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The Stability of Linear Discrete Descriptor Systems in the Sense of Lyapunov: an Overview

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This paper gives a detailed an overview of the work and the results of many authors in the area of Lyapunov stability of particular class of linear discrete descriptor systems. In that sense the discrete Lyapunov equation for discrete implicit systems is of particular interest.
The stability robustness problem has been also treated.
This survey covers the period since 1985 up to now days and has strong intention to present the main concepts and contributions that have been derived during mentioned period through the whole world, published in the respectable international journals or presented on workshops or prestigious conferences.

Key words: **discrete system, descriptive system, linear system, asymptotic stability, Lyapunov stability, robustness.**

Introduction

IT should be noticed that in some systems we must consider their character of dynamic and static state at the same time. *Descriptor systems* (also, referred to as degenerate, singular, generalized, differential - algebraic systems or semi – state) are those the dynamics of which are governed by a mixture of *algebraic* and *difference* equations. Recently many scholars have paid much attention to discrete descriptor systems and have obtained many good consequences. The complex nature of these systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control.

It is well-known that singular systems have been one of the major research fields of control theory.

During the past three decades, singular systems have attracted much attention due to the comprehensive applications in economics as the *Leontief* dynamic model *Silva*, *Lima* (2003), in electrical *Campbell* (1980) and mechanical models *Muller* (1997), etc.

They, also, arise naturally as a linear approximation of systems models, or linear system models in many applications such as electrical networks, *aircraft dynamics*, neutral delay systems, chemical, thermal and diffusion processes, large-scale systems, interconnected systems, economics, optimization problems, feedback systems, robotics, biology, etc.

Discussion of singular systems originated in 1974 with the fundamental paper of *Campbell et al*. (1974) and latter on the anthological paper of *Luenberger* (1977). Since that time, considerable progress has been made in investigating such systems see surveys, *Lewis* (1986) and *Dai* (1989) for linear singular systems, the first results for nonlinear singular systems in *Bajic* (1992).

On the investigation of stability of singular systems, many results in sense of Lyapunov stability have been derived. For example, *Bajic* (1992) and *Zhang et al.* (1999) considered the stability of linear time-varying descriptor systems.

An examples of discrete descriptor systems, in the practice, are significantly less then the corresponding continuous one, but their presence in *economics*, *demography* and some optimal control problems is evident, so their study deserve a careful attention

This paper presents, in a unified way, a collection of results spread out in the literatures and focuses on stability of linear discrete descriptor systems (LDDS).

This paper is not a survey in the usual sense.

We do not try to be exhaustive on the vast literatures concerning this problem. Our object is more to convince the reader of the practical interest of the approach and of the number and the simplicity of the results it leads to. For each aspect we generally give in detail only one result, which is not necessarily the most complete or the most recent one, but is the one which seems to us the most representative and illustrative.

Basic notations

$\mathbb{R}$	–Real vector space;
$\mathbb{C}$	–Complex vector space;
$\mathbf{C}$	–Complex plane;
I	–Unit matrix;
$F=(f_{ij}) \in \mathbb{R}^{n\times n}$	–Real matrix;
F^T	–Transpose of matrix F ;
$F > 0$	–Positive definite matrix;
$F \geq 0$	–Positive semi definite matrix;
$\Re(F)$	–Range of matrix F ;

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N	–Nilpotent matrix;
$\mathbb{N}(F)$	–Null space (kernel) of matrix F;
$\lambda(F)$	–Eigenvalue of matrix F;
$\sigma_{( )}(F)$	–Singular values of matrix F;
$\sigma\{F\}$	–Spectrum of matrix F;
$\|F\|$	–Euclidean matrix norm $\|F\|=\sqrt{\lambda_{\max}(A^T A)}$;
F^D	–Drazin inverse of matrix F;
$\Rightarrow$	–Follows;
$\mapsto$	–Such that.

Stability of regular linear discrete descriptor systems

Generally, the time invariant discrete descriptor control systems can be written as:

$$E\,\mathbf{x}(k+1) = A\mathbf{x}(k), \qquad \mathbf{x}(k_0) = \mathbf{x}_0, \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is a generalized state space (costate, semi-state) vector, $E \in \mathbb{R}^{n \times n}$ is a possibly singular matrix, with $rank\, E = r < n$.

Matrices E and A are of the appropriate dimensions and are defined over the field of real numbers. It should be stressed that, in the general case, the initial conditions for an autonomous and a system operating in the forced regime need not be the same.

System models of this form have some important advantages in comparison with models in the *normal form, e.g. when* $E = I$ and an appropriate discussion can be found in *Bajic* (1992) and *Debeljkovic et al.* (1996. 2005.a, 2205.b).

The complex nature of descriptor systems causes many difficulties *in analytical and numerical treatment* that do not appear when systems in the normal form are considered.

In this sense questions of existence, solvability, and uniqueness are present which must be solved in satisfactory manner.

In the discrete case, the concept of smoothness has little meaning but the idea of consistent initial conditions being these initial conditions x_0, that generate solution sequences $(\mathbf{x}(k) : k \geq 0)$ has a physical meaning.

The fundamental geometric tool in the characterization of the subspace of consistent initial conditions is the subspace sequence

$$W_0 = \mathbb{R}^n,\ W_{j+1} = A^{-1}(EW_j),\ (j \geq 0). \tag{2}$$

Here $A^{-1}(\cdot)$ denotes the inverse image of $(\cdot)$ under the operator A and we will denote by $\mathbb{N}(F)$ and $\Re(F)$ the kernel and range of any operator F respectively.

Lemma 1. The subspace sequence $\{W_0,\ W_1,\ W_2, ...\}$ is nested in the sense that

$$W_0 \supset W_1 \supset W_2 \supset W_3 \supset \cdots. \tag{3}$$

Moreover

$$\mathbb{N}(A) \subset W_j,\ (j \geq 0), \tag{4}$$

and there exists an integer $k \geq 0$ such that

$$W_{k+1} = W_k, \tag{5}$$

and hence $W_{k+1} = W_k$ for $j \geq 1$.

If k^* is the smallest such integer with this property, then

$$W_k \cap \mathbb{N}(E) = \{0\} \qquad (k \geq k^*) \tag{6}$$

provided that $(\lambda E - A)$ is invertible for some $\lambda \in \mathbb{R}$, *Owens*, *Debeljkovic* (1985).

Theorem 1. Under the conditions of *Lemma* 1, $\mathbf{x}_0$ is a consistent initial condition for (1) if $\mathbf{x}_0 \in W_{k^*}$.

Moreover $\mathbf{x}_0$ generates a unique solution $\mathbf{x}(t) \in W_{k^*}$ $(k \geq 0)$ that is real - analytic on $\{k : k \geq 0\}$, *Owens*, *Debeljkovic* (1985).

Theorem 1. is the geometric counterpart of the algebraic results of *Campbell* (1980):

A short and concise, acceptable and understandable explanation of all these questions may be found in the papers of *Debeljkovic* (2001, 2002, 2004).

The survey of updated results for singular (descriptor) systems and a broad bibliography can be found in *Bajic* (1992), *Campbell* (1980, 1982), *Lewis* (1986, 1987), *Debeljkovic et al*. (1996, 2005.a, 2205.b) and in the two special issues of the journal *Circuits, Systems and Signal Procesing* (1986, 1989).

Necessary definitions

Definition 1. The linear discrete descriptor system (1) is said to be regular if $\det(sE - A)$ is not identically zero, *Dai* (1989).

Remark 1. Note that the regularity of matrix pair (E, A) guarantees the existence and uniqueness of solution $\mathbf{x}(\cdot)$ for any specified initial condition, and the impulse immunity avoids impulsive behavior at initial time for inconsistent initial conditions.

It is clear that, for notrivial case $E \neq 0$, impulse immunity implies regularity.

Definition 2. The linear discrete descriptor system (1) is assumed to be *non-degenerate* (or regular), i.e. $\det(zE - A) \neq 0$, *Syrmos et al.* (1995).

Otherwise, it will be called *degenerate*.

If $(zE - A)$ is non-degenerate, we define the *spectrum* of $(zE - A)$, denoted as $\sigma\{E, A\}$ as those isolated values of where $\det(zE - A) \neq 0$ fails to hold.

The usual spectrum of $(zI - A)$ will be denoted as $\sigma\{A\}$.

Note that owing to (possible) singularity of E, $\sigma\{E, A\}$ may contain finite and infinite values of z.

The finite spectrum will be denoted as $\sigma_f\{E, A\}$.

Since the system is regular, then there exist orthogonal transformations U and V such that, *Van Dooren* (1979).

$$U^T EV = \begin{pmatrix} E_f & E_1 \\ 0 & E_\infty \end{pmatrix},\quad U^T AV = \begin{pmatrix} A_f & A_1 \\ 0 & A_\infty \end{pmatrix}, \tag{7}$$

where matrices E_f, A_∞ are nonsingular and E_∞ is *nilpotent*.

The pencil $(zE_f - A_f)$ contains the finite elementary divisors and the pencil $(zE_\infty - A_\infty)$ contains the infinite elementary divisors of $(zE - A)$.

Furthermore, the spectrum of $(zE_f - A_f)$, $\sigma\{E_f, A_f\}$ coincides with $\sigma_f\{E, A\}$.

The structure of the *Jordan* blocks of E_∞ determines the chains at infinity of the pencil $(zE - A)$.

The discrete-time implicit system admits one-sided solutions when $\mathbf{u}(k) = 0$, for every $k \geq 0$, i.e.for every admissible initial condition $\mathbf{x}(k)$ there exists a solution that does not depend on states $\mathbf{x}(k)$ for $k > 0$, if and only if it has trivial chains at infinity.

This is known as the *anticipation phenomenon* described in *Banaszuk et al.* (1987).

For any regular pair (E, A) there exist two real nonsingular matrices M and N such that

$$MEN = \hat{E}\,,\quad MAN = \hat{A}\,, \tag{8}$$

where

$$\hat{E} = \begin{pmatrix} I_r & 0 \\ 0 & J \end{pmatrix},\quad \hat{A} = \begin{pmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{pmatrix}. \tag{9}$$

The pair $(\hat{E}, \hat{A})$, is called the Weierstrass form of (E, A).

In (9) the matrix $J \in \mathbb{R}^{(n-r)\times(n-r)}$ is a nilpotent matrix, and r is the number of finite eigenvalues of (E, A).

Since M and N are nonsingular and $\det(sJ - I_{n-r}) = (-1)^{n-r}$, we have

$$\sigma\{E, A\} = \sigma\{\hat{E}, \hat{A}\} = \sigma\{I_r, A_1\}\,, \tag{10}$$

i.e. the stability of any regular descriptor system (E, A) can be completely determined by that of the state-space system (I_r, A_1) of smaller dimension.

Note that (E, A) is impulse – free if and only if $J = 0$.

If the discrete descriptor system is regular, the transfer function matrix from $\mathbf{u}(k)$ to $\mathbf{y}(k)$ can be uniquely defined as

$$W_{zu}(z) = C(sE - A)^{-1}B + D\,. \tag{11}$$

Any nonzero vector $\mathbf{v}^1$ satisfies $E\mathbf{v}^1 + 0$ is called a *grade* **1** (*infinite generalised*) *eigenvector* of the pair (E, A) and any nonzero vector $\mathbf{v}^k$ $(k \geq 2)$ satisfies $E\mathbf{v}^k = A\mathbf{v}^{k-1}$ is called a grade k (infinite generalised) eigenvector of the pair (E, A).

Definition 3. The linear discrete descriptor system (3.1) is said to be *causal* if (3.1) is regular and $\deg ree\ \det(sE - A) = rank\ E$, *Dai* (1989).

Definition 4. A pair (E, A) is said to be *admissible* if it is regular, impulse-free and stable, *Hsiung*, *Lee* (1999).

Lemma 2. The linear discrete-time descriptor system (3.1) is regular, causal and stable if and only if there exists an invertible symmetric matrix $H \in \mathbb{R}^{n\times n}$ such that the following two inequalities hold

$$E^T HE \geq 0\,, \tag{12}$$

$$A^T HA - E^T HE < 0\,, \tag{13}$$

Xu, *Yang* (1999).

Remark 2. In the case of $E = I$, i.e. the discrete descriptor system (1) reduces to a state-space one.

Lemma 2 coincides with the well-known stability results of discrete-time state-space systems based on Lyapunov inequalities, from this point of view, *Lemma* 2 naturally extends existing results on discrete-time state-space systems to singular ones.

Furthermore, as pointed out that *Lemma* 2 is obtained without assuming the regularity of system (3.1), and it is also given in terms of the coefficient matrices of the whole system, thus, the application of *Lemma* 2 is convenient.

Lemma 3. If the pair (E, A) is not impulse-free, then it has at least one grade 2 eigenvector $\mathbf{v}^2$ such that $E\mathbf{v}^2 = A\mathbf{v}^1$ and $E\mathbf{v}^1 = 0$, where $\mathbf{v}^1$ is the corresponding grade 1 eigenvector.

The following *Lemma* says that the pairs (E, A) and $(\hat{E}, \hat{A})$ share the same properties of regularity and impulse immunity, with $\hat{A}$ being certain linear combination of E and A.

However, the two sets of finite eigenvalues are related in an affine linear manner.

Lemma 4. Let α, $\beta \in \mathbb{R}$ with $\beta \neq 0$.

Let $\hat{A} = \alpha E + \beta A$, then we have the following propreties:

(i) (E, A) is regular and impulse-free if and only if $(\hat{E}, \hat{A})$ is regular and impulse-free

(ii) Each pair of corresponding finite eigenvalues of the two systems is related by $\lambda_i(\hat{E}, \hat{A}) = \alpha + \beta\lambda_i(E, A)$, *Hsiung*, *Lee* (1999).

Stability definitions

Stability plays a central role in the theory of systems and control engineering. There are different kinds of stability problems that arise in the study of dynamic systems, such as *Lyapunov* stability, finite time stability, practical stability, technical stability and BIBO stability.

The first part of this section is concerned with the asymptotic stability of the equilibrium points of *linear discrete descriptor systems.*

As we treat the linear systems this is equivalent to the study of the stability of the systems.

The *Lyapunov* direct method (LDM) is well exposed in a number of very well known references.

Here we present some different and interesting approaches to this problem, including the contributions of the authors of this paper.

Definition 5. Linear discrete descriptor system (1) is said to be stable if (3.1) is regular and all of its finite poles are within Ω(0,1), *Dai* (1989).

Definition 6. The system in (1) is asymptotically stable if all the finite eigenvalues of the pencil $(zE - A)$ are inside the unit circle, and anticipation free if every admissible $\mathbf{x}(0)$ in (3.1) admits one-sided solutions when $\mathbf{u}(k) = 0$, *Syrmos* (1995).

Definition 7. Linear discrete descriptor system (1) is said to be asymptotically stable if, for all consistent initial conditions $\mathbf{x}_0$, we have that $\mathbf{x}(t) \to 0$ as $t \to +\infty$, *Owens*, *Debeljkovic* (1985).

Stability theorems

First we present the fundamental work in the area of stability in the sense of Lyapunov applied to the linear discrete descriptor systems, *Owens*, *Debeljkovic* (1985).

Our attention is restricted to the case of singular (i.e. noninvertible) E and the construction of geometric conditions on $\mathbf{x}_0$ for the existence of causal solutions of (1) in terms of the relative subspace structure of matrices E and A.

The results are hence a geometric counterpart of the algebraic theory of *Campbell* (1980) who established the required form of $\mathbf{x}_0$ in terms of the Drazin inverse and the technical trick of replacing E and A by commuting operators. The ideas in this paper work with E and A directly and commutativity is not assumed.

The geometric theory of consistency leads to a natural class of positive-definite quadratic forms on the subspace containing all solutions. This fact makes possible the construction of a Lyapunov stability theory for linear discrete descriptor systems in the sense that asymptotic stability is equivalent to the existence of symmetric, positive-definite solutions to *a weak form of Liapunov equation.*

Throughout the paper it is assumed that $(\lambda E - A)$ is invertible at all but a finite number of points $\lambda \in \mathbb{C}$ and hence that if a solution $x(k)$, $(k \geq 0)$ of $(x(k) : k = 0,1,...)$ exists for a given choice of $\mathbf{x}_0$, it is unique *Campbell*, (1980).

The linear discrete descriptor systems said to be stable if (3.1) is regular and all of its finite poles are within Ω(0,1), *Dai* (1989), so careful investigation shows there is no need for the matrix A to be invertible, in comparison with continuous case, see *Debeljkovic et al.* (2007) could be noninvertible.

Theorem 2. The linear discrete descriptor system (3.1) is asymptotically stable if, and only if, there exists a real number $\lambda^* > 0$ such that, for all λ in the range $0 < |\lambda| < \lambda^*$, there exists a self-adjoint, positive-definite operator H_λ in $\mathbb{R}^n$ satisfying

$$(A - \lambda E)^T H_\lambda (A - \lambda E) - E^T H_\lambda E = -Q_\lambda , \qquad (14)$$

for some self-adjoint operator Q_λ satisfying the positivity condition

$$\mathbf{x}^T(t) Q_\lambda \mathbf{x}(t) > 0, \quad \forall \mathbf{x}(t) \in W_{k^*} \setminus \{0\} , \qquad (15)$$

Owens, *Debeljkovic* (1985).

Theorem 3. Suppose that matrix A is invertible.

Then the linear discrete descriptor system (3.1) is asymptotically stable if, and only if, there exists a self-adjoint, positive-definite solution H in $\mathbb{R}^n$ satisfying

$$A^T HA - E^T HE = -Q , \qquad (16)$$

where Q is self-adjoint and positive in the sense that

$$\mathbf{x}^T(t) Q \mathbf{x}(t) > 0, \quad \forall \mathbf{x}(t) \in W_{k^*} \setminus \{0\} , \qquad (17)$$

Owens, Debeljkovi (1985).

Theorem 4. The linear discrete descriptor system (3.1) is asymptotically stable if and only if there exists a real number $\lambda^* > 0$ such that, for all λ in the range $0 < |\lambda| < \lambda^*$, there exists a self-adjoint, positive-definite operator H_λ in $\mathbb{R}^n$ satisfying

$$\begin{aligned} &\mathbf{x}^T(t)\left((A - \lambda E)^T H_\lambda (A - \lambda E) - E^T H_\lambda E\right)\mathbf{x}(t) \\ &= -\mathbf{x}^T(t)\mathbf{x}(t), \qquad \forall \mathbf{x}(t) \in W_{k^*} \end{aligned} , \qquad (18)$$

Owens, Debeljkovic (1985).

Corollary 1. If matrix A is invertible, then the linear discrete descriptor system (3.1) is asymptotically stable if and only if (17) holds for $\lambda = 0$ and some self-adjoint, positive-definite operator H_0, *Owens*, *Debeljkovic* (1985).

Now we need some preliminaries to expose the results of *Syrmos et al*. (1995).

Let the generalized Lyapunov function for system (3.1) be of the form

$$V(\mathbf{x}(k)) = \mathbf{x}^T(t) E^T H E \mathbf{x}(t) , \qquad (19)$$

Then

$$\begin{aligned} &\Delta V(\mathbf{x}(k)) = V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) = \\ &= \mathbf{x}^T(k+1) E^T H E \mathbf{x}(k+1) - \mathbf{x}^T(k) E^T H E \mathbf{x}(k) \end{aligned} \qquad (20)$$

and substituting (3.1)

$$\Delta V(\mathbf{x}(k)) = -\mathbf{x}^T(k) E^T Q E \mathbf{x}(k) , \qquad (21)$$

for some Q.

The discussion on structure of the matrix Q will be presented later.

From (21), the generalized *Lyapunov equation* for a discrete implicit system is given by

$$A^T HA - E^T HE + E^T QE = 0 . \qquad (22)$$

In the basis representation implied by (7) we can rewrite (22) as:

$$\begin{aligned} &V^T A^T U U^T H U U^T A V + V^T E^T U U^T H U U^T E V \\ &- V^T E^T U U^T Q U U^T E V = 0 \end{aligned} \qquad (23)$$

Using (7) and replacing E and A by their transformed representation, the generalized Lyapunov equation becomes

$$\begin{aligned} &\begin{pmatrix} A_f^T & 0 \\ A_1^T & A_\infty^T \end{pmatrix} \begin{bmatrix} H_f & H_1 \\ H_1^T & H_\infty \end{bmatrix} \begin{pmatrix} A_f & A_1 \\ 0 & A_\infty \end{pmatrix} \\ &- \begin{pmatrix} E_f^T & 0 \\ E_1^T & E_\infty^T \end{pmatrix} \begin{pmatrix} H_f & H_1 \\ H_1^T & H_\infty \end{pmatrix} \begin{pmatrix} E_f & E_1 \\ 0 & E_\infty \end{pmatrix} \\ &+ \begin{pmatrix} E_f^T & 0 \\ E_1^T & E_\infty^T \end{pmatrix} \begin{pmatrix} Q_f & Q_1 \\ Q_1^T & Q_\infty \end{pmatrix} \begin{pmatrix} E_f & E_1 \\ 0 & E_\infty \end{pmatrix} = 0 \end{aligned} , \qquad (24)$$

where

$$\begin{pmatrix} H_f & H_1 \\ H_1^T & H_\infty \end{pmatrix} = U^T H U , \qquad (25)$$

and

$$\begin{pmatrix} Q_f & Q_1 \\ Q_1^T & Q_\infty \end{pmatrix} = U^T Q U . \qquad (26)$$

Various sub-matrices are partitioned conformably to E_f and E_∞.

On carrying out the calculations for the blocks entries we obtain the following three matrix equations

$$A_f^T H_f A_f - E_f^T H_f E_f + E_f^T Q_f E_f = 0\,, \qquad (27)$$

$$A_f^T H_1 A_\infty - E_f^T H_1 E_\infty + \hat{Q}_1 = 0\,, \qquad (28)$$

$$A_\infty^T H_\infty A_\infty - E_\infty^T H_\infty E_\infty + \hat{Q}_\infty = 0\,, \qquad (29)$$

where

$$\hat{Q}_1 = A_f^T H_f A_1 + E_f^T Q_f E_1 + E_f^T Q_1 E_\infty - E_f^T H_f E_1\,, \qquad (30)$$

$$\begin{aligned}\hat{Q}_\infty &= E_1^T Q_f E_1 + E_1^T Q_1 E_\infty + E_\infty^T Q_1^T E_1 \\ &+ E_\infty^T Q_\infty E_\infty + A_1^T H_f A_1 - E_1^T H_f E_1 \\ &+ A_1^T H_1 A_\infty + E_1^T H_1 E_\infty + A_\infty^T H_1^T A_1 - E_\infty^T H_1^T E_1\end{aligned}\,, \qquad (31)$$

and $\hat{Q}_\infty$ is symmetric.

Furthermore, (27) has the same form as the *Lyapunov equation* (22), while (29) has a similar form and (28) is a general matrix equation.

Clearly, the solution of the *Lyapunov equation* (22) is obtained by solving (27 –29).

Note that the matrices E_f and A_∞ are non-singular.

If we assume that the system admits only *one - sided solutions* for every admissible $\mathbf{x}_0$, we can set E_∞.

In that case the following (simpler) result hold true.

$$A_f^T H_f A_f - E_f^T H_f E_f + E_f^T Q_f E_f = 0\,, \qquad (32)$$

$$A_f^T H_1 A_\infty + A_f^T H_f A_1 + E_f^T Q_f E_1 - E_f^T H_f E_1 = 0\,, \qquad (33)$$

$$\begin{aligned}&E_1^T Q_f E_1 - E_1^T H_f E_1 + A_1^T H_f A_1 \\ &+ A_\infty^T H_1^T A_1 + A_1^T H_1 A_\infty + A_\infty^T H_\infty A_\infty = 0\end{aligned}\,. \qquad (34)$$

Theorem 5. The linear discrete descriptor system (1) is anticipation free if and only if the chains at infinity are trivial, or equivalently if and only if E_∞ in (7) is a null matrix, *Syrmos et al.* (1995).

Theorem 6. The linear discrete descriptor system (1) or matrix pair (E, A) is asymptotically stable and admits one-sided solutions for every admissible initial condition $\mathbf{x}_0$ if and only if for every symmetric positive definite matrix Q, the generalized Lyapunov equation (22) has a positive semi-definite solution P, such that P has n_∞ eigenvalues at zero and $E^T PE$ is unique, *Syrmos et al.* (1995).

From discussion up to this point we can easily draw the following conclusion.

Corollary 2. A system given (E, A) is *asymptotically stable* with no poles at the origin, and admits one-sided solutions for every admissible initial condition $\mathbf{x}_0$ if and only if for every symmetric positive semi-definite matrix Q *the generalized Lyapunov equation* (22) has a unique positive semi-definite solution H, such that matrix H has n_∞ eigenvalues at zero, *Syrmos et al.* (1995).

Theorem 7. Assume that the pair (E, A) has trivial chains at infinity and has all its eigenvalues inside the unit circle, then solution to (28) for any matrix $\hat{Q}_1$ is parameterized by

$$A_f^T H_1 = A_f^T H_f \left(A_f E_f^{-1} E_1 - A_1 \right) A_\infty^{-1}\,, \qquad (35)$$

where matrix H_f is the solution of (32) for some matrix Q_f, *Syrmos et al.* (1995).

Corollary 3. Assume that the pair (E, A) has trivial chains at infinity and and no eigenvalues at the origin and has all finite eigenvalues inside the unit circle, then there exists unique solution to (33) for any Q_1 given by

$$H_1 = H_f \left(A_f E_f^{-1} E_1 - A_1 \right) A_\infty^{-1}\,, \qquad (36)$$

where H_f is the unique solution (32) for some Q_f, *Syrmos et al.* (1995).

Theorem 8. Assume that the pair (E, A) has trivial chains at infinity, then a solution to (34) for any Q_∞ is given by

$$H_\infty = A_\infty^{-T} \left(E_1^T E_f^{-T} A_f^T - A_1^T \right) H_f \left(A_f E_f^{-1} E_1 - A_1 \right) A_\infty^{-1}\,, \qquad (37)$$

where H_1 in (34) is a solution of (33) for some Q_1, *Syrmos et al.* (1995).

Corollary 4. Assume that the pair (E, A) has trivial chains at infinity and no eigenvalues at the origin and has all finite eigenvalues inside the unit circle, then there exists unique solution to (34) for any Q_∞ given by

$$H_\infty = A_\infty^{-T} \left(E_1^T E_f^{-T} A_f^T - A_1^T \right) H_f \left(A_f E_f^{-1} E_1 - A_1 \right) A_\infty^{-1}\,, \qquad (3.38)$$

where H_f is the unique solution (32) for some Q_f, *Syrmos et al.* (1995).

Theorem 9. Assume that the pair (E, A) has trivial chains at infinity and has all its finite eigenvalues inside the unit circle, then there exist a solution to (34) for any Q_∞ is given by

$$H_\infty = H_1^T H_f^{-1} H_1\,, \qquad (39)$$

where H_1 in (33) is a solution of (36) for some Q_1, *Syrmos et al.* (1995).

Corollary 5. Assume that the pair (E, A) has trivial chains at infinity and no eigenvalues at the origin and has all finite eigenvalues inside the unit circle, then there exists unique solution to (34) for any Q_∞ given by

$$H_\infty = H_1^T H_f^{-1} H_1\,, \qquad (40)$$

where H_1 is the unique solution (33) for some Q_1, *Syrmos et al.* (1995).

Theorem 10. A system given (E, A) is *asymptotically stable* and admits one-sided solutions for every admissible initial condition $\mathbf{x}_0$ if and only if for every symmetric positive semidefinite matrix Q the generalized Lyapunov equation (22) has a unique positive semi-definite solution H, such that matrix H has n_∞ eigenvalues at zero and $E^T HE$ is unique *Syrmos et al.* (1995).

Theorem 11. The linear discrete descriptor system (1) is regular, causal and asymptotically stable if and only if for any given $Q > 0$ there exists a unique positive semi-definite solution H to *Lyapunov equation*

$$A^T HA - E^T HE = -E^T QE\,, \qquad (41)$$

satisfying

$$rank\left(E^T HE\right) = rank\, E = r\,, \tag{42}$$

Zhang (1997).

Theorem 12. Suppose that $EA = AE$ and there exists z such that $(zE - A) = I$, then the linear discrete descriptor system (3.1) is regular and asymptotically stable if and only if for any given positive definite matrix H there exist a positive semi-solution to the Lyapunov equation

$$\left(E^h\right)^T A^T \Omega A E^h A - \left(E^{h+1}\right)^T \Omega E^{h+1} = -\left(E^{h+1}\right)^T H E^{h+1} \tag{43}$$

satisfying

$$rank\,\Omega = rank\left(E^{h+1}\right)^T \Omega E^{h+1} = r\,, \tag{44}$$

Zhang (1997).

Theorem 13. The linear discrete descriptor system (1) is regular, causal and asymptotically stable if and only if the Lyapunov function

$$V\left(E\mathbf{x}(k)\right) = \mathbf{x}^T(k) E^T H E\, \mathbf{x}(k)\,, \tag{45}$$

satisfying

$$\Delta V\left(E\mathbf{x}(k)\right) = V\left(E\mathbf{x}(k+1)\right) - V\left(E\mathbf{x}(k)\right) < 0\,, \tag{46}$$

when $E\mathbf{x}(k) \neq \mathbf{0}$, where $H \geq 0$ is unique and fulfills $rank\left(E^T H E\right) = r$, *Zhang et al.* (1998).

Theorem 14. The linear discrete descriptor system (1) is regular, causal and asymptotically stable if and only if the Lyapunov function (45) satisfies Eq. (46), where matrix H satisfies

$$rank\left(E^T H E\right) = r, \quad rank\left(E^T A^T\right) = n\,, \tag{47}$$

Zhang et al. (1998).

Theorem 15. The linear discrete-time descriptor system (1) or matrix pair (*E*, *A*) is admissible if and only if there exists a matrix $H = H^T \in \mathbb{R}^{n\times n}$ satisfying the following generalised *discrete Lyapunov inequality*

$$A^T H A < E^T H E\,, \tag{48}$$

$$E^T H E \geq 0\,, \tag{49}$$

Hsiung, Lee (1999).

Corollary 6. The linear discrete-time descriptor system (1) or matrix pair (*E*, *A*) is regular, impulse-free, and *D*(*c*,*r*)-stable, i.e. $\sigma\{E,\ A\} \subset D(c,r)$, if and only if there exists a matrix $H = H^T \in \mathbb{R}^{n\times n}$ such that

$$\left(c^2 - r^2\right) E^T H E + A^T H A < c\left(E^T H A + A^T H E\right), \tag{50}$$

$$E^T H E \geq 0\,, \tag{51}$$

Hsiung, Lee (1999).

Theorem 16. The regular linear discrete-time descriptor system (1) is asymptotically stable if and only if for any given $Q > 0$ there exists a solution to Lyapunov equation

$$\left(E^h\right)^T H = H^T E^h \geq 0\,, \tag{52}$$

$$\left(E^h\right)^T A^T HA - E^T A^T E^{h+1} = -\left(E^{h+1}\right)^T Q\, E^{h+1}\,, \tag{53}$$

satisfying

$$rank\left(E^h\right)^T H = \deg ree\ \det\left(zE - A\right) = rank\, E^h\,, \tag{54}$$

Zhang et al. (1999).

Theorem 17. The zero solution to the linear discrete descriptor system (1) is asymptotically stable if and only if there exist inverse matrices P and Q such that

$$QEP = \begin{pmatrix} I_1 & 0 \\ 0 & N \end{pmatrix}, \quad QAP = \begin{pmatrix} A_1 & 0 \\ 0 & I_2 \end{pmatrix}, \tag{55}$$

and for any $\lambda \in \sigma\{A_1\}$, $\lambda < 1$, where A_1 is $n_1 \times n_1$ matrix, and I_i is $n_i \times n_i$ unit matrix, N is a nilpotent matrix, $i = 1,\ 2,\ n_2 = n - n_1$, *Liang*, *Ying* (1999).

Theorem 8. The zero solution to the linear discrete descriptor system (1) is asymptotically stable if and only if there exist inverse matrices

$$P = \begin{pmatrix} P_1 & P_2 \end{pmatrix},\ Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \tag{56}$$

such that:

(i) $Q_2 E P_2$ is a nilpotent matrix;

(ii) $\sigma\{Q_1 A P_1\} \subset \Upsilon$, where $\Upsilon = \{|\lambda|, |\lambda| < 1\}$, Q_i, P_i are $n_i \times n$ and $n \times n_i$ matrices, respectively, $i = 1, 2$, $n_2 = n - n_1$, *Liang*, *Ying* (1999).

Theorem 19. The linear discrete-time descriptor system (1) is regular, causal and asymptotically stable if and only if there exists an invertible symmetric matrix H fulfilling the following inequalities:

$$E^T H E \geq 0\,, \tag{57}$$

$$A^T H A - E^T H E < 0\,, \tag{58}$$

Yao et al. (2002).

The system

$$\hat{E}\hat{\mathbf{x}}(k+1) = \hat{A}\hat{\mathbf{x}}(k), \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0\,, \tag{59}$$

is called the system restriction equivalent to the system (3.1) if there exist two non-singular matrices P and Q such that

$$\mathbf{x}(k) = P\hat{\mathbf{x}}(k), \quad QEP = \hat{E}, \quad QAP = \hat{A}\,. \tag{60}$$

Theorem 20. Assume that A is invertible.

Then, the following statements are equivalent:

(i) The system (1) is stable.

(ii) For any positive definite matrix $Q > 0$, the *generalized Lyapunov equation*

$$\hat{A}^T \hat{H} \hat{A} - \hat{E}^T \hat{H} \hat{E} = -\hat{Q}\,, \tag{61}$$

has a unique symmetric solution $\hat{H}$ satisfying

$$In_+\left(\hat{H}\right) = n_1, \quad In_+\left(\hat{H}\right) + In_-\left(\hat{H}\right) = n\,, \tag{62}$$

where $\hat{H} \in \mathbb{R}^{n\times n}$ and $\hat{Q} \in \mathbb{R}^{n\times n}$ are symmetric matrices, *Zhang et al.* (2002).

There exist invertible matrices U, V such that

$$UEV = \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix}, \quad UAV = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{63}$$

then we have the following *Theorem*.

Theorem 21. The regular linear discrete-time descriptor system (1) is regular, causal and asymptotically stable if and only if there exists a positive definite matrix

$$X = V^{-T} \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} V^{-1} > 0\,, \tag{63}$$

fulfill the following conditions

$$A^T \tilde{E}^T X \tilde{E} A - X < 0\,, \tag{64}$$

$$A_{22} A_{22}^T > 0\,, \tag{65}$$

where

$$\tilde{E} = V \begin{pmatrix} I_q & 0 \\ 0 & 0 \end{pmatrix} U\,, \tag{66}$$

Wang et al. (2003).

Theorem 22. The regular linear discrete-time descriptor system (1) is asymptotically stable if and only if for any given $Q > 0$ there exists a unique positive definite solution H to *Lyapunov equation*

$$A^T \hat{E}^T H \hat{E} A - H = -Q\,, \tag{67}$$

where

$$\hat{E} = Q \begin{pmatrix} I_{q1} & 0 \\ 0 & 0 \end{pmatrix} P\,. \tag{68}$$

Wang et al. (2003).

Theorem 23. Suppose that $A\mathbf{x}_e(t) = 0$ and the norm of the eigenvalue of $(zE - A)$ is less than 1, then the linear discrete-time descriptor system (1) is globally asymptotically stable if the balance state $\mathbf{x}_{eq} = 0$, *Yang et.al.* (2003).

Theorem 24. Suppose that for any given real symmetric matrix Q, there exists a unique, positive definite symmetric solution H to the Riccati equation

$$A^T H A - E^T H E = -2Q\,, \tag{69}$$

then the linear discrete descriptor system (1) is globally asymptotically stable if the balance state of the system (1) $\mathbf{x}_{eq} = 0$, *Yang et.al.* (2003).

Lyapunov stability of irregular linear discrete descriptor systems

Introduction

For the purposes of this section we consider a class of linear discrete descriptor systems (LDDS) for which the dynamics are governed by

$$E\mathbf{y}(k+1) = A\mathbf{y}(k)(k = k_0,\ k_0+1,\ldots),\ \mathbf{y}(k_0) = \mathbf{y}_0 \tag{70}$$

with $E, A \in \mathbb{R}^{n\times n}$, where $\mathbf{y}(k) \in \mathbb{R}^n$ is the phase vector (i.e. the generalized state space vector) and where the matrix E may be singular.

The notion of discrete descriptor system was introduced by *Luenberger* (1977).

The complex nature of descriptor systems causes many difficulties in numerical and analytic treatment that do not appear when systems in the normal form are concerned. Qualitative and stability analysis of linear discrete descriptor systems have been treated by several authors. *Campbell*, *Rodrigues* (1985) investigated bounds of response of discrete nonlinear descriptor systems.

With the use of Lyapunov's direct method (LDM) some particular classes of nonlinear nonstationary discrete descriptor systems were studied by *Milic*, *Bajic* (1984) and *Bajic et al.* (1990).

Large-scale discrete descriptor systems were also analysed by *Milic*, *Bajic* (1984).

Owens, *Debeljkovic* (1985) derived some necessary and sufficient conditions for asymptotic stability of a (LDDS) on the basis of geometric considerations.

Owens, *Debeljkovic* (1985) investigated the geometric description of initial conditions that generate solution sequences $\{\mathbf{y}(k) : k = 0,1,\ldots\}$. The results were expressed directly in terms of matrices E and A, and avoid the need to introduce algebraic transformations as a prerequisite for stability analysis.

In that sense, the geometric approach provided a possibility for a basis-free analysis of dynamic properties of this class of systems. However, at that moment, there are no feasible methods for testing the conditions they obtained. The problem stems from the fact that the properties of certain matrices are required to hold only on a linear submanifold of the system's phase space and not in the whole phase space. Later on this problem was overcame by results of *Muller* (1993).

The results presented in the preceding section were established for only *regular* linear discrete descriptor systems.

Preliminaries

Now we turn our attention to the *irregular systems* of the same class of the systems.

The model (70) of a (LDDS) can be transformed into a more convenient form for the intended analysis by a suitable transformation.

For that purpose, consider a (LDDS) given (70).

It is assumed that the matrix E is in the form $E = \operatorname{diag}\left\{I_{n1},\ O_{n2}\right\}$, where I_p and 0_p stand for the $p \times p$ identity matrix and the $p \times p$ null matrix, respectively.

If the matrix E is not in this form, then the transformation $E \to TEQ$, where T and Q are suitable nonsingular matrices *Dai* (1989.a), can convert it to that form, and a broad class of systems (4.1) in this way can be brought to the form

$$\mathbf{x}_1(k+1) = A_1\mathbf{x}_1(k) + A_2\mathbf{x}_2(k), \tag{71.a}$$

$$0 = A_3\mathbf{x}_1(k) + A_4\mathbf{x}_2(k), \tag{71.b}$$

where the column vector $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^n$ is composed of subcolumns $\mathbf{x}_1(k) \in \mathbb{R}^{n1}$ and $\mathbf{x}_2(k) = \mathbb{R}^{n2}$, with

$n = n_1 + n_2$.

Let $k \in \mathcal{K}$ denote the current discrete moment. Here $\mathcal{K} = \{k_0, k_0+1, ...\}$ is a discrete time interval, where k_0 is an integer.

The matrices $A_i\,(i = 1,...,4)$ are of appropriate dimensions. The form (71) for (LDDS) is also known as the second equivalent form *Dai* (1989.a).

For the system (71), $\det E = 0$.

Since the system considered is time-invariant, it is sufficient to consider its current solution value $\mathbf{x}(k)$ as depending only on the current discrete moment k and initial value $\mathbf{x}_0$ at the initial moment k_0.

Hence let $\mathbf{x} = (k, \mathbf{x}_0)$ denote the value of a solution $\mathbf{x}$ of (71), at the moment $k \in \mathcal{K}$, which emanated from $\mathbf{x}_0$ at $k = k_0$. In an abbreviated form the value of solution $\mathbf{x}$ at the moment $k = k_0$ will be denoted by $\mathbf{x}(k)$.

When matrix pencil $\{(sE - A):\ s \in \mathbb{C}\}$ is regular, i.e. when there exists $s \in \mathbb{C}$ such that

$$\det(sE - A) \neq 0, \tag{72}$$

then solutions of (70) exist and are unique for the so-called consistent initial values $\mathbf{y}_0$. Moreover a closed form of the solutions exists *Campbell* (1980).

If A_4 is nonsingular, then the regularity conditions (72) for the system (71) is considerably simplified and reduces to

$$\begin{aligned}&\det\left(c\,I_{n_1} - A_1\right)\det\left(-A_4 - A_3\left(cI_{n_1} - A_1\right)^{-1} A_2\right)\\&= (-1)^{n2}\det A_4 \det\left(\left(cI_{n_1} - A_1\right) + A_2 A_4^{-1} A_3\right) \neq 0\end{aligned} \tag{73}$$

Debeljkovic et al. (1998).

It was proved by *Owens*, *Debeljkovic* (1985) that, under the conditions of an appropriate *Lemma*, $\mathbf{y}_0$ is a consistent initial condition for (70) if $\mathbf{y}_0$ belongs to a certain subspace W_k of consistent initial conditions. Moreover, $\mathbf{y}_0$ generates a discrete solution sequence $\{\mathbf{y}(k): k = 0,1,..\}$ (in this case $k_0 = 0$) such that $\mathbf{y}(k) \in W_k$ for all $k = 0,1,...$.

The subspace is given by $W_k = \mathbb{N}\left(1 - \hat{E}\hat{E}^D\right)$, where E^D is the so-called Drazin inverse of matrix E.

Note that W_k is independent of the particular choice of s, which can be any complex number such that $(sE - A)$ is nonsingular.

Remark 3. The following discussion of the consistent initial values is taken from *Bajic* (1995).

Let us denote the set of the consistent initial values of (71) by $\mathfrak{M}_1$. Consider the manifold $\mathfrak{M} \in \mathbb{R}^n$ determined by (71.b) as $\mathfrak{M} = \left\{\mathbf{x} \in \mathbb{R}^n:\ \ 0 = A_3\mathbf{x}_1 + A_4\mathbf{x}_2\right\}$.

For systems (71) in the general case, $\mathfrak{M}_1 \subseteq \mathfrak{M}$. Thus a consistent value $\mathbf{x}_0 = \begin{bmatrix}\mathbf{x}_{10}\\ \mathbf{x}_{20}\end{bmatrix}$ has to satisfy $\mathbf{0} = A_3\mathbf{x}_{10} + A_4\mathbf{x}_{20}$, or equivalently

$$\mathbf{x}_0 \in \mathfrak{M}_1 \subseteq \mathfrak{M} \equiv \mathbb{N}\left(\left(A_3,\quad A_4\right)\right). \tag{74}$$

However, if

$$rank\left(A_3,\ A_4\right) = rank\ A_4, \tag{75}$$

then $\mathfrak{M}_1 = \mathfrak{M} = \mathbb{N}\left(\left(A_3,\ A_4\right)\right)$, and the determination of $\mathfrak{M}_1$ requires no additional computation, except to convert (71) into the form (71).

In this case, under the assumption that $rank\ A_4 = r \leq n_2$, when $\mathbf{x}_0 \in \mathbb{N}\left(\left(A_3,\ A_4\right)\right)$ (i.e. when $\mathbf{x}_0$ satisfies (71.b), $(n_1 + n_2 - r)$ components of the vector $\mathbf{x}_0$ can be chosen arbitrarily in order to achieve consistent initial conditions of the system governed by (71). More precisely, all components of $\mathbf{x}_{10}$ and some (the case $rank\ A_4 = r \leq n_2$) or none (the case $rank\ A_4 = n_2$) of the components of $\mathbf{x}_{20}$ are free for choice. In both these cases, (71) is reducible to a lower-order normal form system with $\mathbf{x}_1(k)$ as the state variable and for the case $rank\ A_4 = r < n_2$ with some of the components of $\mathbf{x}_2(k)$ as free. In the later case, the reduced-order model will be of the form $\mathbf{x}_1(k+1) = F_1\mathbf{x}_1(k) + F_2\mathbf{x}_2^*(k)$, where F_1 and F_2 are matrices of appropriate dimensions, and where $\mathbf{x}_2^*(k)$ represents a vector composed of those components of $\mathbf{x}_2$ that can be chosen as free.

Thus the existence of solutions is guaranteed for such selected $\mathbf{x}_0$, and $\mathfrak{M}_1 = \mathfrak{M} = \mathbb{N}\left(\left(A_3,\ A_4\right)\right)$.

Note that the uniqueness of solutions is not guaranteed when $rank\ A_4 = r \leq n_2$.

Since the transformation from (70) to (71) is nonsingular, the convergence of solutions $\mathbf{y}(k)$ of (70) toward the origin of the phase space of (70) and that of $\mathbf{x}(k)$ toward the origin of (71) are equivalent problems.

Let $\|(\cdot)\|$ denote the euclidean vector norm or induced matrix norm.

The *potential (weak)* domain of attraction of the null solution $\mathbf{x}(k, \mathbf{0}) = \mathbf{0}\quad (k \in \mathcal{K})$ of (71) is defined by

$$\begin{aligned}\mathcal{A} = \{\mathbf{x}_0 \in \mathfrak{M}_1 :\ &\exists\{\mathbf{x}(k): k = 0,1,...\} \text{ satisfying } (71)\\ &\Rightarrow \mathbf{x}(0) = \mathbf{x}_0 \text{ and } \lim_{k\to\infty}|\mathbf{x}(k)| = 0\}\end{aligned} \tag{76}$$

The term *potential* or *weak* is used because solutions of (71) need not be unique. Thus, for evry $\mathbf{x}_0 \in \mathcal{A}$, there also may exist solutions which do not converge toward the origin.

Our task is to estimate the set $\mathcal{A}$.

Lyapunov's direct method will be used to obtain an underestimate $\mathcal{A}_U$ of the set $\mathcal{A}$ (i.e. $\mathcal{A}_U \subseteq \mathcal{A}$).

Our development will not require the regularity conditions (72) of the matrix pencil $\{(sE - A):\ s \in \mathbb{C}\}$, *Debeljkovic et al.* (1998).

Some other aspects of irregular singular systems were considered by *Dziuria*, *Newcomb* (1987) and *Dai* (1989.b).

Attraction of the origin and its potential domain

This section introduces a stability result which will be employed for the robustness analysis of the attraction of the origin.

We assume that the rank condition (75) holds, which implies $\mathfrak{M}_1 = \mathbb{N}\left(\left(A_3,\ A_4\right)\right)$ for the systems (71).

Consequently, there exists a matrix L which satisfies the matrix equation

$$0 = A_3 + A_4 L, \tag{77}$$

where 0 is the null matrix of the same dimensions as A_3.

From (75) and (77), it follows that, if solutions of (71) exist, then there will be solutions $\mathbf{x}(k)$ whose components satisfy

$$\mathbf{x}_2(k) = L\mathbf{x}_1(k) \quad (k \in \mathcal{K}). \tag{78}$$

Under the rank condition (75), it follows *Bajic* (1995) that, $\mathbb{N}\left(\left(L,\ -I_{n2}\right)\right) \subseteq \mathbf{N}\left(\left(A_3,\ A_4\right)\right)$. To show this, consider an arbitrary $\mathbf{x}^*(k) \in \mathbb{N}\left(\left(L,\ -I_{n2}\right)\right)$, i.e. $\mathbf{x}_2^*(k) = L\mathbf{x}_1^*(k)$, where L is any matrix that satisfies (77).

Then, multiplying (77) from the right by $\mathbf{x}_1(k)$ and using (78), one gets $\mathbf{0} = A_3\mathbf{x}_1^*(k) + A_4 L\mathbf{x}_2^*(k) = A_3\mathbf{x}_1^*(k) + A_4\mathbf{x}_2^*(k)$, which shows that $\mathbf{x}^*(k) \in \mathbb{N}\left(\left(L,\ -I_{n2}\right)\right)$.

Hence $\mathbb{N}\left(\left(L,\ -I_{n2}\right)\right) \subseteq \mathbf{N}\left(\left(A_3,\ A_4\right)\right)$.

Consequently those solutions of (71) that satisfy (78) also have to satisfy the constraints (71.b).

For all solutions of (71) for which (78) holds, the following conclusions are important, *Debeljkovic et al.* (1998).

(i) The solutions of (71) belong to the set $\mathbb{N}\left(\left(L,\ -I_{n2}\right)\right)$.

(ii) If, under the rank conditions (75), the existence of a solution $\mathbf{x}(k)$ of (71) which satisfies (78) and converges to the origin is proved, then the potential domain of attraction of the origin for (71) can be underestimated by

$$\mathcal{A}_{\mathrm{U}} = \mathbf{N}\left(\left[L, -I_{n2}\right]\right) \subseteq \mathcal{A}. \tag{79}$$

Let *pd* and *nd* stand for positive definite and negative definite, respectively.

For the system (71), the Lyapunov function can be selected as

$$V\left(\mathbf{x}(k)\right) = \mathbf{x}_1^{\mathrm{T}}(k) H\mathbf{x}_1(k), \tag{80}$$

where H is pd symmetric.

The total time difference of $V\left(\mathbf{x}(k)\right)$ is defined by the expression

$$\Delta V\left(\mathbf{x}(k)\right) = V\left(\mathbf{x}(\mathrm{k+1})\right) - V\left(\mathbf{x}(k)\right), \tag{81}$$

and its value, calculated along the solutions of (71), is then

$$\begin{aligned}\Delta V\left(\mathbf{x}(k)\right) &= \mathbf{x}_1^{\mathrm{T}}(k+1) H\mathbf{x}_1(k+1) - \mathbf{x}_1^{\mathrm{T}}(k) H\mathbf{x}_1(k)\\ &= \alpha_1 + \alpha_2 + \alpha_2^{\mathrm{T}} + \alpha_3 - \mathbf{x}_1^{\mathrm{T}}(k) H\mathbf{x}_1(k)\end{aligned} \tag{82}$$

where

$$\begin{aligned}\alpha_1 &= \mathbf{x}_1^{\mathrm{T}}(k) A_1^{\mathrm{T}} H A_1 \mathbf{x}_1(k)\\ \alpha_2 &= \mathbf{x}_2^{\mathrm{T}}(k) A_2^{\mathrm{T}} H A_1 \mathbf{x}_1(k)\\ \alpha_3 &= \mathbf{x}_2^{\mathrm{T}}(k) A_2^{\mathrm{T}} H A_2 \mathbf{x}_2(k)\end{aligned}\ . \tag{83}$$

Employing (78) and (82), one obtains

$$\begin{aligned}&\Delta V\left(\mathbf{x}(k)\right) =\\ &\mathbf{x}_1^{\mathrm{T}}(k)\left(\left(A_1 + A_2 L\right)^{\mathrm{T}} H\left(A_1 + A_2 L\right) - H\right)\mathbf{x}_1(k) =\\ &-\mathbf{x}_1^{\mathrm{T}}(k) Z\mathbf{x}_1(k),\end{aligned} \tag{84}$$

where

$$Z = -\left(A_1 + A_2\right)^{\mathrm{T}} H\left(A_1 + A_2\right) + H, \tag{85}$$

note that Z is a real symmetric matrix.

Since H is *pd* matrix, $V\left(\mathbf{x}(k)\right)$ is a *pd* function with respect to $\mathbf{x}_1(k)$.

Thus, if Z is pd, then $V\left(\mathbf{x}(k)\right)$ would tend to zero as $k \to \infty$, provided that the solutions $\mathbf{x}(k)$ exist when $k \to \infty$.

This would imply $\|\mathbf{x}_1(k)\| \to 0$ when $k \to \infty$.

Finding the pair of matrices H and Z to comply with these requirements can be achieved by means of a *discrete Lyapunov matrix equation*

$$A_L^{\mathrm{T}} H A_L - H = -Z, \tag{86}$$

with

$$A_L = A_1 + A_2 L, \tag{87}$$

where Z can be an arbitrary real symmetric pd matrix, and the corresponding H ,which is a symmetric pd matrix, can be found as a unique solution of (86), if and only if A_L is a discrete stable matrix, i.e. a matrix whose eigenvalues lie in the open unit circle of the complex plane.

Note that the order of H is $n_1 \times n_1$, so that (86) can be considered as reduced-order discrete *Lyapunov matrix equation* when compared to the number of the components of $\mathbf{x}(k)$ for the system given (2.1).

Theorem 25. Let (75) hold.

Then the underestimate $\mathcal{A}_{\mathrm{U}}$ of the potential domain $\mathcal{A}$ of attraction of the null solution DLDS (71) is determined by (79), provided that L is any solution of (77) and $A_L = A_1 + A_2 L$ is a discrete stable matrix.

Moreover, $\mathcal{A}_{\mathrm{U}}$ contains more than one element, *Debeljkovic et al.* (1998).

Robustness of attraction property

Physical systems are very often modelled by idealized and simplified models, so that information obtained on the basis of such models is not always sufficiently accurate. This motivates investigating the robustness of properties of the examined system with respect to model inaccuracies.

Quantitative measures of robustness for multivariable systems in the presence of time-dependent nonlinear perturbations were first investigated in *Patel*, *Toda* (1980). The bounds were obtained for the perturbation vector such that the nominal system remains stable.

The stability robustness of linear discrete-time system in the time domain using Lyapunov's approach was treated by *Kolla et al.* (1989). Bounds on linear time-dependent perturbations that maintain the stability of an asymptotically stable nominal system are obtained for both structured and unstructured independent perturbations.

A general overview of results concerning problems of stability robustness in the area of nonlinear time-dependent descriptor systems was published by *Bajic* (1992), while some other robustness results for linear discrete descriptor systems are presented by *Dai* (1989.a).

In the sequel we presents some results on the attraction of the origin for a **(**LLDS).

The results are similar to those given for continuous linear singular systems by *Bajic et al.* (1992). In this section the Lyapunov stability robustness of both *regular* and *irregular* (LLDS) is considered. The bounds on the perturbation matrix are determined so that the attraction property of the origin of the nominal system is preserved for all perturbation matrices of a specific class.

The results presented here rely to some extent on those given by *Bajic* (1995) and *Debeljkovic al.* (1995).

This section, also, gives results on the robustness of attraction of the origin under unstructured and structured perturbation in the model of (LLDS) .

We consider a perturbed version of (4.1.1) which has the form

$$E\mathbf{y}(k+1)=A\mathbf{y}(k)+A_P\mathbf{y}(k),\quad \mathbf{y}(k_0)=\mathbf{y}_0,\qquad (88)$$

where the matrix A_P represents the perturbations in the model.

To analyze the robustness of attraction of the origin of (88), we consider (88) transformed to the form

$$\mathbf{x}_1(k+1)=(A_1+B_1)\mathbf{x}_1(k)+(A_2+B_2)\mathbf{x}_2(k),\quad (89a)$$

$$0=A_3\mathbf{x}_1(k)+A_4\mathbf{x}_2(k)+B_{34}\mathbf{x}(k),\qquad (89.b)$$

where $\mathbf{x}(k)=\begin{bmatrix}\mathbf{x}_1(k)\\ \mathbf{x}_2(k)\end{bmatrix}$ need not represent the original variables $\mathbf{y}(k)$ of the system (5.1), *Debeljkovic et al.* (1998).

To simplify the formulation of results on stability robustness, we introduce the following assumption.

Assumption 1. The matrix B_{34} in (89b) is null.

To perform analysis of robustness for the system (89), we employ the Lyapunov function $V(\mathbf{x}(k))$ denned by (80).

Let the rank condition (75) hold.

Then, taking into account (78) and (83), the expression $\Delta V(\mathbf{x}(k))$ given by (81) along the solutions of (89) is obtained as

$$\begin{aligned}\Delta V(\mathbf{x}(k))&=\mathbf{x}_1^{\mathrm{T}}(k)\left((\Theta_1+\Theta_2L)^{\mathrm{T}}H(\Theta_1+\Theta_2L)\right)\mathbf{x}_1(k)\\ &\quad-\mathbf{x}_1^{\mathrm{T}}(k)H\mathbf{x}_1(k)\\ &=\mathbf{x}_1^{\mathrm{T}}(k)Z_P\mathbf{x}_1(k),\end{aligned}\qquad (90.a)$$

$$\begin{aligned}Z_P&=(\Theta_1+\Theta_2L)^{\mathrm{T}}H(\Theta_1+\Theta_2L)-H\\ \Theta_i&=A_i+B_i,\quad i=1,2\end{aligned},\qquad (90.b)$$

note that Z_P is a real symmetric matrix.

Define the singular values of a real matrix X to be the square roots of the eigenvalues of $X^{\mathrm{T}}X$.

Let $\sigma_M\{X\}$ denote the maximal singular value of a matrix X , while $\lambda_M(S)$ and $\lambda_m(S)$ stand for maximal eigenvalue and minimal eigenvalue of a symmetric matrix S respectively.

Note that $\sigma_M\{S\}$ is also the spectral norm of S .

Now we are in position to state the following result which concerns the *unstructured perturbations* in model (89).

Theorem 26. Let the rank condition (75), *Assumption* 1 and all conditions of *Theorem* 25 hold.

Let Z and H be two real, symmetric, and pd matrices satisfying *discrete Lyapunov matrix equation* (86), and let A_L be the matrix from (86).

Then the underestimate $\mathcal{A}_{\mathrm{U}}$ of the potential domain of attraction of system (**5.2**) is determined by (79) if

$$\sigma_{\mathrm{M}}\{B_L\}<-\sigma_{\mathrm{M}}\{A_L\}+\left(\sigma_{\mathrm{M}}^2\{A_L\}+\frac{\lambda_{\mathrm{M}}(Z)}{\lambda_{\mathrm{M}}(H)}\right)^{\frac{1}{2}},\qquad (91)$$

where $B_L=B_1+B_2L$.

Moreover, $\mathcal{A}_{\mathrm{U}}$ contains more than one element, *Debeljkovic et al.* (1998).

In order to cater for the *structured perturbations* in the model (5.2), we introduce the following assumption.

Assumption 2.

Let $B_L=B_1+B_2L=[b_{ij}:i,j=1,...,n_1]$, where L is any solution of (77).

Let the constraints $|b_{ij}|\le\pi_{ij}$ hold.

Theorem 27. Let the rank condition (75), *Assumption* 1 and *Assumption* 2 and all conditions of *Theorem* 25 hold.

Let Z and H be two real symmetric *pd* matrices satisfying the *discrete Lyapunov matrix equation* (86), and let A_L be the matrix from (86).

Let $\wp=\max_{1\le i,j\le n_1}\wp_{ij}$.

Then the underestimate $\mathcal{A}_{\mathrm{U}}$ of the potential domain of attraction of system (**5.2**) is determined by (79) if

$$\wp<\frac{1}{n_1}\left(-\sigma_{\mathrm{M}}\{A_L\}+\left(\sigma_{\mathrm{M}}^2\{A_L\}+\frac{\lambda_{\mathrm{M}}(Z)}{\lambda_{\mathrm{M}}(H)}\right)^{\frac{1}{2}}\right).\qquad (92)$$

Moreover $\mathcal{A}_{\mathrm{U}}$ contains more than one element, *Debeljkovic et al.* (1998).

Consider the linear descriptive discrete system represented by its state space model in the following form

$$E\mathbf{y}(k+1)=(A)\mathbf{y}(k)+\mathbf{f}_{\mathrm{p}}(\mathbf{y}(k)),\quad \mathbf{y}(k_0)=\mathbf{y}_0\qquad (93)$$

usually with $k_0=0$.

Function $\mathbf{f}_p(\mathbf{y})$ represents the vector of general system perturbations.

Introducing a suitable nonsingular linear transformation

$$T\mathbf{x}(k)=\mathbf{y}(k),\quad \det T\ne 0,\qquad (94)$$

a broad class of linear descriptive discrete systems, (5.6), can be transformed in to the following form

$$\mathbf{x}_1(k+1)=A_1\mathbf{x}_1(k)+A_2\mathbf{x}_2(k)+\mathbf{f}_{1p}(T\mathbf{x}),\qquad (95.a)$$

$$\mathbf{0}=A_3\mathbf{x}_1(k)+A_4\mathbf{x}_2(k)+\mathbf{f}_{2p}(T\mathbf{x}),\qquad (95.b)$$

$$ET=\begin{pmatrix}I&0\\0&0\end{pmatrix},\quad AT=\begin{pmatrix}A_1&A_2\\A_3&A_4\end{pmatrix}\qquad (96)$$

where $\mathbf{x}(t)=\left[\mathbf{x}_1^{\mathrm{T}}(t)\ \ \mathbf{x}_2^{\mathrm{T}}(t)\right]^T \in \mathbb{R}^n$ is the state decomposed vector, with $\mathbf{x}_1^{\mathrm{T}}(t)\in\mathbb{R}^{n1}$ and $\mathbf{x}_2^{\mathrm{T}}(t)\in\mathbb{R}^{n2}$ with $n=n_1+n_2$.

The matrices A_i, $i=1,2...,4$, have appropriate dimensions.

Moreover, it is clear that

$$\det(ET)\det E\det T=0 \tag{97}$$

with $\det T\neq 0$.

Under the applied transformation, the perturbation vector can be expressed as:

$$\begin{aligned}&\mathbf{f}_p(\mathbf{y}(k))=\mathbf{f}_p(T\mathbf{x}(k))\\&=\mathbf{f}(\mathbf{x}(k))=\begin{bmatrix}\mathbf{f}_1(\mathbf{x}(k))\\ \mathbf{f}_2(\mathbf{x}(k))\end{bmatrix}=\begin{bmatrix}\mathbf{f}_{1p}(T\mathbf{x}(k))\\ \mathbf{f}_{2p}(T\mathbf{x}(k))\end{bmatrix}.\end{aligned} \tag{98}$$

The vector $\mathbf{f}(\mathbf{x}(k))$ being decomposed on two subvectors $\mathbf{f}_1(\mathbf{x}(k))$ and $\mathbf{f}_2(\mathbf{x}(k))$.

Stability robustness of linear discrete descriptor systems with unstructured perturbations

Let us consider the system governed by (98) under the following assumption.

Assumption 3.The perturbation vector can be adopted in the following form

$$\mathbf{f}(\mathbf{x}(k))=\left[\mathbf{f}_1^T(\mathbf{x}(k))\ \ \mathbf{0}^T\right]^T. \tag{99}$$

Quasi-Lyapunov function is adopted as in the form given in (80).

Having in mind that H is a symmetric matrix, implying that

$$\mathbf{x}_1^{\mathrm{T}}(k)A_1H\,\mathbf{f}_1(\mathbf{x}(k))=\mathbf{f}_1^T(\mathbf{x}(k))HA_1\mathbf{x}_1(k), \tag{100}$$

and

$$\mathbf{x}_2^{\mathrm{T}}(k)A_2^{\mathrm{T}}HA_1\mathbf{x}_1(k)=\mathbf{x}_1^{\mathrm{T}}(k)A_1^{\mathrm{T}}HA_2\mathbf{x}_2(k), \tag{101}$$

the forward difference is given with

$$\begin{aligned}&\Delta V(\mathbf{x}(k))=\mathbf{x}_1^T(k)\left(A_L^THA_L-H\right)\mathbf{x}_1(k)\\&+2\mathbf{x}_1^T(k)A_L^TH\,\mathbf{f}_1(\mathbf{x}(k))+\mathbf{f}_1^T(\mathbf{x}(k))H\,\mathbf{f}_1(\mathbf{x}(k))\end{aligned}. \tag{102}$$

Theorem 28. Let the rank condition (75), be satisfied and let L be any solution of the matrix equation (77), so that (78) is valid, too.

The perturbation subvector may be adopted in the following manner

$$\mathbf{f}_1(\mathbf{x}(k))=\mathbb{P}\,\mathbf{x}_1(k), \tag{103}$$

with $\mathbb{P}$ being the perturbation matrix.

Let *Assumption* 3 be satisfied.

Then the system (**5.8**) possesses a subset of solutions *covergent to the origin of phase space* if the following condition is satisfied:

$$\begin{aligned}&\sigma_{\max}(\mathbb{P})<-\sigma_{\max}(A_L)\\&+\left(\left(\sigma_{\max}(A_L)\right)^2+\frac{\sigma_{\min}(Z)}{\sigma_{\max}(H)}\right)^{\frac{1}{2}},\end{aligned} \tag{104}$$

$A_L=A_1+A_2L$ being a *discrete stable matrix* (87), with a real symmetric positive definite matrix $H^T=H>0$ being the solution of the *discrete Lyapunov matrix equation* (86) for any symmetric matrix $Z^T=Z>0$, *Debeljkovic et al* (2003).

Remark 4. As well as in the case of time-invariant time continuous systems, *Patel and Toda*, (1980) the constraint given with (104) has its maximum when $Z=I$, i.e. when $\sigma_{\min}(Z)=1$, *Debeljkovic et al* (2003).

Stability robustness of discrete descriptor systems with structured perturbations

Structural independent perturbations

Let the *rank condition* (75) be satisfied, under the condition that matrix L is any solution of the matrix equation (77), so (78) is also fulfilled.

The perturbation subvector, having in mind *Assumption* 3, may be adopted in a convenient form such as (103)

Constants π_{ij} and π are defined in the following manner.

The elements $\wp_{ij}(k)$ of the matrix $\mathbb{P}(k)$ fulfill

$$\wp_{ij}(k)\leq\ \left|\wp_{ij}\right|_{\max}=\pi_{ij},\quad \pi=\max\ \pi_{ij}. \tag{105}$$

Theorem 29. System (**5.8**) is stable if the following condition is satisfied

$$\pi<\frac{1}{\mathrm{n}}\left(-\sigma_{\max}(A_L)+\left(\left(\sigma_{\max}(A_L)\right)^2+\frac{\sigma_{\min}(Z)}{\sigma_{\max}(H)}\right)^{\frac{1}{2}}\right) \tag{106}$$

It should be noted that $\sigma_{\max}(\mathbb{P}_1)=n\pi$ for any matrix all elements of which are π, and it is obvious that $\sigma_{\max}(\mathbb{P})\leq\sigma_{\max}(\mathbb{P}_1)$ since $\pi_{ij}\leq\pi$, so the result follows directly from *Theorem* 28, *Debeljkovic et al* (2003).

Theorem 30. System (5.8) is stable if the following condition is satisfied

$$\begin{aligned}\pi<&-\frac{\sigma_{\max}\left(\mathfrak{A}^T|HA_L|\right)_S}{\sigma_{\max}\left(\mathfrak{A}^T|H|\mathfrak{A}\right)}\\&+\left(\left(\frac{\sigma_{\max}\left(\mathfrak{A}^T|HA_L|\right)_S}{\left(\mathfrak{A}^T|H|\mathfrak{A}\right)}\right)^2+\frac{\sigma_{\min}(Z)}{\sigma_{\max}\left(\mathfrak{A}^T|H|\mathfrak{A}\right)}\right)^{1/2}\end{aligned} \tag{107}$$

the matrix $\mathfrak{A}$ having all positive elements, so that

$$\left|\mathbb{P}(k)\right|\leq\pi\,\mathfrak{A}. \tag{108}$$

Debeljkovic et al (2003).

Remark 5. If the matrix $\mathbb{P}(k)$ is known or we can estimate the maximum values of all elements in (5.18), then the matrix $\mathfrak{A}$ may be formed as: $\mathfrak{A}=\left[\mathrm{u}_{ij}\right]$, $\mathrm{u}_{ij}=\pi_{ij}/\pi$,.

In this case it is obvious that $0\leq\mathrm{u}_{ij}\leq 1$,

If the perturbations $\wp_{ij}$ are not explicitly known, u_{ij} can be used simply as a positive real number.

If the perturbation $\wp_{ij}$ of a_{Lij} elements of the matrix A_L is equal to zero, then directly follows $\mathrm{u}_{ij}=0$, *Debeljkovic et al* (2003).

Structural dependent perturbations

In some classes of problems there are problems with a relatively small number of unknown parameters.

In such cases, the uncertain time-dependent matrix $\mathbb{P}(k)$ may be formed in the following way

$$\mathbb{P}(k) = k_i \mathcal{P}_i , \tag{109}$$

where $\mathcal{P}_i$, are constant matrices and k_i, are uncertain parameters which can vary independently.

Let us define $(mn \times mn)$ and $(n \times n)$ symmetric matrices

$$H_{pp} = \begin{pmatrix} \left(\mathcal{P}_1^T H \mathcal{P}_1\right)_S & \left(\mathcal{P}_1^T H \mathcal{P}_2\right)_S & \cdots & \left(\mathcal{P}_1^T H \mathcal{P}_{\mathrm{m}}\right)_S \\ \left(\mathcal{P}_2^T H \mathcal{P}_1\right)_S & \left(\mathcal{P}_2^T H \mathcal{P}_2\right)_S & \cdots & \left(\mathcal{P}_2^T H \mathcal{P}_{\mathrm{m}}\right)_S \\ \vdots & \vdots & \vdots & \vdots \\ \left(\mathcal{P}_1^T H \mathcal{P}_{\mathrm{m}}\right)_S & \left(\mathcal{P}_2^T H \mathcal{P}_{\mathrm{m}}\right)_S & \cdots & \left(\mathcal{P}_m^T H \mathcal{P}_{\mathrm{m}}\right)_S \end{pmatrix} \tag{110}$$

and

$$H_{api} = \left(A_L^T H \mathcal{P}_i\right)_S \tag{111}$$

Theorem 31. System (**5.8**) with the structural perturbation term given by (109) possesses the subset of solutions converge to the origin of phase space if the following condition is satisfied

$$\sum_{i=1}^{m} |k_i|^2 \sigma_{\max}\left(H_{pp}\right) + 2\sum_{i=1}^{m} |k_i| \sigma_{\max}\left(H_{api}\right) < \sigma_{\min}(Z) \tag{112}$$

or

$$\left|k_{ij}\right| < -\left(\frac{\sigma_{\max}\left(H_{api}\right)}{m\sigma_{\max}\left(\left|H_{pp}\right|\right)}\right) + \left(\left(\frac{\sigma_{\max}\sum_{i=1}^{m}\left|H_{api}\right|}{m\sigma_{\max}\left(\left|H_{pp}\right|\right)}\right)^2 + \frac{\sigma_{\min}(Z)}{m\sigma_{\max}\left(\left|H_{pp}\right|\right)}\right)^{1/2}, \tag{113}$$

Debeljkovic et al (2003).

Conclusion

A detailed, chronological survey of results concerning Lyapunov stability of linear discrete descriptor systems is presented, covering period form 1985 up to 2003.

Moreover the stability robustness consideration of this class of systems is also included both for regular and irregular descriptor systems.

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Stabilnost linearnih diskretnih deskriptivnih sistema u smislu Ljapunova: Pregled rezultata

Ovaj rad daje detaljan hronološki pregled radova i rezultata mnogih autora na polju izučavanja stabilnosti sistema u smislu Ljapunova za posebne klase diskretnih deskriptivnih sistema. U tom smislu diskretna Ljapunovljeva jednačina, izvedena za diskretne implicitne sisteme, je od posebnog značaja.
Problem robusnosti stabilnosti takođe je razmatran.
Ovaj pregled rezultata pokriva period od 1985. god. do današnjih dana i ima jasnu nameru da predstavi osnovne koncepte i glavne doprinose ostvarene u pomenutom periodu u celome svetu a koji su publikovani i respektabilnim međunarodnim časopisima ili saopšteni na prestižnim internacionalnim konferencijama.

Ključne reči: **diskretni sistem, deskriptivni sistem, linearni sistem, asimptotska stabilnost, stabilnost Ljapunova, robusnost.**

Stabilité des systèmes linéaires discrets et descriptifs dans le sens de Lyapunov: tableau des résultats

Ce papier donne un tableau détaillé et chronologique des travaux et des résultats de nombreux auteurs en matière de stabilité des systèmes dans le sens de Lyapunov, pour les classes particulières des systèmes discrets descriptifs. L'équation discrète de Lyapunov, dérivée pour les systèmes discrets implicites est, dans ce sens, d'une importance particulière. Le problème de la robustesse est traîté aussi. Ce tableau des résultats comprend la période depuis 1985 jusqu'à nos jours; son but principal est de présenter les concepts de base et les contributions principales dans le monde entier pour la période citée et qui sont publiés dans les revues internationales renommées ou présentés aux conférences internationales réputées.

Mots clés: **système discret, système descriptif, système linéaire, stabilité asymptotique, stabilité de Lyapunov, robustesse.**