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Dynamic Analysis of the Stress and Strain State of the Spur Gear Pair

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A spur gear pair dynamic model for the gear dynamic contact loading, dynamic contact stress state and dynamic contact strain state analysis is presented. A dynamic model of the gear set with two degrees of freedom is used. The transmission is analyzed using the nonlinear finite elements method where a novel approach for interpreting the results of the stress and strain state using stress and/or strain tensor invariants is developed. For a more general approach, the software for the finite element analysis of the gear set as a whole is developed, using the open source finite elements framework CODE-ASTER/SALOME.

Key words: **stress and strain state, dynamic load, stress and strain state analysis, spur gears, deformation, stress and strain tensor, finite element method.**

Introduction

STARTING with one gear pair, a meshing action is observed using no additional assumptions concerning the gear geometry or the meshing action boundary and load conditions. A numeric experiment is developed simply by using the boundary conditions on the gear-shaft connection, which are assumed to equal zero. The software developed for the purpose of dynamic analysis of the gear-boxes can be used to obtain the following results from such a model:

a) forces that arise in gear contact during the gear meshing action as a result of the incremental quasi-static series of analysis, for each meshing action time step,
b) the comprehensive modal analysis for the gear pair, and the
c) forces obtained in a) will be used further as an input for the nonlinear dynamic analysis. For the purpose of this paper, it is assumed that the obtained force is a harmonic function and in the form:

$$F_n = F_0 \sin \omega t \qquad (1)$$

where F_0 is the resultant force for the observed meshing action time step.

d) Finally, the dynamic analysis of the gear meshing action is done[1]. The contact load as inner dynamic load is taken as an input value obtained in a) and approximated using the harmonic function (as it is described in c)).

The developed software can be modified for the purpose of calculation of different types of geared systems. It is optional which of the mentioned steps will be of the highest priority, and depends, of course, on the purpose of the concrete analysis. That is not of interest here, and therefore will not be discussed.

Finite element model of the spur gear pair

The gears analyzed in this paper are the ones from the deep drilling machine gear set [2,3,4]. The gear geometries are generated using the software developed by the author [9, 10,11] and adapted for the SALOME platform. The obtained geometries are transferred to SALOME mesh generation software and discretized using four node tetrahedral elements there (Fig.1).

Figure 1. Gear set model

The gear geometry parameters are given in Table 1. The mesh parameters of the geared system are:

– Number of nodes: 29777
– Number of elements: 160491
– Number of elements of gear 1 (smaller gear): 33370
– Number of elements of gear 2: 81037

The displacement boundary conditions (Fig.2) are set on the gear-shaft connections nodes. The displacement

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boundary conditions are set to be zero in all degrees of freedom.

The load boundary conditions are generated on the meshing teeth on the edges that are opposite to the contact surfaces of each tooth. The form of the load function is taken according to (1) with the amplitude obtained from the nonlinear static analysis. This analysis helps to determine the form of the amplitude force distribution in the gear mesh. The detailed procedure of how the force is taken from nonlinear static analysis will not be discussed in detail here. The force distribution on other transmission elements is calculated in the same way.

All other meshing conditions are not taken into consideration. The following assumptions are made:

- It is assumed that the contact is without friction
- There is no sliding between the teeth during the meshing action.
- The transmission is functioning in a constant temperature field. All other thermodynamic effects can be neglected.
- The proportional damping is used according to $\mathbf{C} = 2\beta\mathbf{M} + \alpha\mathbf{K}$
- The continuum discretization is done using the tetrahedral finite elements with the linear approximation of the finite element geometry.

Table 1. Constructive transmission parameters

Parameter	Gear 1	Gear 2
Number of teeth	38	65
Modul (m)	0.003	0.003
Pitch circle diameter (m)	0.114	0.195
Addendum circle diameter (m)	0.120	0.201
Base circle diameter (m)	0.1068	0.1878
Measurement teeth number (m)	5	6
Standard angle (degrees)	20	20
Axial distance (m)	0.129	0.129005

Figure 2. Displacement and force boundary conditions

Analytic model of the spur gear transmission

The gear set model in Fig.1 can be simplified using the transmission model in Fig.3 for dynamic analysis. This class of dynamic models can be used for the analysis of the oscillation parameters of the transmission, only if meshing functions of the stiffness and dumping are known. As this is not the case, it is necessary to experimentally obtain the form of the mesh stiffness of the geared mesh, or to use other means like finite elements method to obtain unknown stiffness/ dumping functions. The solution for the differential equations given in (2) is not possible without explicitly defined mesh stiffness and dumping functions, because of the fact that the equations are non-linear functions of time and the solution can be obtained only using numeric methods.

The differential equation of the model in Fig3 are:

$$\begin{aligned} m_1\ddot{x}_1 + k(t)(\dot{x}_1 - \dot{x}_2) + c(t)(x_1 - x_2) &= F_n(t) \\ m_2\ddot{x}_2 - k(t)(\dot{x}_1 - \dot{x}_2) - c(t)(x_1 - x_2) &= -F_n(t) \end{aligned} \quad (2)$$

where:

- m_1 (i=1,2) are equivalent masses of transmission elements,
- $c(t)$ is the gear mesh stiffness function depending on t,
- $k(t)$ is the gear mesh dumping function depending on t.

Shaft stiffness/dumping outside the gears on shaft 2 are not considered.

Stress and strain state analysis using stress and strain tensor invariants

The calculation of the component stresses and strains is done by the nonlinear Finite Element method using the already described procedure.

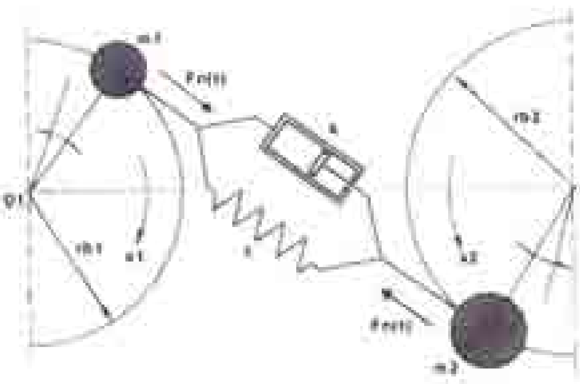

Figure 3. A simplified dynamic model of the gear set in Fig.1

Starting from the known component stresses as the components of tensor **N**, it is possible to obtain stress tensor invariants in every node after the finite elements analysis is done. Since:

$$f(\sigma_0) = |\mathbf{N} - \sigma_0\mathbf{I}| = \begin{vmatrix} \sigma_x - \sigma_0 & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y - \sigma_0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_0 \end{vmatrix} = 0 \quad (3)$$

yields:

$$-(\sigma_0^3 - L_1\sigma_0^2 + L_2\sigma_0 - L_3) = 0 \quad (4)$$

where

$$L_1 = f(\sigma_1, \sigma_2, \sigma_3) = \sigma_1 + \sigma_2 + \sigma_3 = tr(\mathbf{G}), \quad (5)$$

$$\begin{aligned} L_2 &= f(\sigma_1, \sigma_2, \sigma_3) = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3 \\ &= \frac{1}{2}\left(tr(\mathbf{G}^2) - tr(\mathbf{G})^2\right) \end{aligned} \quad (6)$$

$$L_3 = f(\sigma_1, \sigma_2, \sigma_3) = \sigma_1\sigma_2\sigma_3 = \det(\mathbf{G}), \quad (7)$$

and

$$\mathbf{G} = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix}, \tag{8}$$

is a principal stress tensor in the stress principal directions coordinate system at an arbitrary body point.

Using the secular equation in its determinant form (3) or its algebraic form (4) it is possible to obtain the principal stresses as functions of stress tensor invariants L_1, L_2 i L_3, using relations-expressions (5), (6) and (7) as a system of three nonlinear algebraic equations with unknown principal stresses. That yields the functional dependency between the principal stress and the stress invariants in the form: $\sigma_0 = f(L_1, L_2, L_3)$.

When the component stresses of the tensor are known, it is also possible to obtain three sets of three cosines of the main stress directions. The principal stress directions are directions that define the coordinate system of principal directions, where there are no shear stresses in the defined planes.

When the principal stresses are different, $\sigma_1 \neq \sigma_2 \neq \sigma_3$, the maximum values of the shear stresses are found in three normal planes, where the principal axis 1 and one of these planes are collinear, and the angles between the other principal planes and the mentioned plane are 45 degrees. The maximum values of the shear stresses are obtained in the following form:

$$\tau_{(I)} = \pm\frac{\sigma_2 - \sigma_3}{2}, \tau_{(II)} = \pm\frac{\sigma_2 - \sigma_1}{2}, \tau_{(III)} = \pm\frac{\sigma_1 - \sigma_2}{2}. \tag{9}$$

There are no shear stresses in the principal planes. On the other hand, in the maximal shear stress planes, there are not only extreme values of shear stresses, but also normal stress components with values given in the form:

$$\sigma_{(I)} = \frac{\sigma_2 + \sigma_3}{2}, \sigma_{(II)} = \frac{\sigma_1 + \sigma_3}{2}, \sigma_{(III)} = \frac{\sigma_1 + \sigma_2}{2}. \tag{10}$$

As already stated, the principal directions can be obtained from the component stresses using the following relations:

$$\frac{\alpha_0}{K_{31}} = \frac{\beta_0}{K_{32}} = \frac{\gamma_0}{K_{33}} = C_0 \tag{11}$$

or

$$\frac{\alpha_0}{\begin{vmatrix} \tau_{xy} & \tau_{xz} \\ \sigma_y - \sigma_0 & \tau_{yz} \end{vmatrix}} = \frac{\beta_0}{-\begin{vmatrix} \sigma_x - \sigma_0 & \tau_{xz} \\ \tau_{xy} & \tau_{yz} \end{vmatrix}}$$

$$= \frac{\gamma_0}{\begin{vmatrix} \sigma_x - \sigma_0 & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma_0 \end{vmatrix}} = C_0 \tag{12}$$

the system is defined if condition (13) is satisfied:

$$\alpha_0^2 + \beta_0^2 + \gamma_0^2 = 1 \tag{13}$$

The relations (11) and (12) should be taken for s =1,2,3. They give the equations for three main stress directions represented with three direction cosines, with total of nine direction cosines.

Depending on the value of the principal invariants, the properties of the main stresses are defined:

a) if $L_3 > 0$, the roots (principal stresses) given in (8) are positive and different.

b) if $L_3 < 0$, there are negative roots (principal stresses), and

c) if $L_3 = 0$, one of the roots (principal stresses) is equal to zero.

The same discussion can be applied for the strains. Starting from the secular equation for obtaining main dilatations in the form:

$$\begin{vmatrix} \varepsilon_x - \varepsilon_0 & \gamma_{xy} & \gamma_{xz} \\ \gamma_{xy} & \varepsilon_y - \varepsilon_0 & \gamma_{yz} \\ \gamma_{xz} & \gamma_{yz} & \varepsilon_z - \varepsilon_0 \end{vmatrix} = 0 \tag{14}$$

yields:

$$-(\varepsilon_0^3 - J_1\varepsilon_0^2 + J_2\varepsilon_0 - J_3) = 0 \tag{15}$$

where J_1, J_2, i J_3 are known invariants of the strain tensor obtained from known strain components of the geared transmission in every arbitrary point. The solutions of the secular equation (15) are the principal strains given. For the principal directions, the principal strains build the strain tensor in the following matrix:

$$\mathbf{S} = \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{pmatrix} \tag{16}$$

where the strain tensor invariants are given in the form:

$$J_1 = f(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = tr(\mathbf{S}) \tag{17}$$

$$J_2 = f(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_1\varepsilon_3$$

$$= \frac{1}{2}\left(tr(\mathbf{S}^2) - tr(\mathbf{S})^2\right) \tag{18}$$

$$J_3 = f(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \varepsilon_1\varepsilon_2\varepsilon_3 = \det(\mathbf{S}) \tag{19}$$

Analogous to (11), (12) and (13) for the principal stresses, it is possible to obtain the equation of the principal strains by solving the system of equations (17, 18, and 19):

The principal strain directions can be obtained in the following form:

$$\frac{\alpha_0}{\begin{vmatrix} \gamma_{xy} & \gamma_{xz} \\ \varepsilon_y - \varepsilon_0 & \gamma_{yz} \end{vmatrix}} = \frac{\beta_0}{-\begin{vmatrix} \varepsilon_x - \varepsilon_0 & \gamma_{xz} \\ \gamma_{xy} & \gamma_{yz} \end{vmatrix}}$$

$$= \frac{\gamma_0}{\begin{vmatrix} \varepsilon_x - \varepsilon_0 & \gamma_{xy} \\ \gamma_{xy} & \varepsilon_y - \varepsilon_0 \end{vmatrix}} = C_0 \tag{20}$$

if:

$$\alpha_0^2 + \beta_0^2 + \gamma_0^2 = 1 \tag{21}$$

Depending on the value of the principal invariants J_1, J_2 and J_3, the properties of the principal strains (relative deformations) is every body line element at every point are for example:

a) if $J_3 > 0$, the roots (principal strains) of (7) are positive and different and they are extensions,

b) if $J_3 < 0$, there are negative roots (principal strains), are copressions and

c) if $J_3 = 0$, on of the roots (principal strains) equals zero.

Results

Using the above procedures, the results are obtained for the gear set in Fig.1, using the boundary conditions presented in Fig.2 using the gear set parameters given in Table 1.

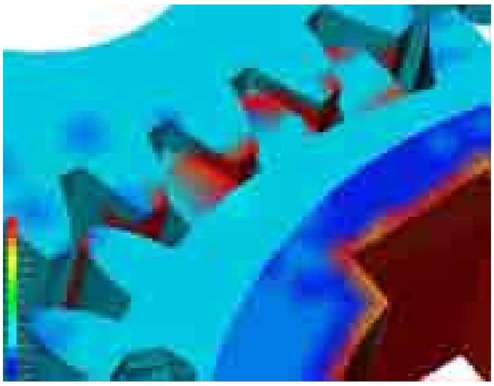

Figure 4. Gear set load distribution in the time step $t = 0.02$

The gear set load distribution along the line of contact for the gear pair given in Fig.4, and distribution of loads on the transmission elements for $t = 0.02$ is given in Fig.5 as an output from the nonlinear dynamic analysis of the geared transmission. The additional results for the strain distribution along the line of contact is given in Fig.8, and the Von Misses stress distribution along the line of contact in each time step (Fig.7) together with the distribution of the Von Misses stress on the transmission elements for $t = 0.02$ is presented in Fig.6.

The given results are the input results for the calculations of principal invariants in each node for each element of the presented gear set. The distribution of the principal stresses in the zone of contact is presented in Figures10 and 11 as a result of the software developed by the author that can be used for the detailed transmission analysis. An example of the principal stress and strain direction calculation for the selected nodes in contact is presented on Fig.9 and corresponding Tables: Table 2, and Table 3.

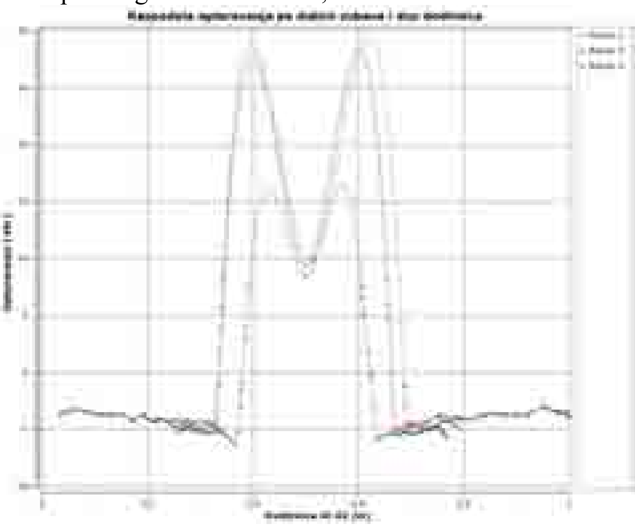

Figure 5. Stress distribution along the line of contact ($t = 0$ - $t = 0.02$)

Figure 6. Stress distribution for $t = 0.02$ by the Von Mises theory

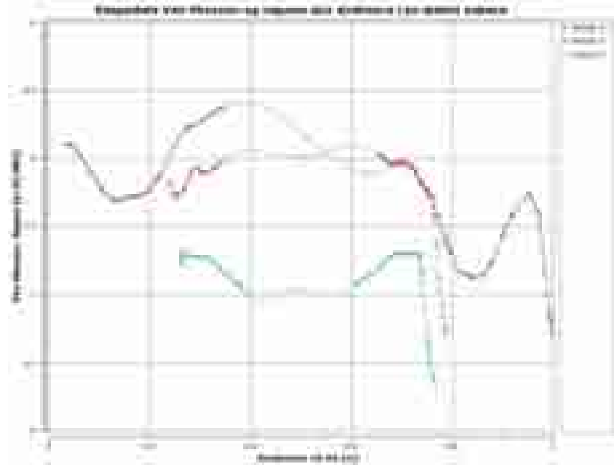

Figure 7. Stress distribution along the line of contact ($t = 0$ – $t = 0{,}02$) obtained by the Von Mises theory

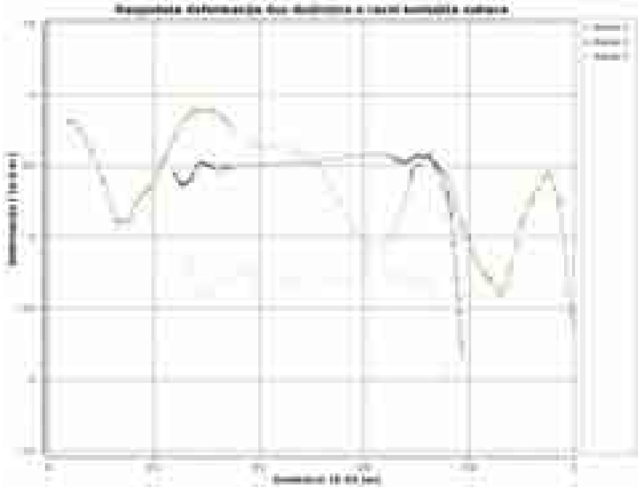

Figure 8. Strain distribution along the line of contact ($t = 0$ - $t = 0.02$)

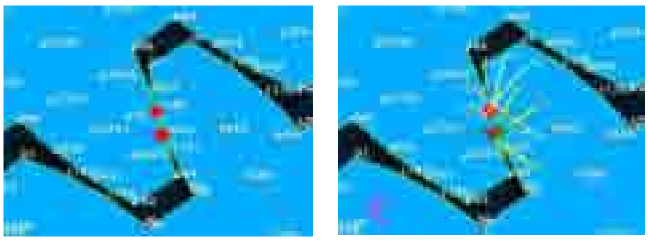

Figure 9. Nodes used for calculating an example of the stress and strain state (left) and example of the graphic representation of the principal stress analysis results in the chosen nodes (right)

Figure 10. Contact strain tensor invariants for $t = 0.02$

Figure 11. Contact stress tensor invariants for t = 0.02

Conclusions

The Finite Element Method is used in this paper to develop a real model of the geared set using the self developed software by the author and using an open source concept of the FEM engineering analysis. The calculated model is very flexible due to the methodology used, and because of the CODE_ASTER/SALOME [12] engine.

Since the conditions in which the dynamic analysis is done are quasistatic, it is to say that the suggested method approximates very well the real working conditions of the transmission. On the other hand, as every dynamic calculation, it is exhaustive and time consuming.

A novel method for the results analysis together with the developed software is used for the analysis of the principal stress and strain state of the geared transmission using principal stress and strain invariants as calculation parameters for the examination of strain and stress state in each point of the body gear.

The software allows a generalized analysis of every system for which this kind of analysis can be useful.

Table 2. Component stresses and strains in node: 3954, principal stress, strain invariants, principal stresses and principal strains and appropriate principal directions, for t=0.02

Node: 3954

σ_x (Pa)	σ_y (Pa)	σ_z (Pa)
3,704E+08	-1,257E+08	4,178E+07
τ_{xy} (Pa)	τ_{xz} (Pa)	τ_{yz} (Pa)
-1,123E+06	4,400E+06	-4,650E+06

α	β	γ
1,57	1,57	5,10E-03
1,56	1,57	8,37E-03
1,57	1,57	1,72E-03

L_1	L_2	L_3
2,87E+09	-3,53E+16	-3,37E+27
σ_1 (Pa)	σ_2 (Pa)	σ_3 (Pa)
-1,39E+09	6,72E+08	3,59E+09

ε_x (μm)	ε_x (μm)	ε_x (μm)
188,363	-118,736	-15,062
γ_{xy} (μm)	γ_{xz} (μm)	γ_{yz} (μm)
-1,390	5,448	-5,757

α	β	γ
1,57	1,46	1,12E-01
1,02	5,50E-01	1,57
1,52	1,57	4,96E-02

J_1	J_2	J_3
5,46E+01	-2,30E+04	-1,53E+04
ε_1 (μm)	ε_2 (μm)	ε_3 (μm)
-1,27E+02	6,63E-01	1,81E+02

Table 3. Component stresses and strains in node: 1180, principal stress,strain invariants, principal stresses and principal strains and appropriate principal directions, in t=0.02

Node: 1180

σ_x (Pa)	σ_y (Pa)	σ_z (Pa)
-4,364E+08	-1,265E+08	-6,712E+07
τ_{xy} (Pa)	τ_{xz} (Pa)	τ_{yz} (Pa)
1,218E+08	1,394+07	1,782E+08

α	β	γ
1,57	1,59	2,37E-02
1,49	1,55	8,27E-02
1,53	1,29	2,79E-01

L_1	L_2	L_3
-6,31E+08	-4,66E+16	3,04E+27
σ_1 (Pa)	σ_2 (Pa)	σ_3 (Pa)
-6,92E+09	-4,24E+08	1,03E+09

ε_x (μm)	ε_x (μm)	ε_x (μm)
180,126	-11,681	48,446
γ_{xy} (μm)	γ_{xz} (μm)	γ_{yz} (μm)
150,750	17,263	220,616

α	β	γ
1,61	1,79	2,26E-01
1,07	6,66E-01	1,98
1,15	5,03-01	1,83E+00

J_1	J_2	J_3
-1,20E+02	-6,40E+04	-3,34E+06
ε_1 (μm)	ε_2 (μm)	ε_3 (μm)
-3,38E+02	6,39E+01	1,54E+02

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Analiza dinamičkog stanja napona i stanja deformacija u spregnutom zupčastom paru

U radu je predstavljen rezultat istraživanja stanja napona i stanja deformacije dinamčki spregnutog zupčastog para koji se koristi za analizu unutrašnjeg dinamičkog opterećenja, vremenski-promenljivog stanja napona i stanja deformacija. Korišćen je model dinamike spregnutih zupčanika sa dva stepena slobode kretanja. Analiza stanja napona i stanja deformacije zupčastog para je urađena korišćenjem novog pristupa za prikaz stanja napona i stanja deformacija u zavisnosti od invarijanti tenzora stanja napona, odnosno tenzora stanja deformacija u pojedinim karakterističnim tačkama zupčanika. Na bazi teorije date u radu, razvijen je *software* za analizu napona i deformacije primenom metode konačnih elemenata koji se bazira na korišćenju softvera CODE-ASTER/SALOME.

Ključne reči: **naponsko stanje, dinamičko naprezanje, analiza napona, zupčasti prenosnik, deformacija, tenzor napona, metoda konačnih elemenata..**

Analyse dynamique de l'état de tension et déformation chez une pair d'engrenages couplés

Le résultat des essais sur l'état de tension et les déformations d'une pair d'engrenages dynamiquement couplés, utilisé pour l'analyse de la charge dynamique interne, de l'état de tension variable dans le temps et l'état de déformation, est présenté dans cet article. On a utilisé le modèle dynamique d'engrenages couplés à deux degrés de liberté des mouvements.L'analyse de l'état de tension et de déformation de pair d'engrenages est faite à l'aide da la nouvelle approche de présentation de l'état de tension et déformations dans les points d'engrenages. A partir de la théorie exposée dans ce travail, on a développé un logiciel pour analyser la tension et les déformations au moyen de la méthode des éléments finis, basée sur l'utilisation du logiciel CODE-ASTER/SALOME.

Mots clés: **état de tension, tension dynamique, analyse de tension, engrenage, déformation, tenseur des contraintes, méthode des éléments finis.**