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Contact Problems Based on the Penalty Method

Snežana Vulović (Eng)[1)]
Miroslav Živković, (Eng)[2)]
Nenad Grujović, (Eng)[2)]

In this paper the contact problems including frictional effects are presented. Contact between two deformable bodies is considered in general. The present approach, based on the Coulomb's frictional law, elastic-plastic tangential slip decomposition and consistent linearization results in quadratic rates of convergence with the help of the Newton-Raphson iterative solution scheme. The results for the verification example are also presented.

Key words: **contact problem, contact strain, friction, finite elements, penalty method, nonlinear analysis.**

Introduction

NUMERICAL analysis of frictional contact problems has been one of the research topics of main interest in recent years [1], [2], [3], [4]. Frictional contact problems arise in many application fields such as metal forming processes, the impact of cars, etc.

The effective application of finite element contact solutions needs a high degree of experience since the general robustness and stability cannot be guaranteed. For this reason the development of more efficient, fast and stable finite element contact discretizations is still a hot topic, especially due to the fact that engineering applications are becoming more and more complex.

In this paper, a framework for contact problems with friction is developed based on the penalty method. The penalty formulation has the advantage for being purely geometrically based and therefore no additional degrees of freedom have to be activated or inactivated. A numerical example is shown to demonstrate that the presented algorithm can be successfully applied to contact problems [6].

Contact kinematics

As the configurations of two bodies coming into the contact are not a priori known, contact represents a nonlinear problem even when the continuum behaves as a linear elastic material.

Using a standard notation in contact mechanics, for each pair of contact surfaces, involved in the problem, we will define slave $\left(\Gamma_C^{(1)}\right)$ and master surfaces $\left(\Gamma_C^{(2)}\right)$, Fig.1. The condition, which must be satisfied, is that any slave particle cannot penetrate the master surface.

Let $\bar{\mathbf{x}}$ be the projection point of the current position of the slave node $\mathbf{x}^k$ onto the current position of the master surface $\Gamma_C^{(2)}$, defined as

$$\frac{\mathbf{x}^k-\bar{\mathbf{x}}(\bar{\xi}^1,\bar{\xi}^2)}{\left\|\mathbf{x}^k-\bar{\mathbf{x}}(\bar{\xi}^1,\bar{\xi}^2)\right\|}\cdot\bar{\mathbf{a}}_\alpha(\bar{\xi}^1,\bar{\xi}^2)=0 \tag{1}$$

where $\alpha=1,2$ and $\bar{\mathbf{a}}_\alpha(\bar{\xi}^1,\bar{\xi}^2)$ are the tangent covariant base vectors at the point $\bar{\mathbf{x}}$. The normal gap or the penetration g_N for the slave node k is defined as the distance between the current positions of this node to the master surface $\Gamma_C^{(2)}$:

$$g_N=(\mathbf{x}^k-\bar{\mathbf{x}})\cdot\bar{\mathbf{n}} \tag{2}$$

where $\bar{\mathbf{n}}$ refers to the normal to the master surface $\Gamma_C^{(2)}$ at point $\bar{\mathbf{x}}$ (Fig.1). The normal will be defined using the tangent vectors at the point $\bar{\mathbf{x}}$

$$\bar{\mathbf{n}}=\frac{\bar{\mathbf{a}}_1\times\bar{\mathbf{a}}_2}{\left\|\bar{\mathbf{a}}_1\times\bar{\mathbf{a}}_2\right\|} \tag{3}$$

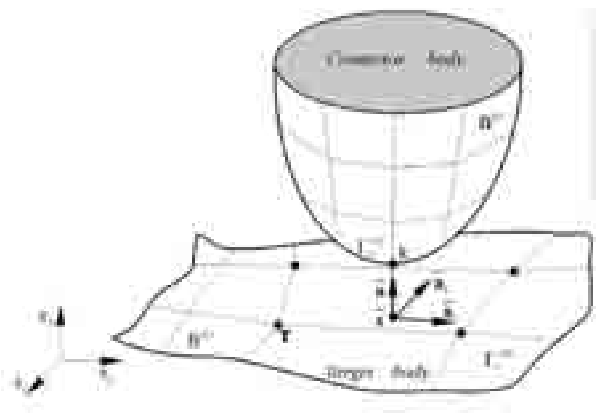

Figure 1. Geometry of the 3D node-to-segment contact element

This gap (2) gives the non-penetration conditions as follows:

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1) Faculty of Information Technology, 11000 Belgrade, SERBIA
2) Faculty of Mechanical Engineering, Sestre Janjić 6, 34000 Kragujevaca, SERBIA

$$\begin{aligned} g_N &= 0 \quad \text{perfect contact;} \\ g_N &> 0 \quad \text{no contact;} \\ g_N &< 0 \quad \text{penetration} \end{aligned} \tag{4}$$

If the analyzed problem is frictionless, function (4) completely defines the contact kinematics. However, if friction is modeled, tangential relative displacement must be introduced. In that case, the sliding path of the node $\mathbf{x}^k$ over the contact surface $\Gamma_C^{(2)}$ is described by total tangential relative displacement as

$$g_T = \int_{t_0}^{t} \|\dot{\mathbf{g}}_T\| dt = \int_{t_0}^{t} \left\| \dot{\bar{\xi}}^{\alpha} \bar{\mathbf{a}}_{\alpha} \right\| dt = \int_{t_0}^{t} \sqrt{\dot{\bar{\xi}}^{\alpha} \dot{\bar{\xi}}^{\beta} a_{\alpha\beta}} dt \tag{5}$$

in the time interval from t_0 to t.

The time derivatives of the parameter $\bar{\xi}^{\alpha}$ in equation (5) can be computed from (1), [4]. We obtain the following result:

$$\bar{a}_{\beta\alpha} \dot{\bar{\xi}}^{\beta} = \left[\dot{\mathbf{x}}^k - \dot{\bar{\mathbf{x}}} \right] \cdot \bar{\mathbf{a}}_{\alpha} = \dot{g}_{T\alpha} \tag{6}$$

where $\bar{a}_{\alpha\beta} = \bar{\mathbf{a}}_{\alpha} \cdot \bar{\mathbf{a}}_{\beta}$ is the metric tensor in the point $\bar{\mathbf{x}}$ of the master surface $\Gamma_C^{(2)}$. From equations (5) and (6) we can express the relative tangential velocity at the contact point

$$\dot{\mathbf{g}}_T = \dot{\bar{\xi}}^{\alpha} \bar{\mathbf{a}}_{\alpha} = \dot{g}_{T\alpha} \bar{\mathbf{a}}^{\alpha} \tag{7}$$

Constitutive equations for contact interface

A contact stress vector $\bar{\mathbf{t}}$ with respect to the current contact interface $\Gamma_C^{(2)}$ can be split into a normal and tangential part.

$$\bar{\mathbf{t}} = \bar{\mathbf{t}}_N + \bar{\mathbf{t}}_T = \bar{t}_N \bar{\mathbf{n}} + \bar{t}_{T\alpha} \bar{\mathbf{a}}^{\alpha} \tag{8}$$

where $\bar{\mathbf{a}}^{\alpha}$ is the contravariant base vector. The stress acts on both surfaces according to the action-reaction principle: $\bar{\mathbf{t}}(\bar{\xi}^1, \bar{\xi}^2) = -\mathbf{t}$ in the contact point $\bar{\mathbf{x}}$. The tangential stress $\bar{t}_{T\alpha}$ is zero in the case of frictionless contact. In the case of contact there is the condition $\bar{t}_N < 0$. If there is no penetration between the bodies, then relations $g_N > 0$ and $\bar{t}_N = 0$ are satisfied. This leads to the statements

$$g_N \geq 0, \quad \bar{t}_N \leq 0, \quad \bar{t}_N g_N = 0 \tag{9}$$

which are known as Kuhn-Tucker conditions. Using the penalty method for normal stress, constitutive equation can be formulated as

$$t_N = \varepsilon_N g_N \tag{10}$$

where ε_N is the normal penalty parameter.

In the tangential direction there is a difference between the rod and the slip. As long as no sliding between two bodies occurs, the tangential relative velocity is zero. If the velocity is zero, also the tangential relative displacement (5) is zero. This state is called a rod case with the following restriction:

$$\dot{\mathbf{g}}_T = \mathbf{0} \Leftrightarrow \mathbf{g}_T = 0 \tag{11}$$

For the rod a simple linear constitutive model can be used to describe the tangential stress.

$$t_{T\alpha}^{stick} = \varepsilon_T g_{T\alpha} \tag{12}$$

where ε_T is the tangential penalty parameter.

A relative movement between two bodies occurs if the static friction resistance is overcome and the loading is large enough so that the sliding process can be kept. The tangential stress vector is restricted as follows:

$$t_{T\alpha}^{sl} = -\mu \left| \mathbf{t}_N \right| \frac{\dot{g}_{T\alpha}^{sl}}{\left\| \dot{\mathbf{g}}_T^{sl} \right\|} \tag{13}$$

where μ is the friction coefficient. In the simplest form of Coulomb's law (13), μ is constant so there is no difference between static and sliding friction.

After the introduction of the rod and slip constraints, we need to introduce an indicator to define whether rod or slip actually takes place. Therefore an indicator function

$$f = \|\mathbf{t}_T\| - \mu |t_N| \tag{14}$$

is evaluated, which respects the Coulomb's model for the frictional interface law. In equation (14) the first term is $\|\mathbf{t}_T\| = \sqrt{t_{T\alpha} \bar{a}^{\alpha\beta} t_{T\beta}}$.

A backward Euler integration scheme and a return mapping strategy are used to integrate the friction equations (14), [1]. If a state of rod is assumed, the trial values of the tangential contact pressure vector $t_{T\alpha}$, and the indicator function f at load step n+1 can be expressed in terms of their values at the load step n as follows

$$t_{T\alpha\ n+1}^{trial} = t_{T\alpha\ n} + \varepsilon_T \Delta g_{T\alpha\ n+1} = t_{T\alpha\ n} + \varepsilon_T \bar{a}_{\alpha\beta} \Delta \xi_{n+1}^{\beta} \tag{15}$$

$$f_{Tn+1}^{trial} = \left\| \mathbf{t}_{Tn+1}^{trial} \right\| - \mu \left| t_{Nn+1} \right| \tag{16}$$

The return mapping is completed by

$$t_{T\alpha\ n+1} = \begin{cases} t_{T\alpha\ n+1}^{trial} & \text{if } f \leq 0 \\ \mu \left| t_{Nn+1} \right| n_{T\alpha\ n+1}^{trial} & \text{if } f > 0 \end{cases} \tag{17}$$

with

$$n_{T\alpha\ n+1}^{trial} = \frac{t_{T\alpha\ n+1}^{trial}}{\left\| \mathbf{t}_{Tn+1}^{trial} \right\|} \tag{18}$$

For both cases, the penalty method can be illustrated as a group of linear elastic springs that force the body back to the contact surface when overlapping or sliding occurs.

Algorithm for frictional contact

For solving a nonlinear equilibrium equation with inequality constraints (4) as a result of contact, we use a standard implicit method. In order to apply Newton's method for the solution system of equilibrium equation, a linearization of the contact contributions is necessary. In this paper, we do not state the linearization procedure for the standard finite element formulation as well as the contact interface law for the normal and tangential part. It could be found in [7], [8].

The tangent stiffness matrix for the normal contact is

$$\mathbf{K}_N = \varepsilon_N \mathbf{N} \mathbf{N}^T \tag{19}$$

The symmetric tangent stiffness matrix for the rod condition is

$$\mathbf{K}_T^{stick} = \varepsilon_T \bar{a}_{\alpha\beta} \mathbf{D}^\alpha \mathbf{D}^{\beta T} \tag{20}$$

where

$$\mathbf{N} = \begin{Bmatrix} \bar{\mathbf{n}} \\ -H_1\bar{\mathbf{n}} \\ -H_2\bar{\mathbf{n}} \\ -H_3\bar{\mathbf{n}} \\ -H_4\bar{\mathbf{n}} \end{Bmatrix} \qquad \mathbf{T}_\beta = \begin{Bmatrix} \bar{\mathbf{a}}_\beta \\ -H_1\bar{\mathbf{a}}_\beta \\ -H_2\bar{\mathbf{a}}_\beta \\ -H_3\bar{\mathbf{a}}_\beta \\ -H_4\bar{\mathbf{a}}_\beta \end{Bmatrix} \qquad \mathbf{D}^\alpha = \bar{a}^{\alpha\beta}\mathbf{T}_\beta . \tag{21}$$

The linearization of $n_{T\alpha\ n+1}^{trial}$ gives (for details see [1])

$$\Delta\left(n_{T\alpha\ n+1}^{trial}\right) = \Delta\left(\frac{t_{T\alpha\ n+1}^{trial}}{\left\|\mathbf{t}_{Tn+1}^{trial}\right\|}\right) = \frac{1}{\left\|\mathbf{t}_{Tn+1}^{trial}\right\|}\left[\delta_\alpha^\beta - n_{T\alpha\ n+1}^{trial} n_{T\ n+1}^{trial\ \beta}\right]\Delta t_{T\beta\ n+1}^{trial} \tag{22}$$

The tangent stiffness matrix for the slip condition is

$$\mathbf{K}_T^{slip} = \mu\varepsilon_N n_{T\alpha\ n+1}^{trial}\mathbf{D}^\alpha\mathbf{N}^T + \frac{\mu\varepsilon_N g_{Nn+1}}{\left\|\mathbf{t}_{Tn+1}^{trial}\right\|}\varepsilon_T \bar{a}_{\beta\gamma}\left[\delta_\alpha^\beta - n_{T\alpha\ n+1}^{trial} n_{T\ n+1}^{trial\ \beta}\right]\mathbf{D}^\alpha\mathbf{D}^{\gamma T} \tag{23}$$

The second term, the tangent matrix is non-symmetric and due to the Coulomb's friction can be viewed as a non-associative constitutive equation.

The frictional contact algorithm based on the penalty method is shown in Table 1.

Table 1. Frictional contact algorithm based on the penalty method

LOOP over all contact segment k

(check for contact (6)) IF $g_N \le 0$ THEN

(the first iteration) IF $i=1$ THEN

set all active nodes to state rod,

$\mathbf{t}_{Tn+1}$ (18), compute matrix $\mathbf{K}_T^{stick}$

ELSE

Compute trial state: $t_{T\alpha\ n+1}^{trial}$ (19) and f_{Tn+1}^{trial} (20)

IF $f_{Tn+1}^{trial} \le 0$ THEN

$t_{T\alpha\ n+1} = t_{T\alpha\ n+1}^{trial}$, compute matrix $\mathbf{K}_T^{stick}$ (40)

GO TO (a)

ELSE

$t_{T\alpha\ n+1} = \mu\left|t_{Nn+1}\right| n_{T\alpha\ n+1}^{trial}$, compute matrix $\mathbf{K}_T^{slip}$ (43)

ENDIF

ENDIF

ENDIF

(a) END LOOP

Example

This example offers a specific verification of the contact algorithm in dynamic analysis conditions, considering the dynamic compatibility for specific elements. The point is in a geometrically quite simple problem, for which there is an analytical solution.

Rods are modeled by 2D (plane stress) and 3D elements. The following geometrical (Fig.2) and material data are presented (without units as in [5]):

	BAR 1	BAR 2
Model	10 elements	40 elements
Young's Modulus	E_1=100000	E_2=6250
Poisson's ratio	ν_1=0	ν_2=0
Density	ρ_1=4	ρ_2=4
Cross-section	A_1=1	A_2=1
Length	L_1=10	L_2=10
Initial velocity	v_1=5	v_2=0

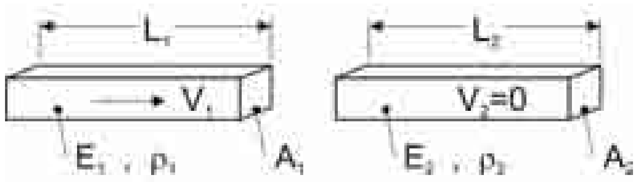

Figure 2. Rod impact model made of different elastic materials

The difference in the element numbers comes from the references for the problem modeling where wave effects appear. The ratio of modules $E_1/E_2 = 16$, according to the equation of stress wave velocity in rods

$$c_{Ei} = \sqrt{E_i/\rho_i} \qquad i=1,2 \tag{a}$$

results in the fact that the other rod has to have four times shorter elements with the chosen time quantity. This way it is possible for the stress wave not to pass more than one element in both rods, in the time increment the boundary value of which is:

$$\Delta t_{cr} = \frac{(L_1/10)}{c_{E1}} = \frac{(L_2/40)}{c_{E2}} = 0.006324 \tag{b}$$

The obtained results with $\varepsilon_N = 10^{11}$ are completely the same in both cases (2D and 3D) and they perfectly match the results obtained by Lagrange's method [5], and the analytical results).

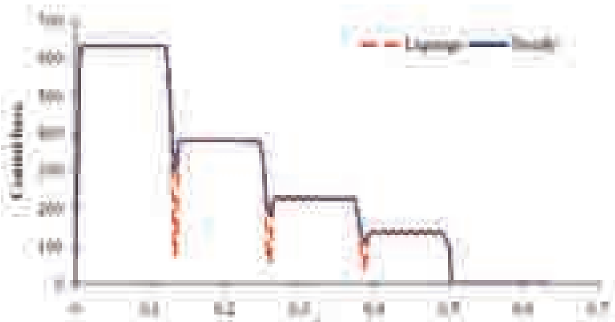

Figure 3. Contact force time histories

Figures 3 - 7 are graphic reviews of time depending on force, displacement, velocity and acceleration of rod points 1 and 2 in the contact. For the sake of clarity, the acceleration diagram is show separately for each section. The result is obtained by the approximately explicit integration (α=0.001001, β=0.502) with the time increment $\Delta t = \Delta t_{cr}$. The whole calculation is done in 100 time increments and quantities on diagrams are shown for each increment. It is seen from the diagram that a better result stability is obtained by the penalty method, which is quite obvious from the acceleration diagram.

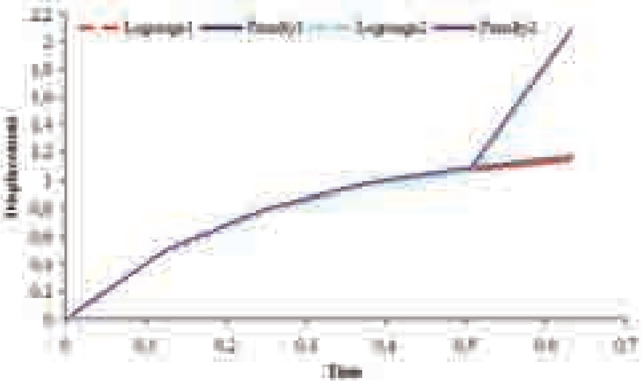

Figure 4. Displacement time histories

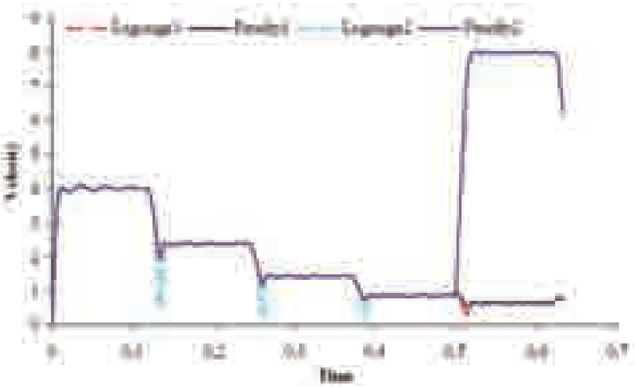

Figure 5. Velocity time histories

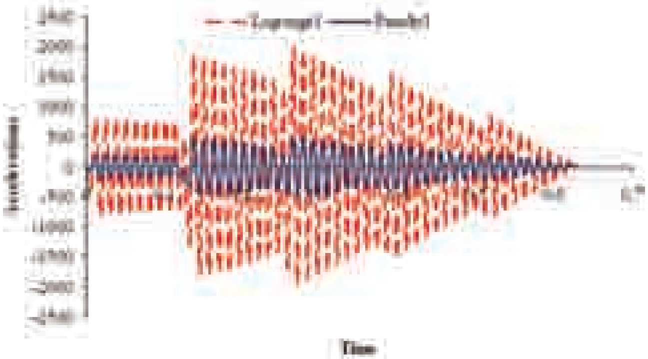

Figure 6. Acceleration time histories, cross-section 1

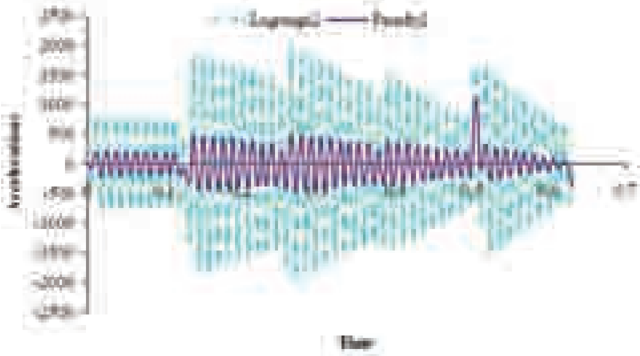

Figure 7. Acceleration time histories, cross-section 2

Conclusions

A model for 3D contact problem based on the penalty method was proposed. Due to the intrinsic similarity between friction and the classical elastic-plasticity [7], the constitutive model for friction can be constructed following the same formalism as in classical elastic-plasticity. The numerical example indicates a possibility of applying the developed method in the analysis of dynamic problems.

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Kontaktni problemi zasnovani na Penalty metodi

U radu su prikazani kontaktni problemi sa trenjem. Razmatran je kontakt između dva deformabilna tela u opštem slučaju. Prikazan prilaz, zasnovan na Kolumbovom zakonu trenja, dekompoziciji kontaktnog napona po analogiji sa elasto-plastičnom dekompozicijom i odgovarajućom linearizacijom rezultira kvadratnom konvergencijom rešenja primenom Njutn-Rapsonove iterativne metode. Na kraju rada su data rešenja za verifikacioni primer.

Ključne reči: **kontaktni problem, kontaktno naprezanje, trenje, konačni element, Panalty metoda, nelinearna analiza.**

Контактные проблемы обоснованы на ПЕНАЛТЮ-методу

В настоящей работе показаны контактные проблемы с трением, а рассматриван контакт между двумя деформированными телами в общем обстоятельстве. Тоже показан подход, обоснован на законе трения Колумба, на разложении контактного напряжения по сходству с упруго-пластическим разложением и соответствующей линейностью, который происходит в результате квадратной сходимости (конвергенции) решения с применением повторяющегося метода Ньютн-Рапсона. В самом конце работы приведены решения для примера проверки.

Ключевые слова: контактная проблемма, контактное напряжение, трение, конечный элемент, ПЕНАЛТЮ-метод, нелинейный анализ.

Problèmes de contact basés sur la méthode PENALTY

Dans ce travail on a présenté les problèmes de contact concernant les effets de friction. On a considéré le contact entre deux corps déformables dans le cas général. L'approche présentée, basée sur la loi de friction selon Coulomb, la décomposition de la tension de contact d'après l'analogie avec la décomposition élastique plastique et la linéarisation correspondante, a pour résultat la convergence carrée de la solution par l'application de la méthode Raphson-Newton. A la fin de ce travail, on a donné les solutions de l'exemple de vérification.

Mots clés: **problème de contact, contrainte de contact, friction, élément fini, méthode Penalty, analyse non linéaire.**