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Survey of the Geometric Approach to the Modern Control Theory

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The geometric approach is a mathematical concept developed to improve the analysis and synthesis of linear multivariable systems. In this article we summarized the foundations for the geometric approach to both linear and singular systems. A mathematical introduction to the controlled and conditioned invariant subspaces has been presented. Additional characteristic subspaces were introduced, such as maximum-controlled and minimum-conditioned ones, as well as output-null invariant subspaces. The solution of the disturbance localization problem with respect to the system stability was described. The mathematical and the geometric tools applied in the control system theory were presented. The geometric approach was systematically applied to the controllability analysis and the pole assignment procedure. The existence and uniqueness of the solutions for singular systems was separately analyzed due to specific system characteristics which are a consequence of the singular matrix in the state space model. The purpose of the article was to give a comprehensive overview of the geometric approach to the control theory.

Key words: **control theory, system control, singular system, system stability, system analysis, geometric approach.**

Introduction

WHEREVER the classic theory of automatic control is not applicable, scientists search for a new technique to solve problems. They define new principles and develop new mathematical theories. As a result we get new aspects, approaches, and theories. In this article we have described the genesis of the geometric theory and some of its significant results. New results have been introduced as well, based on the previous research *Buzurovic*, *Debeljkovic* (2007).

What is the geometric theory? Many authors consider the notion of geometry in the system theory as mutual characteristics of the matrix pencils (A, B) or (A, C) for linear systems or (A, E) for singular systems. Other authors consider the geometric aspect as a study of characteristic subspaces of a system. Some authors accept this notion intuitively without adopting any specific definition. In *Buzurovic* (2000) the descriptive definition of the geometric approach to linear singular systems was presented. The geometric approach (or aspect) should be understood as an approach to a study of the singular systems the purpose of which is to determine and investigate characteristic subspaces which play a crucial role in the analysis of the matrix pencils (A, E).

In the following part, we present the origin and the basic principles of the geometric theory for–linear and linear singular systems as well. To investigate the geometric approach to both the linear and the singular systems it is necessary to understand the linear algebra and the matrix calculus. The basic literature for that is given by *Gantmacher* (1977.*a*.*b*) and *Gelfand* (1950). It is suggested to understand the subspaces theory, linear transformations, singular linear transformations in variant subspaces, dual subspaces as well as generalized pseudo-inversion (Drasin and Moore- Penroso inversions), *Campbell* (1980.*a*), *Campbell* (1980.*b*). Moreover, it is recommended to be familiar with the mathematical principles of the singular decompositions, matrix and norms of subspaces, isomorphism and matrix projection in order to analyze dynamics of singular systems, *Basile*, *Marro* (1992), *Wonham* (1995). Several definitions of the invariant and almost invariant subspaces are given in this article, but broader understanding of these notions is of benefit. The mathematical principles of the geometric concept could be found in *Debeljkovic*, *Buzurovic* (2007).

The geometric approach was first discussed in the articles published by *Basile*, *Marro* (1969.a) and *Wonham*, *Morse* (1970). They discovered that dynamic behavior of the time invariant linear control systems could be investigated based on characteristics of the system invariant subspace of the matrices. As a result, the system behavior could be predicted and the solution of many control problems could be tested by investigating characteristics of the subspaces described. The basic idea of this approach was the application and calculation of subspaces on the computers using algorithms developed for that purpose. With all this in mind, it was shown in the literature that the geometric approach can be used to solve a variety of problems, including finding a control law for systems with feedback, observability problems, disturbance localization, design of the observers, control and tracking, robust control, etc.

It can be concluded that the geometric approach was a mathematical concept developed in order get a better understanding and to give better insight into the most important characteristics of the linear system dynamics. It is mostly represented in the state space domain and used to

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connect characteristics of single and multiple transfer systems. In the literature, the geometric approach did not always rely on algorithms and computer applications, but it should be kept in mind that it was the basic idea when this approach first appeared. The geometric approach was partly or completely unclear due to different notations and definitions of the basic notions. One of the reasons for making this approach hardly acceptable was using unnecessary complicated mathematics introduced by some researchers.

In the following part the geometric concept was applied to investigate the behavior of linear systems. The extension of the theory to the linear singular system has been presented as well.

Literature review of the linear systems

After publishing the first paper on this topic by *Basile*, *Marro* (1969.*a*) the same authors together with *Laschi* published further results in five papers (three in Italian, two in English) applying the geometric theory of *Basile*, *Marro* (1969.*b.c*). They found solutions for disturbance resolving and observability with unknown input. The first paper published by them introduced controlled and controlled-invariant subspaces. The same authors developed stability criteria using finite and half hidden-condition subspaces which make solving stability problems possible. These two authors introduced robust controlled subspaces and applied the described subspaces in computer science. In the books *Basile*, *Marro* (1992) and *Wonham* (1985), new algorithms for dual subspaces were introduced. *Wonham* (1985) is probably the most comprehensive material for the geometric approach application to the linear control system. Unfortunately, this book does not deal with singular systems. Wonham and Morse applied the algorithm for calculation of the maximal controlled invariant subspaces to determine system motion. They solved the coupling problem for mutual independent control, *Wonham*, *Morse* (1970). In the *Wonham* (1985) he used term (A, B) invariant subspace instead of AB controlled invariant subspace. Until that moment, the latter was defined on a different way. Because of this, many authors have considered Wonham responsible for confusion in the terminology. The condition invariant dual object was named AC invariant or CA invariant. As a consequence, the system of the matrices notation was restricted to (A, B, C). In fact, it is considered that the controlled invariant subspace is an element of the family the elements of which belong to the controlled invariant set. The theory of controlled and almost conditioned invariant subspaces appeared in the 80s, *Willems* (1981, 1982). This theory solved control problems for systems with high gain.

Literature review of the singular system

The application of the concept to the singular system started in the beginning of the 80s and reached its peak during the 90s. *Yip*, *Sincovec* (1981) extended the geometric concept given in *Wonham* (1985) to the singular system. The system that they analyzed was not decomposed to the linear part and algebraic equations but it was treated as a compound system. The authors analyzed the existence of solution, controllability and observability of singular systems. *Grassman* (1982) suggested the geometric approach to the solution of the optimization problem in the singular system. As a result, he determined system boundaries where the motion was optimal. An outstanding contribution to this field was made by *Cobb*. His articles (1983.*a*), (1984.*b*) became a milestone for the geometric approach to the singular systems. The author investigated optimal control problems for singular systems without constraints to the initial conditions. As a result he introduced an algorithm for the system gains which eliminated impulse behavior. Also, he investigated the influence of the initial conditions on system dynamics. *Cobb* (1984) is maybe the most important article in the geometric theory. He investigated controllability and observability. As a result, he presented the mutual analysis of the qualitative and the geometric approach to the singular system with non-consistent initial conditions. *Lewis* (1984) discussed the geometric approach to digital singular system decomposition. He mathematically described subspaces for initial and finite distributed systems of the matrix pencils. The article *Malabre*, *Kucera* (1984) introduced geometric characteristics of the system with infinite zeros. The authors investigated the existence of solutions for the linear systems. *Malabre* (1987) used the previous results to extend analyses to singular systems. He defined its invariant subspaces. *Malabre*, *Bonilla* (1993) showed geometric characteristics of the matrix triple (E, A, B). It means that they analyzed system with non-zero initial conditions. Furthermore, they defined the minimal dimension of the system. *Garcia*, *Malabre* (1995) showed that the coupling problem column by column and stability system problem with disturbances analyzed on the same system implied a general solution if and only if each system has an independent solution itself. The result was obtained by the geometrical approach. In the article *Malabre et al*. (1998) the new methodology for pole assignment for singular systems was developed using the geometric method. The same method was applied later to the linear system. *Perdon* (1989) analyzed the influence of the condition invariant sub-module of the singular system with time delay to the observer design problem. *Wyman et al*. (1989) introduced controllability indices together with observability of singular systems. Both techniques, algebraic and geometric one, were applied. As a result, they gave the geometric algorithm for the observability subspace calculation. *Schrader*, *Wyman* (1995) continued with investigation of the controllability for singular systems. They presented the geometric condition for system controllability. In the *Shyman*, *Zhou* (1987) and *Zhou*, *Shyman* et al. (1987) the authors presented the methodology for linear singular system synthesis using proportional and differential feedback with constant gain. The second article analyzed geometric conditions for the controllability and observability of the systems. Their analyses included procedure of the pole assignment. Undoubtedly, an outstanding contribution to the geometry of the singular system was performed by authors *Ozcaldiran*, *Lewis*, *Syrmos* and *Karamancioglu*. They published most of their articles during the 80s and 90s. The article *Ozcaldiran*, *Lewis* (1987) introduced the extension of the pole assignment concept to the linear singular systems using the geometric approach. The method for the system eigenvector calculation was introduced. In the *Lewis*, *Ozcaldiran* (1989) a definition of the output-null subspace was introduced. The relationship between the subspaces mentioned to the other subspaces of the singular systems was defined. The invariant subspaces definitions together with the existence of the solutions were described in the *Karamancioglu*, *Lewis* (1990). *Ozcaldiran*, *Lewis* (1990) analyzed the observability problem for the singular systems. They derived geometric conditions for the three observability classes: effective observability, observability with full and initial conditions, and state trajectory observability. *Lewis*, *Syrmos* (1991) analyzed the effect of the feedback using the geometric approach. The authors solved the problem of the assignment for finite and infinite zeros. They concluded equivalency between pole assignment for the reachable systems and the systems with feedback. Decomposition of the singular systems using direct sums was introduced in *Ozcaldiran* (1991).

In the following part the latest articles in this field will be discussed briefly. *Dam et al.* (1997) applied the geometric approach to the constraint systems where the disturbances influenced singularity of the mathematical models. The article *Ishihara*, *Terra* (2001) introduced controllability and observability analyses for the singular systems with rectangular matrices. The necessary and sufficient conditions for the feedback design were presented, the suggested feedback eliminated the influence of the impulse to the dynamic behavior of the system. *Yu*, *Wang* (2002) analyzed geometric characteristics of decentralized singular systems under the influence of local feedback. The authors used the geometric approach for controllability and observability investigation. *He* (2003) introduced the geometric decomposition of the singular systems and the structure and dynamical behavior of redundant systems. The geometric approach was applied to the singular stochastic system, *Germani et al.* (2004). *Xie*, *Wang* (2004). They presented several criteria for the investigation of different controllability types. The new reachability conditions for the special classes of the singular systems were presented in *Meng*, *Zhang* (2006).

Results in the field of the linear systems

Preliminaries

For the linear transformation it is not always possible to introduce restrictions of the vector spaces $\Re$ and $\Re_1$ because the vectors from $\Re_1$ can be defined only under specific conditions. That characteristic is described by the following definition.

Definition 1: Denote A as a linear transformation defined under the vector subspace $\Re$. Subspace $\Re_1$ of the space $\Re$ is called invariant subspace in relation to A if $\mathbf{x}\in \Re_1$ leads to A$\mathbf{x}\in \Re_1$.
Gantmacher (1977.b).

A similar way to express the previous definition is given below.

Definition 2: The subspace S of vector space $\Re$ is called invariant space of the linear transformation A over S if and only if A$S \subseteq S$.
Gantmacher (1977.b)

Definition 3: For the subspace v of $\mathcal{X} = \Re^n$ is said to be A invariant if $A\mathcal{V} \subset \mathcal{V}$.
Gantmacher (1977.b)

The *A*-invariant subspace plays an important role in analyses of linear systems with zero input. Consider the system:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0\,, \tag{1}$$

Where the vector space is $\mathbf{x}(t) \in \mathcal{X} = \Re^n$, $\mathbf{x}_0 \in \mathcal{X}$, $\mathcal{X}$ and $\mathcal{X}$ is the column subspace of the space vector. *A* je $n \times n$ is the real matrix. Assume that $\mathcal{V}$ is the *A*-invariant subspace. If the $\mathbf{x}(t) \in \mathcal{V}$ it means that speed $\dot{\mathbf{x}}(t) \in \mathcal{V}$, which implies that the state space of the system remains in the subspace $\mathcal{V}$.

***Lemma* 1**: Let the $\mathcal{V} \subset \mathcal{X}$. If for the system (1) with null input $\mathbf{x}_0 \in \mathcal{V}$ is fulfilled, it implies that $\mathbf{x}(t) \in \mathcal{V}$, for each $t \geq 0$ if and only if $\mathcal{V}$ is *A*-invariant subspace.
Buzurović (2000)

Denote by $\mathcal{V}$ an *A*-invariant subspace and denote by this $\{e_1, \ \cdots \ , e_n\}$, base of any vector subspace which fulfils the following

$$span\{e_1, \ \cdots \ , e_v\} = \mathcal{V},\ v \leq n, \tag{2}$$

with the coordinate transformation defined by rule 3.3:

$$\mathbf{x} = [e_1 \ \ ... \ \ e_n]\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix}, x \in \Re^n, \tilde{x}_1 \in \Re^v, \tilde{x}_2 \in \Re^{n-v} \tag{3}$$

Now it can be concluded that with respect to new base, the system can be described as

$$\begin{bmatrix}\dot{\tilde{x}}_1\\ \dot{\tilde{x}}_2\end{bmatrix} = \begin{bmatrix}\tilde{A}_{11} & \tilde{A}_{12}\\ 0 & \tilde{A}_{22}\end{bmatrix}\begin{bmatrix}\tilde{x}_1\\ \tilde{x}_2\end{bmatrix},$$
$$\tilde{x}_1(0) = \tilde{x}_{10}, \quad \tilde{x}_2(0) = \tilde{x}_{20}. \tag{4}$$

It is clear if $\tilde{x}_{20} = 0$, then $\tilde{x}_2(t) = 0$ for any $t \geq 0$, that is $x_0 \in \mathcal{V}$ which implies $x(t)\in \mathcal{V}$ for any $t \geq 0$, which is the statement at *Lemma* 1.

Let $\mathcal{V}$ be an *A*-invariant subspace in $\mathcal{X}$.

Definition 4: *Restriction* $A|\mathcal{V}$ of the linear transformation *A*: $\mathcal{X} \to \mathcal{X}$ (or n × n real matrix *A*) on the subspace $\mathcal{V}$ is the linear transformation from $\mathcal{V}$ to $\mathcal{V}$ denoted by $v \to A\,v$ for each $v \in \mathcal{V}$.
Gantmacher (1977.b)

Definition 5: For any $x \in \mathcal{X}$, $x + \mathcal{V} = \{x + v : v \in \mathcal{V}\}$ is called difference (coset) of *x* modulo $\mathcal{V}$.
Gantmacher (1977.b)

Coset represents a hyper plane with the point *x* on it. The set of the differences by modulo $\mathcal{V}$ is the vector subspace so-called factorized subspace (or denoted subspace) and it is written as $\mathcal{X}/\mathcal{V}$.

Definition 6: *Inductive transformation* $A|\mathcal{X}/\mathcal{V}$ is a linear transformation defined as: $x + \mathcal{V} \to Ax + \mathcal{V}$, $x \in \mathcal{X}$.
Basile, *Marro* (1992).

Definition 7: *A*-invariant subspace $\mathcal{V}$ is internally stabile if $\tilde{A}_{11}$ from the equation (4) is stabile, (all eigenvalues have a negative real part) or equivalent if $A|\mathcal{V}$ is stabile.
Basile, Marro (1992).

Consequently, $x(t)$ converges to the null state when $t\to\infty$ whenever $x_0 \in \mathcal{V}$ if and only if $\mathcal{V}$ is internally stabile.

Definition 8: *A*- invariant subspace $\mathcal{V}$ is *externally stabile* if $\tilde{A}_{22}$ from equation (4) is stabile, or equivalently, if $A|\mathcal{X}/\mathcal{V}$ is stabile.
Basile, Marro (1992).

It means that $x_2(t)$ converges to zero as $t\to \infty$, that is $x(t)$ converges to $\mathcal{V}$ when $t\to \infty$, if and only if $\mathcal{V}$ is externally stabile. It means that eigenvalues of the matrices $\tilde{A}_{11}$ and $\tilde{A}_{22}$ are independent on specific choice of the coordinates as long as equation (2) is satisfied.

Now let us consider a continual invariant linear system described as $\sum := [A, B, C]$ where

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{x}_i(t) &= C\mathbf{x}(t)\end{aligned}, \tag{5}$$

where $\mathbf{x}(t) \in \mathcal{X} := \Re^n$, $\mathbf{u}(t) \in \Re^m$ and $\mathbf{x}_i(t) \in \Re^p$ are the column system state input and the system output during the time respectively, and $t \geq 0$, a $\mathbf{x}_0(t) \in \Re$ is initial state. The matrices A, B and C are system matrices of the constant dimensions. Assume that the function $u(t)$ is a continual function. The system with feedback is investigated. The equation of feedback is given by:

$$\mathbf{u}(t)=F\mathbf{x}(t), \tag{6}$$

so eq. (5) becomes

$$\dot{\mathbf{x}}(t) = (A + BF)\mathbf{x}(t), \quad x(0) = x_0 \tag{7}$$

where F $m\times n$ is real matrix.

Some invariant subspaces are responsible for the system reachability, controllability and observability.

Definition 9: It is said that the state $\tilde{x}$ of system (5) is reachable (controllable) if there is a control which drives the system zero state to the state $\tilde{x}$ ($\tilde{x}$ to the zero state) at the finite time i.e. if control $u(t)$, $0 \leq t \leq t_f$ exists so that is $x(0) = 0$ (or respectively, $\tilde{x}$) and if $x(t_f) = \tilde{x}$ (or respectively 0) for any $0 < t_f < \infty$.

The set of the reachable (or controllable) states forms the subspace which is called *reachable (controllable) subspace* denoted by $\mathcal{V}_{\text{dost}}$ ($\mathcal{V}_{\text{upr}}$). For $n \times n$ real matrix M and subspace $\mathcal{I} \subset \mathcal{X}$ it can be written:

$$\mathcal{R}(M, \mathcal{I}) := \mathcal{I} + M\,\mathcal{I} + \ldots + M^{n-1}\,\mathcal{I}. \tag{8}$$

In the following part, the characteristics of reachable (controllable) subspaces are presented.

***Theorem* 1**: For the continuous time system $\sum := [A, B, C]$ it is satisfied

$$\mathcal{V}_{\text{dost}}=\mathcal{R}(A, \mathcal{R}(B))=\mathcal{R}[B\ AB\ldots A^{n-1}B]=\mathcal{V}_{\text{upr}}. \tag{9}$$

Corollary 1: Subspace $\mathcal{V}_{\text{dost}}= \mathcal{R}(A, \mathcal{R}(B))$ is an A-invariant. Also, it is (A+BF)-invariant subspace for any $m \times n$ real matrix F.

Basile, Marro (1992).

Definition 10: For the matrix pencil (A, B) or the system $\sum := [A, B, C]$ is said to be reachable (controllable) if and only if $\mathcal{V}_{\text{dost}}=\mathcal{X}$($\mathcal{V}_{\text{upr}}=\mathcal{X}$).

Set $\Lambda:=\{\lambda_1 \ldots \lambda_n\}$ of the complex numbers is called a symmetric set for any λ_i which is not a real number, $\lambda_j = \lambda_i^*$ for any j=1,…, n where λ_i^* is complex conjugate value of λ_i . Denoting $\sigma(A+BF)$ as a spectrum i.e. eigenvalues set of the (A+BF), the following theorem can be introduced:

Theorem 2: For any symmetric set $\Lambda:=\{\lambda_1, \ldots, \lambda_n\}$ of the complex numbers $\lambda_1, \ldots, \lambda_n$, exists $m \times n$ real matrix F with characteristics $\sigma(A+BF) = \Lambda$ if and only if the matrix pencil (A, B) is reachable (or controllable).

Corollary 2: Let the $\mathcal{V}_{\text{dost}} = r \leq n$. For any symmetric set $\Lambda:=\{\lambda_1, \ldots, \lambda_n\}$ of the complex numbers $\lambda_1, \ldots, \lambda_n$, exists $m \times n$ matrix F with the characteristics $\sigma(A+BF \mid \mathcal{V}_{\text{dost}}) = \Lambda$.

Proof: The previous equation can be understood as a coordinate transformation using A-invariant subspace of $\mathcal{V}_{\text{dost}}$, equation (4).

The matrix pencil (A, B) is said to be potentially stabile if the real matrix F exists with the characteristics that the pencil (A+BF) has negative real parts.

Consequence 1: Matrix pencil (A, B) is potentially stabile if and only if $\mathcal{V}_{\text{dost}}$ is externally stabile.

Definition 10: The state $\tilde{x}$ of system (6) is said to be unobservable if it produces zero output of the system without input that is if $x(0) = \tilde{x}$ and $u(t) = 0$ for any $t \geq 0$.

The set of the unobservable state forms subspace called unobservable subspace. This subspace is denoted by $\mathcal{V}_{\text{neos}}$.

***Theorem* 3**: The unobservable subspace has the following characteristic:

$$V_{neos} = \mathcal{N}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1}\end{bmatrix} = \mathcal{N}(C) \cap A^{-1}\mathcal{N}(C) \cap \ldots \cap A^{-(n-1)}\mathcal{N}(C). \tag{10}$$

Basile, Marro (1992).

Corollary 3: The subspace $\mathcal{V}_{neos}$ is A-invariant. It is also (A+GC)-invariant for any $n \times p$ real matrix G.

The matrix pencil (A, C) is said to be *observable* if it is $\mathcal{V}_{neos}$ =0. The matrix pencil (A, C) is said to be *detectable* if (A+GC) is stabile for the real matrix G. The following theorem is applied to observability and detectability investigation.

***Theorem* 4**: For any symmetric set $\Lambda:=\{\lambda_1 \ldots \lambda_n\}$ of the complex numbers $\lambda_1, \ldots, \lambda_n$, there exists $n \times p$ real matrix G with the characteristics $\sigma(A+GC) = \Lambda$ if and only if the matrix pencil (A, C) is observable.

Basile, Marro (1992).

Consequence 2: The matrix pencil (A, C) is detectable if and only if $\mathcal{V}_{neos}$ is internally stabile.

For the calculation of the subspaces it is convenient to use the following formulas:

***Lemma* 2**: Let $\mathcal{V}$, $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3 \subset \mathcal{X}$. Then, $\mathcal{V}^{\perp}:=\{x\in \mathcal{X}: x'v = 0,\ \forall\ v\in\mathcal{V}\}$.

- $(\mathcal{V}^{\perp})^{\perp}= \mathcal{V}$,
- $(\mathcal{V}_1 +\mathcal{V}_2)^{\perp}= \mathcal{V}_1^{\perp} \cap\mathcal{V}_2^{\perp}$,
- $(\mathcal{V}_1\cap\mathcal{V}_2)^{\perp}= \mathcal{V}_1^{\perp} + \mathcal{V}_2^{\perp}$,
- $A(\mathcal{V}_1 +\mathcal{V}_2)= A\mathcal{V}_1 + A\mathcal{V}_2$,
- $A(\mathcal{V}_1 \cap\mathcal{V}_2) \subset A\mathcal{V}_1\cap A\mathcal{V}_2$,
- $(A_1 \cap A_2)\ \mathcal{V}= A_1\mathcal{V} + A_2\mathcal{V}$, where A_1 and A_2 are $n\times n$ matrices,
- $(A\mathcal{V}^{\perp}) =A^{-1}\ \mathcal{V}^{\perp}$,
- $\mathcal{V}_1 + (\mathcal{V}_2 \cap \mathcal{V}_3)\subset (\mathcal{V}_1 +\mathcal{V}_2) \cap (\mathcal{V}_1 +\mathcal{V}_3)$,
- $\mathcal{V}_1 \cap (\mathcal{V}_2 + \mathcal{V}_3)\supset (\mathcal{V}_1\cap\mathcal{V}_2) + (\mathcal{V}_1\cap\mathcal{V}_3)$,
- $\mathcal{V}_1\cap(\mathcal{V}_2+\mathcal{V}_3)=\mathcal{V}_2 +\mathcal{V}_1 \cap \mathcal{V}_3$, uz $\mathcal{V}_1 \supset \mathcal{V}_2$.

Hamano (1996).

(A, $\mathcal{R}$(B)) - *controlled and* (A,$\mathcal{N}$(C)) - *conditioned invariant subspaces and duality*

In this part, the new subspaces with relation to system (5) and their characteristics are investigated. According to *Lemma* 1, for linear system (1) with zero input A-invariant subspace there is a subspace with a characteristic that if any system trajectory starts from that subspace it stays there. However, for linear systems with non-zero input A-invariability does not guarantee the previous characteristic. Let $\mathcal{V} \subset \mathcal{X}$. Based on the previous, the new result can be presented.

***Theorem* 5**: The motion of the system is constrained in the subspace $\mathcal{V}$, if and only if $\dot{\mathbf{x}}(t) \in \mathcal{V}$.

Proof: To be fulfilled $\dot{\mathbf{x}}(t) \in \mathcal{V}$ for any $\mathbf{x}(t) \in \mathcal{V}$, it is necessary that $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) = v(t) \in \mathcal{V}$, and that $A\mathbf{x}(t) = v(t) - B\mathbf{u}(t)$ for some $u(t)$ and $v(t)$, which implies $A\mathcal{V} \subset \mathcal{V} + \mathcal{R}(B)$. The opposite is fulfilled as well. The similar result is presented in the following lemma.

***Lemma* 3**: Consider the system given by equation (5). For every system state $x_0 \in \mathcal{V}$, allowable control $u(t)$ exists, $t \geq 0$, for the corresponding system state $x(t) \in \mathcal{V}$ for each $t \geq 0$ if and only if $A\mathcal{V} \subset \mathcal{V} + \mathcal{R}(B)$.(11)

Basile, Marro (1992).

The subspaces which fulfill equation (11) play important roles in the geometric approach and they will be analyzed further on.

Definition 11: The subspace $\mathcal{V}$ is said to be $(A,\mathcal{R}(B))$-*controlled invariant* (subspace) (term (A, B)-invariant subspace is also used), if and only if it is A-invariant modulo $\mathcal{R}(B)$, i.e. if and only if equation (11) is satisfied.

Wonham (1985), *Wonham, Morse* (1970).

Important characteristics of the previously described subspace are given in the following theorem.

***Theorem* 6**: Let $\mathcal{V} \subset \mathcal{X}$. Then $m\times n$ real matrix F exists with the characteristic

$$(A+BF)\mathcal{V} \subset \mathcal{V} \tag{12}$$

if and only if the subspace $\mathcal{V}$ is $(A,\mathcal{R}(B))$-controlled invariant subspace.

Wonham, Morse (1970).

Corollary 4: If the control law (6) applies to system (5) with the corresponding state equation (7), having *Lemma 1* in mind, it can be concluded that if subspace $\mathcal{V}$ is $(A, \mathcal{R}(B))$-controlled invariant, then matrix F exists, so $x(t) \in \mathcal{V}$, for every $t \geq 0$, which ensures $x_0 \in \mathcal{V}$.

Definition 12: The subspace $\mathcal{S}$ of $\mathcal{X}$ is said to be $(A, \mathcal{N}(C))$-*conditioned invariant* (subspace) if and only if:

$$A(\mathcal{S} + \mathcal{N}(C)) \subset \mathcal{S}. \tag{13}$$

Basile, Marro (1969).

Between controlled-invariant and conditioned-invariant subspaces the duality relation exists in the following sense. The calculation of the orthogonal complement from both sides of equation (13) leads to the conclusion that the equation is equivalent to $\{A(\mathcal{S} + \mathcal{N}(C))\}^{\perp} \supset \mathcal{S}^{\perp}$, which is further equivalent to $A'^{-1}(\mathcal{S}^{\perp} + \mathcal{R}(C')) \supset \mathcal{S}^{\perp}$, that is also equivalent to $A'\mathcal{S}^{\perp} \subset \mathcal{S}^{\perp} + \mathcal{R}(C')$. Similarly, equation (11) is fulfilled, if and only if $A'(\mathcal{V}^{\perp} \cap \mathcal{N}(B)') \subset \mathcal{V}^{\perp}$. Based on that, it can be stated:

***Lemma* 4**: Subspace $\mathcal{S}$ is $(A, \mathcal{N}(C))$-conditioned invariant if and only if subspace $\mathcal{S}^{\perp}$ is $(A', \mathcal{N}(C)')$-controlled invariant. Also, the subspace $\mathcal{V}$ is $(A, \mathcal{R}(C))$-controlled invariant if and only if $\mathcal{V}^{\perp}$ is $(A', \mathcal{N}(C)')$-conditioned invariant.

Basile, Marro (1992).

Having the previous lemma in mind, *Theorem 6* can be expressed in the following way.

***Theorem* 7**: Let $\mathcal{S} \subset \mathcal{X}$. Then, $n\times p$ real matrix G exists

$$(A+GC)\mathcal{S} \subset \mathcal{S} \tag{14}$$

if the subspace $\mathcal{S}$ is $(A, \mathcal{N}(C))$-conditioned invariant subspace.

Basile, Marro (1969).

The subspace can be both controlled and conditioned at the same time.

***Lemma* 5**: Let K be $m\times p$ real matrix fulfilling (15): $(A+BKC)\mathcal{V} \subset \mathcal{V}$.(15)

The subspace $\mathcal{V}$ is both $(A, \mathcal{R}(B))$-controlled invariant and $(A, \mathcal{N}(C))$-conditioned invariant subspace, if and only if (15) is satisfied.

Basile, Marro (1992), *Hamano* (1996).

Algebraic properties of controlled- and conditioned-invariant subspaces

The details of algebraic properties of subspaces can be found in *Basile, Marro* (1992).

***Lemma* 6**: If $\mathcal{V}_1$ and $\mathcal{V}_2$ are $(A, \mathcal{R}(B))$-controlled invariants, then $\mathcal{V}_1 + \mathcal{V}_2$ is $(A, \mathcal{R}(B))$-controlled invariant.

Basile, Marro (1992).

Corollary 5: Intersection of the two $(A, \mathcal{R}(B))$-controlled invariant subspaces does not have to be $(A, \mathcal{R}(B))$-controlled invariant.

***Lemma* 7**: Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be $(A, \mathcal{R}(B))$-controlled invariant. Then the $m\times n$ real matrix F exists, which fulfills:

$$(A+BF)\mathcal{V}_i \subset \mathcal{V}_i, \;\; i = 1,2 \tag{16}$$

if and only if $\mathcal{V}_1 \cap \mathcal{V}_2$ is $(A, \mathcal{R}(B))$-controlled invariant.

Basile, Marro (1992).

Because of duality the following can be stated.

***Lemma* 8**: If $\mathcal{S}_1$ and $\mathcal{S}_2$ are $(A, \mathcal{N}(C))$-conditioned invariants, then it is $\mathcal{S}_1 \cap \mathcal{S}_2$.

Basile, Marro (1992).

Corollary 6: Addition of the two $(A, \mathcal{N}(C))$-conditioned invariant subspaces does not have to be $(A, \mathcal{N}(C))$-conditioned invariant.

***Lemma* 9**: Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be $(A, \mathcal{N}(C))$-conditioned invariants. Then the $n\times p$ real matrix G exists, which fulfills

$$(A+BC)\mathcal{S}_i \subset \mathcal{S}_i, \;\; i = 1,2 \tag{17}$$

if and only if $\mathcal{S}_1 + \mathcal{S}_2$ is $(A, \mathcal{N}(C))$-conditioned invariant.

Basile, Marro (1992).

Maximum-controlled and minimum-conditioned invariant subspaces

Let $\mathcal{K} \subset \mathcal{X}$ and let the set of $(A, \mathcal{R}(B))$-controlled invariant subspaces is contained in $\mathcal{K}$. According to *Lemma* 6 the set of $(A, \mathcal{R}(B))$- controlled invariant subspaces is closed to addition. As a result, the set of $(A, \mathcal{R}(B))$-controlled invariant subspaces contained in $\mathcal{K}$ has the maximum element i.e *supremum*. This element is a unique subspace which contains any other $(A, \mathcal{R}(B))$- controlled invariant subspace inside $\mathcal{K}$. That subspace is called *maximum* $(A, \mathcal{R}(B))$-*controlled invariant subspace*. That subspace is denoted by $\mathcal{V}_{\max}(A, \mathcal{R}(B), \mathcal{K})$.

Similarly, let $\mathcal{I} \subset \mathcal{X}$. Taking *Lemma* 8 into account, the set of $(A, \mathcal{N}(C))$-conditioned invariant subspaces which contain $\mathcal{I}$ has a minimum element i.e *infinum*. This unique element contained in all $(A, \mathcal{N}(C))$-conditioned invariant subspaces is called *minimum* $(A,\mathcal{N}(C))$-*conditioned invariant subspace*. This subspace is denoted by $\mathcal{S}_{\min}(A,\mathcal{N}(C),\mathcal{I})$.

The subspaces $\mathcal{V}_{\max}(A,\mathcal{R}(B),\mathcal{K})$ and $\mathcal{S}_{\min}(A,\mathcal{N}(C), \mathcal{I})$

can be calculated in the finite number of iterations *n*. The problem of solution existence can be solved by investigating existence and by calculating these subspaces. In the following part, the algorithm for calculation of subspaces is given, *Basile*, *Marro* (1992).

The algorithm for the calculation of $\mathcal{V}_{\max}(A,\mathcal{R}(B),\mathcal{K})$:

$$\mathcal{V}_{\max}(A,\ \mathcal{R}(B),\ \mathcal{K}) = \mathcal{V}_{\dim_{\mathcal{K}}} \tag{18}$$

Where

$$\mathcal{V}_0 := \mathcal{K} \tag{19}$$

$$\mathcal{V}_i := \mathcal{K} \cap A^{-1}(\mathcal{V}_{i-1} + \mathcal{R}(B)),\ i = 1, ..., \dim\mathcal{K}. \tag{20}$$

Both the proof of the presented algorithm and the algorithm for the matrix *F* calculation can be found in *Basile*, *Marro* (1992), *Wonham* (1985).

Corollary 7: Consecutive subspaces have the following properties:

$$\mathcal{V}_1 \supset \mathcal{V}_2 \supset ... \supset \mathcal{V}_{\dim_{\mathcal{K}}}.$$

1. If $\mathcal{V}_l = \mathcal{V}_{l+1}$, then $\mathcal{V}_l = \mathcal{V}_{l+1}, = ... \mathcal{V}_{\dim_{\mathcal{K}}}$.

The algorithm for the calculation of $\mathcal{S}_{\min}(A,\mathcal{N}(C),\ \mathcal{I})$:

$$\mathcal{S}_{\min}(A,\ \mathcal{N}(C),\ \mathcal{I}) = \mathcal{S}_{n-\dim} \tag{21}$$

where

$$\mathcal{S}_0 := \mathcal{I} \tag{22}$$

$$\mathcal{S}_i := \mathcal{I} + A(\mathcal{S}_{i-1} \cap \mathcal{N}(C)),\ \ i = 1, ..., n - \dim \mathcal{I}. \tag{23}$$

Corollary 8: Consecutive subspaces have the following properties:

2. $\mathcal{S}_1 \subset \mathcal{S}_2 \subset ... \subset \mathcal{S}_{n-\dim\,\mathcal{I}}$.
3. If $\mathcal{S}_l = \mathcal{S}_{l+1}$, then $\mathcal{S}_l = \mathcal{S}_{l+1}, = ... \mathcal{S}_{n-\dim\,\mathcal{I}}$.

Hamano (1996).

Self-constrained (A, $\mathcal{R}$(B)) - *controlled invariant and* (A, $\mathcal{N}$(C)) - *conditioned invariant subspaces. Constrained reachability and observability*

Let the subspace $\mathcal{V}_0$ be $\mathcal{V}_0 \supset (A,\ \mathcal{R}(B))$- controlled invariant subspace contained in the subspace $\mathcal{K}$. Let us consider all possible system (5) trajectories with the variable control vector. State trajectories started from $x_0 \in \mathcal{V}_0$ and they belong to the subspace $\mathcal{K}$. **There exists at least one control for which state trajectories stay into the** $\mathcal{V}_0$ during the system motions. The question is if there is control which influences the system n such a way that all trajectories leave subspace $\mathcal{V}_0$, but in the same time stay inside $\mathcal{K}$. Now, we analyze (A, $\mathcal{R}(B)$)-controlled invariant subspace, which is contained in the $\mathcal{K}$. Assume that it is not possible to find control which transforms any system state from $\mathcal{V}$ out of that subspace. If the state trajectory for any particular case leaves the subspace $\mathcal{V}$, then it can be assumed that the state has to leave the subspace $\mathcal{K}$. The subspace which is (A, $\mathcal{R}(B)$)-invariant contained in $\mathcal{K}$ has the following property:

$$\mathcal{V}_{\max}(A,\ \mathcal{R}(B),\ \mathcal{K}) \cap \mathcal{R}(B) \subset \mathcal{V}. \tag{24}$$

Definition 13: The (A, $\mathcal{R}(B)$)- controlled invariant subspace $\mathcal{V}$, contained in $\mathcal{K}$ is said to be *self-constrained* with respect to $\mathcal{K}$ if and only if for that subspace equation (24) is satisfied.

Basile, *Marro* (1992).

Corollary 9: The left side of equation (24) represents the influence of the control on the system at any specific time when the system state stays inside the subspace $\mathcal{K}$.

An important property of the subspaces defined in the described way is that for each $\mathcal{K}$ at least one control low can be found which will guarantee existence of the self-constrained (A, $\mathcal{R}(B)$)- controlled invariant subspaces with respect to $\mathcal{K}$. More precisely:

***Lemma* 10**: Let F be $m \times n$ real matrix which satisfied the following

$$(A+BF)\mathcal{V}_{\max}(A,\mathcal{R}(B),\mathcal{K}) \subset \mathcal{V}_{\max}(A,\mathcal{R}(B),\mathcal{K}). \tag{25}$$

Then, any self-constrained (A, $\mathcal{R}(B)$)- controlled invariant subspace $\mathcal{V}$ with respect to $\mathcal{K}$ satisfied the equation

$$(A+BF)\mathcal{V} \subset \mathcal{V}. \tag{26}$$

It can be proved that the set of the self-constrained (A, $\mathcal{R}(B)$)-controlled invariant subspaces in $\mathcal{K}$ is closed with respect to the space intersection. Furthermore, if the $\mathcal{V}_1$ are $\mathcal{V}_2$ self-constrained (A, $\mathcal{R}(B)$)-controlled invariant subspaces with respect to $\mathcal{K}$, than it is also $\mathcal{V}_1 \cap \mathcal{V}_2$. Because of the previous fact, $\mathcal{V}$ has a minimum element which is called minimum self-constrained (A, $\mathcal{R}(B)$)- controlled invariant subspaces, denoted by $\mathcal{V}_{og,\min}(A,\ \mathcal{R}(B),\ \mathcal{K})$. The relation which connects subspace $\mathcal{V}_{og,\min}(A,\mathcal{R}(B),\mathcal{K})$ with subspaces $\mathcal{V}_{\max}(A,\mathcal{R}(B),\mathcal{K})$ and $\mathcal{S}_{\min}(A,\mathcal{K},\mathcal{R}(B))$ is given in the following lemma.

Theorem 8:

$$\mathcal{V}_{og,\min}(A,\ \mathcal{R}(B),\ \mathcal{K}) =$$

$$\mathcal{V}_{\max}(A,\mathcal{R}(B),\mathcal{K}) \cap \mathcal{S}_{\min}(A,\mathcal{K},\mathcal{R}(B)) \tag{27}$$

The proof can be found in *Basile*, *Marro* (1992).

The minimum self-constrained (A, $\mathcal{R}(B)$)- controlled invariant subspace is closely related to the characteristic of constrained reachability.

Definition 14: The set of all state trajectories which can be reached from the initial state using the system trajectories constrained in $\mathcal{K}$ is said to be reachable (or reachable subspace; because a set forms a subspace) inside the subspace $\mathcal{K}$ or the *maximum* (A, $\mathcal{R}(B)$)- *reachable subspace* $\mathcal{K}$.

This subspace is denoted by $\mathcal{V}_{dost}(\mathcal{K})$. It is ($A$, $\mathcal{R}(B)$)- control invariant.

***Lema* 11:**

$$\mathcal{V}_{dost}(\mathcal{K}) \subset \mathcal{V}_{\max}(A,\ \mathcal{R}(B),\ \mathcal{K}) \subset \mathcal{K} \tag{28}$$

$$\mathcal{V}_{dost}(\mathcal{V}_{\max}(A,\ \mathcal{R}(B),\ \mathcal{K})) = \mathcal{V}_{dost}(\mathcal{K}). \tag{29}$$

Basile, *Marro* (1992).

***Theorem* 9**: Let F be the real matrix which satisfies (A BF)$\mathcal{V}_{\max} \subset \mathcal{V}_{\max}$, where $\mathcal{V}_{\max} := \mathcal{V}_{\max}(A,\mathcal{R}(B),\mathcal{K})$. Then $\mathcal{V}_{dost}(\mathcal{K}) = \mathcal{V}_{og,\min}(A,\mathcal{R}(B),\mathcal{K}) =$

$$\mathcal{R}(A+BF,\mathcal{R}(B) \cap \mathcal{V}_{\max}). \tag{30}$$

Corollary 10: The expression given by equation (30) represents the minimum $A + BF$ invariant subspace which contains $\mathcal{R}(B) \cap \mathcal{V}_{\max}$.

Basile, *Marro* (1992).

Definition 15: The $(A, \mathcal{N}(C))$-conditioned invariant subspace $\mathcal{S}$, which contains $\mathcal{I}$, is called *self-hidden subspace* with respect to $\mathcal{I}$, if and only if

$$\mathcal{S} \subset \mathcal{S}_{\min}(A, \mathcal{N}(C), \mathcal{I})+\mathcal{N}(C). \qquad (31)$$

***Lema* 12**: Let G be $n\times p$ real matrix which satisfies

$$(A+GC)\mathcal{S}_{\min}(A,\mathcal{N}(C),\mathcal{I})\subset\mathcal{S}_{\min}(A,\mathcal{N}(C),\mathcal{I}). \qquad (32)$$

Then, any$(A,\mathcal{N}(C))$-conditioned invariant subspace $\mathcal{S}$, which contains $\mathcal{I}$, is self-hidden with respect to $\mathcal{I}$, if it is fulfilled

$$(A+GC)\mathcal{S} \subset \mathcal{S}\,. \qquad (33)$$

Basile, *Marro* (1992).

If the subspaces $\mathcal{S}_1$ and $\mathcal{S}_2$ are $(A, \mathcal{N}(C))$-conditioned invariants and contain $\mathcal{I}$ and if they are self-hidden with respect to $\mathcal{I}$, then the subspace $\mathcal{S}_1+\mathcal{S}_2$ is self-hidden. Consequently, the described set has its own maximum element which is called maximum $(A, \mathcal{N}(C))$*conditioned invariant self-hidden subspace*, with respect to $\mathcal{I}$ and it is denoted as $\mathcal{S}_{ss,\max}(A, \mathcal{N}(C),\mathcal{I})$. The subspace $\mathcal{S}_{ss,\max}(A,\mathcal{N}(C),\mathcal{I})$ is in relation to subspaces $\mathcal{V}_{\max}(A,\mathcal{I},\mathcal{N}(C))$ and $\mathcal{S}_{\min}(A,\mathcal{N}(C),\mathcal{I})$ in the following way.

***Theorem* 10**:

$$\mathcal{S}_{ss,\max}(A,\mathcal{N}(C),\mathcal{I})=$$

$$\mathcal{S}_{\min}(A,\mathcal{N}(C),\mathcal{I})+\mathcal{V}_{\max}(A,\mathcal{I},\mathcal{N}(C)) \qquad (34)$$

Basile, *Marro* (1992).

Theorem 11:

$$\mathcal{S}_{ss,\max}(A,\mathcal{N}(C),\mathcal{I})=(\mathcal{N}(C)+\mathcal{S}_{\min})\cap(A+GC)^{-1}$$

$$(\mathcal{N}(C)+\mathcal{S}_{\min})\cap \ldots \cap(A+GC)^{-(n-1)}(\mathcal{N}(C)+\mathcal{S}_{\min}) \qquad (35)$$

where $\mathcal{S}_{\min}:= \mathcal{S}_{\min}(A, \mathcal{N}(C), \mathcal{I})$, and matrix G has the property $(A+GC)\mathcal{S}_{\min} \subset \mathcal{S}_{\min}$.

Basile, *Marro* (1992).

Corollary 11: The right side of equation (35) represents the maximum $A+GC$ invariant subspace which is contained in $\mathcal{N}(C)+\mathcal{S}_{\min}$.

Internal and external stability

In this part, the system stability which is connected with control and conditioned invariant subspaces is investigated.

Definition 16: The $(A, \mathcal{R}(B))$-controlled invariant subspace $\mathcal{V}$ is said to be *internal potentially stabile* if and only if for any initial state $x_0\in\mathcal{V}$ control $u(t)$ exists. In that case, $x(t)\in\mathcal{V}$ for any $t \geq 0$, when $x(t)$ converges to zero state as $t\rightarrow\infty$ or equivalently if and only if $m \times n$ real matrix F exists satisfying $(A+BF)\mathcal{V}\subset\mathcal{V}$ and $(A+BF)|\mathcal{V}$ is stabile.

Definition 17: The $(A, \mathcal{R}(B))$-controlled invariant subspace $\mathcal{V}$ is said to be *externally potentially stabile*, if and only if for any initial state $x_0 \in \mathcal{X}$, there is a control $u(t)$ such that $x(t)$ converges to $\mathcal{V}$ as $t\rightarrow\infty$, or equivalently, if and only if $m \times n$ real matrix F exists satisfying $(A+BF)\mathcal{V}\subset\mathcal{V}$ and $(A+BF)|\mathcal{X}/\mathcal{V}$ is stabile.

External and internal potential stability can be investigated by applying appropriate coordinate transformation. For that purpose it is necessary to define $n \times n$ and $m \times m$ nonsingular set of the matrices $T = [\,T_1\ T_2\ T_3\ T_4]$ and $U = [\,U_1\ U_2]$, respectively. Let $\mathcal{V}$ be $(A, \mathcal{R}(B))$-controlled invariant subspace. T_1 and T_2 are chosen so that $\mathcal{R}(T_1)=\mathcal{V}_{dost}(\mathcal{V})$, and $\mathcal{R}[\,T_1\ \ T_2]=\mathcal{V}$. Then $\mathcal{V}_{dost}(\mathcal{V})=\mathcal{V}\cap\mathcal{S}_{\min}(A,\mathcal{V},\mathcal{R}(B))$. T_3 is chosen to satisfy $\mathcal{R}[\,T_1\ T_2] = \mathcal{S}_{\min}(A, \mathcal{V}, \mathcal{R}(B))$. Also, U_1 is chosen to satisfy $\mathcal{R}(BU_1)= \mathcal{V}\cap \mathcal{R}(B)$. Then it can be written:

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} \\ 0 & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} & \tilde{A}_{34} \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} \end{bmatrix} \qquad (36)$$

$$\tilde{B} = T^{-1}BT = \begin{bmatrix} \tilde{B}_{11} & 0 \\ 0 & 0 \\ 0 & \tilde{B}_{32} \\ 0 & 0 \end{bmatrix} \qquad (37)$$

A null block matrix in the equation for $\tilde{A}$ in the second row appears because $\mathcal{V}_{dost}(\mathcal{V})$ is $(A+BF)$ invariant subspace of F which satisfies $(A+BF)\mathcal{V}\subset\mathcal{V}$. Because of the same reason the second row of matrix $\tilde{B}$ has zero elements. The forth row of the matrix $\tilde{A}$ has zero elements because $\mathcal{V}$ is $(A+BF)$ invariant for some F. Similar can be concluded for the matrix $\tilde{B}$. Let F be chosen to satisfy $(A+BF)\mathcal{V}\subset\mathcal{V}$. Then, together with $\tilde{A}_{31}+\tilde{B}_{32}\tilde{F}_{21}=0$ and $\tilde{A}_{32}+\tilde{B}_{32}\tilde{F}_{22}=0$, it can be written:

$$\tilde{A}+\tilde{B}\tilde{F} = T^{-1}(A+BF)T = \qquad (38)$$

$$\begin{bmatrix} \tilde{A}_{11}+\tilde{B}_{11}\tilde{F}_{11} & \tilde{A}_{12}+\tilde{B}_{11}\tilde{F}_{12} & \tilde{A}_{13}+\tilde{B}_{11}\tilde{F}_{13} & \tilde{A}_{14}+\tilde{B}_{11}\tilde{F}_{14} \\ 0 & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ 0 & 0 & \tilde{A}_{33}+\tilde{B}_{32}\tilde{F}_{23} & \tilde{A}_{34}+\tilde{B}_{32}\tilde{F}_{24} \\ 0 & 0 & \tilde{A}_{43} & \tilde{A}_{44} \end{bmatrix}$$

Together with the 2×4 block matrix $\tilde{F}$.

It can be noticed that $\tilde{A}_{22}$ could be substituted by any element using state feedback with control law which $(A+BF)\mathcal{V}\subset\mathcal{V}$.

***Lemma* 13**:

$$\sigma((A+BF)\mid \mathcal{V}/\mathcal{V}_{dost}(\mathcal{V}))= \sigma(\tilde{A}_{22}) \qquad (39)$$

for any F which satisfies $(A+BF)\mathcal{V}\subset\mathcal{V}$.

Hamano (1996).

Here $(A+BF)|\mathcal{V}/\mathcal{V}_{dost}(\mathcal{V})$ is induced transformation $A+BF$ to $\mathcal{V}/\mathcal{V}_{dost}(\mathcal{V})$. With the previous analysis in mind, it can be noticed that the matrix pencil $(\tilde{A}_{11},\tilde{B}_{11})$ is reachable. The analysis of the statement for *Theorem 2* leads to the conclusion:

***Lemma* 14**: The eigenvalues of $\sigma((A+BF)|\mathcal{V}/\mathcal{V}_{dost}(\mathcal{V}))$ $= \sigma(\tilde{A}_{11}+\tilde{B}_{11}\tilde{F}_{11})$ could be freely assigned by choosing the matrix F appropriately. The matrix F fulfills $(A+BF)\mathcal{V}\subset\mathcal{V}$

The eigenvalues of $(A+BF)|\mathcal{V}/\mathcal{V}_{dost}(\mathcal{V})$ are called *internally unassignable eigenvalues* of the subspace $\mathcal{V}$. The internally unassignable eigenvalues of $\mathcal{V}_{\max}(A,\mathcal{R}(B),\mathcal{N}(C))$ are called the invariant zero of system (5), or the matrix

triple (A,B,C).
Hamano (1996).

***Theorem* 12**: The ($A,\mathcal{R}(B)$)-controlled invariant subspace $\mathcal{V}$ is internally potentially stabile , if and only if all internal unassigned eigenvalues have negative real part.
Basile, *Marro* (1992).

***Theorem* 13**: The ($A,\mathcal{R}(B)$)-controlled invariant subspace $\mathcal{V}$ is externally potentially stabile , if and only if $\mathcal{V}+\mathcal{V}_{\text{dost}}$ is internally potentially stabile.
Basile, *Marro* (1992).

***Lemma* 15**: If the matrix pencil (A, B) is potentially stabile, then all (A, $\mathcal{R}(B)$)-controlled invariant subspaces are externally stabile.
Basile, *Marro* (1992).

Corollary 12: The matrix F could be defined independently of $\mathcal{V}$ or $\mathcal{X}/\mathcal{V}$.

Dual objects to internally and externally stabilizabilities for controlled invariant subspaces are external and internal stabilizabilities for conditioned invariant subspaces, respectively.

Definition 18: The (A, $\mathcal{N}(C)$)-conditioned invariant subspace S is said to be externally potentially stabile, if and only if $n \times p$ real matrix G exists satisfying $(A+GC)S \subset S$ and $(A+GC)|\mathcal{X}/S$ is stabile.

Definition 19: The (A, $\mathcal{N}(C)$)-conditioned invariant subspace S is said to be internally potentially stabile, if and only if $n \times p$ real matrix G exists satisfying $(A+GC)S \subset S$ and $(A+GC)|S$ is stabile.

Choosing eigenvalues for ($A+GC$) is analog to the described procedure for ($A+BF$) and can be found in *Basile*, *Marro* (1992).

Disturbance localization problem

The disturbance localization problem was one of the first problems which were solved by the geometric approach.

When the geometric approach was used to solve this problem, the solution was easily obtained. The time invariant linear system $\sum_d := [A,B,C,D]$ described by equation (40) is considered:

$$\begin{aligned}&\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) + D\mathbf{w}(t),\ \mathbf{x}(0) = \mathbf{x}_0\\ &\mathbf{x}_i(t) = C\mathbf{x}(t)\end{aligned}, \tag{40}$$

where values $\mathbf{x}(t) \in \mathcal{X} := \Re^n$, $\mathbf{u}(t) \in \Re^m$, $\mathbf{x}_i(t) \in \Re^p$ and $\mathbf{w}(t) \in \Re^{m_d}$ are respectively the system state, the input and output of the system, the disturbance vector in time $t \geq 0$, and $\mathbf{x}_0(t) \in \Re^n$ is the initial state, where the matrices A, B, C and D have constant dimensions. Assume that the function $\mathbf{u}(t)$ is continual in the parts, and disturbance cannot be measured. The state feedback (41) is applied to the system:

$$\mathbf{u}(t)=F\mathbf{x}(t), \tag{41}$$

then equation (40) becomes

$$\dot{\mathbf{x}}(t) = (A+BF)\mathbf{x}(t) + D\mathbf{w}(t),\quad x(0) = x_0 \tag{42}$$

where F is the $m \times n$ real matrix. The problem which has to be solved is to choose control law (41) with property that disturbance does not influence the dynamical behavior of the system with feedback (40). Mathematically described, it is necessary that $\mathbf{x}_i(t)=0$, for some $t \geq 0$, and for any $\mathbf{w}(t)$, $t \geq 0$, when $x(0)=x_0=0$. If we go back to *Theorem 1*, it can be seen that for any $t \geq 0$ all possible system states are described by equation (43), with disturbance $w(\tau)$, $0 \leq \tau \leq t$ to system (40).

$$\mathcal{R}(A+BF,\ \mathcal{R}(D))=\mathcal{R}(D)+(A+BF)\mathcal{R}(D)+...$$
$$+(A+BF)^{n-1}\mathcal{R}(D). \tag{43}$$

This problem is called the disturbance localization problem or the decupling problem, *Basile*, *Marro* (1992), *Wonham* (1985). In the algebraic terms it is defined as follows: for given $n \times n$, $n \times n$, $n \times m'$ and $p \times n$ real matrices A, B, D and C, find real $m \times n$ matrix F, which fulfills equation

$$\mathcal{R}(A + BF, \mathcal{R}(D)) \subset \mathcal{N}(C). \tag{44}$$

It is always possible to find the matrix F which satisfied equation (44). The following theorem gives sufficient and necessary conditions for the existence of the matrix F.

***Theorem* 14**: A real $m \times n$ matrix F which satisfied (44) exists if and only if

$$\mathcal{R}(D) \subset \mathcal{V}_{\max}(A,\ \mathcal{R}(B),\ \mathcal{N}(C)). \tag{45}$$

The proof of the *Theorem 14* can be found in *Basile*, *Marro* (1992), and the proof of the sufficient conditions in *Hamano* (1996).

Corollary 13: Let $n^*=\dim\mathcal{V}_{\max}(A,\ \mathcal{R}(B),\ \mathcal{N}(C))$ and let T be a real nonsingular matrix for which the first n^* columns forms a basis of the subspace $\mathcal{V}_{\max}(A,\mathcal{R}(B),\mathcal{N}(C))$. In that case, condition (45) means that the transformation of the coordinates using the real $m \times n$ matrix F can be established. The matrix F fulfills the equation $(A+BF)\ \mathcal{V}_{\max}(A,\ \mathcal{R}(B),\ \mathcal{N}(C)) \subset \mathcal{V}_{\max}(A,\ \mathcal{R}(B),\ \mathcal{N}(C))$. The coordinate transformation is of the form $\mathbf{x}(t)= T\tilde{\mathbf{x}}(t)$. Including equations (40) and (42), it can be derived:

$$\begin{bmatrix}\dot{\tilde{x}}_1(t)\\ \dot{\tilde{x}}_2(t)\end{bmatrix} = \begin{bmatrix}\tilde{A}_{11}+\tilde{B}_1\tilde{F}_1 & \tilde{A}_{12}+\tilde{B}_1\tilde{F}_2\\ 0 & \tilde{A}_{22}+\tilde{B}_2\tilde{F}_2\end{bmatrix}\begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\end{bmatrix} + \begin{bmatrix}\tilde{D}_1\\ 0\end{bmatrix} w(t) \tag{46}$$

$$\mathbf{x}_i(t) = \begin{bmatrix}0 & \tilde{C}_2\end{bmatrix}\begin{bmatrix}\tilde{x}_1(t)\\ \tilde{x}_2(t)\end{bmatrix} \tag{47}$$

where $[\tilde{F}_1\quad \tilde{F}_2] = FT$.

Disturbance localization and stability

At the previous part, disturbance localization is solved without any constrains to the system. In this part, together with the described problem, the system stability is investigated. A new problem can be defined for system (40). It is necessary to find, if the solution exists, the $m \times n$ real matrix F, with the property that equation (44) is satisfied , but the matrix pencil ($A+BF$) should be stabile, i.e. the eigenvalues of the ($A+BF$) have negative real parts. In the trivial case, the latter condition is fulfilled if the matrix pencil (A, B) is potentially stabile, *Basile*, *Marro* (1992).

***Theorem* 15**: The system matrices A, B, C, D are defined similarly to those in part **2.7**. Assume that the matrix pencil (A, B) is potentially stabile. Disturbance localization with the stability condition problem has a solution if and only if equation (45) is fulfilled and if the subspace $\mathcal{V}_{\text{og,min}}(A,\mathcal{R}(B)+\mathcal{R}(D),\mathcal{N}(C))$ is externally potentially stabile.
Basile, *Marro* (1992).

Further stability conditions for investigated the problem can be found in *Wonham* (1985).

Corollary 14: Using the defined values from the *Corollary* 13, $n^*=\dim \mathcal{V}_{og,min}(A, \mathcal{R}(B)+ \mathcal{R}(D),\mathcal{N}(C))$ can be defined in a similar way. Replacing $\mathcal{V}_{max}(A,\mathcal{R}(B), \mathcal{N}(C))$ with $\mathcal{V}_{og,min}(A, \mathcal{R}(B) + \mathcal{R}(D), \mathcal{N}(C))$ it can be calculated that $\sigma(A+ BF \,|\mathcal{V}_{og,min}(A,\mathcal{R}(B)+\mathcal{R}(D),\mathcal{N}(C))) = \sigma(\tilde{A}_{11}+\tilde{B}_1\tilde{F}_1)$ and $\sigma(A + BF \,|\mathcal{X}/\mathcal{V}_{og,min}(A, \mathcal{R}(B)+ \mathcal{R}(D), \mathcal{N}(C))) = \sigma(\tilde{A}_{22}+\tilde{B}_2\tilde{F}_2)$. Consequently, the matrices $\tilde{F}_1$ are $\tilde{F}_2$ chosen to fulfill stability conditions for $\tilde{A}_{11}+\tilde{B}_1\tilde{F}_1$ and $\tilde{A}_{22}+\tilde{B}_2\tilde{F}_2$ and to fulfill $\tilde{A}_{21}+\tilde{B}_2\tilde{F}_1=0$. In that case, the desired value for the matrix F is given as $F=[\tilde{F}_1 \quad \tilde{F}_2]\,T^{-1}$.

The proof for a necessary condition can be found in *Hamano* (1996). To prove the previous statement, the following lemma is used.

***Lemma* 16**: If the externally stabile $(A, \mathcal{R}(B))$-controlled invariant subspace $\mathcal{V}$ which satisfied $\mathcal{V} \subset \mathcal{N}(C)$ and $\mathcal{R}(D)\subset\mathcal{V}$ exists, then the subspace $\mathcal{V}_{og,min}(A, \mathcal{R}(B)+\mathcal{R}(D), \mathcal{N}(C))$ is externally potentially stabile i.e. it is $(A+BF)$ invariant and the subspace $A + BF \,|\mathcal{V}_{og,min}(A, \mathcal{R}(B)+ \mathcal{R}(D), \mathcal{N}(C))$ is stabile for some matrix F.
Basile, *Marro* (1969.b)

Disturbance localization and the dynamic control method

For disturbance localization using the dynamic control method, it is necessary to measure the output values. Furthermore, feedback with the dynamic controller in the forward direction Σ_c should be applied. The goal of the method is the same as in the previous part.

In this case, the system $\sum_{dm}:=[A,B,C,D,C_{mer}]$ described by equation (48) is investigated

$$\begin{aligned}&\dot{\mathbf{x}}(t)=A\mathbf{x}(t)+B\mathbf{u}(t)+D\mathbf{w}(t), \qquad \mathbf{x}(0)=\mathbf{x}_0\\&\mathbf{x}_i(t)=C\mathbf{x}(t)\\&\mathbf{y}_{mer}=C_{mer}\mathbf{x}(t)\end{aligned} \tag{48}$$

where $\mathbf{y}_{mer}(t)\in \mathcal{Y}_{mer}:= \Re^{p_m}$ is the measured system output, C_{mer} is the $p_m\times n$ real matrix. The dynamic controller $\sum_c:=[A_c,B_c,C_c,K_c]$ is defined as follows

$$\begin{aligned}&\dot{\mathbf{x}}_c(t)=A_c\mathbf{x}_c(t)+B_c\mathbf{y}_{mer}(t),\;\; \mathbf{x}_c(0)=\mathbf{x}_{c0}\\&\mathbf{u}(t)=C_c\mathbf{x}_c(t)+K_c\mathbf{y}_{mer}(t)\end{aligned}, \tag{49}$$

where $\mathbf{x}_c(t), \mathbf{x}_{c0} \in \mathcal{X}_c := \Re^{n_c}$. The real matrices A_c, B_c, C_c and K_c have the dimensions $n_c \times n_c$, $n_c \times p_m$, $m \times n_c$ and $m \times p_m$ respectively. A_c, B_c, C_c and K_c are defined in the sense that $y(t) = 0$, for $t \geq 0$ and any $w(t)$, $t \geq 0$, when $x_0 = 0$ and $x_{c0} = 0$. The state values $x(t)$ and $x_c(t)$ converge to zero when $t\to\infty$, for any $\mathbf{x}_0\in\mathcal{X}$ and $\mathbf{x}_{c0}\in\mathcal{X}_c$ with $w(t)=0$, for any $t \geq 0$. Due to convenience, *extended state* is introduced as

$$\hat{x}=\begin{bmatrix}x\\x_c\end{bmatrix}\in\mathcal{X}_c:=\Re^{n+n_c} \tag{50}$$

where $x\in\mathcal{X}$, and $x_c\in\mathcal{X}_c$. Then, the whole system, including $\sum_{dm}$ and $\sum_c$ can be represented as

$$\begin{aligned}&\dot{\hat{\mathbf{x}}}(t)=\hat{A}\hat{\mathbf{x}}(t)+\hat{D}\mathbf{w}(t), \qquad \hat{\mathbf{x}}(0)=\hat{\mathbf{x}}_0\\&\hat{\mathbf{x}}_i(t)=\hat{C}\hat{\mathbf{x}}(t)\end{aligned}, \tag{51}$$

where $\hat{\mathbf{x}}_0=[x_0{}'x_c{}']'$. The matrices are defined on the following way:

$$\begin{aligned}&\hat{A}=\begin{bmatrix}A+BK_cC_{mer} & BC_c\\ B_cC_{mer} & A_c\end{bmatrix},\\&\hat{D}=\begin{bmatrix}D\\0\end{bmatrix} \qquad \hat{C}=\begin{bmatrix}C & 0\end{bmatrix}.\end{aligned} \tag{52}$$

If it is adopted

$$\begin{aligned}&\hat{A}_0=\begin{bmatrix}A&0\\0&0\end{bmatrix}, \qquad \hat{B}_0=\begin{bmatrix}B&0\\0&I\end{bmatrix}\\&\hat{C}_{mer}=\begin{bmatrix}C_{mer}&0\\0&I\end{bmatrix} \qquad \hat{K}_c=\begin{bmatrix}K_c&C_c\\B_c&A_c\end{bmatrix},\end{aligned} \tag{53}$$

it can be calculated

$$\hat{A}=\hat{A}_0+\hat{B}_0\hat{K}_c\hat{C}_{mer} \tag{54}$$

Basile, *Marro* (1992).

Corollary 15: A special case of the controller $\sum_c$ is an observer which has a task to estimate the state space of the system. If the new feedback is applied with the properties given in *Basile*, *Marro* (1992), the described approach can be considered as a system with the asymptotic observer and the state estimation. This approach will be considered now.

The disturbance localization problem using the dynamic disturbance method can be defined as: Find, if any, $n_c = \dim \mathcal{X}_c$, as well as the matrices A_c, B_c, C_c and K_c with appropriate dimensions so that

$$\mathcal{R}(\hat{A}, \mathcal{R}(\hat{D}))\subset\mathcal{N}(\hat{C}), \tag{55}$$

is fulfilled with the stabile matrix $\hat{A}$.

Corollary 16: It can be seen that $\mathcal{R}(\hat{A},\mathcal{R}(\hat{D}))$ is the minimum $\hat{A}$-invariant subspace which contained $\mathcal{R}(\hat{D})$. The first condition given by equation (55) can be transformed as

$$\mathcal{R}(\hat{D})\subset\hat{\mathcal{V}}\subset\mathcal{N}(\hat{C}). \tag{56}$$

The conditions for the described problem to have solutions are given in the following theorem.

***Theorem* 16**: Let the matrix pencil (A, B) be potentially stabile and assume that it is possible to determine (A, C_{mer}). The disturbance localization problem using the dynamic control method together with the system stability problem can be solved if and only if the externally potentially stabile $(A, \mathcal{R}(B))$-controlled invariant subspace $\mathcal{V}$ exists, and the externally potentially stabile $(A, \mathcal{N}(C_{mer}))$-conditioned subspace $\mathcal{S}$ exists, which satisfied equation (57)

$$\mathcal{R}(D)\subset \mathcal{S}\subset\mathcal{V}\subset \mathcal{N}(C). \tag{57}$$

The proof of this theorem and the extension for the conditions presented can be found in *Willems*, *Commault* (1982).

***Theorem* 17**: Assume that the matrix pencil (A, B) is potentially stabile and assume that (A, C_{mer}) can be found. The disturbance localization problem using the dynamic control method together with the system stability problem can be solved if and only if the following conditions are fulfilled:

$$\mathcal{S}_{\min}(A, \mathcal{N}(C_{mer}), \mathcal{R}(D)) \subset \mathcal{V}_{\max}(A, \mathcal{R}(B), \mathcal{N}(C)) \quad (58)$$

$$\mathcal{S}_{\min}(A, \mathcal{N}(C_{mer}), \mathcal{R}(D)) + \mathcal{V}_{\max}(A, \mathcal{R}(D), \mathcal{N}(C) \cap \mathcal{N}(C_{mer}))$$

$$\text{is externally stabile} \quad (59)$$

$$\mathcal{V}_{\text{og,min}}(A, \mathcal{R}(B)+\mathcal{R}(D), \mathcal{N}(C)) + \mathcal{V}_{\max}(A, \mathcal{R}(D), \mathcal{N}(C) \cap$$

$$\mathcal{N}(C_{mer})) \text{ is internally stabile} \quad (60)$$

The proof of the theorem and the conditions for the calculations of the of the matrices A_c, B_c, C_c and K_c can be found in *Hamano* (1996)

Results for the singular systems

In this part we investigate the linear singular system with the zero input. A general case which will be investigated after the system is described by (61) with non-zero input.

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \ \ \mathbf{x}(0) = \mathbf{x}_0 \,, \quad (61)$$

with the square matrix E which is always singular, and E, $A \in \Re^{n\times n}$ are the constant of the matrices, $B \in \Re^{n\times m}$, with $\mathbf{x} \in \Re^n$ and $\mathbf{u} \in \Re^m$.

Invariant subspaces and singular systems

Problem 1: Subspace $\mathcal{V}_c$, of the vector space $\Re^n$ is investigated. Several questions arise:

(i) Does the control $\mathbf{u}(t)$ exist with the property that equation (61) for any $\mathbf{x}_0 \in \mathcal{V}_c$ has at least one solution?

(ii) Does the control $\mathbf{u}(t)$ exists so that for every $\mathbf{x}_0 \in \mathcal{V}_c$ equation (61) has a set of the solutions, so that all solutions have the property $\mathbf{x}(t) \in \mathcal{V}_c$ for any $t \ge 0$?

In the following part, the definitions of the corresponding subspaces for singular systems are given.

Definition 20: The subspace $\mathcal{V}_c$ which satisfied conditions (i) and (ii) described in *Problem 1*, is called *controlled invariant subspace* .
Buzurović (2000).

Problem 2: The subspace $\mathcal{V}_{fc}$ of the vector space $\Re^n$ is given. Does the real $n\times m$ matrix K exists, together with the control $\mathbf{u}(t) = K\mathbf{x}(t)$ which as a result has the existence of the controlled invariant subspace $\mathcal{V}_{fc}$?

Definition 21: The subspace $\mathcal{V}_{fc}$ which satisfied conditions from the *Problem 2* is called *feedback controlled invariant subspace*.
Buzurović (2000).

It is of interest to present the description of the subspaces. The following method of the system analysis is a typical geometric method.

***Theorem* 18**: Assume that $\mathcal{V}$ is subspace of the vector space $\Re^n$, which satisfied the relation $\mathcal{V} \supset \mathcal{N}(E)$, where

$$\mathcal{N}(E) \overset{def}{=} \{\mathbf{x} \in \Re^n \mid E\mathbf{x} = 0\} \quad (62)$$

The following three statements are equivalent:

1. $\mathcal{V}$ is feedback controlled invariant subspace
2. $\mathcal{V}$ is controlled invariant subspace
3. $A\mathcal{V} \subset E\mathcal{V} + \mathcal{R}(B)$

Buzurović (2000).

***Theorem* 19**: Assume that $\mathcal{V}$ is subspace of the vector space $\Re^n$. Define $\mathbf{u}(t) = -K_2\mathbf{x}(t)$ and let $\mathcal{V}^*$ be the maximum subspace in the $\Re^n$ which satisfied

$$(A - BK_2)\mathcal{V}^* \subset E\mathcal{V}^* \quad (63)$$

The following two statements are equivalent:

1. $\mathcal{V}$ is feedback controlled invariant subspace with the control $\mathbf{u}(t) = -K_2\mathbf{x}(t)$
2. $(A - BK_2)\mathcal{V} \subset E\mathcal{V}$, $\mathcal{N}(E) \cap \mathcal{V}^* \subset \mathcal{V}$

It is clear that when the matrix E is nonsingular, the conditions are different. The proof of the given consequences can be found in *Buzurović* (2000).

Consequence 1: Assume that $\mathcal{V}$ is the subspace of the vector space $\Re^n$ and that is $E = I$. The following two statements are equivalent

1. $\mathcal{V}$ is feedback controlled invariant subspace
2. $\mathcal{V}$ is controlled invariant subspace
3. $A\mathcal{V} \subset \mathcal{V} + R(B)$

Buzurović (2000).

Considering *Theorem* 2 the following can be concluded

Consequence 2: $\mathcal{V}^*$ is the feedback maximum controlled invariant subspace with the control $\mathbf{u}(t) = -K_2\mathbf{x}(t)$ if and only if $\mathcal{V}^*$ is a maximum subspace which satisfied (63). It is clear that in the case of the nonsingular matrix E the subspace $\mathcal{V}^* \equiv \Re^n$.

Output-null subspace

To denote subspace, italic letters are used, for instance $\mathcal{M}$, and a corresponding matrix is denoted by M. The range of the matrix M is denoted by $\mathcal{R}(M)$. The null subspace of the matrix M is denoted by $\mathcal{N}(M)$. The upper index -1 denotes the inverse range of the linear operator, and the inverse for the matrices.

The generalized linear dynamic system is considered

$$\begin{aligned} E\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{y} &= C\mathbf{x} + D\mathbf{u} \end{aligned} \quad (64)$$

where $\mathbf{x} \in \Re^n$, $\mathbf{u} \in \Re^m$ and $\mathbf{y} \in \Re^p$ are corresponding vectors. If $detE$=0, the system can be transformed to have the matrix D=0 because the influence of the D can be included into the matrix E. However, because of the symmetry, we will precede with $D \neq 0$.

The subspace $\mathcal{S} \subset \Re^n$ can be defined as the *output-null* $(A, E, \mathcal{B})$*-invariant subspace* of system (64) if it is fulfilled

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{S} \subset \begin{bmatrix} E \\ 0 \end{bmatrix} \mathcal{S} + R\begin{bmatrix} B \\ D \end{bmatrix} \quad (65)$$

The following result clarifies the meaning of the subspace $\mathcal{S}$, given by the generalized Lyapunov (or Sylvester) equation

$$\begin{bmatrix} A \\ C \end{bmatrix} \mathcal{S} \subset \begin{bmatrix} E \\ 0 \end{bmatrix} \mathcal{S}F - \begin{bmatrix} B \\ D \end{bmatrix} G \quad (66)$$

***Theorem* 20**: $\mathcal{S} \subset \Re^n$ is the *output-null* $(A, E, \mathcal{B})$ *invariant subspace* of system (64) if and only if, for any $x(0_-) \in \mathcal{S}$, input $\mathbf{u}(t)$ exists so that the following is fulfilled:

(i) $\mathbf{y}(t) = 0$

(ii) $\mathbf{x}(t) \subset \mathcal{S}$ za $t \geq 0$ and

(iii) (3) $\mathbf{u}(t)$ and $\mathbf{x}(t)$ have Laplace transformation
Ozcaldiran, Lewis (1987)

***Theorem* 4**: $\mathcal{S}$ satisfies (65) if and only if the subspace K exists so that

$$(A+BK)\mathcal{S} \subset E\mathcal{S} \tag{67}$$

$$(C+DK)\mathcal{S}=0 \tag{68}$$

Ozcaldiran, Lewis (1987).

Existence and uniqueness of the solutions

In this part, we analyzed the existence and the uniqueness of the solutions for system (61). The details can be found in *Buzurović* (2000) and *Debeljković, Buzurović* (2007).

It is shown in *Buzurović* (2000) that when $\det(sE - A) \neq 0$, then the existence of the solutions implies uniqueness. In addition, we analyzed the non-uniqueness of the solutions. Before that, new properties will be introduced. They are *causal* and *strict causal* property of the dynamic system with possible non-unique solutions.

Let Ω be set of the allowable control functions which transform the set $[t_0, t_1]$ to $\Re^n$. Usually, $t_1 = \infty$. Let the correspondence of the solutions be the transformation of the set function S from Ω to the trajectories set, where each function $\mathrm{u}(\cdot)$ from Ω is in the relation with the set $S(\mathrm{u}(\cdot))$. Equation (61) is satisfied. Let $S_\tau(\mathrm{u}(\cdot))$ be the restriction of the elements $S(\mathrm{u}(\cdot))$ with respect to the set $[t_0, \tau]$.

Definition 22: Correspondence S is called *strict causal* if for given $\mathrm{u}_1(\cdot)$ and $\mathrm{u}_2(\cdot)$ from Ω it is fulfilled

$$\mathrm{u}_1(\cdot) = \mathrm{u}_2(\cdot),\ \forall t < \tau \ \Rightarrow\ S_\tau(\mathrm{u}_1(\cdot)) = S_\tau(\mathrm{u}_2(\cdot)) \tag{69}$$

For the subspace S is said to be *causal* if equation (64) is satisfied for any $t \leq \tau$. In addition, the strict causal property could be exchanged with the causal property. The set of the strict causal solutions of the system is the maximum strict causal correspondence from the union $\overline{S}$.

For the given $\mathrm{u}(\cdot)$ from Ω, the trajectory $\mathrm{x}(\cdot)$ is called the strict causal solution if it belongs to $\overline{S}(\mathrm{u}(\cdot))$. The strict causal characteristic of the solutions is: system (61) for any t or for any $\tau \in (t_0, t_1)$ starting the motion from $\mathrm{x}(\tau)$ has strict causal solutions for any $\{\mathrm{u}(t), t \geq \tau\}$. In the following part, the non-uniqueness of the solutions is analyzed. However, the impulse modes which can appear will be neglected.

Let $\mathcal{R}(E)$ and $\mathcal{R}(B)$ be the ranges of matrices E and B, given in the subspace $\mathcal{Y} = \Re^r$. Consider the following relation, which is satisfied for the linear subspace $\mathcal{V}$ from $\mathcal{Y}$.

$$A\mathcal{V} \subset E\mathcal{V} \tag{70}$$

Definition 23: *Characteristic subspace* of the matrix pencil (E, B) is the maximum subspace $\mathcal{V}^*$ which satisfied relation (65).
Buzurović (2000).

The defined subspace exists because $\{0\}$ satisfies (65), and this relation is stabile with respect to subspace addition. In the special case of Definition 4, the subspace $\mathcal{V}^*$ could be trivial. This subspace is uniquely connected to the solution of equation (61) with zero input.

***Theorem* 5**: In order for the system described by equation (61) to have strict causal solution on the interval with the arbitrary length for any control sequences $\mathrm{u}(\cdot)$, it is necessary and sufficient that the following equations are fulfilled:

$$\mathcal{R}(B) \subset E\mathcal{V}^* \tag{71}$$

$$\mathrm{x}(0) \in \mathcal{V}^* \tag{72}$$

Buzurović (2000).

The proof of the theorem and more detailed explanations can be found in *Buzurović* (2000) and *Debeljković, Buzurović* (2007).

Definition 24: The characteristic null subspace of the matrix pencil (E, A) is called the subspace $\mathcal{N}$ defined by the expression

$$\mathcal{N} = \mathcal{N}(E) \cap \mathcal{V}^* \tag{73}$$

Let $\dim \mathcal{N} = q$.
Malabre (1987).

Definition 25: The matrix pencil (E, A) is C-regular (regular by columns), if $q=0$, i.e.

$$\mathcal{N} = \{0\}. \tag{74}$$

Malabre (1987).

***Theorem* 21**: Under conditions (66) and (67), the solution of system (61) is unique for any $\mathrm{u}(t)$ if and only if the matrix pencil (E, A) is C-regular.
Buzurović (2000).

Controllability

Let (E,A,B) be a regular system and let $\mathcal{R}(E,A,B)$ be a *controlled subspace*. $\mathcal{R}(E,A,B)$ consists of the system states from $\Re^n$ which are reachable in positive time from the initial condition $x(0-) \cong 0$.

Definition 26: If $\mathcal{R}(E, A, B) = \Re^n$, then the system (E,A,B) is called the *controllable system*.
Buzurović (2000).

If P is the linear transformation on $\Re^n$ and $\mathcal{S}$ is the subspace in $\Re^n$, $\langle P \mid S \rangle$ denotes the subspace $\mathcal{S} + P(\mathcal{S}) + \ldots + P^{n-1}(\mathcal{S})$, i.e. the minimum P invariant subspace which contains $\mathcal{S}$.

***Lemma* 16**: If $(E, A, B) \in \sum(n, m)$ and α is real number which satisfies $\det(\alpha E - A) \neq 0$, then

$$\mathcal{R}(E, A, B) = \left\langle (\alpha E - A)^{-1} E \mid \mathcal{R}\left(\alpha E - A^{-1}\right) B \right\rangle \tag{75}$$

***Theorem* 22**: The generalized system (E, A, B) is controllable if and only if the regular system $(I, (\alpha E - A)^{-1}E, (\alpha E - A)^{-1}B)$ is controllable.

***Theorem* 10**: $(E, A, B) \in \sum_\theta(n, m)$ is controllable if and only if the regular system $\mathcal{R}_0(E, A, B)$ is controllable.
Buzurović (2000).

Indefinite eigenvalues assignments

It is shown in *Gantmacher* (1977.a) that the nonsingular matrices M and N exist with the following property

$$M\left(sE-A\right)N=\begin{bmatrix} sI-L & 0 \\ 0 & sJ-I \end{bmatrix}, \tag{76}$$

where $dimL = r$ and $dimJ = n - r$. The eigenvalues of the matrix L are the same as the eigenvalues of (sE-A), and the matrix J is nilpotent and has null eigenvalues.

Let p be a number of the Jordan blocks for the matrix J with the dimensions greater than 1. Let n_i+1, $i\in(1, 2, ..., p)$ be the dimension of i the Jordan block, $n_i \geq 1$. Therefore, the regular pencil (sE-A) has p indefinite eigenvalues of the order n_i, $i\in(1, 2, ..., p)$, respectively.

Let $\hat{x}=N^{-1}x$ and let $\hat{x}$ be the partition $\hat{x}^T=\begin{pmatrix} x_v^T & x_w^T \end{pmatrix}$ where $x_v \in \Re^r$ and $x_w \in \Re^{n-r}$. Consequently,

$$MB=\begin{bmatrix} B_v \\ B_w \end{bmatrix}\begin{matrix} \}r \\ \}n-r \end{matrix} \tag{77}$$

Moreover, together with equation (77), it is possible to decompose (76) in the following way

$$\dot{x}_v = Lx_v + B_v u \tag{78}$$

$$J\dot{x}_w = x_w + B_w u \tag{79}$$

Considering J as a transformation from $\mathcal{W}$ to $\mathcal{W}$, i.e. J: $\mathcal{W} \rightarrow \mathcal{W}$, with the nilpotent matrix J, it follows that decomposition (80) exists

$$\mathcal{W} = \mathcal{W}_1 \oplus \mathcal{W}_2 \tag{80}$$

such as

$$J=\begin{bmatrix} 0 & 0 \\ 0 & J_2 \end{bmatrix},\ B_w=\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{81}$$

where J_2 contains all elementary dividers of the matrix J of the order greater than 1. From (81) it can be concluded that the controllability for definite eigenvalues of the matrix pencil (sE-A) is equivalent to the existence of the matrix F, such as that eigenvalues of $L+B_v$ are predefined symmetric complex numbers. The modal criterion for controllability of the indefinite eigenvalues for the matrix pencil (sE-A) says that eigenvalues are controllable if the matrix pencil $[sE-A \quad B]$ does not have indefinite eigenvalues.

The following theorem presents sufficient and necessary conditions for the controllability of the indefinite eigenvalues. It is denoted $\mathcal{B}=\mathcal{R}(B)$.

***Theorem* 23**: System (76) is indefinite controllable if one of the conditions is fulfilled:

(i) $\mathcal{R}(J_2) + \mathcal{B}_2 = \mathcal{W}_2$,

(ii) $\langle J_2 | \mathcal{B}_2 \rangle = \mathcal{B}_2 + J_2\mathcal{B}_2 + ... + J_2^{q-1}\mathcal{B}_2 = \mathcal{W}_2$,

(iii) $\mathcal{R}(J) + \mathcal{N}(J) + \mathcal{B}_w = \mathcal{W}$,

(iv) $\mathcal{R}(E) + A\mathcal{N}(E) + \mathcal{B} = \mathcal{H}$

where q is index of nilpotency for the matrix J_2, and consequently for J,

Let γ = rang J = rang J_2 and let us denote $\gamma = \sum_{i\in p} n_i$, where γ is the number of the indefinite eigenvalues. The following result shows the consequence of the conversion from indefinite to definite zeros using state feedback.

***Theorem* 24**: Transformation F: $\mathcal{W} \rightarrow \mathcal{U}$ exists and det (sJ-I-B_wF) = γ if and only if system (76) is infinite controllable.
Buzurović (2000).

Let Λ_i be the set of n symmetric complex numbers and let $\Lambda = U_{i=1}^{q}\Lambda_i$. The following result can be obtained.

***Theorem* 25:** The transformation F: $\mathcal{W} \rightarrow \mathcal{U}$ exists and the subspace of det (sJ-I-B_wF) = 0 is equal to Λ if and only if system (76) is controllable u infinity.
Buzurović (2000).

Further results and more details about the geometric approach can be found in *Buzurović* (2000), *Buzurović*, *Debeljković* (2004), and *Debeljković*, *Buzurović* (2007), as well as in the articles from the reference list.

Conclusion

An overview of the geometric approach to the modern control theory was presented. A comprehensive chronological literature review for both linear and singular systems was given at the beginning of the article. The subspaces which play an important role in the application of the geometric approach were described. The geometric conditions for internal and external stability are described. A solution of the stability problem using the disturbance localization method for the linear system using the geometric approach was introduced. Invariant spaces for singular systems are defined in a different way due to the singular matrix in the state space model. Their characteristics and basic problems of the uniqueness and solutions existence for singular systems were presented. Finally, the directions for eigenstructure assignments are given. However, some proofs of the theorems are omitted on purpose. They can be found in the literature listed.

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Geomtetrijski prilaz u modernoj teoriji upravljanja: Pregled rezultata

Geometrijski prilaz predstavlja matematički koncept razvijen kako bi se unapredila analiza i sinteza linearnih višestruko prenosnih sistema. U ovom radu sumarno je dat pregled osnova geometrijskog prilaza u primeni kako na obične linearne tako in a linearne singularne sisteme. Izložen je matematički uvod ka upravljivim i uslovno invarjantnim potprostorima. Uvedene su i dodatne osobine potprostora, kao što su maksimalno upravljani i minimalno-upravljani, kao i izlazno-nulti potprostori. Opisano je rešenje problema lokalizacije poremećaja u odnosu na stabilnost sistema. Izloženi su i matematički i geometrijski alati koji se danas koriste u teoriji upravljanja. Geometrijski prilaz sistematski se primenjuje za potrebe analize upravljivosti i u podešavanju polova sistema. Postojanje i jedinstvenost rešenja singularnih sistema bile su posebno analizirane imajući u vidu specifične karakteristike koje su neosporne posledica postojanja singularne matrice u modelu tih sistema u prostoru stanja. Osnovni cilj ovog rada je bio da se da jedan opširan pregled dosadašnjih rezultata u okviru teorije geometrijskog prilaza u savremenoj teoriji upravljanja

Ključne reći: **teorija upravljanja, upravljanje sistemom, singularni sistem, stabilnost sistema, analiza sistema, goemetrijski prilaz.**

Геометрический подход в современной теории управления: Осмотр результатов

Геометрический подход собой представляет математическую концепцию, при помощи которой возможно усовершенствовать анализ и синтез линейных многократно передаточных систем. В настоящей работе суммировано представлен осмотр оснований геометрического подхода в применении как на обычных линейных, так и на линейных сингулярных системах. Здесь показан математический ввод к управляемым и к условным инвариантным подпространствам. Введены и добавочные свойства подпространств, подобно максимально управляемым и минимально управляемым, а в том качестве и выходно-нулевым подпространствам. Здесь тоже описано решение проблемы локализации возмущения относительно к устойчивости системы. Тоже представлены и математические и геометрические оборудования, сегодня пользующие в теории управления. Геометрический подход систематически применяется для потребности анализа управляемости и в предназначении полюсов системы. Существование и уникальность решений сингулярных систем были особо анализированы имея в виду специфические характеристики, которые являются бесспорными последствиями наличия сингулярной матрицы в модели этих систем в пространстве состояния. Основной целью настоящей работы было представление одного подробного осмотра прежних результатов в рамках теории геометрического подхода в современной теории управления.

Ключевые слова: **теория управления, управление системой, сингулярная система, устойчивость системы, анализ системы, геометрический подход.**

Approche géométrique dans la théorie moderne de contrôle: tableaux des résultats

L'approche géométrique représente un concept mathématique développé pour améliorer l'analyse et la synthèse des systèmes linéaires multiples. Dans ce papier on a donné un aperçu sommaire des bases de l'approche géométrique dans l'application pour les systèmes linéaires et les systèmes linéaires singuliers. On a exposé l'introduction mathématique vers les sous- espaces contrôlables et variables conditionnellement. On a introduit aussi les caractéristiques supplémentaires des sous-espaces, tels les sous-espaces contrôlés au maximum et minimum ainsi que les sous-espaces zéro de sortie. On a décrit la solution du problème de la localisation de la distribution par rapport à la stabilité du système. Les outils mathématiques et géométriques utilisés actuellement dans la théorie de contrôle ont été présentés. L'approche géométrique est systématiquement appliquée pour les besoins de l'analyse de contrôle et dans l'adaptation des pôles du système. L'existence et l'unité de la solution des systèmes singuliers ont été analysés à part, ayant en vue les caractéristiques spécifiques qui sont conséquence de l'existence d'une matrice singulière dans le modèle de ces systèmes dans l'espace de l'état. Le but principal de ce travail est de donner un compte-rendu détaillée des résultats existants dans le cadre de la théorie de l'approche géométrique de la théorie moderne de contrôle.

Mots clés: **théorie de contrôle, contrôle du système, système singulier, stabilité du système, analyse du système, approche géométrique.**