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New View of the Euler-Bernoulli Equation

Mirjana Filipović, PhD (Eng)[1)]

A special attention is paid to the motion of flexible links in a robotic configuration. The elastic deformation is a dynamic value which depends on the total dynamics of the robot system movements. The Euler-Bernoulli equation should be expanded according to the requirements of the motion complexity of elastic robotic systems. The Euler-Bernoulli equation (based on the existing laws of dynamics) should be supplemented with all the forces (inertial forces, Coriolis, centrifugal forces, gravity forces, environment forces, disturbance forces as well as coupling forces between the present modes) that are participating in the formation of the elasticity moment of the considered mode. This yields the difference in the structure of Euler-Bernoulli equations for each mode. The stiffness matrix is a full matrix as well as a damping matrix. The mathematical model of the actuators also comprises coupling between elasticity forces. A particular integral defined by Daniel Bernoulli should be supplemented with the stationary character of elastic deformation of any point of the considered mode, caused by the present forces. The general form of the mechanism elastic line is a direct outcome of the system motion dynamics, and cannot be described by one scalar equation but by three equations for position and three equations for orientation of every point on that elastic line. The simulation results are shown for a selected robotic example involving the simultaneous presence of elasticity of the gear and of the link (two modes), as well as the environment force dynamics.

Key words: **robotic, motion dynamics, Euler-Bernoulli equations, process modeling, elastic deformation, coupling, stiffness matrix, motion simulation, programmed trajectory.**

Introduction

MODELING and control of elastic robotic systems has been a challenge to researchers in the last three decades. The control of robots with elastic joints in contact with dynamic environment [34] is considered. The feedback control was formed for the robot with flexible links (two-beam, two-joint systems) with distributed flexibility [2], robots with flexible links being also dealt with [17]. The nonlinear control strategy for tip position trajectory tracking of a class of structurally flexible multi-link manipulators is developed [28]. Authors [25], [26] derived dynamic equations of the joint angle, the vibration of the flexible arm, and the contact force.

Author [35] presents an approach to the end point control of elastic manipulators based on the nonlinear predictive control theory. Author [21] designed a control law for local regulation of contact force and position vectors to desired constant vectors. Differently from conventional approaches, authors [4] focus on the design of a rigid part motion control and the selection of bandwidths of a rigid subsystem. Author [22] presents the derivation of the equations of motion for application to mechanical manipulators with flexible links. The equations are derived [23] using Hamilton's principle, and are nonlinear integro-differential equations.

Author [31] discusses the force control problem for flexible joint manipulators. Authors [16] extend the integral manifold approach for the control of flexible joint robot manipulators from the known parameter case to the adaptive case.

A mathematical model of a mechanism with one degree of freedom (DOF), with one elastic gear was defined [30]. Based on the same principle, the elasticity of gears is introduced into the mathematical model in this paper, as in papers [11]-[15] also. However, when the introduction of link flexibility into the mathematical model is concerned, it is necessary to point to some essential problems in this domain.

The LMA ("Lumped-mass approach") is a method which defines motion equation of any point of a considered mechanism. If any link of the mechanism is elastic then we also can define a motion equation of any point of the presented link. We do not know exactly when this approach has been stated. It defines a dynamic equation in any point of a mechanism during movement. The LMA [2], [3] gives the possibility to analyze the motion of the any point of each mode. Papers with this research topic (approach) were rare in robotics journals in the last two decades.

The EBA ("Euler-Bernoulli approach") assumes the use of Euler-Bernoulli equations which appeared in 1750. EBA [4]-[7], [18]-[20], [24]-[26] etc, gives the possibility to analyze a flexible line form of each mode in the course of task realization. The EBA is an approach that is still in the focus of researchers' interest and it was analyzed most often in the last decades.

In the pertinent literature no relationship has been established between the LMA and EBA.

We consider that EBA and LMA, are two comparative methods addressing the same problem but from different aspects [12]-[15].

We consider them as two comparable methods considering the same problem but from different aspects. Using the EBA we obtain the equations of flexible line model of each mode and by setting boundary conditions we obtain model equations of motion at the point of the tip (or

1) Institute of Mihajlo Pupin, Volgina 15, 11060 Belgrade, SERBIA

any other point) of each mode, which is in fact the LMA. These equations are of different types and cannot be combined. As the equation of motion for the mode tip point is essentially the LMA and it follows directly from the equation of a flexible line obtained via the EBA for the preset boundary conditions, it clearly comes out that the structures of these equations are the same (whereas the content of elements of these structures is not the same). Just for this reason, each of the two methods can be used as a check of validity of the other (assuming that no crude approximations have been made).

A mathematical model obtained by any of the methods should satisfy the elementary structure of the models of elastic mechanisms known in the literature [33]. This has been treated in detail in our work, but it is not the only essential problem existing in the pertinent literature.

The Euler Bernoulli equation in the original form as well as its solution were used in the literature [4]-[7], [18]-[20], [24]-[26] etc, published until now as defined [27]. In the meantime, from 1750 when the Euler Bernoulli equation was published until today, our knowledge, especially in the robotics, the oscillation theory and the elasticity theory, has progressed significantly. As a consequence, this paper points out the necessity of the extension of the Euler Bernoulli equation from many aspects.

In the previous literature [4]-[7], [18]-[20], [24]-[26] etc, the general solution of the motion of an elastic robotic system has been obtained by considering flexural deformations as transversal oscillations that can be determined by the method of particular integrals of D. Bernoulli.

It is known that flexural deformations of a body can be caused by:
- disturbance forces, causing an oscillatory motion,
- stationary forces, causing a motion of a stationary character.

We consider that any elastic deformation can be presented by superimposing D. Bernoulli's particular solutions of the oscillatory character and a stationary solution of the forced character. See papers [12]-[15].

The first detailed presentation of the procedure for creating a reference trajectory was given in [1].

In our work we have synthesized a reference trajectory for a robot model including elastic gears and links and the presence of an environment force. The reference trajectory is calculated from the overall dynamic model, when the robot tip is tracking a desired trajectory in a reference regime in the absence of disturbances.

Elastic deformation (of flexible links and elastic gears) is a quantity which is at least partly encompassed by the reference trajectory. It is assumed that all elasticity characteristics in the system (both of stiffness and damping) are "known", at least partly and at that level can be included into the process of defining the reference motion. The reference trajectory thus defined allows the possibility of applying very simple control laws via PD local feedback loops, which ensures reliable tracking of the robotic tip considered in the space of Cartesian coordinates to the level of known elasticity parameters, too.

As far as the working regime of the robot is concerned we think that all forces should participate in generating elastic deformations and that it is a crude approximation to assume that elastic effects are generated only by the gravitational force, or only by the environment force as in [24] and [26], or that Coriolis and centrifugal forces can be neglected altogether that elastic deviations are so small, so that the inertia matrix is not dependent on them, as assumed in [20].

The "Assumed modes technique" [27] was used by all authors in the last 40 years to form the Euler Bernoulli equation of beam. In our paper we form Euler Bernoulli equation but we do not use the "assumed modes technique" in contrast to our contemporaries.

We think that the "assumed modes technique" was and still can be useful in some other research areas but it is used in a wrong way in robotics, theory of oscillations and theory of elasticity.

We assume that the elastic deformation, and also the circular frequency of each mode of the elastic element, is a consequence of the overall dynamics motion of the robotic system.

Let us emphasize once again that in this paper we propose a mathematical model solution that includes in its root the possibility for analyzing simultaneously both present phenomena – the elasticity of gears and the flexibility of links, and the idea originated from [3], but on new principles. We show how the continuously present environment dynamics force affects the behavior of an elastic robot system.

Our future work should be directed to implementating the elasticity of gears and the flexibility of links to any model of rigid robots and also to the model of reconfigurable rigid robots as given in [9], [10] or any other type of mechanisms. The mechanism would be modeled to contain elastic elements and to generate vibrations, which are used for conveying particulate and granular materials in [8].

Dynamics of Change of Elastic Deformations

Meirovitch proposed the "modal technique" more than 40 years ago, in 1967. At that moment it was a big contribution to stimulate scientists to consider new investigations and new ideas. At that time it was a solution of one visionary that explained the Euler-Bernoulli equation in a new manner. It should have been a flywheel in the evolution of the Theory of elasticity and Theory of oscillation as well as Robotics which in this period was in expansion.

The author elaborated a particular application of the Euler-Bernoulli equation supposing that elastic deformation is a quantity defined in advance with respect to amplitude and frequency and, formed in this way, included into a dynamic model. Having in mind the state of engineering and the technical possibilities for data processing at that time and also technical needs at that time, this solution was a very significant contribution to science as well as engineering. Regulations specifying the permissible amplitudes of elastic deformation and the permissible oscillating frequencies of constructions are employed even today in some areas of the construction theory.

The allowed values are mostly the result confirmed by various experiences, measures and scientific solutions also based on [27]. Thus it should be emphasized that such solutions as in [27] should not be denied in any case, but they should further be applied just in areas where it is not possible to form a system model, for example, because of its complexity.

Not finding any other solutions, many researchers in robotics [4]-[7], [18]-[20], [24]-[26] etc, applied the solution [27] in the description of the real dynamics of the robot system elastic deformations, overlooking that the solution [27] was derived under the condition of the elastic deformation defined in advance and by the amplitude and the frequency, or they used many ways to modify the solutions from [27].

By now the authors implemented the elastic deformations as the values on the principles from [27] and they did not get any real values as a result of the robot system movements. Without any other solutions it is obvious that the research in this area was scarce in the last 10 years.

However, today's development of the knowledge about the robot system dynamics modeling enables the establishing and analyzing of new models which will treat the elastic deformation as a dynamic value. Our research field is directed in that way, in order to describe this theory in the real environment, without assumptions i.e. limitations of the elastic deformation that [27] is based on.

With new knowledge collected through generations, the intensive development of new technical areas such as robotics especially strengthened by the development of the data computing process, demanded and enabled that elastic deformation was considered really as a dynamic value which depended on the system parameters. The elastic deformation is a dynamic value by both amplitude and frequency and it is the result of the total system movement i.e. outer and inner, dynamic and static forces. Such elastic deformation should exist in the dynamics of the robot system movements. The synthesis of the robot system dynamics should be processed on the basis of the completely new different principles comparing to [27], with models based on the known, classic dynamics, the elasticity theory and the oscillation theory, where the elastic deformations are described as dynamic values of the inner and outer load which influence the total dynamics of the robot system movements.

The area which we deal with, the robotics, is very important, because the modeling of the robot system movement dynamics with both rigid and elastic elements comes from it directly. The robotics is the area which can offer the solution and it represents the foundation of the further research in many other areas. The reason for that is quite simple: the robotics progressed significantly in the last 40 years. It is important to emphasize the importance of the further researches but now based on new principles which will be set in this paper.

The elastic deformation cannot be defined in advance (with both amplitude and frequency) and put in the system but completely inversely. The elastic deformation is a dynamic value which depends on the total dynamics of the robot system movements. That means that the elastic deformation amplitude and its frequency change depending on the forces (inertial forces, Coriolis, centrifugal forces, gravity forces as well as coupling forces between the present modes, and the play of the environment forces). It, of course, depends on the mechanism configuration, weight, length of the segments of the reference trajectory choice, dynamic characteristics of the motor movements etc.

The elastic deformation exists even in the state of inaction and then it depends on the gravity forces i.e. mechanism configuration. That means that the elastic deformation depends on the robot system characteristics and it can be calculated in any chosen moment. When we conduct the mechanism through the reference trajectory, the elastic deformation also exists but now at the reference level without the influence of the disturbance.

The Euler-Bernoulli equation was written in 1750. It was written by Bernoulli, physicist and Euler, mathematician, his longtime friend and colleague. They did not even dream about the robotics and the knowledge we have now at our disposal. But, although it was made more than 250 years ago, the Euler-Bernoulli equation is still of current interest and it can be connected logically with the contemporary knowledge from the robotics.

I especially emphasize the robotics which, as we have already written, develops very intensively and in this case it imposes the solutions to the other areas such as the oscillation theory and the elasticity theory. It could have been otherwise. However, that is less important. The most important is to see the solution. It is important to establish the continuity in the research through years, decades, centuries. Moreover, if we analyze the past properly, there is hope to build the future. One day our successors will make a critical review of our equations, our assumptions and our interpretations, and we have to be aware of that because we are responsible for what we leave behind ourselves. But, we are not only responsible for the equations that we write but also for the equations to which we „only add" i.e. which we refer to. In that sense the choice is very various.

Source Equations of the Elastic Line

The equation of the elastic line of the beam bending is of the following form:

$$\hat{M}_{1,1} + \beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^2} = 0 . \qquad (1)$$

where $\hat{M}_{1,1}\left[\mathrm{Nm}\right]$ is the load moment, in these source equations encompassing only inertia, $\beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^2} = \hat{\varepsilon}_{1,1}$ is the bending moment and $\beta_{1,1}\left[\mathrm{Nm}^2\right]$ is the flexural rigidity.

The general solution of motion, i.e. the form of transversal oscillations of flexible beams can be found by the method of particular integrals of D. Bernoulli, that is:

$$\hat{y}_{t_{01,1}}(\hat{x}_{1,1}, t) = \hat{X}_{1,1}(\hat{x}_{1,1}) \cdot \hat{T}_{t01,1}(t) . \qquad (2)$$

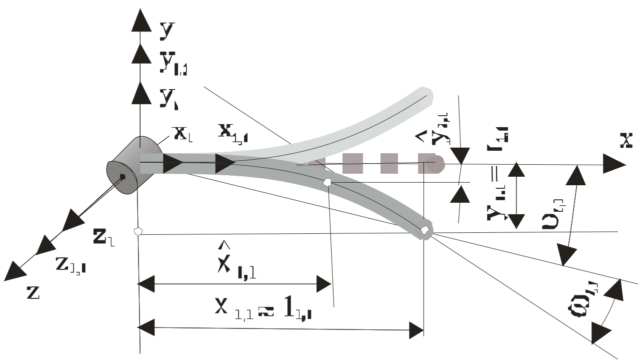

Figure 1. Idealized motion of elastic body according to D. Bernoulli

See Fig.1. By superimposing the particular solutions (2), any transversal oscillation can be presented in the following form:

$$\hat{y}_{t_0 1}(\hat{x}_{1,j}, t) = \sum_{j=1}^{\infty} \hat{X}_{1,j}(\hat{x}_{1,j}) \cdot \hat{T}_{t01,j}(t) . \qquad (3)$$

As already mentioned, Eqs.(1) - (3) were defined under the assumption that the elasticity force is opposed only by the proper inertial force. Besides, it is supposed by definition that the motion in Eq.(1) is caused by an external force, suddenly added and then removed. The solution (2) - (3) of D. Bernoulli satisfies these assumptions.

Bernoulli presumed the horizontal position of the observed body as its stationary state (in this case it matches the position x - axis, see Fig.1). At such presumption, the oscillations happen just around the x - axis. If Bernoulli, at any case, had included the gravity force in its Eq.(1), the situation would have been more real. Then the stationary body position would not have matched the x - axis position, but the body position would have been little lower and the oscillations would have happened around the new stationery position (as presented in Fig.2).

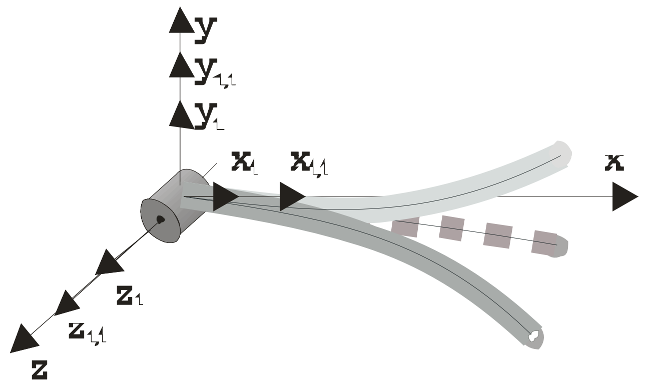

Figure 2. The motion of the elastic body in case of gravity force presence

All marks are the same as in papers [12]-[15].

Equations (1) - (3) need a short explanation that, we think, should be assumed, but which is missing from the original literature [32]. Euler and Bernoulli wrote Eq.(3) based on 'vision'. They did not define the mathematical model of a link with an infinite number of modes, which has a general form of Eq.(4), but they did define the motion solution (shape of elastic line) of such a link, which is presented in Eq.(3). They left the task of link modeling with an infinite number of modes to their successors. Transversal oscillations defined by Eq.(3) describe the motion of elastic beam to which we assigned an infinite number of DOFs (modes), and which can be described by a mathematical model composed of an infinite number of equations, in the form:

$$\hat{M}_{1,j} + \hat{\varepsilon}_{1,j} = 0 \quad j = 1, 2, ..., j, ...\infty \tag{4}$$

The dynamics of each mode is described by one equation. The equations in the model (4) are not of equal structure as our contemporaries, authors of numerous works, presently interpret it. We think that the coupling between the modes involved leads to structural diversity among the equations in the model (4). This explanation is of key importance and is necessary for understanding our further discussion.

Under a mode we understand the presence of coupling between all the modes present in the system. We analyze the system in which the action of coupling forces (inertial, Coriolis', and elasticity forces) exists between the present modes. To differentiate it from "mode shape" or "assumed mode", we could call it a coupled mode or, shorter, in the text to follow, a mode. This yields the difference in the structure of Euler-Bernoulli equations for each mode.

The Bernoulli solution (2) - (3) describes only partially the nature of motion of real elastic beams. More precisely, it is only one component of motion. The Euler Bernoulli equations (1) - (3) should be expanded from several aspects in order to be applicable in a broader analysis of elasticity of robot mechanisms. By supplementing these equations with the expressions that come out directly from the motion dynamics of elastic bodies, they become more complex.

The motion of the considered robotic system mode is far more complex than the motion of the body presented in Fig.1. This means that the equations that describe the robotic system (its elements) must also be more complex than Eqs. (1) - (3), formulated by Euler and Bernoulli. This fact is overlooked, and the original equations are widely used in the literature to describe the robotic system motion. This is very inadequate because valuable pieces of information about the complexity of the elastic robotic system motion are thus lost. Hence, we should especially emphasize the necessity of expanding the source equations for the purpose of modeling robotic systems, and this should be done in the following way:

- *based on the known laws of dynamics, Eq.(1) is to be supplemented by all the forces that participate in the formation of the bending moment of the considered mode. It is assumed that the forces of coupling (inertial, Coriolis, and elastic) between the present modes are also involved, which yields a structural difference between Eqs.(1) in the model (4),
- *Equations (2) - (3) are to be supplemented by the stationary character of the elastic deformation caused by the forces involved.

Euler-Bernoulli equation of a complex robotic system

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Let us consider the motion of the first mode of the given link. The link has in all n_1 modes. The first mode is the bracket (support) of uniformly distributed mass along the mode, loaded by the moment $\hat{M}_{1,1}$. The load moment $\hat{M}_{1,1}$ is composed of all the forces acting on the first mode of the link, and these are inertial forces (own and coupled inertia forces of other modes), centrifugal, gravitational, Coriolis forces (own and coupled), forces due to relative motion of one mode with respect to the other, coupled elasticity forces of other modes, as well as the force of the environment dynamics, which is via the Jacobian matrix transferred to the motion of the first mode. This means that all these forces participate in generating a bending moment i.e. in forming elastic deformation as well as the elasticity line of the first mode. In that case the model of elastic line of the first mode of the elastic link is of the form:

$$\hat{H}_{1,j}\frac{d^2\hat{y}_{1,j}}{dt^2}+\hat{h}_{1,1}+j_{1,1}^T F_{uk}+z_{1,j}\cdot\varepsilon_1+ +\beta_{1,1}\cdot\frac{\partial^2(\hat{y}_{1,1}+\eta_{1,1}\cdot\dot{\hat{y}}_{1,1})}{\partial\hat{x}_{1,1}^2}=0 \tag{5}$$

j - ordinal number of the considered mode, $j=1,\dots\ n_1$.

Vectors in Eq.(5) are:

$\hat{H}_{1,j}\in R^{1xn_1}$ - the vector characterizing the inertia of the first mode.

$\hat{h}_{1,1}\in R^{1x1}$ - centrifugal, gravitational and Coriolis forces of the first mode.

$j_{1,1}\in R^{6x1}$, first row of the Jacobian matrix serving to map the impact of the dynamic force of contact $F_{uk}\in R^{6x1}$ on the behavior of the first mode.

The vector $z_{1,j}\in R^{1xn_1}$ characterizing the effect of elasticity forces of the other modes on the first mode. The $z_{1,j}$ obtained by modeling different link structures (with one, two, three modes).

The moment of bending defined for the tip of any mode of the considered link is:

$$\varepsilon_{1,j}=F_{1,j}\cdot l_{1,j}=C_{s1,j}\cdot r_{1,j}\cdot l_{1,j}+B_{s1,j}\cdot\dot{r}_{1,j}\cdot l_{1,j} \tag{6}$$

The rigidity and damping characteristic for the tip of any mode is designated as $C_{s1,j}$[N/m] and $B_{s1,j}$[Ns/m] respectively, the maximal deflection is $r_{1,j}$ and the mode length is $l_{1,j}$. The vector of bending moments is ε_1.

$$\varepsilon_1=[\varepsilon_{1,1}\ \ \varepsilon_{1,2}\ \ \varepsilon_{1,3}\ \ \varepsilon_{1,4}\ \ \varepsilon_{1,5}\ \ \dots\varepsilon_{1,n_1}]^T.$$

The bending moment defined for an arbitrary point of the first mode is: $\hat{\varepsilon}_{1,1}=\beta_{1,1}\cdot\frac{\partial^2(\hat{y}_{1,1}+\eta_{1,1}\cdot\dot{\hat{y}}_{1,1})}{\partial\,\hat{x}_{1,1}^2}$

The force acting on the formation of the elastic line of an arbitrary mode of the considered link is $\hat{F}_{1,j}$. The load moment $\hat{M}_{1,1}$ from Eq. (5) is defined as:

$$\hat{M}_{1,1}=\hat{H}_{1,J}\frac{d^2\hat{y}_{1,J}}{dt}+\hat{h}_{1,1}+j_{1,1}^t F_{uk}+z_{1,J}\cdot\varepsilon_1 \tag{7}$$

Thus Eq. (5) can be now written in a simpler form:

$$\hat{M}_{1,1}+\hat{\varepsilon}_{1,1}=0 \tag{8}$$

Eq. (8) represents the Euler-Bernoulli's equation *of the first mode.* The same equation was defined under the assumption that the elasticity moment $\hat{\varepsilon}_{1,1}$ is opposed by the load moment $\hat{M}_{1,1}$, which, among the other forces, encompasses also the *coupled elasticity of the other modes*. In a stationary regime of robotic task realization, the mentioned moments that oppose the elasticity moment $\hat{\varepsilon}_{1,1}$ continuously change during the robotic task realization. This system can be also influenced by disturbance forces, which may be of an instant or permanent character.

Therefore, elastic deformations of a given body can be generated by:

- *disturbance* forces, causing oscillatory motion,
- *stationary* forces, causing stationary motion.

By superimposing the particular solutions of oscillatory character and the stationary solution of forced character, any elastic deformation can be presented in the following general form:

$$\hat{y}_{1,1}=\hat{X}_{1,1}(\hat{x}_{1,1})\cdot(\hat{T}_{st1,1}(t)+\hat{T}_{t01,1}(t))= =\hat{a}_{1,1}(\hat{x}_{1,1},\hat{T}_{st1,1},\hat{T}_{t01,1},t) \tag{9}$$

$\hat{T}_{st1,1}(t)$ is the stationary part of elastic deformation caused by stationary forces that may continuously change in time.

In the case when the robot is in state of inaction, then stationary forces are gravity forces. In case when the robot is in the state of motion, then stationary forces are gravity, inertial, centrifugal, Coriolis and of course coupled forces of all forces and environment force (if it is continuous). This means that stationary forces are all forces which change continuously in time.

$\hat{T}_{t01,j}(t)$ is the oscillatory part of elastic deformation as in (2) or (3). This component of elastic deformation is caused by a disturbance force (which acts instantaneously) and can appear in the state of robot inaction and also in state of robot motion.

An environment force can be of:

- disturbing character (for example when the robot is moving without limitation and only in one moment enters into the contact with environment), or
- stationary character (for example when the robot is continuously under the influence of the environment force).

Total motion of the considered mode, defined by the sum of stationary and oscillatory motion, is given by Eq. (9).

Orientation of any point of the first mode is defined by:

$$\psi_{1,1}=\hat{d}_{1,1}(\hat{x}_{1,1},\hat{T}_{st1,1},\hat{T}_{t01,1},t) \tag{10}$$

Like we defined the elastic line model of the first mode by Eq. (5) we can also define the model of the elastic line of the second, third … n_1-th mode of the elastic link. The elastic line model of the first link that has n_1 modes is given in a matrix form by the following equation:

$$\hat{H}_1\cdot\frac{d^2\hat{y}_{1,j}}{dt^2}+\hat{h}_1+j_{e1}^T\cdot F_{uk}+z_1\cdot\varepsilon_1+\hat{\varepsilon}_1=0 \tag{11}$$

Matrices and vectors in Eq. (11) are:

$\hat{H}_1\in R^{n_1xn_1}$ - the matrix characterizing the inertia of the each mode, $\hat{h}_1\notin R^{n_1x1}$ - the vector characterizing the effect of centrifugal, gravitational and Coriolis forces of the each mode,

$j_{e1}\in R^{6xn_1}$ - the Jacobian matrix serving to map the impact of the dynamic force of the contact F_{uk} on the behavior of each mode.

The matrix $z_1\in R^{n_1xn_1}$ characterizing the mutual effect of elasticity forces of the presented modes on each mode.

$$\hat{\varepsilon}_1=\left[\beta_{1,1}\cdot\frac{\partial^2(\hat{y}_{1,1}+\eta_{1,1}\cdot\dot{\hat{y}}_{1,1})}{\partial\hat{x}_{1,1}^2}\ \dots\ \beta_{1,n_1}\cdot\frac{\partial^2(\hat{y}_{1,n_1}+\eta_{1,n_1}\cdot\dot{\hat{y}}_{1,n_1})}{\partial\hat{x}_{1,n_1}^2}\right]^T$$

The load moment $\hat{M}_1$ is defined by:

$$\hat{M}_1 = \hat{H}_1 \cdot \frac{d^2 \hat{y}_{1,j}}{dt^2} + \hat{h}_1 + j_{e1}^T \cdot F_{uk} + z_1 \cdot \varepsilon_1 \tag{12}$$

Eq. (11) can be written in a simpler form as:

$$\hat{M}_1 + \hat{\varepsilon}_1 = 0 \tag{13}$$

Equation (13) represents the Euler-Bernoulli's equation *of the first link.*

To describe the behavior of the one-link robotic system having n_1 modes, the vector Eq. (13) should be supplemented by the mathematical model of the motor. The motor mathematical model can be defined by writing the equation of motion of all the moments that act on the motor shaft. In the case of a rigid robotic system the motor moment is opposed by the mechanism moment. With elastic robotic systems we have a somewhat different situation: the motor moment is opposed by the bending moment of the first elastic mode that comes after the motor and, partly, by the bending moments of the other elastic modes that are connected in series after the motor. All the modes that come after the motor, due to their position, exert certain influence on the motor dynamics.

The effect of the first mode bending moment is defined by the factor $+1/2^0$, of the second by $-1/2^1$, of the third by $+1/2^2$, of the fourth by $-1/2^3$, of the fifth by $+1/2^4$ etc.

We add all these elasticity moments to the motor model because they are just to oppose the rotation moment of the motor shaft. The mathematical model of motor is of the following form:

$$\begin{aligned} u_1 &= R_1 \cdot i_1 + C_{E1} \cdot \dot{\bar{\theta}}_1 \\ C_{M1} \cdot i_1 &= I_1 \cdot \ddot{\bar{\theta}}_1 + B_{u1} \cdot \dot{\bar{\theta}}_1 - S_1 \cdot \varepsilon_{m1} \end{aligned} \tag{14}$$

$R_1\,[\Omega]$ is the rotor circuit resistance; $i_1\,[\mathrm{A}]$ is the rotor current; $C_{E1}\,[V/(\mathrm{rad/s})]$ and $C_{M1}\,[\mathrm{Nm/A}]$ are the proportionality constants of the electromotive force and moment, respectively; $B_{u1}\,[\mathrm{Nm/(rad/s)}]$ is the coefficient of viscous friction; $I_1\,[\mathrm{kgm}^2]$ is the inertia moments of the rotor and the reducer; S_1 is the expression defining the reducer geometry; ε_{m1} is the equivalent elasticity moment that opposes the rotation moment of the motor shaft.

$$\varepsilon_{m1} = z_{m1,j} \cdot \varepsilon_1 .$$

$$z_{m1,j} = \left[+ \frac{1}{2^0} - \frac{1}{2^1} + \frac{1}{2^2} - \frac{1}{2^3} \ldots (-1)^{(n_1-1)} \frac{1}{2^{(n_1-1)}} \right].$$

The vector $z_{m1,j}$ characterizes the influence of the elasticity moment of each mode on the motor dynamics.

It will not be explained how we have obtained expression $C_{M1} \cdot i_1$, $I_1 \cdot \ddot{\bar{\theta}}_1$, $B_{u1} \cdot \dot{\bar{\theta}}_1$ in Eq. (14), because this is already known from the literature. We will explain the procedure of obtaining the equivalent elasticity moment ε_{m1}.

The potential energy in top of j - th mode of the first link is $E_{pels1,j} = \frac{1}{2} C_{s1,j} \cdot \vartheta_{1,j}^2 \cdot l_{1,j}^2$, while the dissipative energy is: $\Phi_{els1,j} = \frac{1}{2} B_{s1,j} \cdot \dot{\vartheta}_{1,j}^2 \cdot l_{1,j}^2$.

All quantities should be expressed in dependence of generalized coordinates.

One of them is also the rotation angle of the motor shaft $\bar{\theta}_1$.

By applying Lagrange's equations on the expressions $E_{pels1,1}$, $E_{pels1,2}$, $\ldots E_{pels1,j}$, $\ldots E_{pels\,1,n_1}$ and $\Phi_{els1,1}$, $\Phi_{els1,2}$, $\ldots$ $\Phi_{els1,j}$, $\ldots$ $\Phi_{els\,1,n_1}$ with respect to the generalized coordinate $\bar{\theta}_1$, we obtain the equivalent elasticity moment $\varepsilon_{m\,1}$, that opposes to the rotation moment of the first motor shaft.

(In case of presence of the elastic gear behind the motor, we have its potential energy: $E_{pel\xi} = \frac{1}{2} \cdot C_\xi \cdot \xi^2$ and dissipative: $\Phi_{el\xi} = \frac{1}{2} \cdot B_\xi \cdot \dot{\xi}^2$ energy. These quantities should also be expressed in dependence of generalized coordinates and Lagrange's equation should be applied.)

All this is explained in detail at modeling of considered example in paper [13] (chapter 4.A).

The overall order of the system (13) - (14) is $n_1 + 1$.

Like we defined the motion of any point on the first mode elastic line by Eqs. (13) - (14), we can also define the motion of any point on the elastic line of the second, third n_1 -th mode of the elastic link.

By superimposing the solution (9) - (10) for all the present modes of the first link and adding to it the dynamics of motor motion that drives it, we obtain the total solution of the system (13) - (14) in the form:

$$\hat{y}_1 = \hat{a}_1(\hat{x}_{1,j}, \hat{T}_{st1,j}, \hat{T}_{t01,j}, \bar{\theta}_1, t) \tag{15}$$

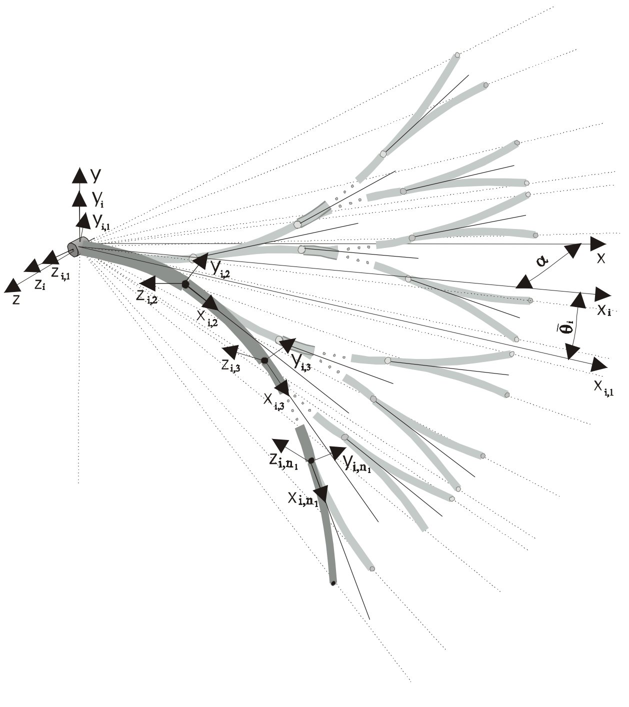

Figure 3. Possible positions of the tip of the elastic line with n_1 modes.

On considering Fig.3 we can see that the position $\hat{x}_1$ should also be defined, which is not only $\sum_{j=1}^{n_1} \hat{x}_{1,j}$ (because the directions of the axes $\hat{x}_{1,1}$, $\hat{x}_{1,2} \ldots \hat{x}_{1,n_1}$ most often do not coincide with the direction of the axis $\hat{x}_1$), but also includes to a significant extent the geometry and characteristics of the mechanism bending, i.e. the mechanism dynamics.

$$\hat{x}_1 = \hat{b}_1(\hat{x}_{1,j}, \hat{T}_{st1,j}, \hat{T}_{t01,j}, \overline{\theta}_1, t) \tag{16}$$

Any form of the elastic line and the pertinent transversal oscillations, as well as the motor motion, can be presented by Eqs. (15) - (16). To this equation one should add also the equation defining the orientation of each point on the elastic line of the link.

$$\hat{\psi}_1 = \hat{d}_1(\hat{x}_{1,j}, \hat{T}_{st1,j}, \hat{T}_{to1,j}, \overline{\theta}_1, t) \tag{17}$$

In Fig.3 we sketched the possible forms of elastic line of the i -th link having n_i modes that appear in the plane x_i - y_i. The plane x_i - y_i is rotated by the angle α, characterizing in the figure the position of the link base with respect to the main coordinate frame x - y - z. In the same figure we presented only some of the possible forms of the elastic line. The link tip can assume very different positions in the plane x_i - y_i.

Let us consider a robotic system with m links, whereby the first link contains n_1 modes, second link contains n_2 modes, … m -th link contains n_m modes. See Fig.4. Model of the elastic line of this complex elastic robotic system is given in the matrix form by the following equation:

$$\hat{H} \cdot \frac{d^2 \hat{y}}{dt^2} + \hat{h} + j_e^T \cdot F_{uk} + z \cdot \Theta \cdot \varepsilon + \hat{\varepsilon} = 0 \tag{18}$$

If we define $k = \sum_{i=1}^{m} n_i$ then we have that

$\hat{H} \in R^{kxk}$ - matrix characterizing the inertia,

$\hat{h} \in R^{kx1}$ - vector of the centrifugal, gravitational and Coriolis forces, $j_e^T \in R^{6xk}$ - Jacobian matrix mapping the effect of the dynamic contact force F_{uk},

$\Theta \in R^{k \times k}$ - matrix characterizing the robot configuration,

$z \in R^{k \times k}$ matrix characterizing the mutual influence of the forces of elastic modes of all the links,

$$\varepsilon = \left[\varepsilon_{1,1} \ \varepsilon_{1,2} \ldots \varepsilon_{1,n_1} \ \varepsilon_{2,1} \ \varepsilon_{2,2} \ldots \varepsilon_{2,n_2} \ \ldots.. \ \varepsilon_{m,n_m} \right]^T .$$

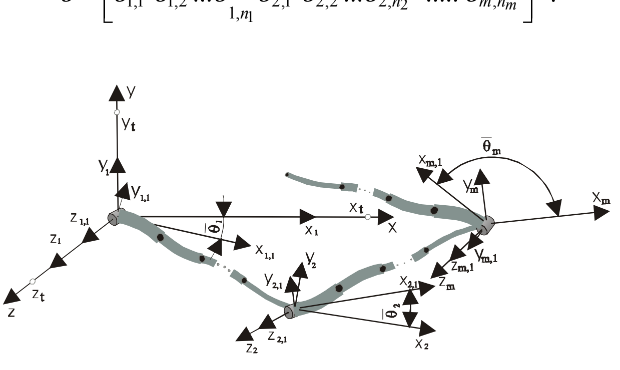

Figure 4. The elastic line of the complex robotic system with m links.

$$\hat{\varepsilon} = \left[\beta_{1,1} \cdot \frac{\partial^2(\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})}{\partial \hat{x}_{1,1}^2} \ldots \beta_{n_m,n_{ml}} \cdot \frac{\partial^2(\hat{y}_{n_m,n_{ml}} + \eta_{n_m,n_{ml}} \cdot \dot{\hat{y}}_{n_m,n_{ml}})}{\partial \hat{x}_{n_m,n_{ml}}^2} \right]^T .$$

If we define from (18) the load moment $\hat{M}$ as:

$$\hat{M} = H \cdot \frac{d^2 \hat{y}}{dt^2} + \hat{h} + j_e^T \cdot F_{uk} + z \cdot \Theta \cdot \varepsilon \tag{19}$$

Eq. (18) can be written in the form:

$$\hat{M} + \hat{\varepsilon} = 0 \tag{20}$$

Equation (20) represents the Euler-Bernoulli's equation *of the overall robotic system.* In order to describe the behavior of a robotic system having m links (each of them containing n_i modes), we have to add to the vector Eq. (20) the mathematical model of all the motors written in a vector form. Let us define it by setting for each motor the equation of motion of all the moments acting about the rotation axis of the given motor. It has the form of the mathematical model of the motor of a rigid robotic system, but the difference being in that the moment of the i -th motor is not opposed by the mechanism moment (as with rigid robotic systems).

The motor moment is opposed by the bending moment of the first elastic mode that comes after the motor, and also in part, by the bending moments of the other elastic modes that are connected in series after the given motor. All the modes after the motor, due to their position, influence the dynamics of the motor motion.

The mathematical model of all m motors can be written in a vector form as:

$$\begin{aligned} u &= R \cdot i + C_E \cdot \dot{\overline{\theta}} \\ C_M \cdot i &= I \cdot \ddot{\overline{\theta}} + B_u \cdot \dot{\overline{\theta}} - S \cdot \varepsilon_m \end{aligned} \tag{21}$$

In Eq. (21) we have m equations of motors.

$$\varepsilon_m = z_m \cdot \Theta \cdot \varepsilon \tag{22}$$

$z_m \in R^{mxk}$ is the matrix characterizing the effect of the elasticity moment of each mode on the motor motion dynamic.

The potential energy in the top of j - th mode of the i - th link is $E_{pels\,i,j} = \frac{1}{2} C_{s\,i,j} \cdot \vartheta_{i,j}^2 \cdot l_{i,j}^2$, while dissipative energy is: $\Phi_{els\,i,j} = \frac{1}{2} B_{s\,i,j} \cdot \dot{\vartheta}_{i,j}^2 \cdot l_{i,j}^2$.

All quantities should be expressed in dependence of generalized coordinates.

Some of them are also the rotation angles of the motor shaft $\overline{\theta}_i$.

By applying Lagrange's equations on the expressions $E_{pels1,1}$, $E_{pels1,2}$, …, $E_{pels\,i,j}$, … , $E_{pels\ m,n_1}$ and $\Phi_{els1,1}$, $\Phi_{els1,2}$, … , $\Phi_{els\,i,j}$, … , $\Phi_{els\ m,n_1}$ with respect to the generalized coordinate $\overline{\theta}_i$, we obtain the equivalent elasticity moment $\varepsilon_{m\,i}$, that opposes to the rotation moment of the i -th motor shaft. $\varepsilon_m = \left[\varepsilon_{m1} \ \varepsilon_{m2} \ldots \varepsilon_{mi} \ldots \varepsilon_{mm} \right]^T$.

The overall order of the system (20) - (21) is $k + m$.

The full model is planned on classical principles of the mechanics [33].

It is known that the robot configuration can substantially influence the mutual position and orientation of elastic lines of particular links (see Fig. 4).

The solution of the system (20) - (21), i.e. the form of its elastic line, can be obtained by superimposing the solutions (15) - (17) for all the links involved in the presence of the dynamics (angle) of rotation of each motor, as well as by taking into account the robotic configuration, i.e. the angle between the axes z_{i-1} and z_i.

$$\begin{aligned}
\hat{y} &= \hat{a}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{t0i,j}, \bar{\theta}, \alpha, t) \\
\hat{x} &= \hat{b}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{t0i,j}, \bar{\theta}, \alpha, t) \\
\hat{z} &= \hat{c}(\hat{x}_{i,j}, T_{sti,j}, T_{t0i,j}, \bar{\theta}, \alpha, t) \\
\hat{\psi} &= \hat{d}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{t0i,j}, \bar{\theta}, \alpha, t) \\
\hat{\xi} &= \hat{e}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{t0i,j}, \bar{\theta}, \alpha, t) \\
\hat{\varphi} &= \hat{f}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{t0i,j}, \bar{\theta}, \alpha, t)
\end{aligned} \tag{23}$$

Thus we defined the position and orientation of each point of the elastic line in the space of the Cartesian coordinates. It should be pointed out that the form of elastic line comes directly out from the dynamics of the system motion.

Simulation Example

Robot starts from point "A" (Fig.5) and moves toward point "B" in the predicted time $T = 2[\mathrm{s}]$. The same example was analyzed as in paper [13] where all mechanisms characteristics were given.

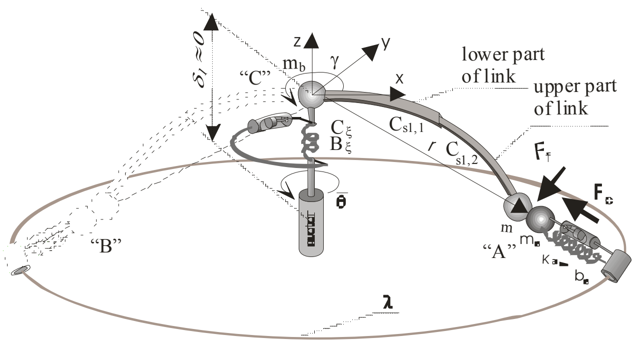

Figure 5. The robotic mechanism.

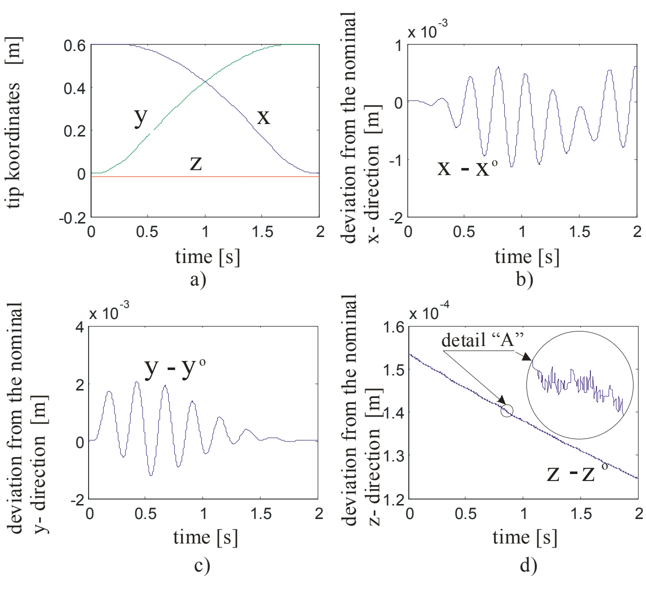

Figure 6. Tip coordinates and the deviation of the position from the reference level.

The dynamics of the environment force [29] is included into the dynamics of system's motion. The adopted velocity profile is trapezoidal $\left(\dot{q}^{o}_{\max} = 0.9817[\mathrm{rad/s}]\right)$, with the acceleration / deceleration period of $0.2 \cdot T$ $\left(\ddot{q}^{o}_{\max} = \pm 2.4544\left[\mathrm{rad/s^2}\right]\right)$.

The same example analyzed as in paper [13] only with somewhat different parameters flexibility.

Elastic deformation is a quantity which is at least partly encompassed by the reference trajectory as explained in [13] (2.1 under 2).

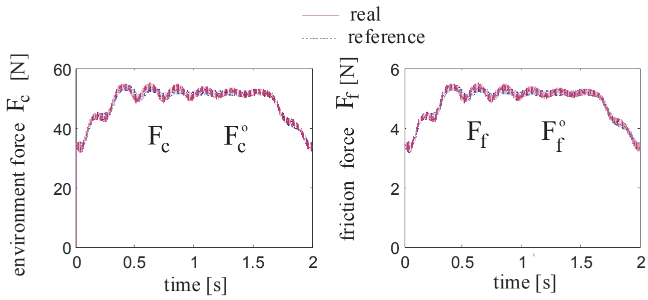

Figure 7. Dynamic force of the environment.

All other characteristics of the system and environment are the same as in paper [13].

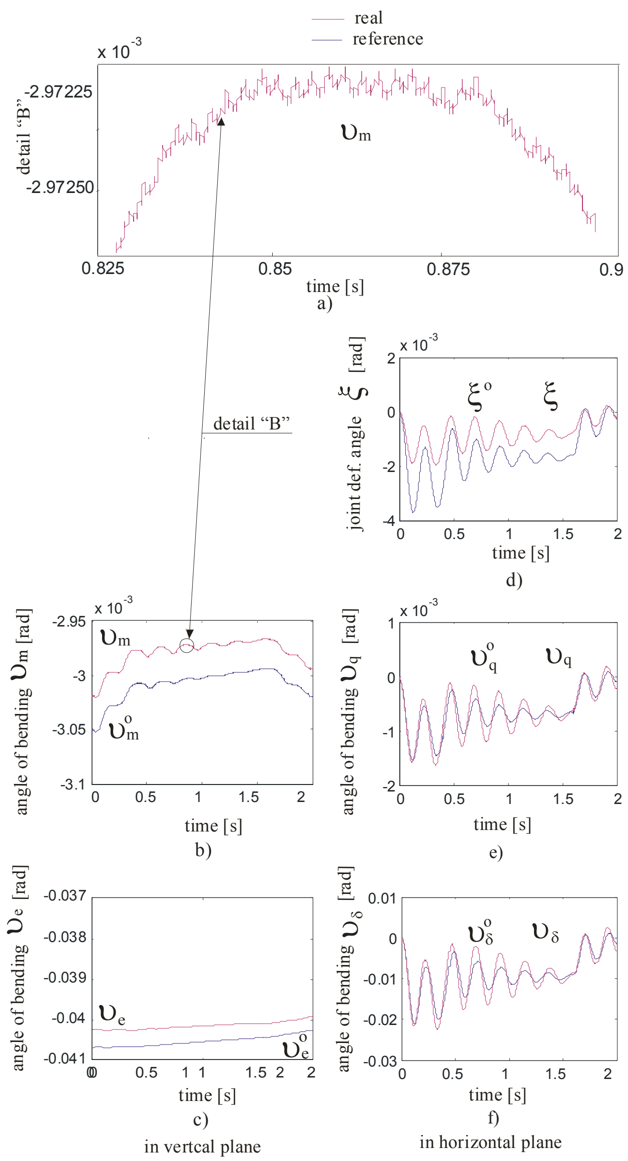

Figure 8. The elastic deformations.

The characteristics of stiffness and damping of the gear in the real and reference regimes are not the same and neither are the stiffness and damping characteristics of the link. $C_{\xi} = 2 \cdot C^{0}_{\xi} \cdot$, $B_{\xi} = 2 \cdot B^{0}_{\xi}$, $C_{s1,1} = 0.99 \cdot C^{0}_{s1,1}$, $B_{s1,1} = 0.99 \cdot B^{0}_{s1,1}$, $C_{s1,2} = 0.99 \cdot C^{0}_{s1,2}$, $B_{s1,2} = 0.99 \cdot B^{0}_{s1,2}$.

As it can be seen from Fig.6 in its motion from point “A” to point “B”, the robot tip tracks well the reference trajectory in the space of the Cartesian coordinates.

As the position control law for controlling local feedback was applied, the tracking of the reference force was directly dependent on the deviation of the position from the reference level (see Fig.7).

The elastic deformations that are taking place in the vertical plane angle of bending of the lower part of the link (first mode) ϑ_m and the angle of bending of the upper part of the link (second mode) ϑ_e, as well as elastic deformations taking place in the horizontal plane:, the angle of bending of the lower part of the link (first mode) ϑ_q, the angle of bending of the upper part of the link (second mode) ϑ_{δ} and the deflection angle of the gear ξ are given in Fig.8.

The rigidity of the second mode is about ten times lower compared with that of the first mode, it is then logical that the bending angle for the second mode is about ten times larger compared to that of the first mode.

The Fig.8a exhibits the wealth of different amplitudes and circular frequencies of the present modes of elastic elements.

Conclusion

It is pointed out that the elastic deformation is the consequence of the total robot system dynamics which is essentially different from, until now, widely used method that implies the adaptation of the “assumed modes technique”.

Based on the EBA, we defined the general form of the Euler-Bernoulli equation of the elastic line of complex elastic robotic system with m segments, and each segment has n_i modes and also the mathematical model of motors which move each link.

The Euler-Bernoulli equation has been expanded from several aspects:

1. Euler-Bernoulli equation (based on the known laws of dynamics) should be supplemented with all the forces that are participating in the formation of the bending moment of the considered mode, which causes the difference in the structure of these equations for each mode.
2. Structure of the stiffness and damping matrix must also have the elements outside the diagonal, because of the existence of strong coupling between the elasticity forces involved.
3. Damping is an omnipresent elasticity characteristic of real systems, so that it is naturally included in the Euler-Bernoulli equation.
4. General form of the transversal elastic deformation is defined by superimposing particular solutions of the oscillatory character (solution of Daniel Bernoulli) and a stationary solution of the forced character (which is a consequence of the forces involved).
1. General form of the elastic line is a direct outcome of the dynamics of system motion-it must be significantly expanded and cannot be represented by one scalar equation but three equations are needed to define the position and three equations to define the orientation of each point on the elastic line.

With elastic robotic systems, the actuator torque is opposed by the bending moment of the first elastic mode, which comes after the motor, and partly by the bending moments of other modes, which are connected in series after the motor considered. All modes coming after the motor, because of their position, exert influence on the dynamics of motor motion. The mathematical model of motor in our paper is connected to the rest of the mechanism via the equivalent elasticity moment.

New structures of the stiffness (and damping) matrix and the mathematical model of actuators appear as a consequence of the coupling between the modes of particular links.

All this has been presented for a relatively simple robotic system that offered the possibility of analyzing the phenomena involved. Through the analysis and modeling of an elastic mechanism we made an attempt to give a contribution to the development of this area.

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Novo viđenje Euler-Bernoulli jednačine

Posebna pažnja je posvećena kretanju elastičnih linkova u robotskoj konfiguraciji. Elastična deformacija je dinamička veličina koja zavisi od ukupne dinamike kretanja robotskog sistema. Euler-Bernoulli jednačinu (koja je dugi niz godina korišćena u literaturi) treba proširiti prema zahtevima složenosti kretanja elastičnih robotskih sistema. Euler-Bernoulli jednačini (zasnovano na postojećim zakonima dinamike) treba dodati sve sile (inercijalne, Coriolisove, centrifugalne, gravitacione, sile okoline, poremećajne kao i sile sprezanja između prisutnih modova) koje učestvuju u formiranju momenta elastičnosti posmatranog moda. To uslovljava različitost u strukturi Euler-Bernoullijevih jednačina za svaki mod. Matrica krutosti je puna matrica kao i matrica prigušenja. Matematički model motora takodje obuhvata kuplovanje izmedju sila elastičnosti. Partikularni integral koji je definisao Daniel Bernoulli treba proširiti stacionarnim karakterom elastične deformacije za bilo koju tačku posmatranog moda uzrokovano prisutnim silama. Opšta forma elastične linije mehanizma direktno proističe iz dinamike kretanja sistema i ne može biti opisana sa jednom skalarnom jednačinom već sa tri jednačine za poziciju i tri jednačine za orjentaciju svake tačke na toj elastičnoj liniji. Simulacioni rezultati su predstavljeni za odabrani primer robota uvodeći simulaciono prisustvo elastičnosti prenosnika i linka (dva moda) kao i dinamiku sila okoline.

Ključne reči: **robotika, dinamika kretanja, Ojler-Bernulijeve jednačine, modelovanje procesa, elastična deformacija, kuplovanje, matrica krutosti, simulacija kretanja, programirana trajektorija.**

Новое представление уравнения Эйлер-Бернулли

Особое внимание посвящёно движению эластичных каналов связи в робототехнических конфигурациях. Эластичная деформация представляет динамическую величину, которая зависит от совокупной динамики движения робототехнической системы. Уравнение Эйлер-Бернулли (долгие годы пользовано в литературе) надо расширить в соответствии с требованиями сложности движений эластичных робототехнических систем. Уравнению Эйлер-Бернулли (обоснованой на уже существующих законах динамики) надо прибавить все силы (инерционные, Кориолиса, центрифугальные, гравитационные, внешные силы, нарушающие силы, а в том числе и силы сопряжения между присутствующими модами), участвующие в формировании момента эластичности рассматриваного мода. А это обуславливает разнообразие в структуре уравнений Эйлер-Бернулли для каждого мода. Матрица жёсткости является полной матрицей, как и матрица демпфирования. Математическая модель двигателя тоже охватывает связывание между силами эластичности. Частичный интеграл, который определил Даниел Бернулли, надо расширить стационарным характером эластичной деформации для любой точки рассматриванного мода, что бывает причиной присутствующих сил. Общая форма эластичной линии механизма прямо исходит из динамики движения системы и не может быть описана только одним скалярным уравнением, но тремья уравнениями для позиции и тремья уравнениями для ориентации каждой точки на этой эластичной линии. Имитирующие результаты представлены для отобранного примера интеллектуального робота, с введением имитирующего присутствия эластичности передаточного механизма и канала связи (два мода), а в том числе и динамики внешних сил.

Ключевые слова: Робототехника, динамика движения, уравнения Эйлер-Бернулли, моделирование процесса, эластичная деформация, связывание, матрица жёсткости, имитация движения, программированая траектория.

Nouvelle vue sur les équations Euler-Bernoulli

L'attention particulière a été portée au mouvement des liens élastiques chez la configuration robotique. La déformation élastique représente une valeur dynamique dépendante de la dynamique totale de mouvement du système robotique. L'équation Euler-Bernoulli (employée pendant longtemps dans la littérature) est à élargir selon les exigences de la complexité du mouvement des systèmes robotiques élastiques. A l'équation Euler-Bernoulli (basée sur les lois existantes de la dynamique) il faut ajouter toutes les forces (initiales, de Corolis, centrifuges, de gravitation, ambiantes, perturbantes ainsi que les force d'attelage entre les modes présents) qui participent à la formation du moment d'élasticité chez le mode observé. Cela conditionne la différence dans la structure des équations Euler-Bernoulli pour chaque mode. La matrice de rigidité est la matrice pleine ainsi que la matrice d'étouffement. Le modèle mathématique du moteur comprend aussi l'attelage entre les forces d'élasticité. L'intégrale particulière définie par Daniel Bernoulli est à élargir par le caractère stationnaire de la déformation élastique pour n'importe quel point du mode observé causé par les forces en présence. La forme générale de la ligne élastique du mécanisme provient directement de la dynamique du mouvement du système et ne peut pas être décrite par une équation scalaire mais par trois équations pour la position et trois équations pour l'orientation de chaque point sur cette ligne élastique. Les résultats de la simulation sont représentés pour le modèle choisi de robot introduisant la présence simulatrice de l'élasticité de transducteur et la liaison (deux modes) ainsi que la dynamique des forces ambiantes.

Mots clés: **robotique, dynamique du mouvement, équations Euler-Bernoulli, modélisation du procédé, déformation élastique, attelage, matrice de rigidité, simulation du mouvement, trajectoire programmée.**