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The Stability of Linear Continuous Singular Systems over the Finite Time Interval: An Overview

Dragutin Lj. Debeljković, PhD (Eng)[1)]
Nemanja S. Višnjić, MSc (Eng)[1)]
Milmir M. Pješćić, MSc (Eng)[1)]

This paper gives a detailed overview of the work and the results of many authors in the area of Non-Lyapunov (finite time stability, technical stability, practical stability, final stability) of particular class of linear systems. The stability robusteness problem has been also treated and presented.
This survey covers the period from 1985 to nowadays and has a strong intention to present the main concepts and contributions that have been derived during the mentioned period through out the world, published in respectable international journals or presented at workshops or prestigious conferences.

Key words: **continuous system, singular system, linear system, system stability, Lyapunov stability, finite time interval.**

Introduction

IT should be noticed that in some systems their character of dynamic and static state must be the considered at the same time. Singular systems (also, referred to as degenerate, descriptor, generalized, differential - algebraic systems or semi – state) are the dynamics of which is governed by a mixture of algebraic and differential equations. Recently many scholars have paid a lot of attention to singular systems and have obtained many positive results. The complex nature of singular systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control.

In practice, there is not only an interest in system stability (e.g. in sense of Lyapunov), but also in the bounds of system trajectories. A system could be stable but completely useless because it possesses undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain sub-sets of state-space, which are *a priori* defined in a given problem. Besides that, it is of particular significance to consider the behavior of dynamic systems only over a finite time interval.

These bound properties of system responses, i.e. the solution of system models, are very important from the engineering point of view. Therefore, numerous definitions of the so-called technical and practical stability were introduced. Roughly speaking, these definitions are essentially based on the predefined boundaries for the perturbation of initial conditions and the allowable perturbation of system response. In engineering applications of control systems, this fact becomes very important and sometimes crucial, for the purpose of characterizing in advance, in quantitative manner, possible deviations of the system response. Thus, the analysis of these particular bound properties of the solutions is an important step, which precedes the design of control signals, when finite time or practical stability control is concerned. In the context of practical stability for linear singular systems, various results were first obtained in *Debeljkovic, Owens* (1985) and *Owens, Debeljkovic* (1986).

Let the linear singular system in *free regime* be governed by:

$$E\dot{\mathbf{x}}(\mathrm{t}) = A\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1}$$

where $E,\, A \in \mathbb{R}^{n\times n}$ are the constant matrices, with E singular, $\mathbf{x}(t) \in \mathbb{R}^n$ is the phase vector (i.e. generalized state-space vector), $\mathbf{x}_0$ is the consistent initial condition.

The systems defined in (1) are usually known as *singular* $(\det E = 0)$, *descriptor* (the way in which the system is initially described), *semi-state, differential-algebraic equations* and *generalized state space systems.* They occur naturally in many physical applications such as electrical networks, aircraft and robot dynamics, neutral delay and large-scale systems, economics and optimization problems, biology, etc.

The survey of the updated results in this area and the broad bibliography can be found in *Campbell* (1980); *Verghese et al.* (1981); *Lewis* (1986); *Campbell* (1990), and in the special issues on semi-state systems of the journal *Circuits, Systems and Signal Processing* (1986, 1989).

Notations and preliminaries

The dynamical behavior of the systems described by (1) is defined over the time interval $J = \{t : t_0 \le t \le t_0 + T\}$, where the quantity T may be either a positive real number or the symbol $+\infty$, so finite time stability and practical stability can be treated simultaneously.

[1)] University of Belgrade, Faculty of Mechanical Engineering, Knjeginje Marije 16, 11120 Belgrade, SERBIA

The time invariant sets, used as bounds of system trajectories, are assumed to be open, connected and bounded.

Let the index β stands for the set of all allowable states of the system and the index α for the set of all initial states of the system, such that the set $S_\alpha \subseteq S_\beta$.

In general, it may be written:

$$S_\rho = \left\{ \mathbf{x} : \left\| \mathbf{x}(t) \right\|_Q < \rho \right\}, \quad \mathbf{x}(t) \in W_k \setminus \{0\} \tag{2}$$

where Q will be assumed to be a symmetric, positive definite, real matrix and where W_k denotes the sub-space of consistent initial conditions generating the smooth solutions. The vector of initial conditions is consistent if there is a continuous, differentiable solution of (1). A geometric treatment *Owens, Debeljkovic* (1985) yields W_k as the limit of the sub-space algorithm:

$$W_0 = \mathbb{R}^n, \quad W_{j+1} = \mathbf{A}^{-1}\left(\mathbf{E}W_j\right), \quad j \geq 0 \tag{3}$$

where $\mathbf{A}^{-1}(\cdot)$ denotes the inverse image of $(\cdot)$ under the operator A.

Campbell et al. (1976) have shown that the sub-space W_k, is the set of vectors satisfying:

$$\left(I - \hat{E}^D \hat{E}\right)\mathbf{x}_0 = \mathbf{0}, \text{ or } W_k = \aleph\left(I - \hat{E}^D \hat{E}\right), \tag{4}$$

where $\hat{E} = \left(cE - \mathbf{A}\right)^{-1} E$. c is any complex scalar such that:

$$\det\left(cE - A\right) \neq 0 \text{ or } W_k \cap \aleph(E) = \{0\} \tag{5}$$

which guarantees uniqueness of the solutions generated by W_k.

The null space[1] of the matrix F is denoted by $\aleph(F)$ and the superscript "D" is used to indicate the Drazin inverse.

If F is $n \times n$ then matrix F^D denotes the Drazin inverse with the following properties:

$$FF^D = F^D F, \quad F^D F F^D = F^D, \quad F^D F^{k+1} = F^k, \tag{6}$$

where k *is* the index of $\mathbf{F}$ defined to be the smallest non-negative integer such that:

$$F^{j+1} = rank\, F^j \tag{7}$$

Let $\|\mathbf{x}\|_{(\cdot)}$ be any vector norm (i. g. $\cdot = 1, 2, \infty$) and $\|(\cdot)\|$ the matrix norm induced by this vector.

Basic notations

$\mathbb{R}$	Real vector space
$\mathbb{C}$	Complex vector space
C	Complex plane
I	Unit matrix
F	$= (f_{ij}) \in \mathbb{R}^{n \times n}$, real matrix
F^T	Transpose of the matrix F
$F > 0$	Positive definite matrix
$F \geq 0$	Positive semi - definite matrix
$\Re(F)$	Range of the matrix F
N	Nilpotent matrix
$\mathbb{N}(F)$	Null space (kernel) of the matrix F
$\lambda(F)$	Eigenvalue of the matrix F
$\sigma_{(\ )}(F)$	Singular values of matrix F
$\sigma\{F\}$	Spectrum of the matrix F
$\|F\|$	Euclidean matrix norm $\| F \| = \sqrt{\lambda_{\max}\left(A^T A\right)}$
F^D	Drazin inverse of the matrix F
$\Rightarrow$	Follows
$\mapsto$	Such that

Matrix measure has been widely used in the literature when dealing with stability of time delay systems.

The matrix measure[2] μ for any matrix $F \in \mathbb{C}^{n \times n}$ is defined as follows

$$\mu(F) \triangleq \lim_{h \to 0^+} \frac{\|I + hF\| - 1}{h}. \tag{8}$$

Also,

$$-\mu(-F) \triangleq \lim_{h \to 0^+} \frac{1 - \|I + hF\|}{h}. \tag{9}$$

From *Mori* (1988), the following inequality holds:

$$-\| F \|_2 \leq -\mu(-F) \leq \mu(F) \leq \| \mathbf{F} \|_2. \tag{10}$$

The matrix measure defined in (8) can be subdefined in three different ways, depending on the norm utilized in its definitions.

$$\mu_1(F) = \sup_k \left(\operatorname{Re}(f_{kk}) + \sum_{\substack{i=1 \\ i \neq k}}^{n} |f_{ik}| \right), \tag{11}$$

$$\mu_2(F) = \frac{1}{2} \max_i \lambda_i \left(F^* + F\right), \tag{12}$$

$$\mu_\infty(F) = \sup_i \left(\operatorname{Re}(f_{ii}) + \sum_{\substack{k=1 \\ k \neq i}}^{n} |f_{ik}| \right). \tag{13}$$

Coppel (1965).

The upper index $*$ denotes the transpose conjugate. In the case *of* $F \in \mathbb{R}^{n \times n}$ it follows $F^* = F^T$, where the index T denotes the transpose.

Consider a linear singular system (LSS) (1).

It is assumed that the matrix E is in the form $E = diag\left(I_{n1}, O_{n2}\right)$. If the matrix E is not in this form, then in many cases it can be transformed to the required form by left multiplication with a (nonsingular) matrix T, and such transformation will not alter the original phase variables $\mathbf{x}(t)$.

[1] In literature the term *Kernel* is very often used, but for the sake of the term *null space* is more correct as it is related clarity, to matrices and the term *Kernel* is related to transformations.

[2] In literature the term *logarithmic matrix norm,* is also used although it can be a negative number which is not a property of a norm.

The resulting (LSS) model will thus be given as

$$\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + A_2\mathbf{x}_2(t) \tag{14}$$

$$\mathbf{0} = A_3\mathbf{x}_1(t) + A_4\mathbf{x}_2(t) \tag{15}$$

where $\mathbf{x}(t) = \left[\mathbf{x}(t)_1^T,\ \mathbf{x}(t)_2^T\right]^T \in \mathbb{R}^n$ is a decomposed vector, with $\mathbf{x}_1(t), \in \mathbb{R}^{n1}$, $\mathbf{x}_2(t) \in \mathbb{R}^{n2}$, and $n = n_1 + n_2$.

The matrices $A_i,\ i = 1,2,3,4$, are of appropriate dimensions. For the system (14 - 15) the $\det E = 0$.

At the expense of changing the original phase variables of the system (1), a much broader class of SLS (1) can be transformed to the form (14-15) using $T^{-1}ET$ transformation of the matrix E, where T is a nonsingular matrix. For that reason we analyze (14-15) instead of the system (1).

The solutions of the (LSS) models in this investigation are continuously differentiable functions of time t which satisfy the considered equations of the model. Since for the (LSS) not all initial values $\mathbf{x}_0$ of $\mathbf{x}(t)$ will generate a smooth solution, those that generate such solutions (continuous to the right) are called consistent.

The value of a particular solution at the moment t, which at the moment $t = 0$ passes through the point $\mathbf{x}_0$, is denoted $\mathbf{x}(t, \mathbf{x}_0)$ in the abbreviated notation $\mathbf{x}(t)$.

Note that if uniqueness of solutions is not guaranteed, then more than one solution of (14-15) can pass through the point x_0 at the moment $t = 0$.

The set of all points S_i, in the phase space $\mathbb{R}^n$, $S_i \subseteq \mathbb{R}^n$, which generates smooth solutions, can be determined via the Drazin inverse technique.

Also, $\|\mathbf{y}(t)\|$ represents the Euclidean norm of the matrix or the vector $\mathbf{y}(t)$.

The solutions of the (LSS) can be unique (them the system is called regular) or nonunique (when the system is regarded as irregular).

Time invariant singular systems

Stability definitions

Definition 1. System (**1**) *is finite time stable* w.r.t $\{J,\alpha,\beta,Q\}$, $\alpha < \beta$, if $\forall\mathbf{x}(t_0) = \mathbf{x}_0 \in W_k$, satisfying $\|\mathbf{x}_0\|_Q^2 < \alpha$, implies $\|\mathbf{x}(t)\|_Q^2 < \beta, \forall t \in J$. *Debeljkovic, Owens* (1985).

Definition 2. System (**1**) is *finite time instable* w.r.t. $\{J,\alpha,\beta,\mathbf{Q}\}$, $\alpha < \beta$, if for $\forall\mathbf{x}(t_0) = \mathbf{x}_0 \in W_k$, satisfying $\|\mathbf{x}_0\|_Q^2 < \alpha$, there is $t^* \in J$ implying $\left\|\mathbf{x}\left(t^*\right)\right\|_Q^2 \geq \beta$, *Owens, Debeljkovic* (1986).

Further on, I_k and 0_k will represent the identity matrix and the null matrix of dimension $k \times k$, respectively.

Let

$$A_G(\gamma) = \left\{\mathbf{x}(t) \in \mathbb{R}^n : \|\mathbf{x}_k(t)\|_G < \gamma\right\},\quad G = G^T > 0.$$

Definition 3. System (**1**) is practically stable w.r.t. $\{J,\alpha,\beta,G\}$ if $\forall\mathbf{x}_0 \in S_\alpha \cap A_G(\alpha)$ at the moment $t = 0$, it follows that $\forall t \in J$, every solution, $\mathbf{x}(t,\mathbf{x}_0) \in A_G(\beta)$, *Debeljkovic et al.* (1995).

If the intention is to consider not all, but only some of the solutions of the system (1), then the solutions can be characterized by

Definition 4. A solution $\mathbf{x}(t,\mathbf{x}_0)$ of the system (**1**) is practically stable w.r.t. $\{J,\alpha,\beta,G\}$ if $\mathbf{x}_0 \in S_\alpha \cap A_G(\alpha)$ at the moment $t = 0$ and $\forall t \in J$, the solution $\mathbf{x}(t,\mathbf{x}_0) \in A_G(\beta)$, *Debeljkovic et al.* (1995).

When the system (14 - 15) is considered, then the more precise characterization of the practical stability than the one given by *Definition* 4, is provided by:

Definition 5. System (**14 - 15**) *is* practically stable w.r.t $\{J,\alpha_1,\alpha_2,\beta_1,\beta_1,I\}$, $\alpha < \beta$, if there is $\mathbf{x}_0 \in \aleph\left(A_3 \quad A_4\right)$, satisfying the conditions

$$\|\mathbf{x}_{10}\|_I^2 < \alpha_1, \text{ and } \|\mathbf{x}_{20}\|_I^2 < \alpha_2$$

implies

$$\|\mathbf{x}(t)\|_I^2 < \beta,\ \ \forall t \in J,\ \ \beta = \beta_1 + \beta_2,$$

Debeljkovic, *et al.*(1992).

Definition 6. System (**14 - 15**) *is* practically unstable w.r.t $\{J,\alpha_1,\alpha_2,\beta_1,\beta_1,I\}$, $\alpha < \beta$, if there is $\mathbf{x}_0 \in \aleph\left(A_3 \quad A_4\right)$, satisfying the conditions

$$\|\mathbf{x}_{10}\|_I^2 < \alpha_1, \text{ and } \|\mathbf{x}_{20}\|_I^2 < \alpha_2$$

and there is a $t^* \in J$, such that implies

$$\left\|\mathbf{x}\left(t^*\right)\right\|_I^2 \geq \beta,\quad t^* \in J,\quad \beta = \beta_1 + \beta_2,$$

Debeljkovic, et al.(1992).

Definition 7. A solution $\mathbf{x}(t,\mathbf{x}_0)$ of the system (**1**) is $\{J,\alpha,\beta_1,\beta_2\}$ - *bounded* if there is $\mathbf{x}_0 \in W_k$ which satisfies.

$$\|\mathbf{x}_{10}\|^2 < \alpha_1 \text{ and } \|\mathbf{x}_{20}\|^2 < \alpha\frac{\beta_2}{\beta_1},$$

implies

$$\|\mathbf{x}_1(t,\mathbf{x}_0)\|^2 < \beta_1,\ \ \underline{\forall}\ \ \|\mathbf{x}_2(t,\mathbf{x}_0)\|^2 < \beta_2 \text{ on } J.$$

Debeljkovic et al. (1992).

Definition 8. A solution $\mathbf{x}(t,\mathbf{x}_0)$ of the system (**1**) is $\{J,\alpha,\beta_1,\beta_2\}$ - *unbounded* if there is $\mathbf{x}\left(t^*,\mathbf{x}_0\right) \in W_k$ which satisfies

$$\|\mathbf{x}_{10}\|^2 < \alpha_1 \text{ and } \|\mathbf{x}_{20}\|^2 < \alpha\frac{\beta_2}{\beta_1},$$

implies

$$\left\|\mathbf{x}_1\left(t^*,\mathbf{x}_0\right)\right\|^2 \geq \beta_1,\ \ \underline{\forall}\ \ \left\|\mathbf{x}_2\left(t^*,\mathbf{x}_0\right)\right\|^2 \geq \beta_2$$

Debeljkovic et al. (1992).

Definition 9. A solution $\mathbf{x}(t,\mathbf{x}_0)$ of the system (**14 - 15**) is $\{J,\alpha,\beta,G\}$ - bounded if and only if $\mathbf{x}_0 \in \aleph\begin{pmatrix} A_3 & A_4 \end{pmatrix}$ and $\|\mathbf{x}_0\|_G^2 < \alpha$, implies $\|\mathbf{x}(t,\mathbf{x}_0)\|_G^2 < \beta$ on J, *Debeljkovic et al.* (1992).

Definition 10. A solution $\mathbf{x}(t,\mathbf{x}_0)$ of the system (**14 - 15**) is $\{J,\alpha,\beta,G\}$ unbounded if and only if there exists a $\mathbf{x}(t^*,\mathbf{x}_0) \in J \times \aleph\begin{pmatrix} A_3 & A_4 \end{pmatrix}$, such that $\|\mathbf{x}_0\|_G^2 < \alpha$, implies $\|\mathbf{x}(t,\mathbf{x}_0)\|_G^2 \geq \beta$, *Debeljkovic et al.* (1992).

Any specific form of the matrix G can be assumed, for example a convenient one is $G = E^T PE$, where $P^T = P$ is an arbitrary pd matrix.

For the purpose of a more convenient analysis (since the matrix E of the system (1) can be of a special structure) it is useful to slightly reformulate the previous definitions as follows:

Definition 11. A solution $\mathbf{x}(t,\mathbf{x}_0)$ of the system (**14 - 15**) is $\{J,\alpha,\beta,G\}$ - bounded if and only if $x_0 \in \aleph\begin{pmatrix} A_3 & A_4 \end{pmatrix}$ and $\|\mathbf{x}_{10}\|^2 < \alpha_1$ and $\|\mathbf{x}_{20}\|^2 < \alpha \dfrac{\beta_2}{\beta_1}$, implies $\left\|\mathbf{x}_1\left(t^*,\mathbf{x}_0\right)\right\|^2 \geq \beta_1, \ \underline{\forall} \ \left\|\mathbf{x}_2\left(t^*,\mathbf{x}_0\right)\right\|^2 \geq \beta_2$ on J, *Debeljkovic et al.* (1993).

Definition 12. A solution $\mathbf{x}(t,\mathbf{x}_0)$ of the system (**14-15**) is $\{J,\alpha,\beta,G\}$ unbounded if and only if there exists a $\mathbf{x}(t,\mathbf{x}_0) \in J \times \aleph\begin{pmatrix} A_3 & A_4 \end{pmatrix}$, such that $\|\mathbf{x}_{10}\|^2 < \alpha_1$ and $\|\mathbf{x}_{20}\|^2 < \alpha \dfrac{\beta_2}{\beta_1}$, implies $\left\|\mathbf{x}_1\left(t^*,\mathbf{x}_0\right)\right\|^2 \geq \beta_1$, or $\left\|\mathbf{x}_2\left(t^*,\mathbf{x}_0\right)\right\|^2 \geq \beta_2$ on J, *Debeljkovic et al.* (1993).

Two comments are necessary at this stage. First note that if all solutions starting from all points of $\aleph\begin{pmatrix} A_3 & A_4 \end{pmatrix} \cap \mathbf{A}_G(\beta)$ are $\{J,\alpha,\beta,G\}$ - bounded then the system considered is practically stable with respect to $\{J,\alpha,\beta,G\}$.

The second comment is that if there is any one solution which is $\{J,\alpha,\beta,G\}$ unbounded, then the system considered is $\{J,\alpha,\beta,G\}$ practically unstable.

Let

$$B_k(\wp) = \left\{\mathbf{x}(t) \in \mathbb{R}^n : \ \|\mathbf{x}_k(t)\| < \wp\right\}, \quad k = 1,\, 2\,.$$

Definition 13. A solution $\mathbf{x}(t,\mathbf{x}_0)$ of the system (**14 - 15**) is practically stable w.r.t. $\{J,\alpha,\beta_1,\beta_2\}$ if $\mathbf{x}_0 \in S_\alpha \cap B_1(\alpha) \cap B_2(\alpha\,\beta_1/\beta_2)$ at the moment $t = 0$ and $\forall t \in J$, the solution $\mathbf{x}(t,\mathbf{x}_0) \in B_1(\beta_1) \cap B_2(\beta_2)$, *Debeljkovic et al.* (1993).

Definition 14. System (1) is practically unstable w.r.t. $\{J,\alpha,\beta,G\}$ if $\exists \mathbf{x}_0 \in S_\alpha \cap A_G(\alpha)$ at the moment $t = 0$, and $\exists t_1 \in J$, and there is a solution $\mathbf{x}(t,\mathbf{x}_0)$ such that $\mathbf{x}(t_1,\mathbf{x}_0) \not\subset A_g(\beta)$, *Debeljkovic et al.* (1993).

Definition 15. A solution $\mathbf{x}(t,\mathbf{x}_0)$ of the system (**1**) is practically unstable w.r.t. $\{J,\alpha,\beta,G\}$ iff $\mathbf{x}_0 \in S_\alpha \cap A_G(\alpha)$ at the moment $t = 0$ and $\exists t_1 \in J$, such that $\mathbf{x}(t_1,\mathbf{x}_0) \not\subset A_g(\beta)$, *Debeljkovic et al.* (1993).

Definition 16. A solution $\mathbf{x}(t,\mathbf{x}_0)$ of the system (**14 - 15**) is practically unstable w.r.t. $\{J,\alpha,\beta_1,\beta_2\}$ if $\mathbf{x}_0 \in S_\alpha \cap B_1(\alpha) \cap B_2(\alpha\,\beta_1/\beta_2)$ at the moment $t = 0$ and $\exists t_1 \in J$, such that $\mathbf{x}(t,\mathbf{x}_0) \not\subset B_1(\beta_1) \cap B_2(\beta_2)$, *Debeljkovic et al.* (1993).

Note that the instability concepts given by previous *Definitions* are also special cases of the general generic qualitative concept introduced in *Bajic* (1992.b).

The concept of interest that is tied directly to the types of stability (or boundedness) property is the so-called potential domain of validity of the respective concept. With regard to the practical stability types defined via proposed *Definitions*, the relevant concept is that of the potential (weak) domain of practical stability. The term potential (weak) is used because for each $\mathbf{x}_0$ which belongs to this domain, it is only guaranteed that there exits at least one solution with the specified practical stability characterization, and that there is no guarantee that all solutions emanating from $\mathbf{x}_0$ possess the required practical stability property.

This problem will be treated later.

Now we turn our attention to the **forced** linear singular systems, described in the following manner

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{16}$$

where $\mathbf{x}(t) \in \mathbb{R}^n, \ \mathbf{u}(t) \in \mathbb{R}^m$.

We strongly underline that the consistent initial conditions for the system operating in free and forced regime need not to be the same.

Definition 17. System given by (**16**) is finite-time stable w.r.t. $\left\{\alpha,\beta,\varepsilon(t),t_0,J\|(\cdot)\|_Q\right\}, \quad \alpha < \beta$, if and only if

$$\|\mathbf{x}_0\|_Q < \alpha, \quad \forall \mathbf{x}_0 \in W_k\,,$$

and

$$\|B\mathbf{u}(t)\| \leq \varepsilon(t), \quad \forall t \in J\,,$$

imply

$$\|\mathbf{x}(t)\|_Q < \beta, \quad \forall t \in J\,,$$

Q being the positive definite matrix on the sub-space of the consistent initial conditions of the system given by (**16**), *Debeljkovic*, *Jovanovic* (1997).

To overcome some mentioned problems concerning the nature of the sub-space W_k for free and for forced singular systems some simplifications have to be done.

Assumption 1. The vector valued function $\mathbf{u}(t)$ has a property that guarantees identical sub-space consistent initial conditions for the system governed by (**1**) as well as for the system given by (**16**).

Proposition 1. If $\varphi(\mathbf{x}) = \mathbf{x}^T(t)M\mathbf{x}(t)$ is a quadratic form on $\mathbb{R}^n$ then it follows that there are numbers $\lambda(M)$ and $\Lambda(M)$, satisfying

$$-\infty < \lambda(M) \le \Lambda(M) < +\infty ,$$

such that

$$\lambda(M) \le \frac{\mathbf{x}^T(t) M \mathbf{x}(t)}{V(\mathbf{x})} \le \Lambda(M), \quad \forall \mathbf{x} \in W_k \setminus \{\mathbf{0}\} .$$

If $M = M^T$ and $\mathbf{x}^T(t) M \mathbf{x}(t) > 0, \quad \forall \mathbf{x} \in W_k \setminus \{\mathbf{0}\}$, then $\lambda(M) > 0$ and $\Lambda(M) > 0$, where $\lambda(M)$ and $\Lambda(M)$ are defined in such way

$$\lambda(M) = \min \left\{ \begin{matrix} \mathbf{x}^T(t) M \mathbf{x}(t), & \mathbf{x} \in W_k \setminus \{\mathbf{0}\}, \\ & \mathbf{x}^T(t) E^T P E \mathbf{x}(t) = 1 \end{matrix} \right\},$$

$$\Lambda(M) = \max \left\{ \begin{matrix} \mathbf{x}^T(t) M \mathbf{x}(t), & \mathbf{x} \in W_k \setminus \{\mathbf{0}\}, \\ & \mathbf{x}^T(t) E^T P E \mathbf{x}(t) = 1 \end{matrix} \right\}.$$

It is convinient to consider, for the purposes of this paper, the agregation function for the system given by (**16**) in the following manner

$$V(\mathbf{x}(t)) = \mathbf{x}^T(t) E^T P E \mathbf{x}(t), \tag{17}$$

with the particular choice $P = I$, I being the identy matrix.

Definition 1. is not suitable for the treatment of singular systems so we use

$$V(\mathbf{x}_0) < \frac{\alpha}{\gamma_1(Q)} < \alpha', \tag{18}$$

instead of $\|\mathbf{x}_0\|_Q^2 < \alpha$ and

$$V(x(t)) < \frac{\beta}{\gamma_2(Q)} < \beta' \tag{19}$$

instead of $\|\mathbf{x}(t)\|_Q^2 < \beta, \quad \forall t \in J$.

It is obvious that (18) and (19) represent sufficient conditions for the conditions of *Definition* 1.

Note that the use of $Q = E^T P E$ leads to $\gamma_1 = \gamma_2 = 1$ and the problem is equivalent.

The matrix Q is chosen to represent physical constraints on the system variables and is assumed to satisfy $Q = Q^T$ and $\mathbf{x}^T(t) Q \mathbf{x}(t) > 0, \ \forall \mathbf{x}(t) \in W_k \setminus \{0\}$.

Stability theorems

Theorem 1. The system is *practicaly stable* with respect to $\{J, \alpha, \beta\}$, $\alpha < \beta$, if the following conditions are satisfied

$$(i) \qquad \beta/\alpha > \frac{\gamma_2(Q)}{\gamma_1(Q)} \tag{20}$$

$$(ii) \qquad \ln \beta/\alpha > \Lambda(Q) + \ln \frac{\gamma_2(Q)}{\gamma_1(Q)}, \qquad \forall t \in J . \tag{21}$$

with $\Lambda(Q)$ as in *Proposition* 1, *Debeljkovic*, *Owens* (1985).

Proof.

Let $\mathbf{x}_0$ be an arbitry consistent initial condition and $\mathbf{x}(t)$ the resulting system trajectory. Then $\mathbf{x}(t) \in W_k \setminus \{0\}$, $\forall t \ge 0$.

Differentiating $V(\mathbf{x}(t))$ along the trajectories of the system yields

$$\begin{aligned} \dot{V}(\mathbf{x}(t)) &= \mathbf{x}^T(t)\left(A^T P E + E^T P A\right)\mathbf{x}(t) \\ &\le \Lambda(Q) \cdot \mathbf{x}^T(t) E^T P E \mathbf{x}(t) \le \Lambda(Q) V(\mathbf{x}(t)) \end{aligned} . \tag{22}$$

Integrating the previous inequality gives

$$V(\mathbf{x}(t)) \le e^{\Lambda(Q)t} \cdot V(\mathbf{x}_0). \tag{23}$$

If $V(\mathbf{x}(t))$ is to be less than β' eg. $V(\mathbf{x}(t)) < \beta'$, for $0 \le t < J$, for all $\mathbf{x}_0$ and $V(\mathbf{x}_0) < \alpha'$ then it is sufficient that

$$e^{\Lambda(Q)t} \alpha' < \beta', \qquad 0 \le t < J . \tag{24}$$

That is

$$\Lambda(Q) t < \ln \frac{\beta'}{\alpha'} \ , \ 0 < t < J . \tag{25}$$

But

$$\frac{\beta'}{\alpha'} = \frac{\beta}{\alpha} \frac{\gamma_1(Q)}{\gamma_2(Q)}, \tag{26}$$

so this is reduced to

$$\ln \frac{\beta}{\alpha} > \Lambda(Q) t + \ln \frac{\gamma_2(Q)}{\gamma_1(Q)} \quad , \quad 0 \le t < J . \tag{27}$$

This is guaranteed by **(i)** and **(ii)** of *Definition* 1.

The condition **(i)** is a consequence of our change in the problem definition.

If $\beta/\alpha > 1$ there is no garantee that $\beta/\alpha > \dfrac{\gamma_2(Q)}{\gamma_2(Q)}$.

If $Q = E^T P E$ in our initial problem then $\gamma_1 = \gamma_2 = 1$. If, however, Q is fixed, we can control this problem by chosing P.

This is illustrated by the next result.

Proposition 2. There is the matrix $P = P^T > 0$, such that $\gamma_1(Q) = \gamma_2(Q) = 1$. *Debeljkovic*, *Owens* (1985).

Proof. Choose c such that $\det(cE - A) \not\equiv 0$.

Set $\hat{E} = (cE - a)^{-1} E$.

Then P can be defined in the following.

$$\begin{aligned} P &= \left(\hat{E}^D (cE - A)^{-1}\right)^* Q \hat{E}^D (cE - A)^{-1} \\ &\quad + \left(\left(\hat{E}\hat{E}^D - I\right)(cE - A)^{-1}\right)^* Q \left(\hat{E}\hat{E}^D - I\right)(cE - A)^{-1} \\ &= P^T > 0 \end{aligned} \tag{28}$$

and

$$\begin{aligned} E^T P E &= \left(\hat{E}^D \hat{E}\right)^T Q \left(\hat{E}^D \hat{E}\right) \\ &\quad + \hat{E}^T \left(\hat{E}\hat{E}^D - I\right)^T Q \left(\hat{E}\hat{E}^D - I\right) \hat{E} \end{aligned} . \tag{29}$$

If $\mathbf{x}(t) \in W_k \setminus \{0\}$ then $\left(I - \hat{E}\hat{E}^D\right)\mathbf{x} = \mathbf{0}$, so

$$\begin{aligned}\left(I-\hat{E}\hat{E}^D\right)E\mathbf{x} &= \left(\hat{E}-\hat{E}\hat{E}^D\hat{E}\right)\mathbf{x}\\ &= \hat{E}\left(I-\hat{E}\hat{E}^D\right)\mathbf{x}=\mathbf{0}\end{aligned}. \tag{30}$$

Hence

$$\begin{aligned}\mathbf{x}(t)^T E^T P E\mathbf{x}(t) &= \mathbf{x}(t)^T\left(\hat{E}^D\hat{E}\right)^T Q\left(\hat{E}^D\hat{E}\right)\mathbf{x}(t)\\ &= \mathbf{x}(t)^T Q\mathbf{x}(t)\end{aligned}, \tag{31}$$

and

$$\gamma_1=\gamma_2=1. \tag{32}$$

with that choice of P , Q.E.D.

Corollary 1. If $\beta/\alpha>1$, there is a choice of P such that

$$\frac{\beta}{\alpha}>\frac{\gamma_2(Q)}{\gamma_1(Q)}. \tag{33}$$

The practical meaning of this result is that condition (i) of *Definition* 1 can be satisfied by an initial choice of free parameters of the matrix P.

Condition (ii) depends also on the system data and hence it is more complex but it is quite legitimate to ask whether we can choose P such that $\Lambda(Q)<0$?

Theorem 2. Suppose that the following rank condition is satisfied

$$rank\left(A_3 \quad A_4\right)=rank\, A_4=r\le n_2 \tag{34}$$

The solutions of the system (**14 - 15**) are practically stable w.r.t. $\{J,\alpha,\beta_1,\beta_2,I\}$ if the following conditisons are fulfilled

$$\gamma_{\max}\cdot t<\ln\frac{\beta_1}{\alpha},\quad \forall t\in J \tag{35}$$

$$\| L\|^2<\frac{\beta_2}{\beta_1}, \tag{36}$$

where

$$\gamma_{\max}=\Lambda_1\left(A_1^T+A_1\right)+\Lambda_2\left(L^TA_2^T+A_2L\right), \tag{37}$$

with the matrix L as any solution to the following matrix equation

$$0=A_3+A_4, \tag{38}$$

Debeljkovic et al. (1992).

Theorem 3. Suppose now that rank condition (34) is satisfied. Then the solutions of the system (**14 - 15**) are *practically unstable* w.r.t. $\{J,\alpha,\beta_1,\beta_2,I\}$ if $\exists\delta\rightarrow 0<\delta<\alpha$ and $\exists t^*\in J$ such that the following conditions are fulfilled

$$\gamma_{\min}\cdot t^*>\ln\frac{\beta_1}{\alpha}, \tag{39}$$

$$\left\| L^{\#}\right\|^2<\frac{\beta_2}{\beta_1}, \tag{40}$$

where

$$\gamma_{\min}=\lambda_1\left(A_1^T+A_1\right)+\lambda_2\left(L^TA_2^T+A_2L\right) \tag{41}$$

with the matrix L being any solution of (38) where

$$L^{\#}=\left(L^TL\right)^{-1}L^T, \tag{42}$$

denotes a general pseudo inverse of the matrix L, *Debeljkovic et al.* (1992).

Now we are coming back to the idea of a *potential* (weak) *domain of practical stability*.

Let $\Omega=\{J,\alpha,\beta,G\}$.

Then, the potential domain of $\{J,\alpha,\beta,G\}$-practical stability is defined as

$$\mathbb{S}(\Omega)=\left\{\begin{matrix}\mathbf{x}_0\in S_\alpha\cap A_G(\alpha):\exists\mathbf{x}(t,\mathbf{x}_0)\text{ such that}\\ \forall t\in \mathrm{J},\quad \mathbf{x}(t,\mathbf{x}_0)\in A_G(\beta)\end{matrix}\right\} \tag{43}$$

In an analogous manner, the potential domains of other practical stability types can be formulated.

For example, the potential domain of the $\{J,\alpha,\beta,G\}$-practical stability for the system (**14-15**) may be defined as

$$\mathbb{S}(\Omega)=\left\{\begin{matrix}\mathbf{x}_0\in S_\alpha\cap B_1(\alpha)\cap B_2(\alpha\beta_1/\beta_2):\\ \exists\mathbf{x}(t,\mathbf{x}_0),\text{ such that}\\ \forall t\in\mathrm{J},\quad \mathbf{x}(t,\mathbf{x}_0)\in B_1(\beta_1)\cap B_2(\beta_2)\end{matrix}\right\}, \tag{44}$$

where $\Omega=\{J,\alpha,\beta_1,\beta_2\}$.

One of the tasks in the analysis will be to estimate the set $\mathbb{S}(\Omega)$ by an underestimate $\mathbb{S}_u(\Omega)$, $\mathbb{S}_u(\Omega)\subseteq\mathbb{S}(\Omega)$, using the LDM.

The exposition in this section is taken from *Bajic* (1995).

It is assumed that the assumption expressed by (34) holds.

The rank assumption (34) implies $S_\alpha=\aleph\left(\left(A_3 \quad A_4\right)\right)$ for the system (**14 - 15**).

In addition, there exists a matrix L which satisfies the matrix equation (38).

From (38) it follows that if the solutions of (**14 - 15**) exist, then there will be the solutions $\mathbf{x}(t)$ the components of which are tied by

$$\mathbf{x}_2(t)=L\mathbf{x}_1(t),\quad \forall t\in J \tag{45}$$

Those solutions of (**14 - 15**) that satisfy (45) also have to satisfy the constraints (**15**).

As $\mathbb{N}\left(\left[L \quad -I_{n2}\right]\right)\subseteq\aleph\left(\left[A_3 \quad A_4\right]\right)=S_\alpha$ when (38) holds, then the following conclusions follow under the assumption (34):

(i) There are solutions of the system (**14 - 15**) which belong to the set $\aleph\left(\left[L \quad -I_{n2}\right]\right)$.

(ii) If the solutions of the system (**14 - 15**) that satisfy (45) are $\{J,\alpha,\beta,G\}$-practically stable, then the potential domain of the $\{J,\alpha,\beta,G\}$-practical stability for the system (**14 - 15**) may be determined by

$$\mathbb{S}_u(\Omega)=\left\{x_0\in\mathbb{R}^n:\mathbf{x}((t))\in\aleph\left(\left[L \quad -I_{n2}\right]\right)\cap A_G(\alpha)\right\} \tag{46}$$

To perform the analysis of practical stability for the system (**14 - 15**), we employ the Lyapunov function

$$V(\mathbf{x}(t))=\mathbf{x}^T(t)G\mathbf{x}(t)=\mathbf{x}_1^TP\mathbf{x}_1,\ G=diag\{P,\ O_{n2}\} \tag{47}$$

where the matrix G is symmetric positive semi-definite (psd), and the matrix P is symmetric and positive definite (pd).

The total time derivative of $V(\mathbf{x}(t))$ along the solutions of (**14** - **15**) is given by

$$\dot{V}(\mathbf{x}(t)) = \mathbf{x}_1^T \left(A_1^T P + PA_1\right)\mathbf{x}_1 + \mathbf{x}_2^T A_2^T P\mathbf{x}_1 + \mathbf{x}_1^T PA_2\mathbf{x}_2 \quad (48)$$

Let $\lambda_M(Z)$ and $\lambda_m(Z)$ denote the maximal and the minimal eigen value of a real symmetric matrix Z, respectively.

The following *Theorem* determines the practical stability of (**14** - **15**).

Theorem 4. Let the rank condition (34) hold and let the matrix G be defined as in (47). Then there are $\Omega = \{J,\alpha,\beta,G\}$ - practically stable solutions of (**14** - **15**) that satisfy (45), if

$$\wp \cdot t \le \ln\frac{\beta}{\alpha}, \quad \forall t \in J, \quad (49)$$

where $\alpha,\beta \in \mathbb{R}$, $\beta \ge \alpha$, and $\wp = \lambda_M(Z)/\lambda_M(P)$ when $\lambda_M(Z) \le 0$ or $\wp = \lambda_M(Z)/\lambda_m(P)$ when $\lambda_M(Z) > 0$, where $Z = (A_1 + A_2L)^T P + P(A_1 + A_2L)$, and L satisfies (38).

Moreover, if $\lambda_M(Z) \le 0$, i.e. if Z is negative semidefinite (nsd), then $J = [0,+\infty[$, $\beta = \alpha$, can be selected.

If $\lambda_M(Z) > 0$, then $J = [0,\ T[$, $T < +\infty$ and $\beta > \alpha$ has to he selected to have $T > 0$, *Debeljkovic et al.* (1995).

Theorem 5. Let the rank condition (34) hold and let the matrix G be defined as in (47). Then there are $\{J,\alpha,\beta_1,\beta_2\}$ - practically stable solutions of (**14** - **15**) that satisfy (45), if

$$\wp t \le \ln\frac{\beta_1}{\alpha}, \quad \forall t \in J, \quad (50)$$

$$\|L\|^2 = \frac{a_2}{a_1}, \quad (51)$$

where $\alpha,\beta \in \mathbb{R}$, $\beta \ge \alpha$, and $\wp = \lambda_M(Z)/\lambda_M(P)$ when $\lambda_M(Z) \le 0$ or $\wp = \lambda_M(Z)/\lambda_m(P)$ when $\lambda_M(Z) > 0$, where $Z = (A_1 + A_2L)^T P + P(A_1 + A_2L)$, with L satisfying (38).

Moreover, if $\lambda_M(Z) \le 0$, i.e. if Z is nsd, then $J = [0,+\infty[$, $\beta = \alpha\phi$, can be selected where $\phi = \lambda_M(P)/\lambda_m(P)$. If $\lambda_M(Z) > 0$ then $J = [0,\ T[$, $T < +\infty$ and $\beta = \alpha\phi$ has to be selected to have $T > 0$, *Debeljkovic et al.* (1995).

Theorem 6. Let $\Omega = \{J,\alpha,\beta_1,\beta_2\}$ and let the conditions of *Theorem* 5 hold.

Then the underestimate $\mathbb{S}_u(\Omega)$ of the potential domain $P(e)$ of the $\{J,\alpha,\beta_1,\beta_2\}$ - practical stability for the system (**14** - **15**) may be defined as

$$\mathbb{S}_u(\Omega) = \left\{ \begin{array}{l} \mathbf{x} = \mathbf{x}(t) \in \mathbb{R}^n : \\ \mathbf{x} \in \aleph\left(\begin{bmatrix} L & -I_{n2} \end{bmatrix}\right) \cap B_1(\alpha) \cap B_2(\alpha\beta_2/\beta_2) \end{array} \right\} \quad (52)$$

where $\mathbb{S}(\Omega)$ is defined by (44), *Debeljkovic et al.* (1995).

The *practical instability* can be concluded from

Theorem 7. Let the rank condition (34) hold and let the matrix G be defined as in (47). Then there are $\{J,\alpha,\beta_1,\beta_2\}$ - *practically unstable solutions* of (**14** - **15**), where $\alpha \le \beta_1$, if the matrix $Z = (A_1 + A_2L)^T P + P(A_1 + A_2L)$, with L satisfying (38), is pd and for some δ, $0 < \delta < \alpha$, there is $(1/\wp)\ln(\beta_1/\varphi\Delta) < T$, where $\varphi = \lambda_m(P)/\lambda_M(P)$, $\wp = \lambda_m(Z)/\lambda_M(P)$, *Debeljkovic et al.* (1995).

Theorem 8. System (**1**) is *finite time stable* w.r.t. $\{J,\alpha,\beta,I\}$, if the following condition is satisfied

$$\Phi_{LSS}(t) < \sqrt{\frac{\beta}{\alpha}}, \quad \forall t \in J, \quad (53)$$

$\Phi_{LSS}(t)$ being the *fundamental matrix* of the linear singular system (**1**), *Debeljkovic et al.* (1997).

Now we apply *Coppel*'s (1965) and the matrix mesure approach.

Theorem 9. System (**1**) is *finite time stable* w.r.t. $\{J,\alpha,\beta,I\}$, if the following condition is satisfied

$$e^{\mu(\Upsilon)\cdot t} < \frac{\beta}{\alpha}, \quad \forall t \in J, \quad (54)$$

where:

$$\Upsilon = \hat{E}^D\hat{A}, \quad \hat{A} = (sE - A)^{-1}A, \quad \hat{E} = (sE - A)^{-1}E \quad (55)$$

Debeljkovic et al. (1997).

Starting with the explicit solution of system (**1**), derived in *Campbell* (1980)

$$\mathbf{x}(t) = e^{\hat{E}^D\hat{A}(t-t_0)}\mathbf{x}_0, \qquad \mathbf{x}_0 = \hat{E}\hat{E}^D\mathbf{x}_0 \quad (56)$$

and differentiating eqtutation (56), one gets

$$\dot{\mathbf{x}}(t) = \hat{E}^D\hat{A}e^{\hat{E}^D\hat{A}\cdot t}\cdot\mathbf{x}_0 = \hat{E}^D\hat{A}\mathbf{x}(t) \quad (57)$$

so only the *regular* singular systems are treated with matrices given (55).

Theorem 10. For the given constant matrix $\hat{E}^D\hat{A}$ any solution of (**l**) satisfy the following inequality

$$\begin{array}{ll} \|\mathbf{x}(t_0)\| e^{-\mu\left(-\hat{E}^D\hat{A}\right)(t-t_0)} \le \|\mathbf{x}(t)\| & \forall t \in J \\ \le \|\mathbf{x}(t_0)\| e^{\mu\left(\hat{E}^D\hat{A}\right)(t-t_0)} & \end{array}, \quad (58)$$

Kablar, Debeljkovic (1998).

Theorem 11. In order that system (**1**) *be finite time stable* w.r.t. $\{J,\alpha,\beta,\mathbf{I}\}$, $\alpha < \beta$, it is *necessary* that the following condition is satisfied

$$e^{-\mu\left(-\hat{E}^D\hat{A}\right)(t-t_0)} < \sqrt{\frac{\beta}{\delta}}, \quad \forall t \in J \quad (59)$$

where $0 < \delta \le \alpha$, *Kablar*, *Debeljkovic* (1998).

Theorem 12. In order that system (**1**) be *finite time instable* w.r.t. $\{J,\alpha,\beta,\mathbf{I}\}$, $\alpha<\beta$, it is *necessary* that there is $t^*\in J$ such that the following condition is satisfied

$$e^{\mu\left(\hat{E}^D\hat{A}\right)\left(t^*-t_0\right)}\geq\sqrt{\frac{\beta}{\alpha}},\qquad t^*\in J\,. \tag{60}$$

Theorem 13. The system (**1**) *is finite time instable* w.r.t. $\{J,\alpha,\beta,\mathbf{I}\}$, $\alpha<\beta$, *if* $\exists\delta,\ 0<\delta\leq\alpha$ and $t^*\in J$ such that the following condition is satisfied

$$e^{-\mu\left(-\hat{E}^D\hat{A}\right)\left(t^*-t_0\right)}<\sqrt{\frac{\beta}{\delta}},\qquad t^*\in J\,. \tag{61}$$

Finally, we use the *Bellman – Gronwall approach* to derive our results.

Lemma 1. Suppose the vector $\mathbf{q}(t,t_0)$ is defined in the following manner

$$\mathbf{q}(t,t_0)=\Phi(t,t_0)\hat{E}^D\hat{E}\mathbf{v}(t_0)\,. \tag{62}$$

So if:

$$E\mathbf{q}(t,t_0)=E\Phi(t,t_0)\hat{E}^D\hat{E}v(t_0) \tag{63}$$

then:

$$\left\|\mathbf{q}(t,t_0)\right\|^2_{E^TE}\leq\left\|v(t_0)\right\|^2_{E^TE}e^{\Lambda_{\max}(M)(t-t_0)}, \tag{64}$$

where:

$$\Lambda_{\max}(M)=\max\left\{\begin{matrix}\mathbf{q}^T(t,t_0)M\,\mathbf{q}(t,t_0):\mathbf{q}(t,t_0)\in W_k\setminus\{0\},\\ \mathbf{q}^T(t,t_0)E^TE\,\mathbf{q}(t,t_0)=1\end{matrix}\right\} \tag{65}$$

$$M=A^TE+E^TA\,, \tag{66}$$

$$\mathbf{v}(t_0)=\mathbf{q}(t_0,t_0)\,, \tag{67}$$

Debeljkovic, Kablar (1999).

Lemma 2. If equations (62) and (63) holds, then

$$\left\|\mathbf{q}(t,t_0)\right\|^2_{E^TE}\geq\left\|\mathbf{v}(t_0)\right\|^2_{E^TE}e^{\Lambda_{\min}(M)(t-t_0)}, \tag{68}$$

where,

$$\Lambda_{\min}(M)=\min\left\{\begin{matrix}\mathbf{q}^T(t,t_0)M\,\mathbf{q}(t,t_0):\mathbf{q}(t,t_0)\in W_k\setminus\{0\},\\ \mathbf{q}^T(t,t_0)E^TE\,\mathbf{q}(t,t_0)=1\end{matrix}\right\} \tag{69}$$

and

$$M=A^TE+E^TA\,, \tag{70}$$

$$\mathbf{v}(t_0)=\mathbf{q}(t_0,t_0)\,, \tag{71}$$

Debeljkovic, Kablar (1999).

Using this approach the results of *Theorem* 1 can be preformulate in the following manner.

Theorem 14 (1). The system given by (1) is *finite time stable* w.r.t. $\left\{J,\alpha,\beta,\|(\cdot)\|^2_Q\right\}$, $a<\beta$, if the following condition is satisfied:

$$e^{\Lambda_{\max}(M)\cdot(t-t_0)}<\frac{\beta}{\alpha},\quad\forall t\in J\,, \tag{72}$$

with $\Lambda_{\max}(M)$ given (65) and the matrix M with (66), *Debeljkovic*, *Kablar* (1999).

Proof. The solution of the system given by (**1**) is

$$\mathbf{x}(t)=A\Phi(t,t_0)\hat{E}^D\hat{E}\mathbf{x}_0\,, \tag{73}$$

as well as

$$E\mathbf{x}(t)=EA\Phi(t,t_0)\hat{E}^D\hat{E}\mathbf{x}_0\,. \tag{74}$$

Applying *Lemma* 1, it is easy to see

$$\left\|\mathbf{x}(t,t_0)\right\|^2_{E^TE}\leq\left\|\mathbf{x}_0\right\|^2_{E^TE}e^{\Lambda_{\max}(M)(t-t_0)}, \tag{75}$$

and, by using *Definition* 1, with particular choice $Q=E^TE$, one can get

$$\left\|\mathbf{x}(t,t_0)\right\|^2_{E^TE}\leq\alpha\cdot e^{\Lambda_{\max}(M)\cdot(t-t_0)}, \tag{76}$$

and finally, by using the main condition of *Theorem* 14, namely eq. (72) one gets

$$\begin{aligned}\left\|\mathbf{x}(t,t_0)\right\|^2_{E^TE}&\leq\alpha\cdot e^{\Lambda_{\max}(M)\cdot(t-t_0)}\\ &<\alpha\cdot\frac{\beta}{\alpha},\quad\forall t\in J\end{aligned}, \tag{77}$$

i.e.

$$\left\|\mathbf{x}(t)\right\|^2_Q<\beta,\quad\forall t\in J \tag{78}$$

which had to be proved and is identical to the result derived in *Debeljkovic, Owens* (1985). **Q.E.D.**

Forced Linear Time Invariant Singular Systems

In the veiw of *Definition* 17, the following result can be presented.

Assumption 1: The vector valued function $\mathbf{u}(t)$ has a property that guarantees the identical sub-space consistent initial conditions for the system governed by (**1**) as well as for the system given by (2).

Theorem 15. Suppose that *Assumption* 1 holds. Then, the system governed by (16) is finite-time stable w.r.t. $\left\{\alpha,\beta,\varepsilon(t),t_0,T,\|(\cdot)\|_Q\right\}$, $\alpha<\beta$ if

$$\alpha e^{\frac{1}{2}\Lambda(M)\cdot(t-t_0)}+\int_{t_0}^{t}\varepsilon(\kappa)e^{\frac{1}{2}\Lambda(M)\cdot(t-\kappa)}d\kappa<\beta,\ \forall t\in J\,, \tag{79}$$

the matrix M given by

$$M=M^T=A^TE+E^TA \tag{80}$$

and

$$\Lambda(M)=\max\left\{\begin{matrix}\mathbf{x}^T(t)M\mathbf{x}(t),\ \ \mathbf{x}\in W_k\setminus\{\mathbf{0}\},\\ \mathbf{x}^T(t)E^TPE\mathbf{x}(t)=1\end{matrix}\right\} \tag{81}$$

Debeljkovic, Jovanovic (1997).

Time varying singular systems

Let the time-varying linear singular system (TVLSS) be governed by

$$E(t)\dot{\mathbf{x}}(t)=A(t)\mathbf{x}(t),\qquad\mathbf{x}(t_0)=\mathbf{x}_0\,, \tag{82}$$

in the free regime, and by

$$E(t)\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t) + B(t)\mathbf{u}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0^* \quad (83)$$

in the forced regime, where $E(t),\ A(t) \in \mathbb{R}^{n\times n}$ are time-varying matrices, with $E(t)$ singular. $B(t) \in \mathbb{R}^{n\times m}$.

Stability definitions

Definition 18 The system governed by (**82**) is *finite time stable* w.r.t. $\{J,\alpha,\beta,Q(t)\}$, $\alpha<\beta$ if there is $\mathbf{x}_0 \in W_k$ which satisfying $\|\mathbf{x}_0\|_Q^2 < \alpha$ implies $\|\mathbf{x}(t)\|_Q^2 < \beta,\ \forall t \in J$, *Kablar*, *Debeljkovic* (1998).

Remark 1. The quadratic form $\|\mathbf{x}(t)\|_Q^2$, is defined by:

$$\|\mathbf{x}(t)\|_Q^2 = \mathbf{x}^T(t)Q(t)\mathbf{x}(t) \quad (84)$$

where $Q(t)$ is the positive definite matrix on consistent initial set, satisfying $Q(t) = E^T(t)P(t)E(t)$, where $P(t) = P^T(t) > 0$ is an arbitrarily specified matrix.

Definition 19. The system governed by (**83**) is *finite time stable* w.r.t. $\{J,\alpha,\beta,\varepsilon(t),Q(t)\}$, $\alpha<\beta$,

$Q(t) = Q^T(t) > 0$, , if there is a consistent initial condition $\mathbf{x}_0 \in W_k$ and a vector valued function $\mathbf{u}(t)$, which, satisfying $\|\mathbf{x}_0\|_Q < \alpha,\ \|B(t)\mathbf{u}(t)\|_Q \le \varepsilon(t)$ implies $\|\mathbf{x}(t)\|_Q < \beta,\ \forall t \in J$, *Kablar*, *Debeljkovic* (1998).

Definition 20. The system governed by (**82**) is *finite time instable* w.r.t. $\{J,\alpha,\beta,Q(t)\}$, $\alpha<\beta$ iff $\forall \mathbf{x}(t_0) = \mathbf{x}_0 \in W_k$, satisfying $\|\mathbf{x}_0\|_Q^2 < \alpha$, there is $t^* \in J$ implying $\|x(t^*)\|_Q^2 \ge \beta$, *Kablar*, *Debeljkovic* (1999).

Definition 21 The system governed by (**83**) is *finite time instable* w.r.t. $\{J,\alpha,\beta,\varepsilon(t),Q(t)\}$, $\alpha<\beta$, $Q(t) = Q^T(t) > 0$, if $\forall \mathbf{x}(t_0) = \mathbf{x}_0^* \in W_k$ and the vector valued function $\mathbf{u}(t)$, satisfying $\|\mathbf{x}_0^*\|_Q < \alpha$, $\|B(t)\mathbf{u}(t)\|_Q \le \varepsilon(t)$, $\forall t \in J$, there is $t^* \in J$ implying $\|\mathbf{x}(t^*)\|_Q \ge \beta$, *Kablar*, *Debeljkovic* (1999).

Stability Theorems

Theorem 16. The system governed by (**82**) is *finite time stable* w.r.t. $\{J,\alpha,\beta,Q(t)\}$, $\alpha<\beta$, $Q(t) = Q^T(t) > 0$ if the following conditions are satisfied:

$$\alpha e^{\int_{t_0}^{t} \Lambda_{\max}(M(t))dt} < \beta, \qquad \forall t \in J \quad (85)$$

where:

$$\Lambda_{\max}(M(t)) = \max\left\{ \begin{array}{c} \mathbf{x}^T(t)M(t)\mathbf{x}(t) : \mathbf{x}(t) \in W_k \setminus \{\mathbf{0}\} \\ \mathbf{x}(t)E^T(t)P(t)E(t)\mathbf{x}(t) = 1 \end{array} \right\} \quad (86)$$

and the matrix $M(t)$ is defined by

$$M(t) = \left(\begin{array}{l} A^T(t)P(t)E(t) + \\ \dot{E}(t)P(t)E(t) + \\ E^T(t)\dot{P}(t)E(t) + \\ E^T(t)P(t)\dot{E}(t) + \\ E^T(t)P(t)A(t) \end{array} \right) \quad (87)$$

Kablar, Debeljkovic (1998).

Theorem 17. The system governed by (**83**) is *finite time stable* w.r.t. $\{J,\alpha,\beta,\varepsilon(t),Q(t)\}$, $\alpha<\beta$, if the following condition is satisfied:

$$\alpha e^{\frac{1}{2}\int_{t_0}^{t} \Lambda(M(t))dt} + \int_{t_0}^{t} \varepsilon(\tau) e^{-\frac{1}{2}\int_{t}^{\tau} \Lambda(M(t))dt} d\tau < \beta, \ \forall t \in J \quad (88)$$

where: $\Lambda_{\max}(M(t))$ is given by (86) and the matrix $M(t)$ is defined with (87), *Kablar*, *Debeljkovic* (1998).

Theorem 18. The system governed by (**83**) *is finite time instable* w.r.t. $\{J,\alpha,\beta,Q(t)\}$, $\alpha<\beta$, $Q(t) = Q^T(t) > 0$ if there is, δ, $0<\delta<\alpha$, and $t^* \in J$ such that the following condition is satisfied:

$$\delta \cdot e^{\int_{t_0}^{t^*} \lambda(M(t))dt} \ge \beta, \quad (89)$$

where:

$$\begin{aligned} \lambda(M(t)) &= \lambda_{\min}(M(t)) \\ &= \min\left\{ \begin{array}{c} \mathbf{x}^T(t)M(t)\mathbf{x}(t) : \mathbf{x}(t) \in W_k \setminus \{\mathbf{0}\} \\ \mathbf{x}(t)E^T(t)P(t)E(t)\mathbf{x}(t) = 1 \end{array} \right\}, \end{aligned} \quad (90)$$

and the matrix $M(t)$ is defined by:

$$M(t) = \left(\begin{array}{l} A^T(t)P(t)E(t) \\ +\dot{E}(t)P(t)E(t) \\ +E^T(t)\dot{P}(t)E(t) \\ +E^T(t)P(t)\dot{E}(t) + E^T(t)P(t)A(t) \end{array} \right) \quad (91)$$

Kablar, Debeljkovic (1998).

Robustness stability consideration

Considerable attention has been focused in recent years on design of controllers for multivariable linear systems so that certain system properties are preserved under various classes of perturbations occurring in the system.

Patel and Toda (1980) first reported on the robustness bounds on unstructured perturbations of linear continuous-time systems.

Yedavalli and Liang (1985) improved Patel's result for linear perturbations with a known structure and proposed similarity transformation method to reduce robustness bounds conservatism.

In this part of the paper, the *non-Lyapunov* (practical, finite time) stability robustness consideration of linear, both *regular* and *irregular*, linear singular systems is addressed using the Lyapunov approach. The bounds of the unstructured perturbation vector function, for different representations of singular systems, that maintain the nominal system practical stability are presented.

Regarding *stability robustness consideration,* it is convenient to represent the model (1) in the following form:

$$E\dot{\mathbf{y}}(t) = A\mathbf{y}(t) + \mathbf{f}_p(\mathbf{y}(t)), \quad \mathbf{y}(t_0) = \mathbf{y}_0, \tag{92}$$

where $\mathbf{f}_p(\mathbf{y}(t))$ is the perturbation vector or in the following form

$$\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + A_2\mathbf{x}_2(t) + \mathbf{x}_{p1}(\mathbf{x}(t)), \tag{93}$$

$$\mathbf{0} = A_3\mathbf{x}_1(t) + A_4\mathbf{x}_2(t) + \mathbf{f}_{p2}(\mathbf{x}(t)), \tag{94}$$

with:

$$\mathbf{f}_p(\mathbf{y}) = \mathbf{f}_p(T\mathbf{x}(t)) = \mathbf{f}_p(\mathbf{x}(t)) = \begin{bmatrix} \mathbf{f}_{p1}(\mathbf{x}(t)) \\ \mathbf{f}_{p2}(\mathbf{x}(t)) \end{bmatrix}, \tag{95}$$

Theorem 19. The system (92) is practically stable w.r.t. (J,α,β,G), $\alpha < \beta$, if the following conditions are satisfied:

$$\frac{(\Lambda(M) + 2\eta)}{t} < \ln\frac{\beta}{\alpha}, \quad \forall t \in J, \tag{96}$$

$$\| \mathbf{f}_p(\mathbf{y}(t)) \| \le \eta \cdot \| E\mathbf{y}(t) \|, \quad \eta = const., \tag{97}$$

where

$$\Lambda(M) = \max\left\{ \begin{matrix} \mathbf{x}^T(t)M\mathbf{x}(t), \ \mathbf{x} \in W_k \setminus \{\mathbf{0}\}, \\ \mathbf{x}^T(t)E^TPE\mathbf{x}(t) = 1 \end{matrix} \right\} \tag{98}$$

Debeljkovic et al. (1995.a).

Assumption 2. The vector perturbation function $\mathbf{f}_p(\mathbf{x}(t))$ satisfies the following condition

$$\mathbf{f}_{p2}(\mathbf{x}(t)) \equiv \mathbf{0}, \qquad \mathbf{f}_p(\mathbf{x}(t)) = \left(\mathbf{f}_{p1}^T\mathbf{x}(t) \ \ \mathbf{0}^T\right)^T, \tag{99}$$

so it can be written

$$\mathbf{f}_p(\mathbf{x}(t)) = \left(\begin{bmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{bmatrix}\right) = f_{p1}\left(\begin{bmatrix} \mathbf{x}_1(t) \\ L\mathbf{x}_1(t) \end{bmatrix}\right) = \mathbf{f}_{p1}(\mathbf{x}_1(t)). \tag{100}$$

Theorem 20. Suppose *Assumption* 2 and (34) and (38) hold and let $rank\, F < n_2$, where the matrix F is defined by

$$F\mathbf{x}(t) = \begin{pmatrix} A_3 & A_4 \\ L & -I \end{pmatrix}\mathbf{x}(t) = \mathbf{0}. \tag{101}$$

Then the solutions of (**93 - 94**), different from the null solution $\mathbf{x}(t) \equiv \mathbf{0}$, are *practically stable* w.r.t. $\{J,\alpha,\beta_1,\beta_2,G,\eta\}$, $\alpha \le \beta_1$, if the following conditions are satisfied:

$$\ln\frac{\beta_1}{\alpha} \ \ge \ 2(\Lambda(A_s) + \eta)t, \quad \forall t \in J, \tag{102}$$

$$\| L \|^2 \le \frac{\beta_2}{\beta_1}, \tag{103}$$

$$\| \mathbf{f}_{p1}(\mathbf{x}_1(t)) \| \ \le \ \eta \cdot \| \mathbf{x}_1(t) \|, \tag{104}$$

where the matrix A_s is defined as follows

$$A_s = \frac{1}{2}\left((A_1 + A_2L)^T + (A_1 + A_2L)\right), \tag{105}$$

and $\Lambda(\cdot)$ with (98), *Debeljkovic et al.* (1995.a).

To analyze the robustness of practical stability let us consider the perturbed linear singular system which can be presented in the form

$$\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + A_2\mathbf{x}_2(t) + B_1\mathbf{x}_1(t) + B_2\mathbf{x}_2(t), \tag{106}$$

$$\mathbf{0} = A_3\mathbf{x}_1(t) + A_4\mathbf{x}_2(t), \tag{107}$$

where $\mathbf{x}(t) = \left[\mathbf{x}_1^T(t) \ \ \mathbf{x}_2^T(t)\right]^T$ need not represent the original phase variables of the nominal system.

The vectors $B_1\mathbf{x}_1(t)$ and $B_2\mathbf{x}_2(t)$, represent the model perturbation.

The matrices B_1 and B_2 need not be known completely and for our robustness analysis we require the knowledge only of the bounds of their norms.

To simplify the formulation of the stability robustness results, we introduce the following assumption:

Assumption 3. Let L be a matrix which satisfies (38), let ε_i, $i = 1,2,3$, be positive numbers, and let $\|B_1\| \le \varepsilon_1$, $\|B_2\| \le \varepsilon_2$, $\|L\| \le \varepsilon_3$.

Now define a new matrix Z as

$$Z = (A_1 + A_2L)^T P + P(A_1 + A_2L) + \varepsilon I_{n1} \tag{108}$$

with

$$\varepsilon = 2\lambda_M(P)(\varepsilon_1 + \varepsilon_2\varepsilon_3) \tag{109}$$

With the so-redefined matrix Z, we can state the results of stability robustness, expressed in terms of constraints given in *Assumption* 3, using analogs of *Theorems* 4 - 7.

Theorem 21. Let the rank condition (34) and *Assumption* 3 hold, and let the matrix G be defined as in (47). Then there are $\{J,\alpha,\beta,G\}$ - practically stable solutions of (**106 - 107**) that satisfy (45), if

$$\eta \cdot t \le \ln\frac{\beta}{\alpha}, \quad \forall t \in J, \tag{110}$$

where $\alpha,\beta \in \mathbb{R}$, $\beta \ge \alpha$, and $\eta = \lambda_M(Z)/\lambda_M(P)$ when $\lambda_M(Z) \le 0$ or $\eta = \lambda_M(Z)/\lambda_m(P)$ where $\lambda_M(Z) > 0$, where the matrix Z is defined by (108 -109), and L satisfies (38). Moreover, if $\lambda_M(Z) \le 0$, i.e. if Z is nsd, then $J = [0,+\infty[$, $\beta = \alpha$, can be selected. If $\lambda_M(Z) > 0$, then $J = [0,T[$, $T < +\infty$ and $\beta > \alpha$ has to be selected to have $T > 0$, *Debeljkovic et al.* (1995.b).

Theorem 22. Let the rank condition (34) and *Assumption* 3 hold, and let the matrix G be defined as in (47). Then there are $\{J,\alpha,\beta_1,\beta_2\}$ - practically stable solutions of (**106 - 107**) that satisfy (45), if

$$\eta \cdot t \le \ln\frac{\beta}{\alpha}, \quad \forall t \in J, \tag{24}$$

$$\| L \|^2 = \frac{\beta_2}{\beta_1} \tag{25}$$

where $\alpha, \beta \in \mathbb{R}$, $\beta \geq \alpha$, and $\eta = \lambda_M(Z)/\lambda_M(P)$ when $\lambda_M(Z) \leq 0$ or $\eta = \lambda_M(Z)/\lambda_m(P)$ when $\lambda_M(Z) > 0$, where the matrix Z is defined by (108 - 109) with L satisfying (38). Moreover, if $\lambda_M(Z) \leq 0$, i.e. if Z is nsd, then $J = [0, +\infty[$, $\beta = \alpha\phi$, can be selected where $\phi = \lambda_m(P)/\lambda_M(P)$. If $\lambda_M(Z) > 0$, then $J = [0, T[$, $T < +\infty$ and $\beta \geq \alpha\phi$ has to be selected to have $T > 0$, *Debeljkovic et al.* (1995.b).

Conclusion

The main features of finite-time stability have been extended to singular (semistate, descriptor) systems. The derived results represent the sufficient condition for stabilty of such systems, based on Liapunov-like functions and their properties on sub-space of consistent initial conditions. In particular these functions need not have: (**a**) properties of positivity in the whole state space and (b) negative derivatives along the system trajectories.

Simple sufficient algebraic conditions are derived for testing the existence of SLS the solutions which have a specific "practical stability" characterization of boundedness properties. The estimate of the potential (weak) region of practical stability is obtained. The results could serve as a basis for further development of a similar analysis for general SLS, as well for nonlinear and time-variable, and time-descrete descriptor systems.

Some other results have been derived using the matrix measure approach, e,g. using Coppels inequality which significantly simplfies some analysis.

Finally the Bellma – Gronwall approach was used to show that some of the previous results can be derived in more simplified manner.

For a particular class of (SLS) simple sufficient conditions constraints for the existence of solutions with specific practical stability, and practical instability are derived.

The estimate of the potential domain of the practical stability is obtained. The results are adapted to cater for the robustness of the practical stability for a class of perturbed (LSS). The results obtained could serve as a basis for further development of the practical stability analysis and the robustness consideration of (LSS).

The similar problems have been presented and solved for a class of time varying linear singular systems.

Finally, it should be noted that the derived results are *independent of the system index.* The system index is a measure of system singularity.

An ordinary differential equation is of index zero. The existence of algebraic equations in a singular system description guarantees that the system index is at least one and the index increase implies more complex behavior.

For more information about this important characteristic of singular systems, an interested reader is directed to read an excellent paper *Campbell* (1990, 1995) and also *Campbell*, *Marszalek* (1996).

Mathematically, it will be useful to simplify the results gathered for a non-homogenous case. For now, it seems that usage of well-known Bellman-Gronwall lemma could be a good method. Some results for a homogenous case are derived. With their extension to a non-homogeneus case, a satisfactory solution of this problem can be expected.

One of the main problems here is finding matrices P, and, consequently, $P(t)$.

Some procedures are given in *Owens*, *Debeljkovic* (1985) for a time-invariant case and in *Bajic* (1992) for a general nonlinear time-varying case, but the algorithmic approach is not available. In practical usage, the assumption that $P = I$ gives sufficiently good results.

Therefore, finding a successful method of calculating the matrix P, and, consequently, $P(t)$ the derived results more will make valuable.

References

[1] APLEVICH,J.D.: *Implicit Linear Systems*, Springer Verlag, Berlin, 1991.

[2] BAJIC,B.V.: *Partial Stability of Motion of Semi-State Systems*, Int. J. of Control, 44(5), (1986) 1383-1394,

[3] BAJIĆ,V.B.: *Lyapunov Function Candidates for Semi-State Systems*, Int. J. Control, **46** (6) (1987) 2171–2181.

[4] BAJIC,B.V.: *Generic Stability and Boundedness of Semistate Systems*, IMA Journal of Mathematical Control and Information, **5** (2) (1988) 103–115.

[5] BAJIC,B.V.: *Non-linear functions and stability of motions of implicit differential systems*, International Journal of Control, Vol. 52, No. 5, (1990) 1167-1187.

[6] BAJIC,B.V.: *Algebraic conditions for stability of linear singular systems*, Proceedings of the 1991 IEEE International Symposium on Circuits and Systems, Vol. 2 (General Circuits and Systems), June 11-14, (1991) 1089-1092,

[7] BAJIC,B.V.: *Lyapunov's Direct Method in the Analysis of Singular Systems and Networks*, Shades Technical Publication, Hillerest, Natal, RSA, 1992.

[8] BAJIC,B.V., DEBELJKOVIĆ,D., GAJIĆ,Z., PETROVIĆ,B.: *Weak Domain of Attraction and Existence of Solutions Convergent to the Origin of the Phase Space of Singular Linear Systems*, University of Belgrade, ETF, Series: Automatic Control, (**1**) (1993) 53–62.

[9] BAJIC,B.V., DEBELJKOVIĆ,D., GAJIĆ,Z.: *Existence of Solution Converging Toward the Origin of the Phase Space of Singular Linear Systems*, Proc. SAUM, Kragujevac, Yugoslavia (June) 1992. 334–348

[10] BAJIC,B.V., DEBELJKOVIĆ,D., GAJIĆ,Z.: *Existence of Solution Converging Toward the Origin of the Phase Space of Singular Linear Systems*, AMSE Conference on System Analysis, Control and Design, Lyon, France (1994) 171–184.

[11] BAJIC,B.V., MILIC,M.M.: *Theorems on the Bounds of Solutions of Semi-State Models*, Int. J. Control, 43(3): 2183-2197, 1986.

[12] BAJIC,B.V., MILIC,M.M.: *Extended Stability of Motion of Semi-State Systems*, Int. J. of Control, 46(6): 2183-2197, 1987.

[13] BOUKAS,E.K., LIU,Z.K.: *Delay-Dependent Stability Analysis of Descriptor Linear Continuous-Time System*", IEEE Proc.Control Theory, 150(4), 2003.

[14] CAMPBELL,S.L.: *Singular systems of differential equation*, Pitman, London, 1980.

[15] CAMPBELL,S.L.: *Singular Systems of Differential Equations II*, Pitman, Marshfield, Mass., 1982.

[16] CAMPBELL,S.L.: *Index two linear time-varying singular systems of differential equations*, SIAM Journal Alg. Disc. Meth., Vol. 4, (1983) 311-326.

[17] CAMPBELL,S.L.: *Nonlinear Time-Varying Generalized State-Space Systems: An Overview*, Proceedings of 23rd Conference on Decision and Control, Las Vegas, NV, (1984) 268-273.

[18] CAMPBELL,S.L.: *Index two linear time-varying singular systems of differential equation*", SIAM Journal Alg. Disc. Meth.. Vol. 4, (1986) 311-326.

[19] CAMPBELL,S.L.: *Cosistent Initial Conditions for Linear Time Varying Singular Systems*, In: Frequency Domain and Slate Space Methods/or Linear Systems, C. I. Byrnes and A. Lindquist, Elsevier Science Publishers B. V. (North-Holand), 1986.

[20] CAMPBELL,S.L.: *Local Realizations of Time Varying Descriptor Systems*, Proceedings of 26th Conference on Decision and Control, (1987) 1129-1130.

[21] CAMPBELL,S.L.: *A general form for solvable linear time varying singular systems of differential equations*, SIAM Journal Mathematical Anal.. Vol. 18, (1987) 1101 - 1115.

[22] CAMPBELL,S.L.: *Descriptor Systems in the 90's"*, Proceedings of the 29th Conference on Decision and Control, Honolulu, Hawaii, (1990) 442 - 447.

[23] CAMPBELL,S.L.: *A Survey of Time Varying and Nonlinear Descriptor Control Systems*" Proceedings 1992 Intern, Syrup, On Implicit and Nonlinear Systems, Automation and Robotics, Res. Inst. Arlington, TX, (1992) 356-363.

[24] CAMPBELL,S.L.: *Uniqueness of Completions for Linear Time Varying Differential Equations*, Linear Algebra and its Applications, Vol. 161, (1992) 55-67.

[25] CAMPBELL,S.L.: *High-Index of Differential Algebraic Equations*, Mech. Struct. & Mach., Vol. 23, (1995), 199-222.

[26] CAMPBELL,S.L., MEYER,C.D., ROSE,N.J.: *Application of Drazin Inverse to Linear System of Differential Equations*, SIAM J. Appl. Math. Vol. 31, (1974), 411 – 425.

[27] CAMPBELL,S.L., PETZOLD,L.R.: *Canonical Forms and Solvable Systems of Differential Equations*, SIAM Journal Alg. Disc. Meth. ,Vol. 4, No. 4, (1983) 517-521.

[28] CAMPBELL,S.L., GRIEPENTROG,E.: *Solvability of General Differential Algebraic Equations*, SIAM Journal Sci. Comp., Vol. 16, no. 2, (1995) 257-270.

[29] CAMPBELL,S.L., Marszalek,W.: *The Index of an Infinite Dimensional System*, Mathematical Modeling of Systems, Vol. 1, (1996)1-25.

[30] CHEN,H-G., KUANG,W.H.: *Improved quantitativemeasures of robustness for multivariable systems*, IEEE Trans. Autom. Contr., 1994, AC - 39, No.4, 807-810

[31] DAI,L.: *Singular Control Systems,* Lecture Notes in Control and Information Sciences, Springer, Berlin, 118, 1989.a

[32] Dai,L.: *The Difference between Regularity and Irregularity in Singular Systems*, Circuits, Systems and Signal Processing, Vol. 8, No. 4, pp. 435-444, 1989.b.

[33] DEBELJKOVIĆ,D.LJ.: *The Survey of Stability Results on Singular and Implicit Systems*, Proc. 4th Conference SAUM, Kragujevac, Yugoslavia (1992) 301–315.

[34] DEBELJKOVIĆ,D.LJ.: *Stability of Linear Autonomous Singular Systems in the sense of Lyapunov: An Overview* NTP (YU), (in Serbian), Vol. LI, No. 3, (2001), 70 – 79.

[35] DEBELJKOVIĆ,D.LJ.: S*ingular Control Systems*, Scientific Review, Series: Science and Engineering, Vol. 29 – 30, (2002), 139 – 162**.**

[36] DEBELJKOVIĆ,D.LJ.: *Singular Control Systens***,** Dynamics of Continuous, Discrete and Impulsive Systems, (Canada), Vol. 11, Series A: Math. Analysis, No. 5 – 6 (2004), 691 – 706.

[37] DEBELJKOVIĆ,D.LJ., OWENS,D.H.: *On practical stability of singular systems*, Proc. Melecon Conf .85, October 85, Madrid (Spain), (1985) 103 – 105.

[38] DEBELJKOVIĆ,D.LJ., BAJIĆ,V.B., GAJIĆ,Z., PETROVIĆ,B.: *Boundedness and Existence of Solutions of Regular and Irregular Singular Systems*, Publications of Faculty of Electrical Eng. Series: Automatic Control, Belgrade (YU), (1) (1993) 69–78.

[39] DEBELJKOVIĆ,D.LJ., BAJIĆ,V.B., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B.: *Quantitative Measures of Robustness of Generalized State Space Systems*, Proc. AMSE Conference on System Analysis, Control and Design, Lyon, France (1994.a) 219–230.

[40] DEBELJKOVIĆ,D.LJ., BAJIĆ,V.B., GRGIC,A.U., MILINKOVIC,S.A.: *Further results in non-Lypunov stability robustness of generalized state – space systems*, 1st IFAC Workshop on New Trends in Design of Control Systems, Smolenice, Bratislava (Slovac Republic), September 7 – 10, (1994.b) 255 – 260.

[41] DEBELJKOVIĆ,D.LJ., BAJIĆ,V.B., GRGIC,A.U., MILINKOVIC,S.A.: *Non-Lyapunov stability and instability robustne**ss** consideration for linear singular systems*, Proc. 3 – rd ECC, Roma (Italy), September 5 – 8, (1995), 1373 – 1379.

[42] DEBELJKOVIĆ,D.LJ., MILINKOVIC,S.A., JOVANOVIĆ,M.B.: *Application of singular system theory in chemical engineering: Analysis of process dynamics*, International Congress of Chemical and Process Eng., CHISA 96, Prague (Czech Rep.), 25-30 August, 1996.

[43] DEBELJKOVIĆ,D.LJ. JOVANOVIC,M.R.: *Non-Lyapunov stability consideration of linear descriptor systems operating under perturbing forces*, AMSE – Advances in Modeling and Analysis, (France), Part: C, Vol. 49, No. 1 – 2, (1997) 1 – 8.

[44] DEBELJKOVIĆ,D.LJ., LAZAREVIC,M.P., KORUGA,DJ., TOMAŠEVIC,S.: *Finite time stability of singular systems operating under perturbing forces: Matrix measure approach*, Proc. AMSE Conference, Melbourne (Australia), October 29 – 31, (1997) 447 – 450.

[45] DEBELJKOVIĆ,D.LJ., KORUGA,DJ., MILINKOVIC,S.A., JOVANOVIC,M.R., JACIC,LJ.A.: *Finite time stability of linear descriptor systems*, Proc. MELECON 98, Tel – Aviv (Israel), May 18 – 20, Vol. 1, (1998), 504 – 508.

[46] DEBELJKOVIĆ,D.LJ., MILINKOVIC,S.A., JOVANOVIC,M.R. JACIC,LJ.A., KORUGA,DJ.: *Further results on Non-Lyapunov stability of time delay systems***,** Preprints 5th IFAC Symposium on Low Cost Automation, Shenyang (China), September 8 – 10 (1998), TS13 6 – 10.

[47] DEBELJKOVIĆ,D.LJ., KABLAR,N.A.: *On necessary and sufficient conditions of linear singular systems stability operating over finite time interval*, Proc. XII CBA , Uberlandia (Brazil), September 14 – 18 Vol. IV, (1998), 1241 – 1246.

[48] DEBELJKOVIC,LJ.D., KABLAR,N.A.: *Finite time stability robustness of time varying linear singular systems*, Proc. ASCC 2000, July 4 – 7, , Shanghai, (China), (2000) 826 – 829.

[49] DEBELJKOVIC,LJ.D., KABLAR,N.A.: *Further results on finite time stability of discrete linear singular systems: Bellman – Gronwall approach*, Proc. APCCM (The 4th Asia – Pacific Conf. on Control and Measurements), 9 – 12 July, Guilin (China),(2000) D.10.

[50] DEBELJKOVIC,LJ.D., KORUGA,DJ.: *Non-Lyapunov Stability of Linear Singular Systems: A Quite New Approach*, Proc. APCCM 2002, July 8 – 12, (2002), Dali, Lijiang (China), CD-ROM, also in Proc. CBA XIV, September 2 – 5, (2002), Natal (Brazil), CD-Rom.

[51] DEBELJKOVIC,LJ.D., ZHANG,Q.L.: *Dynamic analysis of generalized nonautonomous state space systems*, Scientific Technical Review, 2003, Vol.LIII, No.1, pp.30 – 40.

[52] DEBELJKOVIC,D.LJ., MILINKOVIC,S.A., JOVANOVIC,M.B.: *Continuous Singular Control Systems,* GIP Kultura, Belgrade, 2005.a.

[53] DEBELJKOVIC,D.LJ., MILINKOVIC,S.A., JOVANOVIC,M.B., .JACIC,LJ.A.: *Discrete Descriptor Control Systems*, GIP Kultura, Belgrade, 2005.b.

[54] HU,G., SUN,J.T.: *Stability Analysis for Singular Systems With Time-Varying*, Journal of Tongji University, 31(4), (2003) pp.481-485.

[55] ISIHARA,J.Y., TERRA,M.H.: *On the Lyapunov Theorem for Singular Systems*, IEEE Trans. Automat. Control **AC**-47 (11) (2000.b), pp.1926 – 1930.

[56] KABLAR,A.N., DEBELJKOVIC,D.LJ.: *Non-Lyapunov stability of linear singular systems: Matrix measure approach*, Proc. MNTS – Mathematical Theory of Networks and Systems, Padova (Italy), July 6 – 10, (1998.a) – Presented lecture.

[57] KABLAR,A.N., DEBELJKOVIC,D.LJ.: *Non-Lyapunov stability of linear singular systems: Matrix measure approach*, Preprints 5th IFAC Symposium on Low Cost Automation, Shenyang (China), September 8 – 10 (1998.b), TS13 16 – 20.

[58] KABLAR,A.N., DEBELJKOVIC,D.LJ.: *Finite time stability of time varying singular systems*, Proc. CDC 98, Florida (USA), December 10 – 12 (1998), 3831 – 3836.

[59] KABLAR,A.N., DEBELJKOVIC,D.LJ.: *Finite time instability of time varying linear singular systems*, Proc. ACC 99, San Diego (USA), June 2 – 4 (1999) 1796 – 1800.

[60] KABLAR,A.N., DEBELJKOVIC,D.LJ.: *Finite time stability of linear singular systems : Bellman – Gronwall approach*, Proc. ACC 99, San Diego (USA), June 2 – 4 (1999) 1803 – 1806.

[61] LEWIS,E.L.: *A Survey of Linear Singular Systems*, Circuits, Systems and Signal Processing, 5 (1) (1986) 3-36.

[62] LI,Y.Q., LIU,Y.Q.: *Stability of Solution of Singular Systems with Delay*, Control Theory and Application, 15 (4) (1998) 542-550.

[63] LIU,Y.Q., LI,Y.Q.: *Stability of Solutions of a Class of Time-Varying Singular Systems With Delay*, Control and Decision, 12 (3) (1997) 193-197.

[64] LIANG,J.R.: *Analysis of Stability for Descriptor Discrete Systems With Time-Delay*, Journal of Guangxi University (Nat. Sc. Ed.), 25 (3) (2000) 249-251.

[65] LIANG,J.R.: *The Asymptotic Stability and Stabilization for Singular Systems with Time Delay*, System Engineering and Electronics, 23 (2) (2001) 62-64.

[66] LIANG,J.R., YING,Y.R.: *Analysis of Stability for Singular Discrete Linear Systems*, Journal of Shanxi Normal University (Natural Sciences Edition), 27 (4) (1999)17-21.

[67] LUENBERGER,D.G.: *Dynamic Equations in Descriptor Form*, IEEE Trans. Automat. Control, 22 (3) (1977) 312-321.

[68] MASUBUCHI,I., KAMITANE,Y., OHARA,A. et al.: *Control for Descriptor Systems – A Matrix Inequalities Approach*, Automatica, 33(4) (1997) 669-673.

[69] MASUBICHI,I., SHIMEMURA,E.: *An LMI Condition for Stability of*

Implicit Systems, Proceedings of the 36th Conference on Decision and Control, San Diego, California USA, 1997.

[70] MASUBUCHI,I., KAMITANE,Y., OHARA,A.: *Control for Descriptor Systems – A Matrix Inequalities Approach*, Automatica, 33(4): (1997) 669-673.

[71] MEN,B., ZHANG,Q.L., LI,X., YANG,C., CHEN,Y.: *The Stabilty of Linear Descriptor Systems*, Intern. Journal of Inform. and System Science Vol.2, No.3, (2006) 362 – 374.

[72] MILIC,M.M., BAJIC,V.B.: *Stability Analysis of Singular Systems*, Circuits Systems Signal Process, 8(3) (1989) 267-287.

[73] MULLER,P.C.: *Linear Mechanical Descriptor Systems: Identification, Analysis and Design*, Preprints of IFAC, Conference on Control of Independent Systems, Belfort, France, (1997) 501-506.

[74] OWENS,H.D., DEBELJKOVIC,D.LJ.: *Consistency and Liapunov Stability of Linear Descriptor Systems: A Geometric Analysis*, IMA Journal of Mathematical Control and Information, (**2**), (1985), 139-151.

[75] PANDOLFI,L.: *Controlabillity and Stabilization for Linear System of Algebraic and Differential Equations*, Jota 30 (4) (1980) 601 – 620.

[76] PATEL,R.V., TODA,M.: *Quantitative measures of robustness for multivariable system*", Proc. Joint Contr. Conf., San Francisco, CA (1980), CA, TP 8-A.

[77] SILVA,M.S., DE LIMA,T.P.: *Looking for nonnegative solutions of a Leontief dynamic model*, Linear Algebra, 364 (2003) 281-316.

[78] SYRMOS,V.L., MISRA,P., ARIPIRALA,R.: *On the Discrete Generalized Lyapunov Equation*, Automatica, 31 (2) (1995) 297-301.

[79] TAKABA,K.: *Robust Control Descriptor System With Time-Varying Uncertainty*, Int. J. of Control, 71 (4) (1998) 559 – 579.

[80] TSENG,H.C., KOKOTOVIC,P.V.: *Optimal Control in Singularly Perturbed Systems: The Integral Manifold Approach*, IEEE Proc. on CDC, Austin, TX (1988) 1177–1181.

[81] WANG,Q., ZHANG,Q.L., YAO,B.: *Lyapunov Equation With Positive Definite Solution for Discrete Descriptor Systems*, Journal of Natural Sciences of Heilongjiang University, 20 (1) (2003) 50-54.

[82] WU,H.S., MIZUKAMI,K.: *Lyapunov Stability Theory and Robust Control of Uncertain Descriptor Systems*, Int. Journal of System Sci., 26 (10) (1995)1981-1991.

[83] XU,S., YANG,C.: *An Algebraic Approach to the Robust Stability Analysis and Robust Stabilization of Uncertain Singular Systems*, Int. J. System Science, Vol.31, (2000.a) 55–61.

[84] XU,S., YANG,C.: *H_∞ State Feedback Control for Discrete Singular Systems*, IEEE Trans. Automat. Control AC-45 (6) (2000.b) 1405 – 1409.

[85] XU,S., YANG,C., NIU,Y., LAM,J.: *Robust Stabilization for Uncertain Discrete Singular Systems*, Automatica, Vol.37, (2001.a) 769 – 774.

[86] XU,S., LAM,J. YANG,C.: *Quadratic Stability and Stabilization of Uncertain Linear Discrete – time Systems with State Delay*, Systems Control Lett. (43), (2001.b) 77–84.

[87] XU,S., DOOREN,P.V. STEFAN,R., LAM,J.: *Robust Stability and Stabilization for Singular Systems with State Delay and Parameter Uncertainty*, IEEE Trans. Automat. Control AC-47 (7) (2002) 1122 – 1128.

[88] XU,S., LAM,J., YANG,C.: *Robust H_∞ Control for Discrete Singular Systems with State Delay and Parameter Uncertainty*, Dynamics of Continuous, Discrete and Impulsive Systems, Vol. 9, No. 4, (2002) 539 - 554

[89] YANG,D.M., ZHANG,Q.L., YAO,B.: *Descriptor System*s, Science Publisher, Beijing, 2004.

[90] YANG,Z.M., WANG,S.K. WU,M.X., LIU,L.H.: *The Criteria Stability for a Class Discrete Generalized Systems*, J. of Xianyung Teachers' College, 18 (4)(2003)12-13,

[91] YAO,B., ZHANG,Q.L., YANG,D.M., WANG,F.Z.: *Analysis and Control of Asymptotic Stability for Discrete Descriptor Systems*, Journal of Northeastern University (Natural Sciences), 23(4) (2002) 315-317.

[92] YEDAVALLI,R.K.: *Improved Measures of Stability Robustness for Linear State Space Models*, IEEE Trans. Autom. Contr., AC-30, No. 6, (1985) 577-579.

[93] YEDAVALLI,R.K., LIANG,Z.: *Reduced Conservatism in Stability Robustness Bounds by State Transformation*, IEEE Trans. Autom. Contr.,AC-31, No 9, (1986) 863-866.

[94] YUE,X.N., ZHANG,X.N.: *Analysis of Lyapunov Stability for Descriptor Systems*, Journal of Liaoning Educational Institute, 15 (5) (1998) 42-44.

[95] ZHANG,Q.L.: *The Lyapunov Method of Descriptor Systems Structure*, System and Science and Maths, 14 (2) (1994) 117-120,

[96] ZHANG,Q.L.: *Dispersal Control and Robust Control for Descriptor Systems*, Northwestern Industrial University Publisher, 1997.

[97] ZHANG,Q.L., XU,X.H.: *Structural Stability and Linear Quadratic Control for Discrete Descriptor Systems*, Proc. of the Asian Control Conference Tokyo, 7 (1994) 27-30.

[98] ZHANG,Q.L., DAI,G.Z., LAM,J., ZHANG,L.Q.: *The Asymptotic Stability and Stabilization of Descriptor Systems*, Acta Automatica Sinica, 24 (2) (1998.a.) 208-312.

[99] ZHANG,Q.L., DAI,G.Z., XU,X.H., XIE, X.K.: *Analysis and Control of Stability for Discrete Descriptor Systems via Lyapunov Methods*, Acta Automatica Sinica, 24 (5) (1998.b.) 622-629.

[100] ZHANG,Q.L., LAM,J., ZHANG,Q.L.: *Lyapunov and Riccati Equations of Discrete-Time Descriptor Systems*, IEEE Trans. on Automatic Control, 44(11) (1999) 2134-2139.

[101] ZHANG,Q.L., LAM,J., ZHANG,Q.L.: *Generalized Lyapunov Equation for Analyzing the Stability of Descriptor Systems*, Proc. of the 14-th World Congress of IFAC (1999.a) D-2b-01-4:19-24.

[102] ZHANG,Q.L., ZHANG,L.Q., NIE,Y.Y.: *Analysis and Synthesis of Robust Stability for Descriptor Systems*, Control Theory and Applications, 16(4) (1999.b) 525-528,

[103] ZHANG,J.H., YU,G.R.: *Analysis of Stability for Continuous Generalized Systems*, Journal of Shenyang Institute of Aeronautical Engineering, Shenyang, 19(2) (2002) 67 -69.

[104] ZHANG,Q.L., LAM,J., ZHANG,L.Q.: *Generalized Lyapunov Equation for Analyzing the Stability of Descriptor Systems*, Proceedings of the 14th World Congress of IFAC, D-2b-01-4:19-24, (1999.c).

[105] ZHANG,Q.L., LAM,J., ZHANG,L.Q.: *New Lyapunov and Riccati Equations for Discrete - Time Descriptor Systems*, Proceedings of the 14th World Congress of IFAC, D-2b-01-2: 7-12, (1999.d.)

[106] ZHANG,Q.L., LIU,W.Q., HILLE,D.: *Lyapunov Approach to Analysis of Discrete Descriptor Systems*, Systems Control Letters, 45: 237-247, (2002.a.)

[107] ZHANG,Q.L., WANG,Q., CONG,X.: *Investigation of Stability of Descriptor Systems*, Journal of Northeastern University (Natural Sciences), 23(7):624-627, (2002.b.)

[108] ZHANG,X.M., WU,M., HE,Y.: *On Delay-Dependent Stability for Linear Descriptor Systems With Delay*, Journal of Circuits and Systems, 8(4):3-7, 2003.

[109] ZHOU,K., KHARGONEKAR,P.P.: *Stability Robustness Bounds for Linear State-Space Models with Structure Uncertainty*, IEEE Trans. Autom. Contr., **AC-32**, No. 7, (1987) 621-623.

[110] ZHU,J.D., MA,S., CHEN,Z.L.: *Singular LQ Problem for Nonregular Descriptor Systems*, 47(7) (2002) 1128-1133.

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Stabilnost linearnih kontinualnih singularnih sistema na konačnom vremenskom intervalu: Pregled rezultata

Ovaj rad daje detaljan pregled rada i rezultata mnogih autora na polju izučavanja neljapunopvske stabilnost (stabilnost na konačnom vremenskom intervalu, praktična stabilnost, konačna stabilnost) posebne klase linearnih sistema. Problem robusnosti stabilnosti takođe je razmatran i izložen.
Ovaj pregled rezultata pokriva period od 1985. godine do današnjih dana i ima izraženu nameru da predstavi glavne koncepte i doprinose na ovom polju stvorene u celom svetu a u pomenutom periodu, koji su objavljeni u vodećim međunarodnim časopisima ili prezentovani na prestižnim konferencijama ili worksopovima.

Ključne reči: **kontinualni sistem, singularni sistem, linearni sistem, stabilnost sistema, stabilnost Ljapunova, konačni vremenski interval.**

Устойчивость линейных непрерывных сингулярных систем в конечном временном интервале: Обзор и анализ результатов

Настоящая работа даёт подробный обзор и анализ результатов многих авторов в области исследования неляпуновой устойчивости (устойчивость на конечном временном интервале, практическая устойчивость, конечная устойчивость) особого класса линейных систем. Проблемма крепкости устойчивости тоже здесь рассматривана и растолкована.
Этот обзор и анализ результатов охватывают период с 1985-ого года до сих пор и у него выразительное намерение представить концепции и вклад в этой области созданые в целом мире в упомянутом периоде и опубликованные в передовых международных журналах или показаны и представлены на выдающихся конференциях или в мастерских.

Ключевые слова: непрерывная система, сингулярная система, линейная система, устойчивость системы, устойчивость Ляпунова, конечный временной интервал.

Stabilité des systèmes linéaires continus singuliers chez l'intervalle temporelle finie: tableaux des résultats

Ce papier présente un tableau détaillé des travaux et des résultats de nombreux auteurs dans le domaine des études sur la stabilité de non-Lyapunov (stablité chez l'intervalle temporelle finie, stabilité pratique, stabilité finie) de la classe particulière des systèmes linéaires. Le problème de la robustesse de la stabilité est aussi considéré et exposé. Ce tableau des résultats comprend la période depuis 1985 jusqu'à nos jours et son but est de présenter les concepts principaux et les contributions dans ce domaine dans le monde entier pendant la période citée et qui sont publiées dans les principales revues internationalles ou présentées aux conférences prestigieuses ou workshops.

Mots clés: **système continu, système singulier, système linéaire, stabilité du système, stabilité de Lyapunov, intervalle temporelle finie.**