UDK: 651.511.26.012:681.5.013
COSATI: 12-01

Asymptotic Stability Analysis of Particular Classes of Linear Time-Delay Systems: A New Approach

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This paper offers new, necessary and sufficient conditions for the delay-dependent asymptotic stability of systems of the form $\mathbf{x}(k+1)=A_0\mathbf{x}(k)+A_1\mathbf{x}(k-h)$ and $\dot{\mathbf{x}}(t)=A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau)$. The time-dependent criteria are derived by Lyapunov's direct method. Two matrix equations have been derived: matrix polynomial equation and continuous (discrete) Lyapunov matrix equation. Also, modifications of the existing sufficient conditions of convergence of Traub and Bernoilli algorithms for computing the dominant solvent of the matrix polynomial equation are derived. These results have been extended to large scale systems as well. Numerical computations are performed to illustrate the results obtained.

Key words: **continuous system, discrete system, linear system, system stability, asymptotic stability, Lyapunov stability, time delay system, time delay.**

Introduction

THE problem of investigation of time delay systems has been exploited over many years.

The existence of pure time lag, regardless if it is present in the control or/and the state, may cause an undesirable system transient response, or even instability. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc.

Consequently, the problem of the stability analysis for this class of systems has been one of the main interests for many researchers. In general, the introduction of time delay factors makes the analysis much more complicated.

In the existing stability criteria, mainly two methods of approach have been adopted.

Namely, one direction is to contrive the stability condition which does not include the information on the delay, and the other is the method which takes it into account.

The former case is often called the delay - independent criteria and generally provides simple algebraic conditions. Numerous reports have been published on this matter, with particular emphasis on the application of Lyapunov's second method or on using the concept of the matrix measure Mori et al. (1981), Mori (1985), Hmamed (1986), Lee et al. (1986), Alastruey, De La Sen (1996).

The majority of stability conditions in the literature available, of both continual and discrete time delay systems, are sufficient conditions independent of time delay.

Only a small number of works provide both necessary and sufficient conditions, Lee, Dianat (1981), Xu, et al. (2001) and Boutayeb, Darouach (2001), which are in their nature mainly dependent on time delay.

The results concerning Lyapunov stability, for non-delay time systems, are well documented in a number of known references, and, for the sake of brevity, are omitted here.

A discussion of the problem of investigation of linear *discrete time delay* systems and their Lyapunov stability should point out that there are not too many results dealing with this problem so we turn our attention, in the sequel, only to this class of systems.

Namely, Koepcke (1965), was the first who paid attention to this class of systems solving a synthesis problem of controlling the systems governed by linear differential – difference equations. It has been shown, in the same paper, that such systems are equivalent to infinite dimensional difference equations the matrix elements of which can be calculated readily by recursive formulas. Some results, concerning stability in the sense of Lyapunov, were also derived. The problem of finding an optimal control in linear discrete systems with time delays in both the state variables and control were studied in Chung (1967, 1969).

The method of orthogonal projection was used to derive the equations for optimal estimating the state of a non-stationary linear discrete system with multiple delays in Premier, Vacroux (1969). A Kalman - type filter with the necessary recursive error and cross error matrix equations were also derived. The linear – quadratic tracking problem was discussed, for the first time, in Pindyck (1972), for a discrete – time systems with the time delay incorporating in inputs.

Several sufficient conditions for asymptotic stability of linear discrete – delay systems were presented in the paper of Mori et al. (1982). Since these conditions are independent of delay and possess simple forms, they provide useful tools to check system stability at the first stage.

The study of stabilization problem for general decentralized large - scale linear continuous and discrete time delay systems using local feedback controllers were

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presented by Lee, Radovic (1987).

The local feedback controls were assumed to be memory less. In that sense, the sufficient stabilization conditions were established.

The problem of delays in interconnections, for the same class of systems, was studied latter in Lee, Radovic (1988).

The paper of Trinh, Aldeen (1995) presents some new sufficient conditions for robust and *D*-stability of discrete – delay perturbed systems. It has been shown that these results are less conservative than those reported in literature, particularly to Mori et. al (1982).

Based on a derived algebraic inequality a criterion to guarantee the robust stabilization and state estimation for perturbed discrete - time – delay large scale systems was proposed in Wang, Mau (1995).

That criterion is independent of time delay and does not need the solution of Lyapunov or Riccati equation.

The organization of this chapter is as follows.

In Section **2** we present a new, necessary and sufficient conditions for *delay-dependent* asymptotic stability of systems of particular class of continuous and discrete time delay systems.

Moreover, we show that in the paper of Lee, Diant (1981), where it is asserted that the derivative sign of a Lyapunov function (*Lemma*) and thereby the asymptotic stability of the system (*Theorems* 1 and 2) can be determined based on the knowledge of only one or any solution of the particular nonlinear matrix equation, those statements are incorrect.

To improve those results we propose new formulations of the *Lemma* and *Theorems* 1 and 2.

Further extensions of these results to the class of continuous and discrete large scale time delay are presented in Section **3**. A particular case of two and more subsystems is also investigated.

All theoretical results are supported by suitably chosen numerical examples.

Section **4** discusses and summarizes contributions.

Time delay systems

Throughout this chapter we use the following notation.

$\mathbb{R}$ and $\mathbb{C}$ denote real (complex) vector space or the set of real (complex) numbers, $\mathbb{T}^+$ denotes the set of all non-negative integers, λ^* means conjugate of $\lambda \in \mathbb{C}$ and F^* conjugate transpose of matrix $F \in \mathbb{C}^{n\times n}$.

The superscript T denotes transposition. For a real matrix F the notation $F > 0$ means that the matrix F is positive definite. $\lambda_i(F)$ is the eigenvalue of the matrix F such that $\{\lambda \mid \det(F-\lambda I)=0\}$.

The spectrum of the matrix F is denoted with $\sigma(F)$ and the spectral radius with $\rho(F)$.

CONTINUOUS TIME DELAY SYSTEMS

For the sake of completness, we present the following result Lee, Dianat (1981).

Consider the class of continuous time-delay systems described by

$$\begin{gathered}\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) \\ \mathbf{x}(t) = \varphi(t), \quad -\tau \le t < 0\end{gathered}, \tag{1}$$

Lemma 1. Lee, Dianat (1981).

Let the system be (1) and let $P_1(t)$, a characteristic matrix of dimension $(n\times n)$, be continuous and differentiable in $[0,\ \tau]$ and 0 elsewhere, and a set

$$\begin{aligned}V(\mathbf{x}_t,\tau) = &\left(\mathbf{x}(t) + \int_0^h P_1(\tau)\mathbf{x}(t-\tau)\,\mathrm{d}\tau\right)^T \times \\ &\times P_0 \cdot \left(\mathbf{x}(t) + \int_0^h P_1(\tau)\mathbf{x}(t-\tau)\,\mathrm{d}\tau\right)\end{aligned}, \tag{2}$$

where $P_0 = P_0^* > 0$ is Hermitian and $\mathbf{x}_t(\theta) = \mathbf{x}(t+\theta)$, $\theta \in [-\tau,\ 0]$.

If

$$P_0\left(A_0 + P_1(0)\right) + \left(A_0 + P_1(0)\right)^* P_0 = -Q, \tag{3}$$

$$\dot{P}_1(\kappa) = \left(A_0 + P_1(0)\right)P_1(\kappa), \quad 0 \le \kappa \le \tau, \tag{4}$$

where $P_1(\tau) = A_1$ and $Q = Q^* > 0$ is Hermitian, then

$$\dot{V}(\mathbf{x}_t,\ \tau) = \frac{d}{dt}V(\mathbf{x}_t,\ \tau) < 0 \tag{5}$$

Eq. (2) defines Lyapunov's function for the system (1) and * denotes conjugate transpose of the matrix.

In the paper Lee, Dianat (1981) it is emphasized that the key to the success in the construction of a Lyapunov function corresponding to the system (1) is the existence of at least one solution $P_1(t)$ of (4) with the boundary condition $P_1(\tau) = A_1$.

In other words, it is required that the nonlinear algebraic matrix equation

$$e^{\left(A_0 + P_1(0)\right)\tau} P_1\left(0\right) = A_1 \tag{6}$$

has at least one solution for $P_1(0)$.

Theorem 1. Lee, Dianat (1981).

Let the system be described by (1). If for *any* given positive definite Hermitian matrix Q there exists a positive definite Hermitian matrix P_0, such that

$$P_0\left(A_0 + P_1(0)\right) + \left(A_0 + P_1(0)\right)^T P + Q = 0 \tag{7}$$

where for $\kappa \in [0,\ \tau]$, $P_1(\kappa)$ satisfies

$$\dot{P}_1(\eta) = \left(A_0 + P_1(0)\right)P_1(\eta), \tag{8}$$

with the boundary condition $P_1(\tau) = A_1$ and $P_1(\tau) = 0$ elsewhere, then the system is asymptotically stable.

Theorem 2. Lee, Dianat (1981).

Let the system be described by (1) and furthermore, let (6) has a solution for $P_1(0)$, which is nonsingular.

Then the system is asymptotically stable if (8) of Theorem 1 is satisfied.

The necessary and sufficient conditions for the stability of the system are derived by Lyapunov's direct method through construction of the corresponding "energy" function. This function is known to exist if a solution $P_1(0)$ of the algebraic nonlinear matrix equation $A_1 = \exp\left(A_0 + P_1(0)\right)P_1(0)$ can be determined.

It is asserted, there, that a derivative sign of a Lyapunov function (*Lemma* 1) and thereby asymptotic stability of the system (*Theorem* 1 and *Theorem* 2) can be determined based on the knowledge of *only one* or *any* solution of the particular nonlinear matrix equation.

We now demonstrate that *Lemma* 1 should be improved since it does not take into account all possible solutions for (6).

The counter example, based on our approach and supported by the Lambert function application, is given in Stojanovic, Debeljkovic (2006).

Remark 1.

If we introduce a new matrix,

$$R \triangleq A_1 + P_1(0) \tag{9}$$

then condition (3) reads

$$P_0 R + R^* P_0 = -Q \tag{10}$$

which presents a well-known Lyapunov's equation for the system without time delay.

This condition will be fulfilled if and only if R is a stable matrix i.e. if

$$\operatorname{Re}\lambda_i(R) < 0 \tag{11}$$

holds, Stojanovic, Debeljkovic (2005).

Remark 2. Stojanovic, Debeljkovic (2005)

Eq. (6) expressed through the matrix R can be written in a different form as follows,

$$R - A_0 - e^{-R\tau} A_1 = 0 \tag{12}$$

and there follows

$$\det\left(R - A_0 - e^{-R\tau} A_1\right) = 0 \tag{13}$$

Substituting the matrix variable R by the scalar variable s in (11), the characteristic equation of the system (1) is obtained as

$$f(s) = \det\left(sI - A_0 - e^{-s\tau} A_1\right) = 0 \tag{14}$$

Let us denote

$$\Sigma \triangleq \{ s \mid f(s) = 0 \} \tag{15}$$

a set of all characteristic roots of the system (1), Stojanovic, Debeljkovic (2005).

The necessity for the correctness of desired results forced us to propose new formulations of Lemma 1 and Theorem 1 and Theorem 2.

Lemma 1.a Stojanovic, Debeljkovic (2006).

Suppose that there exist(s) the solution(s) $P_1(0)$ of (6) and let the Lyapunov's function be (2).

Then, $\dot{V}(\mathbf{x}_t, \tau) < 0$ if and only if for any matrix $Q = Q^* > 0$ there exists a matrix $P_0 = P_0^* > 0$ such that (3) holds for **all** solution(s) $P_1(0)$.

Remark 3.

The necessary condition of *Lemma* 1.a follows directly from the proof of *Theorem* 2 in Lee, Dianat (1981), Stojanovic, Debeljkovic (2006)

Theorem1.a Stojanovic, Debeljkovic (2006).

Suppose that there exist(s) the solution(s) of $P_1(0)$ of (6).

Then, the system (1) is asymptotically stable if for any matrix $Q = Q^* > 0$ there exists a matrix $P_0 = P_0^* > 0$ such that (3) holds for all solutions $P_1(0)$ of (6), Stojanovic, Debeljkovic (2006).

Theorem 2.a Stojanovic, Debeljkovic (2006).

Suppose that there exist (s) the solution(s) $P_1(0)$ of (6). If the system (1) is asymptotically stable, then the following statements are equivalent:

1. For any matrix $Q = Q^* > 0$ there exists a matrix $P_0 = P_0^* > 0$ such that the (3) holds for all solutions $P_1(0)$ of (6).
2. The condition $\operatorname{Re}\lambda_i\left(A_1 + P_1(0)\right) < 0$ holds for all solutions of $P_1(0)$ of (6).

Remark 4.

Theorem 1.a contains the sufficient and *Theorem* 2.a the necessary condition of stability.

The mentioned conditions of stability are formulated together in the following *Theorem*, Stojanovic, Debeljkovic (2006).

Theorem 3. Stojanovic, Debeljkovic (2006).

Suppose that there exist(s) the solution(s) $P_1(0)$ of (6).

Then, the system (1) is asymptotically stable **if and only** if any of the two following statements holds:

1. For any matrix $Q = Q^* > 0$ there exists a matrix $P_0 = P_0^* > 0$ such that (3) holds for all solutions $P_1(0)$ of (6).
2. The condition $\operatorname{Re}\lambda_i\left(A_1 + P_1(0)\right) < 0$ holds for all solutions $P_1(0)$ of (6).

Remark 5.

The statements *Lemma* 1.a and *Theorems* 1.a and *Theorems* 2.a require that corresponding conditions are fulfilled for any solution $P_1(0)$ of (6) or R of (12). These matrix conditions are analogous to the following known scalar condition of asymptotic stability: System (1) *is asymptotically stable* if and only if *the condition* $\operatorname{Re} s < 0$ holds for **all** solutions s of (14), Stojanovic, Debeljkovic (2006).

Remark 6.

From the preceding theorems, the following practical question is imposed: how can all possible solutions $P_1(0)$ of (6) be numerically computed? This problem cannot be directly numerically solved because the number of solutions $P_1(0)$ is not known beforehand, and can be very large (infinite), Stojanovic, Debeljkovic (2006).

However, in order to examine the stability of the system more efficiently, the mentioned numerical problem can be replaced by a new, numerically simpler problem that reads:

a) (12) is solved instead of (6), and

b) computations are done for the solution $R_{\max}$ of (12) whose spectrum contains the eigenvalue $\lambda_{\max} \in \Sigma$ with a maximal real part.

Step b) in the last problem requires investigations of new numerical algorithms for direct computations of the matrix $R_{\max}$ from nonlinear (exponential) matrix eq. (12).

To the authors' knowledge, such algorithms have not been presented in literature so far.

At present, use is being made of algorithms based on various standard optimization methods and they demand initial guesses of solution for a given equation.

On the basis of *Remark* 6, it is possible to reformulate *Theorem* 3 in the following way:

Theorem 4. Stojanovic, Debeljkovic (2006).

Suppose that there exists the solution $R_{\max}$ of (6).

Then, the system (1) is asymptotically stable if and only if any of the two following equivalent statements holds:

1. For any matrix $Q = Q^* > 0$ there exists a matrix $P_0 = P_0^* > 0$ such that (10) holds for the solution $R_{\max}$.
2. $\operatorname{Re} \lambda_i (R_{\max}) < 0$.

DISCRETE TIME DELAY SYSTEMS

In the sequel we propose new, necessary and sufficient conditions for delay-dependent asymptotic stability of systems of the form $\mathbf{x}(k+1) = A_0 \mathbf{x}(k) + A_1 \mathbf{x}(k-h)$. The time-dependent criteria are derived by Lyapunov's direct method and are exclusively based on the maximal and dominant solvents of a particular matrix polynomial equation.

Two matrix equations have been derived: matrix polynomial equation and discrete Lyapunov matrix equation.

It has been demonstrated that, if a dominant solvent can be computed by Traub or Bernoulli algorithm, a decrease in the number of computations is to be expected in favor of the derived stability criteria compared with the existing ones.

Modifications of the existing sufficient conditions of convergence of Traub and Bernoilli algorithms for computing the dominant solvent of the matrix polynomial equation are derived as well.

Introduction

The stability problem of linear systems with time delays has been investigated by many researchers, (see references).

It is obvious that there are much more published papers in the area of continuous than discrete time delay systems.

Certainly, one of the basic reasons for that lies in the fact that discrete time delay systems are of *finite dimensions* so the *equivalent systems* of considerably *high order* can be easily built, Malek-Zavarei, Jamshidi (1987), Gorecki et al. (1989).

The basic inspiration for our investigation is based on paper Lee, Dianat (1981), however, the stability of discrete time delay systems is considered herein.

In this paper, we first propose a modification of the existing sufficient condition for non-singularity of the block Vandermonde matrix $V(S_1, \ldots, S_{h+1})$.

This condition has a weaker hypothesis than a similar condition from Dennis et al. (1976) and represents the generalization of the results presented in Kim (2000).

It has been then demonstrated that the condition of non-singularity of the block Vandermonde matrix $V(S_2, \ldots, S_{h+1})$ is the direct outcome of the non-singularity of the block matrix $V(S_1, \ldots, S_{h+1})$.

Likewise, we have arrived at a new sufficient condition for the convergence of Traub and Bernoilli algorithms.

This condition has a weaker hypothesis than a similar condition in Dennis et al. (1978).

At the end, we propose new necessary and sufficient conditions for delay dependent stability of discrete linear time delay systems, which, as distinguished from the criterion based on the eigenvalues of the equivalent system matrix Gantmacher (1960), use matrices of considerably lower dimensions.

Preliminaries

A linear, discrete time-delay system can be represented by the difference equation

$$\mathbf{x}(k+1) = A_0 \mathbf{x}(k) + A_1 \mathbf{x}(k-h) \tag{16}$$

with an associated function of the initial state

$$\mathbf{x}(\theta) = \boldsymbol{\psi}(\theta), \qquad \theta \in \{-h, -h+1, \ldots, 0\} \tag{17}$$

Eq. (16) is referred to as homogenous or the unforced state equation.

The vector $\mathbf{x}(k) \in \mathbb{R}^n$ is a state vector and $A_0, A_1 \in \mathbb{R}^{n \times n}$ are constant matrices of appropriate dimensions, and pure system time delay is expressed by the integers $h \in \mathbb{T}^+$.

System (16) can be expressed with the following representation without delay, Malek-Zavarei, Jamshidi (1987), Gorecki et al. (1989).

$$\mathbf{x}_{eq}(k) = \left[\mathbf{x}^T(k-h) \mathbf{x}^T(k-h+1) \;\; \mathbf{x}^T(k) \right] \in \mathbb{R}^N$$

$$\mathbf{x}_{eq}(k+1) = A_{eq} \mathbf{x}_{eq}(k), \qquad N \hat{=} n(h+1) \tag{18}$$

$$A_{eq} = \begin{bmatrix} 0 & I_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & I_n \\ A_1 & 0 & \ldots & A_0 \end{bmatrix} \in \mathbb{R}^{N \times N}$$

The system defined by (18) is called the equivalent system, while the matrix A_{eq}, is the matrix of the equivalent system.

The characteristic polynomial of system (16) is given with:

$$f(\lambda) \hat{=} \det M(\lambda) = \sum_{j=0}^{n(h+1)} a_j \lambda^j, \quad a_j \in \mathbb{R}, \qquad M(\lambda) = I_n \lambda^{h+1} - A_0 \lambda^h - A_1 \tag{19}$$

Denote with

$$\Omega \hat{=} \{ \lambda \mid f(\lambda) = 0 \} = \lambda(A_{eq}) \tag{20}$$

the set of all characteristic roots of system (16).

The number of these roots amounts to $n(h+1)$.

A root λ_m of Ω with the maximal module:

$$\lambda_m \in \Omega : \ |\lambda_m| = \max |\lambda(A_{eq})| \tag{21}$$

let us call the maximal root (eigenvalue). Note that there can exist a number of maximal roots of Ω.

If the scalar variable λ in the characteristic polynomial is replaced by the matrix $X \in \mathbb{C}^{n \times n}$ the two following monic matrix polynomials are obtained

$$M(X) = X^{h+1} - A_0 X^h - A_1 \tag{22}$$

$$F(X) = X^{h+1} - X^h A_0 - A_1 \tag{23}$$

It is obvious that $F(\lambda) = M(\lambda)$.

For the matrix polynomial $M(X)$, the matrix of the equivalent system A_{eq} represents the *block companion matrix*.

A matrix $S \in \mathbb{C}^{n \times n}$ is a *right solvent* of $M(X)$, Dennis et al. (1976) if

$$M(S) = 0 \tag{24}$$

If

$$F(R) = 0 \tag{25}$$

then $R \in \mathbb{C}^{n\times n}$ is a *left solvent* of $M(X)$, Dennis et al. (1976).

We will further use the matrix S to denote the right solvent and the matrix R to denote the left solvent of $M(X)$.

In the present paper the majority of presented results start from the left solvents of $M(X)$.

In contrast, in the existing literature the right solvents of $M(X)$ were mainly studied.

The mentioned discrepancy can be overcome by the following *Lemma*.

Lemma 2. The conjugate transpose value of the left solvent of $M(X)$ is also, at the same time, the right solvent of the following matrix polynomial

$$\mathcal{M}(X) = X^{h+1} - A_0^T X^h - A_1^T \tag{26}$$

Proof. Let R be the right solvent of $M(X)$.

Then it holds

$$\begin{aligned}\mathcal{M}(R^*) &= (R^*)^{h+1} - A_0^T (R^*)^h - A_1^T \\ &= (R^{h+1} - A_0 R^h - A_1)^* = F^*(R) = 0\end{aligned} \tag{27}$$

so R^* is the right solvent of $\mathcal{M}(X)$ **Q.E.D**

Conclusion 1. Based on *Lemma* 2, all characteristics of the left solvents of $M(X)$ can be obtained by the analysis of the conjugate transpose value of the right solvents of $\mathcal{M}(X)$.

The following proposed factorization of the matrix $M(\lambda)$ will help us to understand better the relationship between the eigenvalues of left and right solvents and the roots of the system.

Lemma 3. The matrix $M(\lambda)$ can be factorized in the following way

$$\begin{aligned}M(\lambda) &= \left(\lambda^h I_n + (S - A_0)\sum_{i=1}^{h} \lambda^{h-i} S^{i-1}\right)(\lambda I_n - S) \\ &= (\lambda I_n - R)\left(\lambda^h I_n + \sum_{i=1}^{h} \lambda^{h-i} R^{i-1}(R - A_0)\right)\end{aligned} \tag{28}$$

Proof.

$$\begin{aligned}&M(\lambda) - M(X) = \lambda^{h+1} I_n - X^{h+1} - A_0(\lambda^h I_n - X^h) = \\ &= \left(\sum_{i=0}^{h} \lambda^{h-i} X^i - A_0 \sum_{i=0}^{h-1} \lambda^{h-1-i} X^i\right)(\lambda I_n - X)\end{aligned} \tag{29}$$

If S is a right solvent of $M(X)$, from (19) follows (28).

Similarly, if R is a left solvent of $M(X)$, from

$$\begin{aligned}&M(\lambda) - F(X) = \\ &= (\lambda I_n - X)\left(\lambda^h I_n + \sum_{i=1}^{h} \lambda^{h-i} X^{i-1}(X - A_0)\right)\end{aligned} \tag{30}$$

follows (28). **Q.E.D**

Conclusion 2 From (19) and (28) follows $f(S) = f(R) = 0$, e.g. the characteristic polynomial $f(\lambda)$ is an *annihilating polynomial* for the right and left solvents of $M(X)$.

Therefore, $\lambda(S) \subset \Omega$ and $\lambda(R) \subset \Omega$ hold.

The eigenvalues and the eigenvectors of the matrix have a crucial influence on the existence, enumeration and characterization of solvents of the matrix equation (24), Dennis et al. (1976), Pereira (2003).

Definition 1. Dennis et al. (1976), Pereira (2003).

Let $M(\lambda)$ be a matrix polynomial in λ.

If $\lambda_i \in \mathbb{C}$ is such that $\det M(\lambda_i) = 0$, then we say that λ_i is a *latent root* or an *eigenvalue* of $M(\lambda)$.

If a nonzero $\mathbf{v}_i \in \mathbb{C}^n$ is such that

$$M(\lambda_i)\mathbf{v}_i = \mathbf{0} \tag{31}$$

then we say that v_i is a (right) *latent vector* or a (right) *eigenvector* of $M(\lambda)$, corresponding to the eigenvalue λ_i.

The eigenvalues of the matrix $M(\lambda)$ correspond to the characteristic roots of the system, i.e. eigenvalues of its block companion matrix A_{eq}, Dennis et al. (1976). Their number is $n \cdot (h+1)$.

Since $F^*(\lambda) = \mathcal{M}(\lambda^*)$ holds, it is not difficult to show that matrices $M(\lambda)$ and $\mathcal{M}(\lambda)$ have the same spectrum.

In the papers Dennis et al. (1976, 1978), Kim (2000) and Pereira (2003) some sufficient conditions for the existence, enumeration and characterization of the right solvents of $M(X)$ were derived.

They show that the number of solvents can be *zero*, *finite* or *infinite*.

For the needs of system stability (16) only the so-called maximal solvents are usable, the spectrums of which contain the maximal eigenvalue λ_m. A special case of the maximal solvent is a so-called dominant solvent, Dennis et al. (1976), Kim (2000), which, unlike maximal solvents, can be computed in a simple way.

Definition 2. Every solvent S_m of $M(X)$, the spectrum $\sigma(S_m)$ of which contains the maximal eigenvalue λ_m of Ω is a *maximal solvent*.

Definition 3. Dennis et al. (1976), Kim (2000).

The matrix A dominates the matrix B if all the eigenvalues of A are greater, in modulus, than those of B.

In particular, if the solvent S_1 of $M(X)$ dominates the solvents $S_2, \dots, S_l$ we say it is a *dominant solvent*. (Note that a dominant solvent cannot be singular.)

Conclusion 3. The number of maximal solvents can be greater than one. A dominant solvent is at the same time the maximal solvent, too.

The dominant solvent S_1 of $M(X)$, under certain conditions, can be determined by the *Traub*, Dennis et al. (1978) and *Bernoulli iteration* Dennis et al. (1976), Kim (2000).

Main results

We will further provide improvements for some existing sufficient conditions related to non-singularity of the block Vandermonde matrix and the existence of a dominant solvent.

The following *Lemma* gives a sufficient condition for the regularity of the block Vandermonde matrix and has a weaker hypothesis than *Theorem* 6.1 in Dennis et al. (1976).

This *Lemma* represents the generalization of the corresponding result presented in Kim (2000).

Lemma 4. If $S_1,\ldots,S_{h+1}$ are solvents of $M(X)$ with $\sigma(S_1)\cap\ldots\cap\sigma(S_{h+1})=\varnothing$ then $V(S_1,\ldots,S_{h+1})$ is nonsingular.

Proof. It is derived by the generalization of the proof given in Kim (2000), for the case h=1. **Q.E.D.**

It is demonstrated by the following *Lemma* that the condition of the non-singularity of the matrix $V(S_2,\ldots,S_{h+1})$ is superfluous, since it results directly from the non-singularity of the matrix $V(S_1,\ldots,S_{h+1})$.

Lemma 5 If the block Vandermonde matrix $V(S_1,\ldots,S_{h+1})$ is nonsingular, then $V(S_2,\ldots,S_{h+1})$ is also nonsingular.

Proof. If the block Vandermonde matrix $V(S_2,\ldots,S_{h+1})$ is nonsingular, then

$$\begin{aligned}\det\begin{bmatrix} I & I & \cdots & I \\ S_1 & S_2 & \cdots & S_{h+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_1^h & S_2^h & \cdots & S_{h+1}^h\end{bmatrix} &= (-1)^{nh}\det V(S_2,\ldots,S_{h+1})\times \\ \times\det\left\{S_1^h-\begin{bmatrix}S_2^h & \cdots & S_{h+1}^h\end{bmatrix}V^{-1}(S_2,\ldots,S_{h+1})\begin{bmatrix} I \\ \vdots \\ S_1^h\end{bmatrix}\right\}\end{aligned} \tag{32}$$

From $\det V(S_1,...,S_{h+1})\neq 0$, follows $V(S_2,...,S_{h+1})\neq 0$, so $V(S_2,...,S_{h+1})$ is nonsingula, when $V(S_1,...,S_{h+1})$ is regular. **Q.E.D.**

By combining *Lemma* 4 - 5 one can modify some existing conditions for convergence of *Traub* and *Bernoulli* algorithms presented in Dennis et al. (1978).

These conditions have a weaker hypothesis than the conditions given in Dennis et al. (1978).

Lemma 6. If $M(X)$ is a matrix polynomial of a degree (h+1) such that

(i) it has the solvents $S_1,\ldots,S_{h+1}$

(ii) S_1 is a dominant solvent

(iii) $\sigma(S_1)\cap\ldots\cap\sigma(S_{h+1})=\varnothing$

then *Traub* and *Bernoulli* algorithms Dennis et al. (1978) converge.

Proof. The first two conditions of this *Lemma* are identical with conditions (i)-(ii) of *Theorem* 2.1 and *Theorem* 3.2 in Dennis et al. (1978).

From *Lemmas* 4 - 5 follows that $V(S_1,\ldots,S_{h+1})$ and $V(S_2,\ldots,S_{h+1})$ are nonsingular, whereby the third condition of *Theorem* 2.1 and *Theorem* 3.2 in Dennis et al. (1978) has been fulfilled too.

So, Traub and Bernoulli algorithms converge to a dominant solvent. **Q.E.D.**

Similarly to the definition of the right solvents S_m and S_1 of $M(X)$, the definitions of both the maximal left solvent, R_m, and the dominant left solvent, R_1, of $M(X)$can be provided.

These left solvents of $M(X)$ are used in a number of theorems to follow.

Owing to *Lemma* 2, they can be determined by proper right solvents of $\mathcal{M}(X)$.

Generally, all aforementioned about the existence, enumeration and characterization of the right solvents of $M(X)$, holds also for the right solvents of $\mathcal{M}(X)$, therefore for the left solvents of $M(X)$, too.

Necessary and sufficient conditions for asymptotic stability of linear discrete time-delay systems (16) are to follow.

Theorem 5. Stojanovic, Debeljkovic (2008.b).

Suppose that there exists at least one left solvent of $M(X)$ and let R_m denote one of them.

Then, linear discrete time delay system (16) is *asymptotically stable* if and only if for any matrix $Q=Q^*>0$ there exists a *Hermitian* matrix $P=P^*>0$ such that

$$R_m^* P R_m - P = -Q \tag{33}$$

Proof. Define the following vector discrete functions

$$\mathbf{x}_k=\mathbf{x}(k+\theta),\qquad \theta\in\{-h,-h+1,\ldots,0\} \tag{34}$$

$$\mathbf{z}(\mathbf{x}_k)=\mathbf{x}(k)+\sum_{j=1}^{h}T(j)\mathbf{x}(k-j) \tag{35}$$

where $T(k)\in\mathbb{C}^{n\times n}$ is, in general, a time varying discrete matrix function.

The conclusion of the theorem follows immediately by defining the Lyapunov functional for system (16) as

$$V(\mathbf{x}_k)=\mathbf{z}^*(\mathbf{x}_k)P\mathbf{z}(\mathbf{x}_k),\quad P=P^*>0 \tag{36}$$

It is obvious that $\mathbf{z}(\mathbf{x}_k)=0$ if and only if $\mathbf{x}_k=0$, so it follows that $V(\mathbf{x}_k)>0$ for $\forall\mathbf{x}_k\neq 0$.

The forward difference of (36), along the solutions of system (16) is

$$\begin{aligned}\Delta V(\mathbf{x}_k)&=\Delta\mathbf{z}^*(\mathbf{x}_k)P\mathbf{z}(k)\\ &+\mathbf{z}^*(\mathbf{x}_k)P\Delta\mathbf{z}(\mathbf{x}_k)+\Delta\mathbf{z}^*(\mathbf{x}_k)P\Delta\mathbf{z}(\mathbf{x}_k)\end{aligned} \tag{37}$$

A difference of $\Delta\mathbf{z}(\mathbf{x}_k)$ can be determined in the following manner

$$\Delta\mathbf{z}(\mathbf{x}_k)=\Delta\mathbf{x}(k)+\sum_{j=1}^{h}T(j)\Delta\mathbf{x}(k-j) \tag{38}$$

with

$$\Delta\mathbf{x}(k)=(A_0-I_n)\mathbf{x}(k)+A_1\mathbf{x}(k-h) \tag{39}$$

and

$$\begin{aligned}\sum_{j=1}^{h}T(j)\Delta\mathbf{x}(k-j)&=T(1)[\mathbf{x}(k)-\mathbf{x}(k-1)]+\\ &+T(h)[\mathbf{x}(k-h+1)-\mathbf{x}(k-h)]\end{aligned} \tag{40}$$

Then simple manipulations lead to

$$\begin{aligned}&\sum_{j=1}^{h}T(j)\Delta\mathbf{x}(k-j)=T(1)\mathbf{x}(k)-T(h)\mathbf{x}(k-h)+\\ &(T(2)-T(1))\mathbf{x}(k-1)+(T(h)-T(h-1))\cdot\\ &\mathbf{x}(k-h+1)\end{aligned} \tag{41}$$

Define a new matrix R by

$$R = A_0 + T(1) \tag{42}$$

If

$$\Delta T(h) = A_1 - T(h) \tag{43}$$

then $\Delta \mathbf{z}(\mathbf{x}_k)$ has a form

$$\Delta \mathbf{z}(\mathbf{x}_k) = (R - I_n)\mathbf{x}(k) + \sum_{j=1}^{h}\left[\Delta T(j)\cdot \mathbf{x}(k-j)\right] \tag{44}$$

If one adopts $\Delta T(j) = (R - I_n)T(j), \quad j = 1, 2, \ldots, h$ (45)

then $\Delta \mathbf{z}(\mathbf{x}_k)$ becomes

$$\Delta \mathbf{z}(\mathbf{x}_k) = (R - I_n)\mathbf{z}(\mathbf{x}_k) \tag{46}$$

Therefore, (37), becomes

$$\Delta V(\mathbf{x}_k) = \mathbf{z}^*(\mathbf{x}_k)\left(R^* P R - P\right)\mathbf{z}(\mathbf{x}_k) \tag{47}$$

It is obvious that if the following equation is satisfied

$$R^* P R - P = -Q, \quad Q = Q^* > 0 \tag{48}$$

then $\Delta V(\mathbf{x}_k) < 0, \ \mathbf{x}_k \neq 0$.

In the Lyapunov matrix eq. (48), of all possible solvents R of $M(X)$, only one of maximal solvents is of importance, for it is the only one that contains the maximal eigenvalue $\lambda_m \in \Omega$ (*Conclusion* 2), which has dominant influence on the stability of the system. So, (33) represents the stability *sufficient condition* for the system given by (16). The matrix $T(1)$ can be determined in the following way.

From (45), it follows

$$T(h+1) = R^h\, T(1) \tag{49}$$

and using (42-43) one can get (25), and for the sake of brevity, instead of the matrix $T(1)$, one introduces a simple notation T.

If a solvent which is not maximal is integrated into Lyapunov equation, it may happen that there will exist positive definite solution of Lyapunov matrix eq. (33) although the system is not stable (see *Example* 4).

Conversely, if the system (16) is asymptotically stable then all roots $\lambda_i \in \Omega$ are located within the unit circle. Since $\sigma(R_m) \subset \Omega$, follows $\rho(R_m) < 1$, so the positive definite solution of Lyapunov matrix eq. (33) exists (*necessary condition*). **Q.E.D.**

Corollary 1. Suppose that there exists at least one maximal left solvent of $M(X)$ and let R_m denote one of them. Then, system (16) is asymptotically stable if and only if $\rho(R_m) < 1$, Stojanovic, Debeljkovic (2008.b).

Proof. Follows directly from *Theorem*5. **Q.E.D.**

Conclusion 4. *Corollary* 1 may be proved in the following way.

From *Conclusion* 2 follows $\sigma(R) \subset \ \subset \Omega = \lambda(A_{eq})$ and based on the properties of the maximal solvent R_m it follows $\rho(R_m) = \rho(A_{eq})$.

So, if the maximal solvent is discrete stable then A_{eq} will be also a discrete stable matrix and vice versa.

Corollary 2. Suppose that there exists a dominant left solvent R_1 of $M(X)$, Stojanovic, Debeljkovic (2008.b).

Then, system (16) is asymptotically stable if and only if $\rho(R_1) < 1$.

Proof. Follows directly from *Corollary* 1, since a *dominant solution* is, at the same time, a *maximal solvent*. **Q.E.D.**

Conclusion 5 In the case when the dominant solvent R_1 may be deduced by *Traub* or *Bernoulli* algorithm, *Corollary* 3 represents a quite simple method.

If the aforementioned algorithms are not convergent but still there exists at least one of maximal solvents R_m, then one should use *Corollary* 1.

The maximal solvents may be found, for example, using the concept of eigenpars, Pereira (2003).

If there exists no maximal solvent R_m, then the proposed necessary and sufficient conditions *cannot be used* for system stability investigation.

Conclusion 6. For some time delay systems it holds

$$\begin{aligned} &\dim(R_1) = \dim(R_m) = \\ &= \dim(A_i) = n \ll \dim(A_{eq}) = n(h+1) \end{aligned}$$

For example, if time delay amounts to $h = 100$, and the row of matrices of the system is $n = 2$, then: $R_1, R_m \in \mathbb{C}^{2\times 2}$ and $A_{eq} \in \mathbb{C}^{202\times 202}$.

To check the stability by the eigenvalues of the matrix A_{eq}, it is necessary to determine 202 eigenvalues, which is not numerically simple. On the other hand, if a dominant solvent can be computed by *Traub* or *Bernoulli* algorithm, *Corollary* 2 requires a relatively small number of additions, subtractions, multiplications and inversions of the matrix format of only 2×2.

So, in the case of great time delay in the system, by applying *Corollary* 2, a smaller number of computations is to be expected compared with a traditional procedure of examining the stability by the eigenvalues of the companion matrix A_{eq}.

An accurate number of computations for each of the mentioned methods requires additional analysis, which is not the subject of this paper.

Numerical examples

Example 1. Let us consider linear discrete system with delayed state (16) with

$$A_0 = \begin{bmatrix} 0.1 & 0.3 \\ 0.1 & -0.15 \end{bmatrix}, \ A_0 = \begin{bmatrix} 0.3 & 0.4 \\ 0.2 & 0.25 \end{bmatrix}, \ h = 1$$

and let us check the stability properties of the system under consideration, based on the application of *Theorem* 1, *Corollaries* 1 and 2.

Application of Theorem 5 By the left solvents S_i of $\mathcal{M}(X)$, applying the concept of eigenpar Pereira (2003), the left solvents R_i of $M(X)$ are calculated:

$$R_1 = S_1^* = \begin{bmatrix} 3.548 & 4.759 \\ 2.408 & -3.39 \end{bmatrix}, R_2 = S_2^* = \begin{bmatrix} -1.812 & 2.490 \\ -1.171 & 1.604 \end{bmatrix},$$

$$R_3 = S_3^* = \begin{bmatrix} 0.453 & 0.576 \\ 0.342 & 0.326 \end{bmatrix}, R_4 = S_4^* = \begin{bmatrix} 0.402 & 0.620 \\ 0.388 & 0.287 \end{bmatrix},$$

$$R_5 = S_5^* = \begin{bmatrix} -0.345 & -0.502 \\ -0.191 & -0.394 \end{bmatrix}, R_6 = S_6^* = \begin{bmatrix} -0.386 & -0.417 \\ -0.167 & -0.443 \end{bmatrix},$$

The solvents R_1, R_3 and R_4 are the maximal solvents, since they contain the eigenvalue $\lambda_m = 0.838 \in \Omega$.

From the solved Lyapunov eq. (33), for example, $R_m = R_1$ and $Q = I_2$, we can conclude that the system under consideration is asymptotically stable.

Application of Corollary 1 By adopting, for example, $R_m = R_3$ as a maximal solvent, we conclude that in equation $\rho(R_m) = 0.838 < 1$ is satisfied, therefore the observed system is asymptotically stable.

Application of Corollary 2 If for a set of $h+1=2$ solvents, we choose R_1 and R_2, the conclusion is that R_1 is a dominant solvent, whereby the condition has been fulfilled $\det(V(R_1, R_2)) \neq 0$.

Therefore, the *Traub* or *Bernoulli* algorithm can be used for the determination of a dominant solvent.

By *Traub* algorithm, after only three iterations upon the matrices G_i [13] and three iterations upon X_i [13] (3+3), identical value, as above calculated, was obtained for dominant solvent R_1.

Similarly, by applying *Bernoulli* algorithm, after 12 iterations upon X_i [13], an identical value, as above calculated, was obtained for the dominant solvent R_1.

Since $\rho(R_1) = 0.838 < 1$, based on *Corollary* 2, it follows that the system under consideration is *asymptotically stable*.

Example 2. Let us consider linear discrete systems with delayed state (16), with

$$A_0 = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},\ h = 1.$$

and let us check the stability properties of the system under consideration.

Application of Corollary 1.

The left solvents R_i of $M(X)$ are

$$R_1 = \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix},\ R_2 = \begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix},\ R_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\lambda(R_1) = \{-1, 1\}$, $\lambda(R_2) = \{-1, 0\}$ and $\lambda(R_3) = \{1, 0\}$ *there exists no dominant solvent*, but all the three solvents are the maximal ones.

Because $\rho(R_i) = 1$, $1 \le i \le 3$, based on *Corollary* 1, the system is not asymptotically stable.

Example 3. Let us consider linear discrete systems with delayed state (16) with

$$A_0 = \begin{bmatrix} 7/10 & 1/2 \\ 1/2 & 17/10 \end{bmatrix},\ A_1 = \begin{bmatrix} -1/75 & -1/3 \\ 1/3 & 49/75 \end{bmatrix}$$

There are two left solvents of matrix polynomial eq. (25):

$$R_1 = \begin{bmatrix} 19/30 & 1/6 \\ -1/6 & 29/30 \end{bmatrix},\ R_2 = \begin{bmatrix} 1/5 & 1/3 \\ -1/3 & 11/15 \end{bmatrix}$$

Since $\lambda(R_1) = \left\{\frac{4}{5}, \frac{4}{5}\right\}$, $\lambda(R_2) = \left\{\frac{2}{5}, \frac{2}{5}\right\}$, the dominant solvent is R_1.

As we have $V(R_1, R_2)$ nonsingular, *Traub* or *Bernoulli* algorithm may be used.

Application of Corollary 2.

Only after (4+3) iterations for *Traub* and 17 iterations for *Bernoulli* algorithm, a dominant solvent can be found with an accuracy of 10^{-4}.

Since $\rho(R_1) = \frac{4}{5} < 1$, based on *Corollary* 2, it follows that the system under consideration is *asymptotically stable*.

Example 4. Let us consider linear discrete systems with delayed state (16), with

$$A_0 = \begin{bmatrix} 17/6 & -11/6 \\ 1/3 & 2/3 \end{bmatrix},\ A_1 = \begin{bmatrix} -5/3 & 17/12 \\ -2/3 & 5/12 \end{bmatrix},\ h = 1.$$

The eigenvalues of matrices $M(X)$ are given with $\{0.5,\ 0.5,\ 0.5,\ 2\} = \Omega$.

There is *only one* solvent of matrix polynomial eq. (25):

$$R = \begin{bmatrix} 12/7 & 1/7 \\ -4/7 & 16/7 \end{bmatrix}$$

with $\lambda(R) = \{0.5, 0.5\}$.

It can be seen that there exist no dominant and maximal solvents of (25), so the proposed stability conditions *cannot be applied*.

If we, disregarding the assumption on the existence of the maximal solvent R_m, apply *Corollary* 1, based on $\rho(R) = 0.5 < 1$, we would arrive at a wrong conclusion that the system is asymptotically stable.

But, the system is unstable since it possesses a characteristic root $\lambda_m = 2 > 1$.

All numerical examples are taken from Stojanovic, Debeljkovic (2008.b).

Large scale Time delay systems

CONTINUOUS LARGE SCALE TIME DELAY SYSTEMS

This paper offers new necessary and sufficient conditions for the *delay-dependent* asymptotic stability of the linear continuous large scale time delay systems. The obtained conditions of stability are expressed by nonlinear system of matrix equations and the Lyapunov matrix equation for an ordinary linear continuous system without delay. This condition is not conservative, however, it requires somewhat more complex numerical computations.

Introduction

In the past two decades, a considerable interest has been permanently shown in the problem of asymptotic stability of continuous large scale time delay systems. The stabilization problem for large scale time delay systems with or without perturbations is studied in Suh, Bein (1982), Lee, Radovic (1982, 1987), Kolla, Farison (1991). Wang et al. (1995) extended the results of Lee, Radovic (1982) to the problems of stabilization, estimation and robustness. Moreover, Wang, Mau (1997) derived a much more concise and less conservative result other than Wang et al. (1995). Hu (1994) and Trinh, Alden (1995.b) have synthesized some decentralized controllers to stabilize the whole system. Xu (1995) provides a new criterion for delay-independent stability of linear large scale time delay systems by employing an improved Razumikhin-type theorem and M-matrix properties.

In Trinh, Alden (1997.b), by employing a Razumikhin-type theorem, a robust stability criterion for a class of linear system subject to delayed time-varying nonlinear perturbations is given. New sufficient conditions for delay -

independent asymptotic stability of large scale systems are presented by Huang et al. (1995) using the properties of matrix norm and measure. It is shown that the presented approach simplifies the stability problem.The basic aim of the above mentioned works was to obtain only sufficient (S) conditions for stability of large scale time delay systems. It is notorious that those conditions of stability are more or less conservative.

In contrast, the major result of our investigations are necessary and sufficient (NS) conditions of asymptotic stability of continuous large scale time delay autonomous systems (see Lee, Diant (1981) for similarly results for time delay systems). The obtained (NS) conditions are expressed by nonlinear system of matrix equations and the Lyapunov matrix equation for an ordinary linear continuous system without delay. Those conditions of stability are delay-dependent and are not conservative.

Unfortunately, viewed mathematically, they require somewhat more complex numerical computations.

Main Results

Consider linear continuous large scale time delay *autonomous* systems composed of N interconnected subsystems.

Each subsystem is described as:

$$\dot{\mathbf{x}}_i(t) = A_i\mathbf{x}_i(t) + \sum_{j=1}^{N} A_{ij}\mathbf{x}_j(t-\tau_{ij}),\ 1 \le i \le N \tag{50}$$

with an associated function of the initial state $\mathbf{x}_i(\theta) = \varphi_i(\theta),\ \theta \in [-\tau_{m_i},\ 0],\quad 1 \le i \le N$. $\mathbf{x}_i(t) \in \mathbb{R}^{n_i}$ is the state vector, $A_i \in \mathbb{R}^{n_i \times n_i}$ denotes the system matrix, $A_{ij} \in \mathbf{R}^{n_i \times n_j}$ represents the interconnection matrix between the i-th and the j-th subsystems, and τ_{ij} is the constant delay.

For the sake of brevity, we first observe system (50) made up of *two subsystems* $(N = 2)$.

For this system, we derive new necessary and sufficient delay-dependent conditions for stability, by Lyapunov's direct method. The derived results are then extended to the linear continuous large scale time delay systems with *multiple subsystems*.

a) Large scale systems with two subsystems

Theorem 6. Given the following system of matrix equations (SME)

$$\mathcal{R}_1 - A_1 - e^{-\mathcal{R}_1\tau_{11}}A_{11} - e^{-\mathcal{R}_1\tau_{21}}S_2A_{21} = 0 \tag{51}$$

$$\mathcal{R}_1S_2 - S_2A_2 - e^{-\mathcal{R}_1\tau_{12}}A_{12} - e^{-\mathcal{R}_1\tau_{22}}S_2A_{22} = 0 \tag{52}$$

where A_1, A_2, A_{12}, A_{21} and A_{22} are the matrices of system (50) for $N = 2$, n_i represents the subsystem orders and τ_{ij} represents pure time delays of the system.

If there exists a solution of SME (51-52) upon the unknown matrices $\mathcal{R}_1 \in \mathbf{C}^{n_1 \times n_1}$ and $S_2 \in \mathbf{C}^{n_1 \times n_2}$, then the eigenvalues of matrix $\mathcal{R}_1$ belong to a set of roots of the characteristic equation of system (50) for $N = 2$.

Proof. By introducing the time delay operator $e^{-\tau s}$, system (50) can be expressed in the form

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} A_1 + A_{11}e^{-\tau_{11}s} & A_{12}e^{-\tau_2 s} \\ A_{21}e^{-\tau_{21}s} & A_2 + A_{22}e^{-\tau_{22}s} \end{bmatrix}\mathbf{x}(t) = A_e(s)\mathbf{x}(t),$$
$$\mathbf{x}(t) = \begin{bmatrix}\mathbf{x}_1^T(t) & \mathbf{x}_2^T(t)\end{bmatrix}^T \tag{53}$$

Let us form the following matrix

$$F(s) = \begin{bmatrix} F_{11}(s) & F_{12}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix} = sI_{n_1+n_2} - A_e(s)$$
$$= \begin{bmatrix} sI_{n_1} - A_1 - A_{11}e^{-\tau_{11}s} & -A_{12}e^{-\tau_{12}s} \\ -A_{21}e^{-\tau_{21}s} & sI_{n_2} - A_2 - A_{22}s^{-\tau_{22}s} \end{bmatrix} \tag{54}$$

Its determinant is

$$\det F(s) = \det \begin{bmatrix} F_{11}(s) & F_{12}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix}$$
$$= \det\begin{bmatrix} F_{11}(s) + S_2F_{21}(s) & F_{12}(s) + S_2F_{22}(s) \\ F_{21}(s) & F_{22}(s) \end{bmatrix} \tag{55}$$
$$= \det\begin{bmatrix} G_{11}(s,S_2) & G_{12}(s,S_2) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \det G(s,S_2)$$

$$G_{11}(s,S_2) = sI_{n1} - A_1 - A_{11}e^{-\tau_{11}s} - S_2A_{21}e^{-\tau_{21}s} \tag{56}$$

$$G_{12}(s,S_2) = sS_2 - S_2A_2 - A_{12}e^{-\tau_{12}s} - S_2A_{22}e^{-\tau_{22}s} \tag{57}$$

Relations (55-57) were obtained by applying a finite sequence of elementary row operations of type 3 over the matrix $F(s)$ Lancaster, Tismenetsky (1985). Transformational matrix S_2 is unknown for the time being, but a condition determining this matrix will be derived in the further text.

The characteristic polynomial of system (50) for $N = 2$, defined by

$$f(s) \mathrel{\hat{=}} \det(sI_N - A_e(s)) = \det G(s,S_2) \tag{58}$$

is independent of the choice of the matrix S_2, because the determinant of the matrix $G(s,S_2)$ is invariant with respect to the elementary row operation of type 3, Lancaster, Tismenetsky (1985).

Let us designate a set of roots of the characteristic equation of system (50) by $\sum \mathrel{\hat{=}} \{ s \mid f(s) = 0 \}$.

Substituting the scalar variable s by the matrix X in $G(s,S_2)$ we obtain

$$G(X,S_2) = \begin{bmatrix} G_{11}(X,S_2) & G_{12}(X,S_2) \\ G_{21}(X) & G_{22}(X) \end{bmatrix} \tag{59}$$

If there exist the transformational matrix S_2 and the matrix $\mathcal{R}_1 \in \mathbf{C}^{n_1 \times n_1}$ such that $G_{11}(\mathcal{R}_1,S_2) = 0$ and $G_{12}(\mathcal{R}_1,S_2) = 0$ is satisfied, i.e. if (51-52) hold, then

$$f(\mathcal{R}_1) = \det G_{11}(\mathcal{R}_1,S_2) \cdot \det G_{22}(\mathcal{R}_1) = 0 \tag{60}$$

So, the characteristic polynomial (58) of system (50) is an annihilating polynomial [14] for the square matrix $\mathcal{R}_1$, defined by (51-52). In other words, $\sigma(\mathcal{R}_1) \subset \sum$. **Q.E.D.**

Theorem 7. Given the following SME

$$\mathcal{R}_2 - A_2 - e^{-\mathcal{R}_2\tau_{12}} S_1 A_{12} - e^{-\mathcal{R}_2\tau_{22}} A_{22} = 0 \quad (61)$$

$$\mathcal{R}_2 S_1 - S_1 A_1 - e^{-\mathcal{R}_2\tau_{11}} S_1 A_{11} - e^{-\mathcal{R}_2\tau_{21}} A_{21} = 0 \quad (62)$$

where A_1, A_2, A_{12}, A_{21} and A_{22} are the matrices of system (50) for $N = 2$, n_i represents the subsystem orders and τ_{ij} represents the time delays of the system.

If there exists a solution of SME (61-62) upon the unknown matrices $\mathcal{R}_2 \in \mathbf{C}^{n_2\times n_2}$ and $S_1 \in \mathbf{C}^{n_2\times n_1}$, then the eigenvalues of matrix $\mathcal{R}_2$ belong to a set of roots of the characteristic equation of system (50) for $N = 2$.

Proof. The proof is similar with the proof of *Theorem* 6. **Q.E.D.**

Corollary 3. If system (50) is asymptotically stable, then the matrices $\mathcal{R}_1$ and $\mathcal{R}_2$, defined by SME (51-52) and (61-62), respectively, are stable ($\operatorname{Re}\lambda(\mathcal{R}_i) < 0$, $1 \le i \le 2$).

Proof. If system (50) is asymptotically stable, then $\forall s \in \Sigma, \operatorname{Re} s < 0$. Since $\sigma(\mathcal{R}_i) \subset \Sigma$, $1 \le i \le 2$, it follows that $\forall\, \lambda \in \sigma(\mathcal{R}_i)$, $\operatorname{Re}\lambda < 0$, i.e. the matrices $\mathcal{R}_1$ and $\mathcal{R}_2$ are stable. **Q.E.D.**

Definition 4. The matrix $\mathcal{R}_1$ ($\mathcal{R}_2$) is referred to as a *solvent* of SME (51-52) or (61-62).

Definition 5. Each root λ_m of the characteristic eq. (58) of system (50) which satisfies the following condition: $\operatorname{Re}\lambda_m = \max \operatorname{Re} s,\ s \in \Sigma$ will be referred to as the *maximal root* (*eigenvalue*) of system (50).

Definition 6 Each solvent $\mathcal{R}_{1m}$ ($\mathcal{R}_{2m}$) of SME (51-52) or (61-62), the spectrum of which contains the maximal eigenvalue λ_m of system (50), is referred to as the *maximal solvent* of SME (51-52) or (61-62).

Theorem 8. Stojanovic, Debeljkovic (2005).

Suppose that there exists at least one maximal solvent of SME (61-62) and let $\mathcal{R}_{1m}$ denote one of them. Then, system (50), for $N = 2$, is asymptotically stable if and only if for any matrix $Q = Q^* > 0$ there exists the matrix $P = P^* > 0$ such that

$$\mathcal{R}_{1m}^* P + P\,\mathcal{R}_{1m} = -Q \quad (63)$$

Proof. (*Sufficient condition*) Similarly Lee, Diant (1981), define the following vector continuous functions

$$\mathbf{x}_{ti} = \mathbf{x}_i(t+\theta), \quad \theta \in [-\tau_{m_i},\ 0], \quad (64)$$

$$\mathbf{z}(\mathbf{x}_{t1}, \mathbf{x}_{t2}) = \sum_{i=1}^{2} S_i \left(\mathbf{x}_i(t) + \sum_{j=1}^{2} \int_0^{\tau_{ji}} T_{ji}(\eta)\mathbf{x}_i(t-\eta)\,d\eta \right) \quad (65)$$

where $T_{ji}(t) \in \mathbf{C}^{n_i\times n_i}$, $j = 1,2$ are some time varying continuous matrix functions and $S_1 = I_{n_1}$, $S_2 \in \mathbf{C}^{n_1\times n_2}$.

The proof of the theorem follows immediately by defining the Lyapunov functional for system (50) as

$$V(\mathbf{x}_{t1}, \mathbf{x}_{t2}) = \mathbf{z}^*(\mathbf{x}_{t1}, \mathbf{x}_{t2}) P\, \mathbf{z}(\mathbf{x}_{t1}, \mathbf{x}_{t2}), \quad P = P^* > 0 \quad (66)$$

The derivative of (66), along the solutions of system (50) is

$$\begin{aligned}\dot V(\mathbf{x}_{t1}, \mathbf{x}_{t2}) &= \dot{\mathbf{z}}^*(\mathbf{x}_{t1}, \mathbf{x}_{t2})\, P\, \mathbf{z}(\mathbf{x}_{t1}, \mathbf{x}_{t2}) \\ &+ \mathbf{z}^*(\mathbf{x}_{t1}, \mathbf{x}_{t2})\, P\, \dot{\mathbf{z}}(\mathbf{x}_{t1}, \mathbf{x}_{t2})\end{aligned} \quad (67.a)$$

$$\dot{\mathbf{z}}(\mathbf{x}_{t1}, \mathbf{x}_{t2}) = \sum_{i=1}^{2} S_i \left(\dot{\mathbf{x}}_i(t) + \sum_{j=1}^{2} \frac{d}{dt} \int_o^{\tau_{ji}} T_{ji}(\eta)\mathbf{x}_i(t-\eta)\,d\eta \right) \quad (67.b)$$

From

$$\begin{aligned}\frac{d}{d\eta}\left[T_{ji}(\eta)\mathbf{x}_i(t-\eta)\right] &= T_{ji}'(\eta)\mathbf{x}_i(t-\eta) - \\ &- \frac{d}{dt} T_{ji}(\eta)\mathbf{x}_i(t-\eta)\end{aligned} \quad (68)$$

follows

$$\begin{aligned}\frac{d}{dt}\int_0^{\tau_{ji}} T_{ji}(\eta)\, \mathbf{x}_i(t-\eta)\, d\eta &= \int_0^{\tau_{ji}} T_{ji}'(\eta)\, \mathbf{x}_i(t-\eta)\, d\eta \\ &+ T_{ji}(0)\, \mathbf{x}_i(t) - T_{ji}(\tau_{ji})\, \mathbf{x}_i(t-\tau_{ji})\end{aligned} \quad (69)$$

Therefore

$$\begin{aligned}\dot{\mathbf{z}}(\mathbf{x}_{t1}, \mathbf{x}_{t2}) = \sum_{i=1}^{2} \Bigg\{ & S_i \left(A_i + \sum_{j=1}^{2} T_{ji}(0) \right) \mathbf{x}_i(t) \\ & + \sum_{j=1}^{2} \left(S_j A_{ji} - S_i T_{ji}(\tau_{ji}) \right) \mathbf{x}_i(t-\tau_{ji}) \\ & + \sum_{j=1}^{2} \int_0^{\tau_{ji}} S_i T_{ji}'(\eta)\mathbf{x}_i(t-\eta)\,d\eta \Bigg\}\end{aligned} \quad (70)$$

If we define new matrices

$$\mathcal{R}_i = A_i + \sum_{j=1}^{2} T_{ji}(0), \quad i = 1,2 \quad (71)$$

and if one adopts

$$S_i T_{ji}(\tau_{ji}) = S_j A_{ji}, \quad i,j = 1,2 \quad (72)$$

$$S_i T_{ji}'(\eta) = \mathcal{R}_1 S_i T_{ji}(\eta), \quad S_i \mathcal{R}_i = \mathcal{R}_1 S_i, \quad i,j = 1,2 \quad (73)$$

then

$$\dot{\mathbf{z}}(\mathbf{x}_{t1}, \mathbf{x}_{t2}) = \mathcal{R}_1 \mathbf{z}(\mathbf{x}_{t1}, \mathbf{x}_{t2}) \quad (74)$$

$$\dot V(\mathbf{x}_{t1}, \mathbf{x}_{t2}) = \mathbf{z}^*(\mathbf{x}_{t1}, \mathbf{x}_{t2})\left(\mathcal{R}_1^* P + P\mathcal{R}_1\right)\mathbf{z}(\mathbf{x}_{t1}, \mathbf{x}_{t2}) \quad (75)$$

It is obvious that if the following equation is satisfied

$$\mathcal{R}_1^* P + P\mathcal{R}_1 = -Q < 0, \quad (76)$$

then $\dot V(\mathbf{x}_{t1}, \mathbf{x}_{t2}) < 0$, $\forall \mathbf{x}_{ti} \ne \mathbf{0}$.

In the Lyapunov matrix eq. (63), of all possible solvents $\mathcal{R}_1$ only one of maximal solvents $\mathcal{R}_{1m}$ is of importance, because it contains the maximal eigenvalue $\lambda_m \in \Sigma$, which has dominant influence on the stability of the system.

If a solvent, which is not maximal, is integrated into Lyapunov eq. (63), it may happen that there will exist positive definite solution of this equation, although the system is not stable.

(*Necessary condition*) Let us assume that system (50) for $N=2$ is asymptotically stable, i.e. $\forall s \in \Sigma$, $\operatorname{Re} s < 0$ hold. Since $\sigma(\mathcal{R}_{1m}) \subset \Sigma$ follows $\operatorname{Re}\lambda(\mathcal{R}_{1m}) < 0$ (see *Corollary* 1) and the positive definite solution of Lyapunov matrix eq. (63) exists.

From (72-73) it follows

$$S_j A_{ji} = e^{\mathcal{R}_1 \tau_{ji}} S_i T_{ji}(0), \; S_1 = I_{n_1}, \; i=1,2, \; j=1,2 \quad (77)$$

Using (71) and (77), for $i=1$, we obtain (51).

Multiplying (71) (for $i=2$) from the left by the matrix S_2 and using (73) and (77) we obtain (52).

Taking a solvent with the eigenvalue $\lambda_m \in \Sigma$ (if it exists) as a solution of the system of eqs. (51-52), we arrive at the maximal solvent $\mathcal{R}_{1m}$. **Q.E.D.**

Theorem 9. Stojanovic, Debeljkovic (2005).

Suppose that there exists at least one maximal solvent of SME (61-62) and let $\mathcal{R}_{2m}$ denote one of them. Then, system (50), for $N=2$, is asymptotically stable if and only if for any matrix $Q=Q^* > 0$ there exists the matrix $P=P^*>0$ such that

$$\mathcal{R}_{2m}^* P + P \mathcal{R}_{2m} = -Q \quad (78)$$

Proof. The proof is almost identical to that given for *Theorem* 8. **Q.E.D.**

Conclusion 7. Consider a following linear continuous system without time delay

$$\dot{\mathbf{x}}(t) = \mathcal{R}_{im} \mathbf{x}(t) \quad (79)$$

where the matrix $\mathcal{R}_{im}$ is defined by SME (51-52), for $i=1$, or by SME (61-62), for $i=2$, respectively.

Applying *Theorem* 8 or *Theorem* 9, the investigation of the stability of large scale time delay system (50) reduces to investigating the stability of corresponding system (79) *without* delay.

The dimension of system (50) is infinite, while the dime-nsion of corresponding system (79) is finite and equals n_i.

Conclusion 8. The proposed criteria of stability are expressed in the form of necessary and sufficient conditions and as such do not possess conservatism unlike the existing sufficient criteria of stability.

Conclusion 9. To the authors' knowledge, in the available literature, there are no adequate numerical methods for direct computations of the maximal solvents $\mathcal{R}_{1m}$ or $\mathcal{R}_{2m}$. Instead, using various initial values for solvents $\mathcal{R}_i$, we determine $\mathcal{R}_{im}$ by applying minimization methods based on nonlinear least squares algorithms (see *Example* 5).

b) Large scale system with multiple subsystems

Theorem 10. Given the following system of matrix equations

$$\begin{aligned} &\mathcal{R}_k S_i - S_i A_i - \sum_{j=1}^{N} e^{-\mathcal{R}_k \tau_{ji}} S_j A_{ji} = 0 \\ &S_i \in \mathbf{C}^{n_k \times n_i}, \quad S_k = I_{n_k}, \quad 1 \le i \le N \end{aligned} \quad (80)$$

for a given k, $1 \le k \le N$, where A_i and A_{ji}, $1 \le i \le N$, $1 \le j \le N$ are the matrices of system (50) and τ_{ji} is time delay in the system.

If there is a solvent of (80) upon the unknown matrices $\mathcal{R}_k \in \mathbf{C}^{n_k \times n_k}$ and S_i, $1 \le i \le N$, $i \ne k$, then the eigenvalues of the matrix $\mathcal{R}_k$ belong to a set of roots of the characteristic equation of system (50).

Proof. The proof of this theorem is a generalization of proof of *Theorem* 6 or *Theorem* 7. **Q.E.D.**

Theorem 11. Suppose that there exists at least one maximal solvent of (80) *for the given* k, $1 \le k \le N$ and let $\mathcal{R}_{km}$ denote one of them.

Then, linear discrete large scale time delay system (50) is asymptotically stable if and only if for any matrix $Q=Q^*>0$ there exists the matrix $P=P^*>0$ such that

$$\mathcal{R}_{km}^* P \mathcal{R}_{km} - P = -Q \quad (81)$$

Proof. The proof is based on generalization of the proof for *Theorem* 8 and *Theorem* 9.

It is sufficient to take arbitrary *N*instead of $N=2$.**Q.E.D.**

Numerical example

Example 5. Consider the following continuous large scale time delay system with delay interconnections

$$\begin{aligned} \dot{x}_1(t) &= A_1 x_1(t) + A_{12} x_2(t-\tau_{12}) \\ \dot{x}_2(t) &= A_2 x_2(t) + A_{21} x_1(t-\tau_{21}) + A_{23} x_3(t-\tau_{23}) \\ \dot{x}_3(t) &= A_3 x_3(t) + A_{31} x_1(t-\tau_{31}) + A_{32} x_2(t-\tau_{32}) \end{aligned} \quad (82)$$

$$A_1 = \begin{bmatrix} -6 & 2 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & -10.9 \end{bmatrix}, \; A_{12} = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 0 & 3 \\ -2 & 1 & 2 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1.87 & 4.91 & 10.30 \\ -2.23 & -16.51 & -24.11 \\ 1.87 & -3.91 & -10.30 \end{bmatrix}, A_{21} = \begin{bmatrix} -1 & 0 & -2 \\ 3 & 0 & 5 \\ 1 & 0 & 2 \end{bmatrix}$$

$$A_{23} = \begin{bmatrix} -1 & -1 \\ 3 & 2 \\ 1 & 1 \end{bmatrix}, \; A_3 = \begin{bmatrix} -18.5 & -17.5 \\ -13.5 & -18.5 \end{bmatrix},$$

$$A_{31} = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \end{bmatrix}, \; A_{32} = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 2 & 0 \end{bmatrix}.$$

Applying *Theorem* 10 to the given system, for $k=1$, the following SME is obtained

$$\begin{aligned} &\mathcal{R}_1 - A_1 - e^{-\mathcal{R}_1 \tau_{21}} S_2 A_{21} - e^{-\mathcal{R}_1 \tau_{31}} S_3 A_{31} = 0 \\ &\mathcal{R}_1 S_2 - S_2 A_2 - e^{-\mathcal{R}_1 \tau_{12}} A_{12} - e^{-\mathcal{R}_1 \tau_{32}} S_3 A_{32} = 0 \\ &\mathcal{R}_1 S_3 - S_3 A_3 - e^{-\mathcal{R}_1 \tau_{23}} S_2 A_{23} = 0 \end{aligned} \quad (82)$$

If for pure system time delays we adopt the following values: $\tau_{12}=1$, $\tau_{21}=1$, $\tau_{23}=1$, $\tau_{31}=1$ and $\tau_{32}=1$, by applying the nonlinear least squares algorithms, we obtain a great number of solutions upon $\mathcal{R}_1$ which satisfy SME (83).

Among those solutions is a maximal solution:

$$\mathcal{R}_{1m} = \begin{bmatrix} -1.6105 & -3.3299 & -8.7623 \\ -6.8446 & -23.2023 & -67.2638 \\ 2.8542 & 8.4472 & 21.1500 \end{bmatrix}$$

and its belonging transformational matrix:

$$S_2 = \begin{bmatrix} 18.42 & 2.33 & 14.44 \\ -3.99 & 1.76 & -6.13 \\ -0.17 & -1.20 & 0.45 \end{bmatrix}, \; S_3 = \begin{bmatrix} 0.44 & -0.78 \\ -0.80 & -0.97 \\ 0.58 & 0.39 \end{bmatrix}$$

The eigenvalues of the matrix $\mathcal{R}_{1m}$ amount to: $\lambda_1 = -0.5059$, $\lambda_{2,3} = -1.5785 \pm j\,8.8824$, wherefrom it follows that the maximal eigenvalue of the given system is $\lambda_m = \lambda_1$.

To check the obtained value for λ_m, from the characteristic equation of system (82), by applying minimization methods, we arrived at the identical value for λ_m. For initial guesses λ_m values were taken from a set of complex numbers with a large real part in order to detect the maximal eigenvalue λ_m of the given system.

Since $\operatorname{Re}\lambda_m < 0$, based on *Theorem* 11, a considered large scale time delay system is asymptotically stable.

If now for pure time delay we adopt the following values: $\tau_{12} = 5$, $\tau_{21} = 2$, $\tau_{23} = 4$, $\tau_{31} = 5$ and $\tau_{32} = 3$, by using the identical procedure as in the previous case, we arrive at the following value for the maximal solvent:

$$\mathcal{R}_{1m} = \begin{bmatrix} -0.0484 & -0.0996 & 0.0934 \\ 0.2789 & -0.3123 & 0.2104 \\ 1.1798 & -1.1970 & -0.3798 \end{bmatrix}$$

The eigenvalues of the matrix $\mathcal{R}_{1m}$ amount to: $\lambda_1 = -0.2517$, $\lambda_{2,3} = -0.2444 \pm j\,0.3726$.

Therefore, for a maximal eigenvalue λ_m one of the values from the set $\{\lambda_2, \lambda_3\}$ can be adopted.

Based on *Theorem* 11, it follows that the large scale time delay system is *asymptotically stable.*

DISCRETE LARGE SCALE TIME DELAY SYSTEMS

In the sequel we will established new necessary and sufficient conditions for the asymptotic stability of a particular class of large-scale linear discrete time-delay systems.

The time-dependent criteria are derived by Lyapunov's direct method and are based on the exact solution of a particular system of monic matrix polynomial equations. It has been demonstrated that with large time delays of the system and a great number of subsystems *N*, a decrease in the number of computations is to be expected in favor of the derived stability criteria compared with a traditional procedure of examining the stability by eigenvalues of the equivalent matrix $\mathcal{A}$.

In order to make the results of this work more applicable in practice, some proposals for more appropriate numerical methods of determining the maximal solvent $\mathcal{R}_{\ell m}$ should be made.

Introduction

A large-scale dynamic system with time delay can usually be characterized by a large number of state variables and complex interaction between subsystems. Recently, the stability and stabilization problem of large-scale systems with delays has been considered by Lee, Radovic (1987, 1988), Hu (1994), Trinh, Alden (1995.b), Xu (1995), Huang, et al. (1995), Lee, Hsien (1997), Wang , Mau (1997) and Park (2002).

Most related works treated the stabilization problem in the continuous-time case. Since most modern control systems are controlled by a digital computer, it is natural to deal with the problem in a discrete-time domain.

The majority of stability conditions in the available literature, of both continual and discrete time-delay systems, are sufficient conditions independent of time delay. Only a small number of works provide both necessary and sufficient conditions which are in their nature mainly dependent on time delay.

Basic inspiration for our investigation is based on the paper Lee, Diant (1981). In this paper the necessary and sufficient conditions of linear continuous systems with one delay have been derived. Necessary and sufficient conditions for the asymptotic stability of linear discrete large-scale systems with multiple delays are considered herein. The obtained conditions of stability are derived by Lyapunov's direct method. But first it is necessary to solve the system of matrix polynomial equations (SMPE) upon an appropriate matrix integrated into discrete Lyapunov equation.

The obtained conditions of stability are not conservative in a traditional sense like the majority of results reported in the available literature. In case that a solution of SMPE exists, it is always possible by using those results to find out if the system is stable or not. The mentioned restriction (solubility of SMPE) can be taken as a conditional (non-classical) conservativeness.

Compared to the traditional method of investigating the stability by the equivalent matrix of the system (see *Lemma* 1), the advantage of this method is in a lower number of numerical computations and its disadvantage is in the impossibility of applying this method in the situations when there is no adequate solution of SMPE.

Preliminaries

Consider a large-scale linear discrete time-delay systems composed of N interconnected S_i.

Each subsystem S_i, $1 \le i \le N$ is described as

$$\mathsf{S}_i: \; \mathbf{x}_i(k+1) = A_i \mathbf{x}_i(k) + \sum_{j=1}^{N} A_{ij} \mathbf{x}_j\left(k - h_{ij}\right) \tag{83}$$

with an associated function of initial state

$$\mathbf{x}_i(\theta) = \boldsymbol{\psi}_i(\theta), \quad \theta \in \left\{-h_{m_i}, -h_{m_i}+1\,, \ldots, 0\right\} \tag{84}$$

where $\mathbf{x}_i(k) \in \mathbb{R}^{n_i}$ is state vector, $A_i \in \mathbb{R}^{n_i \times n_i}$ denotes the system matrix, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ represents the interconnection matrix between the *i*-th and the *j*-th subsystems and the constant delay $h_{ij} \in \mathbb{T}^+$.

In the following lemma necessary and sufficient condition for the asymptotic stability of system (83) has been given, expressed via eigenvalues the so-called *equivalent matrix* $\mathcal{A}$. This condition is based upon the fact that the observed system is finite-dimensional. The order of this system is very high and time delay dependent.

Lemma 7. System (83) will be asymptotically stable if and only if

$$\rho(\mathcal{A}) < 1 \tag{85}$$

holds, where the matrix

$$\begin{aligned} &\mathcal{A}=\left[\mathcal{A}_{ij}\right]\in\mathbb{R}^{N_e\times N_e}, \\ &N_e=\sum_{i=1}^{N} n_i\left(h_{m_i}+1\right),\quad h_{m_i}=\max_j h_{ji} \end{aligned} \tag{86}$$

is defined in the following way

$$\begin{array}{c} \begin{array}{cccccccc} 1 & & \cdots & h_{ii}+1 & & & & \\ \downarrow & & & \downarrow & & & & \end{array} \\ \mathcal{A}_{ii}=\left[\begin{array}{cccccc|c} A_i & 0 & \cdots & A_{ii} & \cdots & 0 & 0 \\ \hline I_{n_i} & 0 & \cdots & 0 & \cdots & 0 & 0 \\ 0 & I_{n_i} & \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & I_{n_i} & 0 \end{array}\right]\in\mathbb{R}^{n_i\left(h_{m_i}+1\right)\times n_i\left(h_{m_i}+1\right)} \end{array} \tag{87}$$

$$\begin{array}{c} \begin{array}{ccccc} 1 & \cdots & h_{ij}+1 & & \\ \downarrow & & \downarrow & & \end{array} \\ \mathcal{A}_{ij}=\left[\begin{array}{ccccc} 0 & \cdots & A_{ij} & \cdots & 0 \\ \hline 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{array}\right]\in\mathbb{R}^{n_i\left(h_{m_i}+1\right)\times n_j\left(h_{m_j}+1\right)} \end{array} \tag{88}$$

where A_i and A_{ij}, $1\le i\le N$, $1\le j\le N$, are the matrices of system (83).

Proof. It is not difficult to demonstrate that system (83) can be given in the following equivalent form

$$\begin{aligned} &\hat{\mathbf{x}}(k+1)=\mathcal{A}\,\hat{\mathbf{x}}(k) \\ &\hat{\mathbf{x}}(k)=\left[\,\hat{\mathbf{x}}_1^T(k)\;\;\hat{\mathbf{x}}_2^T(k)\;\;\cdots\;\;\hat{\mathbf{x}}_N^T(k)\right]^T \\ &\hat{\mathbf{x}}_i(k)=\left[\,\mathbf{x}_i^T(k)\;\;\mathbf{x}_i^T(k-1)\;\;\cdots\;\;\mathbf{x}_i^T\left(k-h_{m_i}\right)\right]^T \\ &1\le i\le N \end{aligned} \tag{89}$$

wherefrom a given condition for asymptotic stability follows directly. **Q.E.D.**

Main results

Using Lyapunov's direct method, necessary and sufficient stability time-delay dependent conditions for system (83), are derived.

Prior to it, we demonstrate that the spectrum of the matrix, which is integrated into Lyapunov equation, is a subset of the spectrum of the matrix $\mathcal{A}$, i.e. a set of characteristic roots of system (83).

Theorem 12. Given the following system of monic matrix polynomial equations (SMPE)

$$\begin{aligned} &\mathcal{R}_\ell^{\,h_{m_i}+1}S_i-\mathcal{R}_\ell^{\,h_{m_i}}S_iA_i-\sum_{j=1}^{N}\mathcal{R}_\ell^{\,h_{m_i}-h_{ji}}S_jA_{ji}=0 \\ &S_i\in\mathbb{C}^{n_\ell\times n_i},\quad S_\ell=I_{n_\ell},\quad h_{m_i}=\max_j h_{ji},\quad 1\le i\le N \end{aligned} \tag{90}$$

for a given ℓ, $1\le\ell\le N$, where A_i and A_{ji}, $1\le i\le N$, $1\le j\le N$ are matrices of system (83) and h_{ji} is time delay in the system.

If there is a solution of SMPE (90) upon the unknown matrices $\mathcal{R}_\ell\in\mathbb{C}^{n_\ell\times n_\ell}$ and S_i, $1\le i\le N$, $i\ne\ell$, then $\lambda(\mathcal{R}_\ell)\subset\lambda(\mathcal{A})$ holds, where the matrix $\mathcal{A}$ is defined by (86-88).

Proof. By introducing the time-delay operator z^{-h}, system (83) can be expressed in the following form

$$\begin{aligned} &\mathbf{x}(k+1)=A_e(z)\,\mathbf{x}(k), \\ &A_e(z)=\begin{bmatrix} A_1+A_{11}z^{-h_{11}} & A_{12}z^{-h_{12}} & \cdots & A_{1N}z^{-h_{1N}} \\ A_{21}z^{-h_{21}} & A_2+A_{22}z^{-h_{22}} & \cdots & A_{2N}z^{-h_{2N}} \\ \vdots & \vdots & \ddots & \vdots \\ A_{N1}z^{-h_{N1}} & A_{N2}z^{-h_{N2}} & \cdots & A_N+A_{NN}z^{-h_{NN}} \end{bmatrix} \end{aligned}$$

$$\mathbf{x}(k)=\left[\mathbf{x}_1^T(k)\;\;\mathbf{x}_2^T(k)\;\;\cdots\;\;\mathbf{x}_N^T(k)\right]^T \tag{91}$$

Let us form the following matrix.

$$\begin{aligned} F(z)&=zI_{N_e}-A_e(z)=\begin{bmatrix} F_{11}(z) & F_{12}(z) & \cdots & F_{1N}(z) \\ F_{21}(z) & F_{22}(z) & \cdots & F_{2N}(z) \\ \vdots & \vdots & \ddots & \vdots \\ F_{N1}(z) & F_{N2}(z) & \cdots & F_{NN}(z) \end{bmatrix} \\ &=\begin{bmatrix} zI_{n_1}-A_1-A_{11}z^{-h_{11}} & -A_{12}z^{-h_{12}} & \cdots & -A_{1N}z^{-h_{1N}} \\ -A_{21}z^{-h_{21}} & zI_{n_2}-A_2-A_{22}z^{-h_{22}} & \cdots & -A_{2N}z^{-h_{2N}} \\ \vdots & \vdots & \ddots & \vdots \\ -A_{N1}z^{-h_{N1}} & -A_{N2}z^{-h_{N2}} & \cdots & zI_{n_N}-A_N-A_{NN}z^{-h_{NN}} \end{bmatrix} \end{aligned} \tag{92}$$

If we add to the arbitrarily chosen ℓ - th block row of this matrix the rest of its block rows previously multiplied from the left by the matrices $S_j\ne 0$, $1\le j\le N$, $j\ne\ell$ respectively, we obtain

$$\begin{aligned} &\det F(z)= \\ &=\det\begin{bmatrix} F_{11}(z) & F_{12}(z) & \cdots & F_{1N}(z) \\ \vdots & \vdots & \cdots & \vdots \\ F_{\ell 1}(z)+\sum\limits_{\substack{j=1\\ j\ne\ell}}^{N}S_jF_{j1}(z) & F_{\ell 2}(z)+\sum\limits_{\substack{j=1\\ j\ne\ell}}^{N}S_jF_{j2}(z) & \cdots & F_{\ell N}(z)+\sum\limits_{\substack{j=1\\ j\ne\ell}}^{N}S_jF_{jN}(z) \\ \vdots & \vdots & \ddots & \vdots \\ F_{N1}(z) & F_{N2}(z) & \cdots & F_{NN}(z) \end{bmatrix} \end{aligned} \tag{93}$$

After multiplying i-th of the block column, $1\le i\le N$, of the preceding matrix by $z^{h_{m_i}}$ and after integrating the matrix $S_\ell=I_{n_\ell}$, the determinant of the matrix $F(z)$ equals

$$\begin{aligned} \det F(z)&=z^{-\sum_{i=1}^{N}n_ih_{m_i}}\det\begin{bmatrix} z^{h_{m_1}}F_{11}(z) & \ldots & z^{h_{m_N}}F_{1N}(z) \\ \vdots & \vdots & \vdots \\ z^{h_{m_1}}\sum\limits_{j=1}^{N}S_jF_{j1}(z) & \ldots & z^{h_{m_N}}\sum\limits_{j=1}^{N}S_jF_{jN}(z) \\ \vdots & \ddots & \vdots \\ z^{h_{m_1}}F_{N1}(z) & \ldots & z^{h_{m_N}}F_{NN}(z) \end{bmatrix}= \\ &=z^{-\sum_{i=1}^{N}n_ih_{m_i}}\det\begin{bmatrix} G_{11}(z) & G_{12}(z) & \ldots & G_{1N}(z) \\ \vdots & \vdots & \vdots & \vdots \\ G_{\ell 1}(z,S) & G_{\ell 2}(z,S) & \ldots & G_{\ell N}(z,S) \\ \vdots & \vdots & \vdots & \vdots \\ G_{N1}(z) & G_{N2}(z) & \ldots & G_{NN}(z) \end{bmatrix}= \\ &=z^{-\sum_{i=1}^{N}n_ih_{m_i}}\det G(z,S),\quad S=\{S_1,\ldots,S_N\} \end{aligned} \tag{94}$$

The $\boldsymbol{\ell}$-th block row of the $N\times N$ block matrix $G(z,S)$ is defined by

$$G_{\ell i}\left(z,S\right)=z^{h_{mi}+1}S_i-z^{h_{mi}}S_iA_i-\sum_{j=1}^{N}z^{h_{mi}-h_{ji}}S_jA_{ji}, \quad 1\le i\le N, \quad S_\ell=I_{n_\ell} \tag{95}$$

The relation (93) was obtained by applying a finite sequence of elementary row operations of type 3 over the matrix $F(z)$, Lancaster, Tismenetsky (1985).

The mentioned sequence of elementary row operations can be expressed in an equivalent form by the following nonsingular matrix

$$E_\ell=\begin{bmatrix} I_{n_1} & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ S_1 & \cdots & S_\ell & \cdots & S_N \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & \cdots & 0 & \cdots & I_{n_N}\end{bmatrix}\begin{array}{l}\leftarrow 1\\ \vdots \\ \leftarrow \ell \\ \vdots \\ \leftarrow N\end{array} \tag{96}$$

that multiplies the matrix $F(z)$ from the left.

The transformation matrices $S_1,\cdots,S_N$, with the exception of the matrix $S_\ell=I_{n_\ell}$, are unknown for the time being, but in the further text a condition will be derived that the unknown matrices are determined upon.

The characteristic polynomial of system (83), Gorecki et al. (1989)

$$g(z)\hat{=}\det G(z,S)=\sum_{j=0}^{N_e}a_jz^j \tag{97}$$

where

$$N_e=\sum_{i=1}^{N}n_i\left(h_{m_i}+1\right), a_j\in\mathbb{R}, 0\le j\le N_e \tag{98}$$

does not depend on the choice of the transformation matrices $S_1,\cdots,S_N$), Lancaster, Tismenetsky (1985).

Let us denote

$$\Sigma\hat{=}\left\{\,z\mid g(z)=0\right\} \tag{99}$$

a set of all characteristic roots of system (83).

This set of roots equals the set $\lambda(\mathcal{A})$.

Substituting a scalar variable z by the matrix $X\in\mathbb{C}^{n_\ell\times n_\ell}$ in $G(z,S)$, a new block matrix is obtained $G(X,S)$.

If there exist the transformation matrices S_i, $1\le i\le N$, $i\ne\ell$ and the solvent $\mathcal{R}_\ell\in\mathbb{C}^{n_\ell\times n_\ell}$ such that for the ℓ-th block row of $G(X,S)$ holds

$$G_{\ell i}\left(\mathcal{R}_\ell,S\right)=0,\quad 1\le i\le N \tag{100}$$

i.e. holds (91), then
and

$$g\left(\mathcal{R}_\ell\right)=0 \tag{101}$$

Therefore, the characteristic polynomial of system (83) is an annihilating polynomial for the square matrix $\mathcal{R}_\ell$ and $\lambda(\mathcal{R}_\ell)\subset\Sigma$ holds. The mentioned assertion holds $\forall\ell,\ 1\le\ell\le N$. **Q.E.D.**

Definition 7. The matrix $\mathcal{R}_\ell$ is referred to as a *solvent* of eq. (90) for the given ℓ, $1\le\ell\le N$.

From (90) for the given ℓ, $1\le\ell\le N$, the transformation matrices S_j $1\le j\le N$ and the solvent $\mathcal{R}_\ell$ are computed, the latter being used further for examining the stability of system (83).

Definition 8. The characteristic root λ_m of system (83) with the maximal module:

$$\lambda_m\in\Sigma:\ \left|\lambda_m\right|=\max\left|\Sigma\right|=\max_i\left|\lambda_i\left(\mathcal{A}\right)\right| \tag{102}$$

will be referred to as the *maximal root* (*eigenvalue*) of system (83).

Definition 9. Each solvent $\mathcal{R}_{\ell m}$ of SMPE (90), for the given ℓ, $1\le\ell\le N$, the spectrum of which contains the maximal eigenvalue λ_m of system (83), is referred to as the *maximal solvent* of (90).

Theorem 13. Stojanovic, Debeljkovic (2008.a).

Suppose that there exist at least one ℓ, $1\le\ell\le N$, that there exists *at least one* maximal solvent of SMPE (90) and let $\mathcal{R}_{\ell m}$ denote one of them. Then, linear discrete large-scale time-delay system (83) is asymptotically stable if and only if for any matrix $Q=Q^*>0$ there exists the matrix $P=P^*>0$ such that

$$\mathcal{R}_{\ell m}^*\,P\,\mathcal{R}_{\ell m}-P=-Q \tag{103}$$

Proof. Define the following vector discrete functions

$$\begin{aligned}&\mathbf{x}_{ki}=\mathbf{x}_i\left(k+\theta\right),\\ &\theta\in\left\{-h_{m_i},-h_{m_i}+1,\ \ldots\ ,0\right\},\quad 1\le i\le N\end{aligned} \tag{104}$$

$$\begin{aligned}&\mathbf{v}\left(\mathbf{x}_{k1},\cdots,\mathbf{x}_{kN}\right)\\ &=\sum_{i=1}^{N}S_i\left[\mathbf{x}_i\left(k\right)+\sum_{j=1}^{N}\sum_{l=1}^{h_{ji}}T_{ji}\left(l\right)\mathbf{x}_i\left(k-l\right)\right]\end{aligned} \tag{105}$$

where $T_{ji}(k)\in\mathbb{C}^{n_i\times n_i}$, $1\le j\le N$, $1\le i\le N$ are, in general, some time-varying discrete matrix functions and $S_\ell=I_{n_\ell}$, $S_i\in\mathbb{C}^{n_\ell\times n_i}$, $1\le i\le N$, $i\ne\ell$.

The conclusion of the theorem follows immediately by defining Lyapunov functional for system (83) as

$$\begin{aligned}&V\left(\mathbf{x}_{k1},\cdots,\mathbf{x}_{kN}\right)=\mathbf{v}^*\left(\cdot,\cdots,\cdot\right)P\,\mathbf{v}\left(\cdot,\cdots,\cdot\right),\\ &P=P^*>0\end{aligned} \tag{106}$$

It is obvious that $V(\cdot,\cdots,\cdot)>0$ for $\forall\mathbf{x}_{ki}\ne\mathbf{0}$, $1\le i\le N$.

The forward difference of (106), along the solutions of system (83) is

$$\begin{aligned}\Delta V\left(\cdot,\cdots,\cdot\right)&=\Delta\,\mathbf{v}^*\left(\cdot,\cdots,\cdot\right)P\,\mathbf{v}\left(\cdot,\cdots,\cdot\right)\\ &+\mathbf{v}^*\left(\cdot,\cdots,\cdot\right)P\,\Delta\,\mathbf{v}\left(\cdot,\cdots,\cdot\right)\\ &+\Delta\,\mathbf{v}^*\left(\cdot,\cdots,\cdot\right)P\,\Delta\,\mathbf{v}\left(\cdot,\cdots,\cdot\right)\end{aligned} \tag{107}$$

A difference of $\mathbf{v}(\cdot,\cdots,\cdot)$ can be determined in the following manner

$$\begin{aligned}&\Delta\,\mathbf{v}\left(\cdot,\cdots,\cdot\right)=\\ &\sum_{i=1}^{N}S_i\left[\Delta\,\mathbf{x}_i\left(k\right)+\sum_{j=1}^{N}\sum_{l=1}^{h_{ji}}T_{ji}\left(l\right)\Delta\,\mathbf{x}_i\left(k-l\right)\right]\end{aligned} \tag{108}$$

with

$$\Delta \mathbf{x}_i(k) = \left(A_i - I_{n_i}\right)\mathbf{x}_i(k) + \sum_{j=1}^{N} A_{ij}\mathbf{x}_j\left(k - h_{ij}\right) \quad (109)$$

and

$$\begin{aligned} \sum_{j=1}^{N}\sum_{l=1}^{h_{ji}} T_{ji}(l)\, \Delta \mathbf{x}_i(k-l) &= \sum_{j=1}^{N} T_{ji}(1)\mathbf{x}_i(k) \\ &+ \sum_{j=1}^{N} T_{ji}\left(h_{ji}\right)\mathbf{x}_i\left(k-h_{ji}\right) \\ &+ \sum_{j=1}^{N}\sum_{l=1}^{h_{ji}-1} \Delta T_{ji}(l)\mathbf{x}_i(k-l) \end{aligned} \quad (110)$$

Then

$$\Delta \mathbf{v}(\cdot,\cdots,\cdot) = \sum_{i=1}^{N} S_i \begin{bmatrix} \left(A_i - I_{n_i} + \sum_{j=1}^{N} T_{ji}(1)\right)\mathbf{x}_i(k) \\ + \sum_{j=1}^{N} T_{ji}\left(h_{ji}\right)\mathbf{x}_i\left(k-h_{ji}\right) \\ + \sum_{j=1}^{N}\sum_{l=1}^{h_{ji}-1} \Delta T_{ji}(l)\mathbf{x}_i(k-l) \\ + \sum_{j=1}^{N} A_{ij}\mathbf{x}_j\left(k-h_{ij}\right) \end{bmatrix} \quad (111)$$

It is obvious that for the last member in the sum (111) holds

$$\begin{aligned} \sum_{i=1}^{N}\sum_{j=1}^{N} S_i A_{ij}\mathbf{x}_j\left(k-h_{ij}\right) &= \sum_{j=1}^{N}\sum_{i=1}^{N} S_i A_{ij}\mathbf{x}_j\left(k-h_{ij}\right) = \\ &= \sum_{i=1}^{N}\sum_{j=1}^{N} S_j A_{ji}\mathbf{x}_i\left(k-h_{ji}\right) \end{aligned} \quad (112)$$

and if we define new matrices

$$\mathcal{R}_i = A_i + \sum_{j=1}^{N} T_{ji}(1), 1 \le i \le N \quad (113)$$

then $\Delta \mathbf{v}(\cdot,\cdots,\cdot)$ has a form

$$\begin{aligned} \Delta \mathbf{v}(\cdot,\cdots,\cdot) &= \sum_{i=1}^{N} S_i\left(\mathcal{R}_i - I_{n_i}\right)\mathbf{x}_i(k) + \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N}\left(S_j A_{ji} - S_i T_{ji}\left(h_{ji}\right)\right)\mathbf{x}_i\left(k-h_{ji}\right) + \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{l=1}^{h_{ji}-1} S_i \Delta T_{ji}(l)\mathbf{x}_i(k-l) \end{aligned} \quad (114)$$

If

$$\begin{gathered} S_j A_{ji} - S_i T_{ji}\left(h_{ji}\right) = S_i \Delta T_{ji}\left(h_{ji}\right) \\ 1 \le i \le N, \quad 1 \le j \le N \end{gathered} \quad (115)$$

then

$$\begin{aligned} \Delta \mathbf{v}(\cdot,\cdots,\cdot) &= \sum_{i=1}^{N}\Big[S_i\left(\mathcal{R}_i - I_{n_i}\right)_i \mathbf{x}_i(k) \\ &+ \sum_{j=1}^{N}\sum_{l=1}^{h_{ji}} S_i \Delta T_{ji}(l)\mathbf{x}_i(k-l)\Big] \end{aligned} \quad (116)$$

If one adopts

$$S_i\left(\mathcal{R}_i - I_{n_i}\right) = \left(\mathcal{R}_\ell - I_{n_\ell}\right)S_i, \quad 1 \le i \le N \quad (117)$$

$$\begin{gathered} S_i \Delta T_{ji}(l) = \left(\mathcal{R}_\ell - I_{n_\ell}\right)S_i T_{ji}(l) \\ 1 \le i \le N, \quad 1 \le j \le N \end{gathered} \quad (118)$$

then

$$\Delta \mathbf{v}(\cdot,\cdots,\cdot) = \left(\mathcal{R}_\ell - I_{n_\ell}\right)\mathbf{v}(\cdot,\cdots,\cdot) \quad (119)$$

and

$$\Delta V(\cdot,\cdots,\cdot) = \mathbf{v}^*(\cdot,\cdots,\cdot)\left(\mathcal{R}_\ell^{\,*} P \mathcal{R}_\ell - P\right)\mathbf{v}(\cdot,\cdots,\cdot) \quad (120)$$

It is obvious that if the following equation is satisfied

$$\mathcal{R}_\ell^{\,*} P \mathcal{R}_\ell - P = -Q, \qquad Q = Q^* > 0 \quad (121)$$

then $\Delta V(\cdot,\cdots,\cdot) < 0$, $\forall \mathbf{x}_{ki} \ne \mathbf{0}$, $1 \le i \le N$.

In Lyapunov matrix eq. (103), of all possible solvents $\mathcal{R}_\ell$ of (90), only one of maximal solvents $\mathcal{R}_{\ell m}$ is of importance, for it is the only one that contains the maximal eigenvalue $\lambda_m \in \Sigma$ (*Definition* 9), which has dominant influence on the stability of the system. If a solvent which is not maximal is integrated into Lyapunov eq. (103), it may happen that there will exist a positive definite solution of this equation, although the system is not stable.

Accordingly, condition (103) represents the *sufficient condition* of the stability of system (83).

If it exists, the maximal solvent $\mathcal{R}_{\ell m}$ can be determined in the following way.

From (115) and (118) we obtain

$$\begin{gathered} S_j A_{ji} = \mathcal{R}_\ell^{\,h_{ji}} S_i T_{ji}(1) \\ S_\ell = I_{n_\ell}, \quad 1 \le i \le N, \quad 1 \le j \le N \end{gathered} \quad (122)$$

Multiplying i-th equation of the system of matrix eq. (113) from the left by the matrix $\mathcal{R}_\ell^{\,h_{mi}} S_i$ and using (117) and (122), we obtain eq. (90). Taking a solvent with the eigenvalue $\lambda_m \in \Sigma$ (if it exists) as a solution of the system of eq. (90), we arrive at the maximal solvent $\mathcal{R}_{\ell m}$.

Conversely, if system (84) is asymptotically stable, then $\forall \lambda_i \in \Sigma$, $|\lambda_i| < 1$. Since $\lambda(\mathcal{R}_{\ell m}) \subset \Sigma$, it follows that $\rho(\mathcal{R}_{\ell m}) < 1$, therefore the positive definite solution of Lyapunov matrix eq. (103) exists (*necessary condition*). **Q.E.D**

Corollary 4. Suppose that for the given ℓ, $1 \le \ell \le N$, there exists the matrix $\mathcal{R}_\ell$ being a solution of SMPE (90).

If system (84) is asymptotically stable, then the matrix $\mathcal{R}_\ell$ is discrete stable ($\rho(\mathcal{R}_\ell) < 1$).

Proof. If system (83) is asymptotically stable, then $\forall z \in \Sigma |\, z\,| < 1$. Since $\lambda(\mathcal{R}_\ell) \subset \Sigma$, it follows that $\forall \lambda \in \lambda(\mathcal{R}_\ell)$, $|\lambda| < 1$, i.e. the matrix $\mathcal{R}_\ell$ is discrete stable. **Q.E.D.**

Conclusion 10. To the authors' knowledge in the available literature there are no adequate numerical methods for direct computations of maximal solvents of SMPE of type (90). One can arrive at each individual solution for the mentioned equations by applying minimization methods that require initial guesses. In addition, the convergence of those solutions is directly dependent on initial guesses.

By analogy to conditions for the existence and enumeration of solvents of matrix polynomials given in literature, it is to be expected that the number of solvents of (91) can be zero, finite or infinite.

Conclusion 11. It follows from the aforementioned, that it makes no difference which of the matrices $\mathcal{R}_{\ell m}$, $1 \le \ell \le N$ we use for examining the asymptotic stability of system (83). The only condition is that there exists at least one matrix for at least one ℓ. Otherwise, it is impossible to apply *Theorem* 13.

Conclusion 12. The dimension of system (83) amounts to $N_e = \sum_{j=1}^{N} n_j \left(h_{m_j} + 1\right)$. Conversely, if there exists a maximal solvent, the dimension of $\mathcal{R}_{\ell m}$ is much smaller and amounts to n_ℓ. That is why our method is superior over a traditional procedure of examining the stability by the eigenvalues of the matrix $\mathcal{A}$.

The disadvantage of this method reflects in the probability that the obtained solution need not be a maximal solvent and it cannot be known ahead if a maximal solvent exists at all.

Hence the proposed methods are at present of greater theoretical than of practical significance.

Numerical example

Consider the large-scale linear discrete time-delay system consisting of three subsystems described by Lee, Radovic (1987)

$$\mathrm{S}_1:\ x_1(k+1) = A_1 x_1(k) + B_1 u_1(k) + A_{12} x_2(k - h_{12}),$$

$$\mathrm{S}_2:\ x_2(k+1) = A_2 x_2(k) + B_2 u_2(k) + A_{21} x_1(k - h_{21}) + A_{23} x_3(k - h_{23}),$$

$$\mathrm{S}_3:\ x_3(k+1) = A_3 x_3(k) + B_3 u_3(k) + A_{31} x_1(k - h_{31}),$$

with

$$A_1 = \begin{bmatrix} 0.8 & 0.6 \\ 0.4 & 0.9 \end{bmatrix},\ B_1 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix},\ A_{12} = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0 & 0.1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.7 & 0 & -0.5 \\ -0.1 & 6 & -0.1 \\ -0.6 & 1 & 0.8 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & -0.1 \\ 0.1 & 0.2 \\ 0 & 0.1 \end{bmatrix},$$

$$A_{21} = \begin{bmatrix} -0.1 & -0.2 \\ 0.3 & 0.1 \\ 0.1 & 0.2 \end{bmatrix},\ A_{23} = \begin{bmatrix} -0.1 & 0 \\ 0.2 & -0.2 \\ 0.1 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 0.1 \\ -0.1 & 0.8 \end{bmatrix},\ B_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},\ A_{31} = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}$$

The overall system is stabilized by employing a local memory-less state feedback control for each subsystem

$$\mathbf{u}_i(k) = K_i \mathbf{x}_i(k),$$

where for the matrix of decentralized gains is now adopted

$$K_1 = \begin{bmatrix} -6 & -7 \end{bmatrix},\ K_2 = \begin{bmatrix} -7 & -45 & 10 \\ 4 & -4 & -4 \end{bmatrix},\ K_3 = \begin{bmatrix} -5 & -1 \\ 1 & -4 \end{bmatrix}$$

Substituting the inputs into this system, we obtain the equivalent closed loop system representations

$$\mathrm{S}_i:\ \mathbf{x}_i(k+1) = \hat{A}_i\, \mathbf{x}_i(k) + \sum_{j=1}^{3} A_{ij} \mathbf{x}_j(k - h_{ij}),\quad 1 \le i \le 3$$

where $\hat{A}_i = A_i + B_i K_i$.

This closed loop system, in its type of model, corresponds to system (83), therefore it is possible to apply the previously obtained results for examining its stability.

For time delay in the system, let us adopt: $h_{12} = 5$, $h_{21} = 2$, $h_{23} = 4$ and $h_{31} = 5$.

Applying *Theorem* 12 to the given closed loop system, we obtain the following SMPE for $\ell = 1$

$$\mathcal{R}_1^{\,6} - \mathcal{R}_1^{\,5} \hat{A}_1 - \mathcal{R}_1^{\,3} S_2 A_{21} - S_3 A_{31} = 0,$$

$$\mathcal{R}_1^{\,6} S_2 - \mathcal{R}_1^{\,5} S_2 \hat{A}_2 - A_{12} = 0,$$

$$\mathcal{R}_1^{\,5} S_3 - \mathcal{R}_1^{\,4} S_3 \hat{A}_3 - S_2 A_{23} = 0.$$

Solving this SMPE by minimization methods, we obtain

$$\mathcal{R}_1 = \begin{bmatrix} 0.6001 & 0.3381 \\ 0.6106 & 0.3276 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0.0922 & 1.3475 & 0.5264 \\ 0.0032 & 1.3475 & 0.4374 \end{bmatrix},\ S_3 = \begin{bmatrix} 0.6722 & -0.3969 \\ 1.3716 & -1.0963 \end{bmatrix}$$

The eigenvalue with the maximal module of the matrix $\mathcal{R}_1$ equals 0.9382.

Since the eigenvalue λ_m of $\mathcal{A} \in \mathbb{R}^{40\times 40}$ also has the same value, we conclude that the solvent $\mathcal{R}_1$ is a maximal solvent ($\mathcal{R}_{1m} = \mathcal{R}_1$). Applying *Theorem* 13, we arrive at the condition $\rho(\mathcal{R}_{1m}) = 0.9382 < 1$ wherefrom we conclude that the observed closed loop large-scale time-delay system is asymptotically stable.

The difference in dimensions of matrices $\mathcal{R}_1 \in \mathbb{R}^{2\times 2}$ and $\mathcal{A} \in \mathbb{R}^{40\times 40}$ is rather high, even with relatively small time delays (the greatest time delay in our example is 5). So, in the case of great time delays in the system and a great number of subsystems N, by applying the derived results, a smaller number of computations are to be expected compared with a traditional procedure of examining the stability by eigenvalues of the matrix $\mathcal{A}$.

An accurate number of computations for each of the mentioned method require additional analysis, which is not the subject of this paper.

Conclusion

We have presented new, necessary and sufficient, conditions for the asymptotic stability of a particular class of linear continuous and discrete time delay systems. Moreover, these results have been extended to large scale systems covering the cases of two and multiple existing subsystems.

The time-dependent criteria were derived by Lyapunov's direct method and are exclusively based on the maximal and dominant solvents of particular matrix polynomial equations. It can be shown that these solvents exist only under certain conditions, which, in a sense, limits the applicability of the method proposed. The solvents can be calculated using generalized Traub`s or Bernoulli`s algorithms.

Both of them possess significantly smaller number of flops counts than the standard algorithm.

Improving the converging properties of the used algorithms for these purposes, may be a particular research topic in the future.

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Received: 16.03.2008.

Asimptotska stabilnost posebnih klasa linearnih sistema sa čistim vremenskim kašnjenjem: potpuno novi prilaz

U ovom radu izlažu se novi potrebni i dovoljni uslovi asimptotske stabilnosti posebne klase linearnih sistema sa čistim vremenskim kašnjenjem čije su vektorske diferencijalne jednačine stanja date sa: $\mathbf{x}(k+1)=A_0\mathbf{x}(k)+A_1\mathbf{x}(k-h)$ i $\dot{\mathbf{x}}(t)=A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau)$. U tom smislu izvedeni su uslovi koji uzimaju u obzir iznos čisto vremenskog kašnjenja a koristeći dobro poznatu tehniku Druge Ljapunovljeve metode. Dve matrične jednačine su izvedene i to: matrična polinomijalna jednačina i posebna kontinualna (diskretna) matrična jednačina Ljapunova. Takođe su date modifikacije postojećih dovoljnih uslova konvergencije Traub-ovog i Bernoilli-jevog algoritma za sračunavanje dominantnog solventa matričnog polinoma. Ovi su rezultati dalje prošireni na velike sisteme. Izloženi su i odgovarajući numerički primeri sa ciljem da se potkrepe i ilustruju dobijeni rezultati.

Ključne reči: **kontinualni sistem, diskretni sistem, linearni sistem, stabilnost sistema, asimptotska stabilnost, stabilnost Ljapunova, sistem sa kašnjenjem, vremensko kašnjenje.**

Устойчивость асимптоты линейных систем особого класса со чистой временной задержкой Новый подход

В настоящей работе выведены и представлены новые нужные и довольные условия асимптотической устойчивости особого класса линейных систем со чистой временной задержкой, чьи векториальные дифференциальные уравнения состояния представлены в форме: $\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-h)$ и $\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau)$. В том смысле выведены условия, учитывающие сумму чистой временной задержки и пользуясь хорошо известной техникой второго метода Ляпунова. Два матричных уравнения выведены, и то: матричное многочленное уравнение и особое непрерывное (дискретное) матричное уравнение Ляпунова.
Здесь тоже представлены модификации сущесвующих довольных условий сходимости алгорифмов Трауба и Бернулли для вычисления доминирующего солвента матричного полинома. Эти результаты дальше распространены на большие системы. Здесь тоже представлены и соответствующие численные примеры с целью усиления и иллюстрации полученых результатов.

Ключевые слова: **Непрерывная система, дискретная система, линейная система, устойчивость системы, устойчивость асимптоты, устойчивость Ляпунова, система со временной задержкой, временная задержка.**

La stabilité asymptotiques des classes particulières des systèmes linéaires à délai temporel pur: nouvelle approche

Dans ce papier on expose les nouvelles conditions, nécessaires et suffisantes, de la stabilité asymptotique de classe particulière des systèmes linéaires à délai temporel pur dont les équations différentielles vectorielles sont données par: $\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-h)$ et $\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau)$. Dans ce sens on a dérivé les conditions qui considèrent le délai temporel pur en utilisant la technique connue de la deuxième méthode de Lyapunov. On a donné aussi les modifications des conditions suffisantes de la convergence de l'algorithme de Traub et Bernoilli pour calculer le solvant dominant du polynôme de matrice. Ces résultats ont été ensuite appliqués aux grands systèmes. Les exemples numériques correspondants sont présentés pour illustrer les résultats obtenus.

Mots clés: **système continu, système discret, système linéaire, stabilité du système, stabilité asymptotique, stabilité de Lyapunov, système à délai, délai temporel.**