}\omega_1^2 \text{ and } \eta_{21}^{(2)} = 1 - \frac{m_1}{2c_1}\omega_2^2 \qquad (139)$$

In this way the particular integrals (80) from the homogeneous part of the differential equations have the form

$$x_1(t) = A_1^{(1)}\cos\left(\omega_1 t - \alpha_1\right) + A_1^{(2)}\cos\left(\omega_2 t - \alpha_2\right)$$

$$x_2(t) = \eta_{21}^{(1)}A_1^{(1)}\cos\left(\omega_1 t - \alpha_1\right) + \eta_{21}^{(2)}A_1^{(2)}\cos\left(\omega_2 t - \alpha_2\right) \qquad (140)$$

By using the initial conditions of motion of the double-mass vibro-impact system dynamics $x_1(0) = 0,\ \ x_2(0) = 0,$ $\dot{x}_1(0) = \dot{x}_{10}$ and $\dot{x}_1(0) = 0$ we get the integrational constants - eigen amplitudes $A_1^{(1)}$ and $A_1^{(2)}$ for the first and the second eigen circular frequency in the following form:

$$A_1^{(1)} = \frac{\eta_{21}^{(2)}\dot{x}_{10}}{\omega_1\left[\eta_{21}^{(2)} - \eta_{21}^{(1)}\right]},\ A_1^{(2)} = \frac{\eta_{21}^{(1)}\dot{x}_{10}}{\omega_2\left[\eta_{21}^{(2)} - \eta_{21}^{(1)}\right]} \qquad (141)$$

while the angles of the phase difference are $\alpha_1 = \alpha_2 = \frac{\pi}{2}$ because $\tan\alpha_1 = \tan\alpha_2 = \infty$.

The expressions of the generalized coordinate of free oscillations of the observed double-mass vibro-impact system have the form

$$x_{1h}(t)=\frac{\eta_{21}^{(2)}\dot{x}_{10}}{\omega_1\left[\eta_{21}^{(2)}-\eta_{21}^{(1)}\right]}\cos\left(\omega_1 t-\frac{\pi}{2}\right)+ +\frac{\eta_{21}^{(1)}\dot{x}_{10}}{\omega_2\left[\eta_{21}^{(2)}-\eta_{21}^{(1)}\right]}\cos\left(\omega_2 t-\frac{\pi}{2}\right)$$

$$x_{2h}(t)=\eta_{21}^{(1)}\frac{\eta_{21}^{(2)}\dot{x}_{10}}{\omega_1\left[\eta_{21}^{(2)}-\eta_{21}^{(1)}\right]}\cos\left(\omega_1 t-\frac{\pi}{2}\right)+ +\eta_{21}^{(2)}\frac{\eta_{21}^{(1)}\dot{x}_{10}}{\omega_2\left[\eta_{21}^{(2)}-\eta_{21}^{(1)}\right]}\cos\left(\omega_2 t-\frac{\pi}{2}\right) \tag{142}$$

Forced oscillations of the observed double-mass vibro-impact system.

The expressions of the generalized coordinate of the forced oscillations of the observed double-mass vibro-impact system have the form:

$$x_{1P}=D_1\cos\Omega_1 t+D_2\cos\Omega_2 t$$

$$x_{2P}=E_1\cos\Omega_1 t+E_2\cos\Omega_2 t \tag{143}$$

where the unknown coefficients - integrational constants D_1, D_2, E_1, E_2, have the following forms:

$$D_1=\frac{\left(2c_1+c_2-m_2\Omega_1^2\right)F_{01}}{\left(2c_1+c_2-m_2\Omega_1^2\right)\left(2c_1-m_1\Omega_1^2\right)-4c_1^2}$$

$$D_2=\frac{\left(2c_1+c_2-m_2\Omega_2^2\right)F_{02}}{\left(2c_1+c_2-m_2\Omega_2^2\right)\left(2c_1-m_1\Omega_2^2\right)-4c_1^2}$$

$$E_1=\frac{2c_1F_{01}}{\left(2c_1+c_2-m_2\Omega_1^2\right)\left(2c_1-m_1\Omega_1^2\right)-4c_1^2} \tag{144}$$

$$E_2=\frac{2c_1F_{02}}{\left(2c_1+c_2-m_2\Omega_2^2\right)\left(2c_1-m_1\Omega_2^2\right)-4c_1^2}$$

so the particular solutions of the two frequency forced oscillations of the observed double-mass vibro-impact system are:

$$x_{1P}=\frac{\left(2c_1+c_2-m_2\Omega_1^2\right)F_{01}}{\left(2c_1+c_2-m_2\Omega_1^2\right)\left(2c_1-m_1\Omega_1^2\right)-4c_1^2}\cos\Omega_1 t+ +\frac{\left(2c_1+c_2-m_2\Omega_2^2\right)F_{02}}{\left(2c_1+c_2-m_2\Omega_2^2\right)\left(2c_1-m_1\Omega_2^2\right)-4c_1^2}\cos\Omega_2 t$$

$$x_{2P}=\frac{2c_1F_{01}}{\left(2c_1+c_2-m_2\Omega_1^2\right)\left(2c_1-m_1\Omega_1^2\right)-4c_1^2}\cos\Omega_1 t+ +\frac{2c_1F_{02}}{\left(2c_1+c_2-m_2\Omega_2^2\right)\left(2c_1-m_1\Omega_2^2\right)-4c_1^2}\cos\Omega_2 t \tag{145}$$

The principle of movement of the observed double-mass vibroimpact system is:

$$x_1(t)=x_{1h}(t)+x_{1p}(t)$$

$$x_2(t)=x_{2h}(t)+x_{2p}(t) \tag{146}$$

The impact contact condition is

$$x_2(t)=x_1(t) \tag{147}$$

According to this condition the time t_{ul_1} when the first impact contact (collision) will occur is determined by:

$$x_2(t_{ul_1})-x_1(t_{ul_1})=0\,. \tag{148}$$

The speed just before the first stroke is determined when we put the time t_{ul_1} in the pattern of the speed of the firs material particle of the mass m_1.

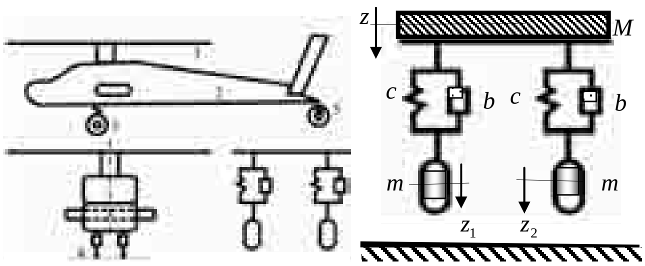
$$v_{ul_1}=\dot{x}_1(t_{ul_1})=\dot{x}_{1h}(t_{ul_1})+\dot{x}_{1P}(t_{ul_1}) \tag{149}$$

After the determination of the principle of system motion of the observed double-mass vibro-impact system, the time when the first impact contact (collision) occurs and the velocity just before the first impact contact (collision) can be individually derived for some particular examples as well as a more detailed analysis of the motion of the previously mentioned system dynamics after the impact contact (collision).

Concluding remarks

In the monograph [44] written by Ragulskene V. L.: stereo mechanical metods are presented applied to the solutions of the vibroimpact system dynamics.

In references [4] and [6] writen by Bapat, C. N., Popplewell N. we can see a model of the helicopter presented in Fig.14. The title *Several similar vibroimpact systems* also indicates that there are possibilities for considering mathematical analogy and phenomenological mapping of identification of same models applied in the different technological processes and principles of vibroimpact machine works.



a* b*

Figure 14. Helicopter a* and model of the rolling systems for airplane landing gears b*.

Fig.14 represents a helicopter a* and a model of the rolling systems for airplane landing gears. The abstracted model of the vibro-impact dynamics of the landing gear rolling system (b*) is partly possible to be investigated as the models presented in Part 2.3. as well as in Part 2.4. or as two subsystems coupled.

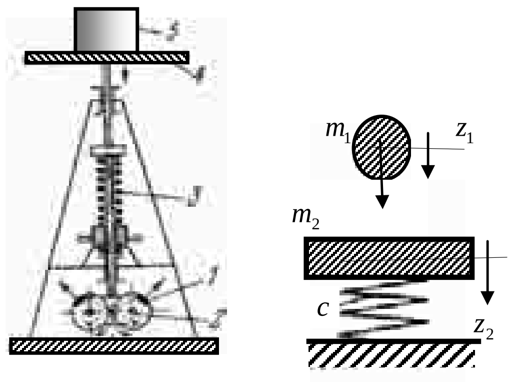


Figure 15. Vibroimpact platform a* and a model of the abstracted real system of the vibroimpact platform b*.

In reference [32] written by Kobrinskii A.E. And Kobrinskii A.A., a vibroimpact platform presented in Fig.15.a* and a model of the abstracted real system of the vibroimpact platform b* are considered. We can see that the presented model is similar to an example presented in Part 2.1. with modifications enabling the motion of the impact limiter.

It can be concluded that for the investigation of the vibro-impact system dynamics, it is very important to identify all dynamic singular processes in the no impact system dynamics which is a basis for the vibro-impact dynamics in civil engineering. It can be also seen that impacts introduce into system dynamics discontinuity expressed as alternations of the velocity directions and this jumps is strong nonlinearity of the vibro-impact system dynamics.

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Modeli tehnoloških procesa na bazi vibroudarne dinamike

U radu je predstavljena serija modela tehnoloških procesa, na kojima se baziraju realno izvedene inženjerske konstrukcije sa odgovarajućim modelima vibroudarne dinamike, koje su u osnovi njihovog funkcionisanja. Izabrani su karakteristični modeli, koji se javljaju u naučnim monografijama vodećih naučnika i istraživača iz oblasti vibroudarne dinamike i za svaki model su data osnovna analitička rešenja za bazne sisteme na kojima se zasniva odgovarajući model vibroudarne dinamike. Autori ovog rada sistematizovali su moguća analitička rešenja sa odgovarajućim početnim uslovima i uslovima udara, odnosno sudara za izabrane primere i obogatili ih nizom sopstvenih doprinosa vizuelizacijom u vidu faznih portreta, grafika kinetičke, potencijalne i totalne mehaničke energije sistema, kao i amplitudno-frekventnim i fazno'frekventnim krivim. Kao primeri za analizu karakteristične vibroudarne diname izabrani su modeli: vibroudarni čekić, ručno rotacioni čekić, štampač, vibronabijač. mehanizam točkova za sletanje aviona ili helikoptera i slično. Za svaki od apstrahovanih modela napisana je odgovarajuća diferencijalna jednačina, integro-differencijalna jednačina ili diferencijalna jednačina necelog (frakcionog) reda, odnosno sistem diferencijalnih jednačina, kojima su pridruženi početni uslovi i uslovi udara i prikazana metodologija odredjivanja kinetičkih parametara udara u trenutku, neposredno pre i posle, udara i prikazana vizuelizacija istih u faznoj ravni.

Ključne reči: **udar, udarno dejstvo, vibracije, vibroudarna dinamika, dinamika procesa, diferencijalne jednačine.**

Модели технологических процессов на основе виброударной динамики

В настоящей работе представлена серия моделей технологических процессов, на которых базируются реально выведённые инженерные конструкции со соответствующими моделями виброударной динамики, находящиеся в основании их функционирования. Избраны характерные модели, появляющиеся в научных монографиях ведущих-передовых учёных и исследователей из области виброударной динамики и для каждой модели даны основные аналитические решения для базисной системы, на которых обосновывается соответствующая модель виброударной динамики. Авторы настоящей работы систематизировали возможные аналитические решения со соответствующими начальными условиями и условиями удара, т.е. столкновения за избраные примеры и обогащали их рядом собственных выходов визуализацией в виде фазовых портретов, графиков кинетической, потенциальной и совокупной механической энергии системы, подобно и амплитудно-частотным и фазово- частотным кривым. Примерами для анализа характерной виброударной динамики избраны модели: виброударный молоток, ручной вращающийся молоток, вибротрамбовка, типограф, механизм колёс для посадки самолёта или вертолёта и.т.п. Для каждой из отвлечённых моделей написано соответствующее дифференциальное уравнение, интегродифференциальное уравнение или дифференциальное уравнение неполного (фракционного) порядка, т.е. система дифференциальных уравнений, к которым присоединены исходные условия и условия удара и показана методология определения кинетических параметров удара в моменте непосредственно перед и после удара и показана их визуализация в фазовой плоскости.

Ключевые слова: **Удар, ударное действие, колебания (вибрации), виброударная динамика, динамика процесса, дифференциальные уравнения.**

Modèles des processus technologiques basés sur la dynamique vibratoire impact

Une série de modèles des processus technologiques servant de base pour les constructions réelles faites avec les modèles correspondants de la dynamique vibratoire impact qui sont à la base de leur fonctionnement sont présentés dans ce papier. On a choisi les modèles caractéristiques qui figurent dans les monographies des savants renommés et des chercheurs dans le domaine de la dynamique vibratoire impact et on a donné les solutions analytiques de base pour les systèmes basiques sur lesquels est fondé le modèle correspondant de la dynamique citée. Les auteurs de cette étude ont systématisé les solutions analytiques possibles avec les correspondantes conditions initiales d'impact, concernant la collision pour les exemples choisis, et les ont enrichis par une série de propres contributions à l'aide de la visualisation en forme des portraits de phase , de graphique de cinétique potentielle et totale de l'énergie mécanique du système, ainsi que par les courbes amplitude fréquence et courbes phase fréquence. Comme exemples pour l'analyse des propriétés de la dynamique vibratoire impact on a choisi les modèles suivants : marteau vibratoire impact, marteau rotatif à main, le train d'atterrissage des avions et des hélicoptères, etc. Pour chaque modèle cité on a écrit l'équation différentielle correspondante ou l'équation différentielle fractionnelle où l'on a ajouté les conditions initiales et d'impact et on a exposé la méthodologie pour la détermination des paramètres cinétiques de l'impact juste avant, pendant et après l'impact et on a présenté leur visualisation dans le plan de phase.

Mots clés: **impact, effet d'impact, dynamique vibratoire impact, dynamique de processus, équations différentielles.**