UDK: 681.511.26:517.922
COSATI: 12-01

The Stability of Linear Continuous Singular Systems in the sense of Lyapunov: An Overview

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This paper gives a detailed overview of the work and results obtained by many authors in the area of Lyapunov stability of a particular class of linear systems. The stability robustness problem has also been treated.
This survey, covering the period from 1980 up to the present days, aims at presenting the main concepts and contributions derived over the mentioned period worldwide, published in the respectable international journals or presented on workshops or prestigious conferences.

Key words: **continuous system, singular system, linear system, system stability, Lyapunov stability, asymptotic stability.**

Introduction

IN certain systems, their character of dynamic and static state must be considered. Singular systems (also, referred to as degenerate, descriptor, generalized, differential - algebraic systems or semi - state) are those the dynamics of which are governed by a mixture of algebraic and differential equations. Recently, many scholars have paid much attention to singular systems which brought about numerous conveniences. The complex nature of singular systems - causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control.

It is well-known that singular systems have been among major research fields of the control theory. Over the past three decades, singular systems have attracted much attention due to the comprehensive applications in economics as the *Leontief* dynamic model *Silva*, *Lima* (2003), in electrical *Campbell* (1980) and mechanical models *Muller* (1997), etc.

Furthermore, they arose naturally, as a linear approximation of system models, or linear system models in many applications such as electrical networks, *aircraft dynamics*, neutral delay systems, chemical, thermal and diffusion processes, large-scale systems, interconnected systems, economics, optimization problems, feedback systems, robotics, biology, etc.

Discussion of singular systems originated in 1974 with the fundamental paper of *Campbell et al.* (1974) and latter on the antological paper of *Luenberger* (1977). Since then, considerable progress has been made in investigating such systems; see surveys of *Lewis* (1986) and *Dai* (1989) for linear singular systems and the first results for nonlinear singular systems in *Bajic* (1992).

Through investigation of stability of singular systems, many results in the sense of Lyapunov stability have been derived. For example, *Bajic* (1992) and *Zhang et al.* (1999) considered the stability of linear time-varying descriptor systems.

This paper presents, in a unified way, a collection of results found in references and focuses on the stability of linear continuous systems (LCSS).

This paper is not a survey in the usual sense.

The paper does not attempt to be exhaustive of the vast resources concerning this problem. The objective is more to convince the reader of the practical interest of the approach and of the number and simplicity of the results it leads to.

For each aspect, only one result is generally given in detail, the one which is not necessarily the most complete or the most recent one, but is the one which seems the most representative and illustrative.

Basic notations

R	– Real vector space
$\mathbb{C}$	– Complex vector space
$\mathbf{C}$	– Complex plane
I	– Unit matrix
F	$- = (f_{ij}) \in \mathbb{R}^{n\times n}$, real matrix
F^T	– Transpose of matrix F
$F > 0$	– Positive definite matrix
$F \geq 0$	– Positive semi definite matrix
$\Re(F)$	– Range of matrix F
N	– Nilpotent matrix
$\mathbb{N}(F)$	– Null space (kernel) of matrix F
$\lambda(F)$	– Eigenvalue of matrix F
$\sigma_{(\ )}(F)$	– Singular values of matrix F
$\sigma\{F\}$	– Spectrum of matrix F
$\lVert F\rVert$	– Euclidean matrix norm $\lVert F\rVert = \sqrt{\lambda_{\max}\left(A^T A\right)}$
F^D	– Drazin inverse of matrix F

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$\Rightarrow$ – Follows
$\mapsto$ – Such that

Stability of linear continuous singular systems

The stability of linear time invariant continuous singular systems

Generally, the time invariant continuous singular control systems can be written as:

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0(t), \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is a generalized state space (co-state, semi-state) vector, $\mathbf{u}(t) \in \mathbb{R}^l$ is a control vector, $E \in \mathbb{R}^{n\times n}$ is a possibly singular matrix, with $\operatorname{rank} E = r < n$.

Matrices E and A are of the appropriate dimensions and are defined over the field of real numbers. System given by eq. (1) is operating in a free regime and no external forces are applied on it.

It should be stressed that, in general, the initial conditions for an autonomous and system operating in the forced regime need not be the same.

System models of this form have some important advantages in comparison with the models in the *normal form, e.g. when* $E = I$ and an appropriate discussion can be found in *Bajic* (1992) and *Debeljkovic et al.* (1996. 2005.a, 2205.b).

The complex nature of singular systems causes many difficulties *in analytical and numerical treatment* that do not appear when systems in the normal form are considered.

In this sense, questions of existence, solvability, uniqueness and smoothness are present, which must be solved in a satisfactory manner.

A short and concise, acceptable and understandable explanation of all these questions may be found in the papers of *Debeljkovic* (2001, 2002, 2004).

The survey of updated results for singular systems and a broad bibliography can be found in *Bajic* (1992), *Campbell* (1980, 1982), *Lewis* (1986, 1987), *Debeljkovic et al.* (1996, 2005.a, 2205.b) and in the two special issues of the journal *Circuits, Systems and Signal Processing* (1986, 1989).

Stability definitions

Stability plays a central role in the theory of systems and control engineering. There are different kinds of stability problems that arise in the study of dynamic systems, such as *Lyapunov* stability, finite time stability, practical stability, technical stability and BIBO stability. The first part of this section is concerned with the asymptotic stability of the equilibrium points of *linear continuous singular systems.*

When linear systems are treated, this is equivalent to the study of the stability of the systems.

The *Lyapunov* direct method (LDM) is exploited in a number of very well known references. Here some different and interesting approaches to this problem are presented, including the contributions of the authors of this paper.

***Definition* 1.** System (1) is *regular* if there exist $s \in \mathbf{C}$ $\det(sE - A) \neq 0$, *Campbell et al.* (1974).

***Definition* 2.** Eq. (1) is exponentially stable if two positive constants α, β such that for every solution of Eq. (1) can be found, *Pandolfi* (1980).

***Definition* 3.** The system given by eq. (1) will be termed *asymptotically stable* if, for all consistent initial conditions $\mathbf{x}_0$, $\mathbf{x}(t) \to 0$ as $t \to \infty$, *Owens, Debeljkovic* (1985).

***Definition* 4.** The system given by eq. (1) is *asymptotically stable* if all roots of $\det(sE - A)$, *i.e.* all finite eigenvalues of this matrix pencil, are in the open left - half complex plane, and system under consideration is *impulsive free* if there is no $\mathbf{x}_0$ such that $\mathbf{x}(t)$ exhibits discontinuous behaviour in the free regime, *Lewis* (1986).

***Definition* 5.** The system given by eq. (1) is called *asymptotically stable* if and only if all finite eigenvalues $\lambda_i, i = 1,\ldots,n_1$, of the matrix pencil $(\lambda E - A)$ have negative parts, *Muller* (1993).

***Definition* 6.** The equilibrium $\mathbf{x} = 0$ of the system given by eq. (1) is said to be *stable* if for every $\varepsilon > 0$, and for any $t_0 \in T$, there exists a $\delta = \delta(\varepsilon, t_0) > 0$, such that $\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon$ holds for all $t \geq t_0$, whenever $\mathbf{x}_0 \in W_k$ and $\|\mathbf{x}_0\| < \delta$, where T denotes time interval such that $T = [t_0, +\infty)$, $t_0 \geq 0$, *Chen*, *Liu* (1997).

***Definition* 7.** The equilibrium $\mathbf{x} = 0$ of a system given by eq. (1) is said to be *unstable* if there exists a $\varepsilon > 0$ and $t_0 \in T$, for any $\delta > 0$, such that there exists a $t^* \geq t_0$, for which $\|\mathbf{x}(t^*, t_0, \mathbf{x}_0)\| \geq \varepsilon$ holds, although $\mathbf{x}_0 \in W_k$ and $\|\mathbf{x}_0\| < \delta$, *Chen*, *Liu* (1997).

***Definition* 8.** The equilibrium $\mathbf{x} = 0$ of a system given by eq. (1) is said to be *attractive* if for every $t_0 \in T$, there exists an $\eta = \eta(t_0) > 0$, such that $\lim_{t\to\infty} \mathbf{x}(t, t_0, \mathbf{x}_0) = 0$, whenever $\mathbf{x}_0 \in W_k$ and $\|\mathbf{x}_0\| < \eta$, *Chen, Liu* (1997).

***Definition* 9**. The equilibrium $\mathbf{x} = 0$ of a singular system given by eq. (1) is said to be *asymptotically stable* if it is *stable* and *attractive*, *Chen*, *Liu* (1997).

***Definition* 10.** System (1) is asymptotically stable if positive numbers α, β satisfying $\|\mathbf{x}(t)\|_2 \leq \alpha e^{-\beta t}\|\mathbf{x}(0)\|_2$, $t > 0$ exist. *Yang et al.* (2004).

***Definition* 5.** is equivalent to $\lim_{t\to+\infty} \mathbf{x}(t) = 0$

***Lemma* 1.** The equilibrium $\mathbf{x} = 0$ of a linear singular system given by eq. (1) is *asymptotically stable* if and only if it is *impulsive-free* and $\sigma(E, A) \subset \mathbf{C}^-$, *Chen*, *Liu* (1997).

***Lemma* 2.** The equilibrium $\mathbf{x} = 0$ of a system given by eq. (1) is *asymptotically stable* if and only if it is *impulsive-free and* $\lim_{t\to\infty} \mathbf{x}(t) = 0$, *Chen*, *Liu* (1997).

Due to the system structure and complicated solution, the regularity of the systems is the condition to make the solution of singular control systems exist and be unique. Moreover, if the consistent initial conditions are applied, the closed form of solutions can be established.

If system (1) is *regular*, it is a restricted system equivalent (r.s.e.) to:

$$\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) \tag{2.a}$$

$$N\dot{\mathbf{x}}_2(t) = \mathbf{x}_2(t) \tag{2.b}$$

and the system is *impulse free* when $N = 0$.

Stability theorems

***Theorem* 1.** Eq. (1), with $A = I$, I being the identity matrix, is *exponentially stable* if and only if the eigenvalues of *E* have non positive real parts, *Pandolfi* (1980).

***Theorem* 2.** Let I_Ω be the matrix that represents the operator on $\mathbb{R}^n$ which is the identity on Ω and the zero operator on Λ.

Eq. (1), with $A = I$, is stable if an $n \times n$ matrix P exists and is the solution of the matrix equation:

$$E^T P + PE = -I_\Omega \triangleright, \tag{3}$$

with the following properties:

$$P = P^T,$$

$$P\mathbf{q} = 0,\ \mathbf{q} \in \Lambda, \tag{4}$$

$$\mathbf{q}^T P\mathbf{q} > 0,\ \mathbf{q} \neq 0,\ \mathbf{q} \in \Omega,$$

where:

$$\Omega = W_k = \aleph(I - EE^D), \tag{5.a}$$

$$\Lambda = \aleph(EE^D), \tag{5.b}$$

where W_k is the subspace of consistent initial conditions, *Pandolfi* (1980). $\aleph$ denotes the kernel or null space of the matrix ( ).

***Theorem* 3**. The system given by eq. (1) is *asymptotically stable* if and only if:
A is invertible a positive-definite, self-adjoint operator P on $\mathbb{R}^n$ exists, such that:

$$A^T PE + E^T PA = -Q, \tag{6}$$

where Q is self-adjoint and positive in the sense that:

$$\mathbf{x}^T(t) Q\mathbf{x}(t) > 0 \text{ for all } \mathbf{x} \in W_{k^*} \setminus \{0\} \tag{7}$$

Owens, Debeljkovic (1985).

***Theorem* 4.** The system given by eq. (1) is *asymptotically stable* if and only if:
A is invertible and a positive-definite, self-adjoint operator P exists, such that:

$$\mathbf{x}^T(t)\left(A^T PE + E^T PA\right)\mathbf{x}(t) = -\mathbf{x}^T(t) I\mathbf{x}(t),$$

for all

$$\mathbf{x} \in W_{k^*}. \tag{8}$$

where W_{k^*} denotes the subspace of consistent initial conditions, *Owens, Debeljkovic* (1985).

***Theorem* 5.** Let (E, A) be regular and (E, A, C) be observable.

Then (E, A) is *impulsive free* and *asymptotically stable* if and only if a positive definite solution P to:

$$A^T PE + E^T PA + E^T C^T CE = 0, \tag{9}$$

exists and if P_1 and P_2 are two such solutions, then $E^T P_1 E = E^T P_2 E$, *Lewis* (1986).

***Theorem* 6**. If there are symmetric matrices P, Q satisfying:

$$A^T PE + E^T PA = -Q, \tag{10}$$

and if:

$$\mathbf{x}^T E^T PE\mathbf{x} > 0,\ \mathbf{x} \in S_1\mathbf{y}_1 \neq 0, \tag{11}$$

$$\mathbf{x}^T Q\mathbf{x} \geq 0,\ \forall \mathbf{x} = S_1\mathbf{y}_1, \tag{12}$$

then the system described by eq. (1) is *asymptotically stable* if:

$$rank\begin{pmatrix} sE - A \\ S_1^T Q \end{pmatrix} = n,\ \forall s \in \mathbf{C}, \tag{13}$$

and marginally stable if the condition given by eq. (13) does not hold, *Muller* (1993).

***Theorem* 7.** The equilibrium $\mathbf{x} = 0$ of a system given by eq. (1) is *asymptotically stable,* if an $n \times n$ symmetric positive definite matrix P exists, such that along the solutions of the system, given by eq. (1), the derivative of function $V(E\mathbf{x}) = (E\mathbf{x})^T P(E\mathbf{x})$, is a negative definite for the variates of $E\mathbf{x}$, *Chen*, *Liu* (1997)

***Theorem* 8**. If an $n \times n$ symmetric, positive definite matrix P exists, such that along with the solutions of the system, given by eq. (1), the derivative of the function $V(E\mathbf{x}) = (E\mathbf{x})^T P(E\mathbf{x})$ i.e. $\dot{V}(E\mathbf{x})$ is a positive definite for all variates of $E\mathbf{x}$, then the equilibrium $\mathbf{x} = 0$ of the system given by eq. (1) is *unstable, Chen*, *Liu* (1997).

***Theorem* 9**. If an $n \times n$ symmetric, positive definite matrix P exists, such that along with the solutions of the system, given by eq. (1), the derivative of the function $V(E\mathbf{x}) = (E\mathbf{x})^T P(E\mathbf{x})$ i.e. $\dot{V}(E\mathbf{x})$ is negative semi definite for all variates of $E\mathbf{x}$, then the equilibrium $\mathbf{x} = 0$ of the system, given by eq. (1), is *stable, Chen*, *Liu* (1997).

***Theorem* 10.** Let (E, A) be regular and (E, A, C) be impulse observable and finite dynamics detectable. Then (E, A) is stable and impulse-free if and only if a solution (P, H) to the generalized *Lyapunov equations* (GLE) exists:

$$A^T P + H^T A + C_y{}^T C_y = 0, \tag{14}$$

$$H^T E = E^T P \geq 0, \tag{15}$$

Takaba et al. (1995).

Some assumptions and preliminaries are needed for further exposures.

Let forced linear continuous singular systems (LCSS) represented by:

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),\quad \mathbf{x}(t_0) = \mathbf{x}_0, \tag{16}$$

$$\mathbf{y}(t) = C\mathbf{x}(t), \tag{17}$$

be considered.

Suppose that matrices E and A commute and that $EA = AE$.

Then, a real number λ exists such that $(\lambda E - I) = A.$,

otherwise, from the property of regularity, Eqs. (16) and (17) may be multiplied by $(\lambda E - A)^{-1}$ so a system that satisfies the above assumption and keeps the stability the same as the original system can be obtained.
It is well known that there always exists linear non-singular transformation, with invertible matrix T, such that:

$$\left(TET^{-1} \quad TAT^{-1}\right) = \left\{diag\left(E_1 \quad E_2\right) \quad diag\left(A_1 \quad A_2\right)\right\} \tag{18}$$

where E_1 is of full rank and E_2 is a nilpotent matrix, satisfying:

$$E_2^h \neq 0, \ E_2^{h+1} = 0, \ h \geq 0. \tag{19}$$

In addition, it is evident:

$$A_1 = \lambda E_1 - I_1, \ A_2 = \lambda E_2 - I_2. \tag{20}$$

The system, given by Eqs. (16) and (17), is (r.s.e.) to:

$$E_1 \dot{\mathbf{x}}_1(t) = A_1 \mathbf{x}_1(t) + B_1 \mathbf{u}(t), \tag{21}$$

$$E_2 \dot{\mathbf{x}}_2(t) = A_2 \mathbf{x}_2(t) + B_2 \mathbf{u}(t), \tag{22}$$

where $\mathbf{x}^T(t) = \left[\mathbf{x}_1^T(t) \quad \mathbf{x}_2^T(t)\right]$.

***Lemma* 4.** The system, given by Eqs. (16) and (17), is *asymptotically stable* if and only if the "*slow*" *sub-system*, eq. (16) is asymptotically stable, *Zhang et al.* (1998.a).

***Lemma* 5.** $\mathbf{x}_1 \neq 0$ is equivalent to $E^{h+1}\mathbf{x} \neq \mathbf{0}$, *Zhang et al.* (1998.a).

Lyapunov function is defined as:

$$V\left(E^{h+1}\mathbf{x}(t)\right) = \mathbf{x}^T(t)\left(E^{h+1}\right)^T P E^{h+1}\mathbf{x}(t), \tag{23}$$

where: $P > 0, \ P \in \mathbb{R}^{n \times n}$ satisfying: $V(E^{h+1}\mathbf{x}) > 0$ if $E^{h+1}\mathbf{x} \neq \mathbf{0}$, when $V(0) = 0$.

From Eqs. (16), (17) and (22), bearing in mind that $EA = AE$, can be obtained:

$$\left(E^h\right)^T A^T P E^{h+1} + \left(E^{h+1}\right)^T P A E^h = -\left(E^{h+1}\right)^T W E^{h+1} \tag{24}$$

where $W > 0, \ W \in \mathbb{R}^{n \times n}$.

Eq. (24) is said to be *Lyapunov* equation for a system given by Eqs. (16) and (17).
Denoted with

$$\deg \det\left(sE - A\right) = rank\, E_1 = r. \tag{25}$$

***Theorem* 11**. The system, given by Eqs. (16) and (17), is *asymptotically stable* if and only if for any matrix $W > 0$, eq. (24) has a solution $P \geq 0$ with a positive external exponent *r*, Zhang *et al.* (1998.a).

***Theorem* 12.** The system, given by Eqs. (16) and (17), is *asymptotically stable* if and only if for any given $W > 0$ the *Lyapunov* eq. (24) has the solution $P > 0$, *Zhang et al* (1998.a).

The conclusion is the same as in the case of the very well known *Lyapunov* stability theory if E is of full rank.

If matrix E is singular then there is more than one solution.

It should be noted that the results of the preceding theorems are very similar in some way and are derived only for *regular linear continuous singular systems*.

First, some basic statements which will be used in the sequel will be given here, *Isihara*, *Terra* (2002).

System (1) with a $n \times n$ matrix E is called *regular* if $\det\left(sE - A\right) \neq 0$ for some $s \in \mathbf{C}$.

Regular system (1) is said to be:

i) *stable* if all roots of $\det\left(sE - A\right) = 0$ are in the open left-half plane;
ii) *impulse free* if it exhibits no impulsive behaviour;
iii) *finite-dynamics detectable* if (O1) holds;
iv) *finite-dynamics observable* if (O2) holds;
v) *impulse observable* if (O3) holds;
vi) *S-observable* (Following Lewis (1986), it shall be said *observable*) of (O2) and (O3) hold.
vii) *C-observable* if (O2) and (O4) hold

where (O1)-(O4) conditions are given by:

$$(\mathrm{O1}) \ rank\begin{pmatrix} sE - A \\ C \end{pmatrix} = n, \ Re(s) \geq 0, \tag{26}$$

$$(\mathrm{O2}) \ rank\begin{pmatrix} sE - A \\ C \end{pmatrix} = n, \ for \ all \ s \in \mathbf{C}, \tag{27}$$

$$(\mathrm{O3}) \ rank\begin{pmatrix} E & A \\ 0 & C \\ 0 & E \end{pmatrix} = n + rank\, E, \tag{28}$$

$$(\mathrm{O4}) \ rank\begin{pmatrix} E \\ C \end{pmatrix} = n. \tag{29}$$

It is immediately seen that C - observability implies S - observability.

***Theorem* 13.** Let $E, A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{p \times n}$ be given by (a Weierstrass form of some regular system $\left(\tilde{E}, \ \tilde{A}, \ \tilde{C}\right)$:

$$E = \begin{pmatrix} I_q & 0 \\ 0 & \Lambda \end{pmatrix}, \tag{30}$$

$$A = \begin{pmatrix} J & 0 \\ 0 & I_{n-q} \end{pmatrix}, \tag{31}$$

$$C = \left(C_F \quad C_\infty\right), \tag{32}$$

where I_q denotes a $q \times q$ identity matrix, J corresponds to the finite zeros of $\left(sE - A\right)$, Λ is nilpotent ($\Lambda^k = 0, \ \Lambda^{k-1} \neq 0$ for some integer $k > 0$), and $\left(E, \ A, \ C\right)$ is observable.

Then $\left(E, A\right)$ is stable and impulse free if and only if a positive-definite solution P to the following Lyapunov equation exists:

$$A^T P E + E^T P A + E^T C^T C E = 0. \tag{33}$$

Moreover, if P and P' are two such solutions, then $E^T P' E = E^T P E$, *Isihara*, *Terra* (2002).

***Theorem* 14**. Let $\left(E, A\right)$ be regular and consider the following generalized Lyapunov equation (GLE):

$$A^T P E + E^T P A + E^T Q E = 0. \tag{34}$$

The following applies:

If there exist matrices $P \geq 0$ and $Q > 0$ satisfying the GLE (34), then (E, A) is impulse free and stable;

If (E, A) is impulse free and stable, then for each $Q > 0$ there exist $P > 0$ solution of GLE (34). Furthermore, $E^T PE \geq 0$ is unique for each $Q > 0$, *Isihara, Terra* (2002).

***Theorem* 15.** Let $E, A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$ be such that (O1) and (O3) ((O2) and (O3)) are satisfied. Consider also a matrix $E_0 \in \mathbb{R}^{n\times(n-r)}$ of full-column rank such that $E^T E_0 = 0$, where $r = rank\, E$.

The following statements are equivalent:

i) system (E, A) is regular, impulse free and stable;

ii) there exists a solution $P, Q \in \mathbb{R}^{n\times n} \times \mathbb{R}^{(n-r)\times n}$ with $P \geq 0$ (> 0) to the following GLE:

$$A^T PE + E^T PA + C^T C + A^T E_0 Q + Q^T E_0^T A = 0, \quad (35)$$

iii) there exists a solution $P, Q \in \mathbb{R}^{n\times n} \times \mathbb{R}^{(n-r)\times n}$ with $E^T PE \geq 0$ ($E^T PE \geq 0$ and $rank\left(E^T PE\right) = rank\, E$) to GLE (35), *Isihara*, *Terra* (2002).

***Theorem* 16**. Let $E, A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{p\times n}$ be such that the system (E, A) is regular, impulse free and stable, and consider the following statements:

i) The Lyapunov equation:

$$A^T PE + E^T PA + E^T C^T CE = 0, \quad (36)$$

has a solution P such that $E^T PE \geq 0$ and $rank\left(E^T PE\right) = r$;

ii) The Lyapunov equation (35) has a solution P such that $E^T PE \geq 0$ and $rank\left(E^T PE\right) = r$.

iii) The Lyapunov equation:

$$A^T PE + E^T PA + C^T C = 0, \quad (37)$$

has a solution P such that $E^T PE \geq 0$ and $rank\left(E^T PE\right) = r$;

iv) The Lyapunov equation:

$$\begin{aligned} E^T X = X^T E \geq 0, \\ A^T X + X^T A + C^T C = 0, \end{aligned} \quad (38)$$

has a solution X such that $E^T X \geq 0$ and $rank\left(E^T X\right) = r$.

Then:

1. $(E,\, A,\, CE)$ is observable ((O2) and (O3) hold with C replaced by CE) if and only if i) holds.
2. $(E,\, A,\, C)$ is observable if and only if any one of the statements ii)-iv) holds, *Isihara, Terra* (2002).

Linear singular continuous irregular singular systems

In order to investigate the stability of irregular singular systems, the following results can be used, *Bajic at al.* (1992).

For this case, the *linear continuous singular system* is used in the suitable canonical form, i.e.:

$$\dot{\mathbf{x}}_1(t) = A_1 \mathbf{x}_1(t) + A_2 \mathbf{x}_2(t), \quad (39)$$

$$0 = A_3 \mathbf{x}_1(t) + A_4 \mathbf{x}_2(t). \quad (40)$$

Herewith, the existence of solutions which converge toward the origin of the systems phase-space for regular *and irregular singular linear systems* is examined.

By a suitable non-singular transformation, the original system is transformed to a convenient form.

This form of system equations enables development and easy application of *Lyapunov's diect method* (LDM) for the intended existence analysis for a subclass of solutions.

In the case when the existence of such solutions is established, an underestimation of the weak domain of the attraction of the origin is obtained based on *symmetric positive definite solutions of a reduced order matrix Lyapunov equation.*

The estimated weak domain of attraction consists of points of the phase space, which generates at least one solution convergent to the origin.

Let, as before, the *subset of the consistent initial conditions* of eqs. (39) and (40) be denoted by W_{k^*}.

Also, consider the manifold $\mathfrak{M} \subseteq \mathbb{R}^{n\times n}$ determined by eq. (40) as $m = m = \aleph\left(\begin{pmatrix} A_3 & A_4 \end{pmatrix}\right)$.

For the system governed by Eqs. (39) and (40) the set W_{k^*} of the consistent initial values is equal to the manifold $\mathfrak{M}$, that is $W_{k^*} = \mathfrak{M}$.

It is easy to see, that the convergence of the solutions of system given by eq. (1) and system, given by Eqs. (39) and (40), toward the origin is an equivalent problem, since the proposed transformation is nonsingular.

Thus, for the null solution of Eqs. (39) and (40), the weak domain of attraction is going to be estimated.

The weak domain of attraction of the null solution $\mathbf{x}(t) = 0$ of the system given by Eqs. (39) and (40) is defined by:

$$D \overset{\Delta}{=} \left\{ \mathbf{x}_0 \in \mathbb{R}^n : \mathbf{x}_0 \in \mathfrak{M},\ \exists \mathbf{x}(t, \mathbf{x}_0),\ \lim_{t\to\infty} \| \mathbf{x}(t, \mathbf{x}_0) \| \to 0 \right\} \quad (41)$$

The term weak is used because solutions of eq. (39) and eq. (40) need not be unique, and thus for every $\mathbf{x}_0 \in D$ there may also exist solutions which do not converge towards the origin.

In this case $D = \mathfrak{M} = W_{k^*}$, the weak domain of attraction may be thought of as the weak global domain of attraction Note that this concept of global domain of attraction used in the paper, differs considerably with respect to the global attraction concept known for state variable systems, *Bajić et al.* (1992), *Debeljković et al.* (1996).

The task is to estimate the set D.

LDM will be used to obtain the underestimate D_e of the set D (i.e. $D_e \subseteq D$).

The development will not require *the regularity condition* of the matrix pencil $(sE - A)$.

For the systems in the form of Eqs. (39) and (40) the *Lyapunov-like function* can be selected as:

$$V\left(\dot{\mathbf{x}}(t)\right) = \mathbf{x}_1^T(t) P \mathbf{x}_1(t), \quad P = P^T > 0, \quad (42)$$

where P will be assumed to be a positive definite and real matrix.

The total time derivative of V along the solutions of Eqs. (39) and (40) is then:

$$V(\dot{\mathbf{x}}(t)) = \mathbf{x}_1^T(t)\left(A_1^T P + PA_1\right)\mathbf{x}_1(t) + \mathbf{x}_1^T(t)PA_2\mathbf{x}_2(t) + \mathbf{x}_2^T(t)A_2^T P\mathbf{x}_1(t), \quad (43)$$

A brief consideration of the attraction problem shows that if eq. (43) is negative definite, for every $\mathbf{x}_0 \in \boldsymbol{W}_{k^*}$, $\|\mathbf{x}_1(t)\| \to 0$ as $t \to \infty$.

Then $\|\mathbf{x}_2(t)\| \to 0$ as $t \to \infty$, for all those solutions for which the following connection between $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ holds:

$$\mathbf{x}_2(t) = L\mathbf{x}_1(t), \ \forall t \in \mathbb{R} \quad (44)$$

The main question is if the relation eq. (44) can be established so as not to contradict the constraints.

Since this is not possible for irregular singular linear systems, then the task to establish the relation eq. (44) has to be reformulated so that it does not pose to many additional novel constraints to eq. (40).

In order to use this fact for the analysis of the attraction problem efficiently, the following consideration that also proposes a construction of the matrix L is introduced.

Let eq. (44) hold.

Assume that the rank condition:

$$rank\begin{pmatrix} A_3 & A_4 \end{pmatrix} = rank\, A_4 \le r \le \boldsymbol{n}_2, \quad (45)$$

is satisfied.

Then a matrix L exists, *Tseng and Kokotovic* (1988), as any solution of the matrix equation:

$$0 = A_3 + A_4 L, \quad (46)$$

where 0 is a null matrix of dimensions the same as A_3.

Based on eqs. (44), (46) and (40), it becomes evident that whenever a solution $x(t)$ fulfils eq. (44), then it also has to fulfil eq. (40).

The implications of the introduced equations can be investigated in more detail.

When they hold, then all solutions of the system Eqs. (39) and (40), which satisfy eq. (44), are subject to algebraic constraints

$$Fx(t) = \begin{pmatrix} A_3 & A_4 \\ L & -I \end{pmatrix} x(t) = 0. \quad (47)$$

Assuming that $\dot{V}(\mathbf{x}(t))$ determined by Eq. (43) is a negative definite, the following conclusions are important:

1. The solution of eqs. (39) and (40) has to belong to the set

$$\aleph\left(\begin{pmatrix} A_3 & A_4 \end{pmatrix}\right) \cap \aleph\left(\begin{pmatrix} L & -I \end{pmatrix}\right). \quad (48)$$

2. If rank $F = n$ then judgement on the domain of attraction of the null solution is not possible on the basis of the adopted approach, or more precisely, in this case the estimate of the weak domain D of attraction is a singleton:

$$\left\{x(t) \in \aleph\left(\begin{bmatrix} A_3 & A_4 \end{bmatrix}\right) : x(t) \equiv 0\right\}. \quad (49)$$

3. If rank

$$F < n, \quad (50)$$

then the estimates of the weak domain of attraction needs to be a singleton and it is estimated as

$$D_e = \left\{x(t) \in \mathbb{R}^n : x(t) \in \aleph\left(\begin{pmatrix} A_3 & A_4 \end{pmatrix}\right) \cap \aleph\left(\begin{bmatrix} L - I \end{bmatrix}\right)\right\} \subseteq D. \quad (51)$$

Now eqs. (43) and (44) are employed to obtain:

$$\dot{V}(\mathbf{x}(t)) = \mathbf{x}_1^T(t)\left(\left(A_1 + A_2 L\right)^T P + P\left(A_1 + A_2 L\right)\right)\mathbf{x}_1(t), \quad (52)$$

which is a negative definite with respect to $\mathbf{x}_1(t)$ if and only if:

$$\Omega^T P + P\Omega = -Q, \quad \Omega = A_1 + A_2 L, \quad (53)$$

where Q is real a symmetric positive definite matrix.

The following result can now be stated.

***Theorem* 17.** Let the rank condition eq. (45) hold and let rank $F < n$, where the matrix F is defined in eq. (47).

Then, the underestimate D_e of the weak domain D of the attraction of the null solution of the system given by eqs. (39) and (40), is determinate by eq. (51), providing $(A_1 + A_2 L)$ is a Hurwitz matrix.

If D_e is not a singleton, then there are solutions of Eqs. (39) and (40) different from null solution, $x(t) \equiv 0$, which converge towards the origin as time $t \to +\infty$, *Bajic et al.* (1992).

A presentation of other results in chronological order follows. The results refer to regular singular systems only.

***Theorem* 18**. A regular system (1) is asymptotically stable if

$$\sigma(E, A) \subset \mathbf{C}^- \quad (54)$$

where

$$\mathbf{C}^- = \left\{s \mid s \in \mathbf{C}, \ \operatorname{Re}(s) < 0\right\}, \quad \sigma(E, A) \subset \mathbf{C}^- \quad (55)$$

stands for the field of finite poles, *Yang et al.* (2004).

This theorem is the most direct and basic criterion according to the singular system stability definition and the system response expression.

The stability problem is changed to an algebraic one whether $\sigma(E, A) \subset \mathbf{C}^-$ is satisfied. And obviously, the system stability depends on the coefficient matrices (E, A) of the system. Therefore, stability is the system characteristic and reflects the system structure characteristics. Furthermore, this theorem is valuable. However, the precise pole calculation is difficult if the order of the system (1) is high.

In applications, the stability criteria obtained by *Lyapunov direct method* is always used.

The main idea of Lyapunov method consists of introducing Lyapunov function $V(t, \mathbf{x}(t))$ and its derivative along the system and determining the stability by using the character of $V(t, \mathbf{x}(t))$ and its derivative $dV(t, \mathbf{x}(t))/dt$. The advantage of this method lies in avoiding the difficulties of getting the system state response and uses the character of $V(t, \mathbf{x}(t))$ and

$dV\left(t,\mathbf{x}(t)\right)/dt$ to solve the problems directly.

A series of results is obtained by constructing different Lyapunov functions.

***Theorem* 19.** The system (1) is regular, impulse free and asymptotically stable if and only if a matrix V satisfying the following equations exists:

$$V^T A + A^T V = -Q\,, \tag{56.a}$$

$$E^T V = V^T E \geq 0\,, \tag{56.b}$$

for any positive definite Q, *Zhang et al.* (1999.a).

***Theorem* 20.** The system (1) is regular, impulse free and asymptotically stable if and only if a unique positive semi-definite solution V to the Lyapunov equation exists:

$$A^T VE + E^T VA = -E^T QE\,, \tag{57}$$

satisfying

$$rank\left(E^T V\, E\right) = rank\, V = r\,, \tag{58}$$

Zhang. (1997).

***Theorem* 21.** The system (1) is regular, impulse free and asymptotically stable if and only if the Lyapunov function

$$V\left(E\mathbf{x}(t)\right) = \mathbf{x}^T(t) E^T V\, \mathbf{x}(t) \tag{59}$$

satisfying

$$\frac{dV\left(E\mathbf{x}(t)\right)}{d\mathbf{x}(t)} < 0\,, \tag{60}$$

where $\mathbf{x}(t) \neq 0$ and V satisfying

$$VE + E^T V \geq 0,\;\; rank\left(E^T V\right) = rankE\,, \tag{61}$$

Zhang, Yu (2002).

***Theorem* 22.** The system (1) is regular, impulse free and asymptotically stable if and only if a symmetric solution V to the following exists

$$A^T VE + E^T VA = -E^T QE\,, \tag{62}$$

satisfying:

$$E^T V\, E \geq 0, \quad rank\left(E^T V\, E\right) = rank\, E = r\,, \tag{63}$$

where W is a symmetric matrix, and satisfying

$$E^T QE \geq 0,\;\; rank\left(E^T QE\right) = rank\, E = r\,, \tag{64}$$

Zhang et al. (1999.d).

***Theorem* 23.** The system (1) is regular, impulse free and asymptotically stable if and only if a matrix V satisfying the following exists:

$$V^T A + A^T V < 0\,, \tag{65}$$

$$E^T V = V^T E \geq 0\,, \tag{66}$$

Masubuchi et al. (1997).

Theorems 19 – 23 get the sufficient and necessary condition of *regular* singular systems, impulse free and stable by Lyapunov direct method (LDM).

LDM is different from the normal linear system, so the construction of Lyapunov function for (LCSS) is not only a trivial generalization.

Generally, the Lyapunov function candidates for (LCSS) depend on $\mathbf{x}(t)$ in a special way, that is $V = V\left(E\mathbf{x}(t)\right)$. Because of the complexity of LDM, there always exists a constraint together with the Lyapunov function to the analyze stability of (LCSS).

Theorems 19 - 20 and *Theorem* 22 are analyzed by Lyapunov equation, *Theorem* 21 is analyzed by Lyapunov function and *Theorem* 23 is analyzed by Lyapunov inequality.

Theorems 21 - 23 include a restriction of the matrix rank. This kind of restriction is solved easily in low order systems and hard to deal with if the order is high, while *Theorem* 19 and *Theorem* 23 are most useful in these cases.

It is well known that the geometry method is a useful tool to study the problem of the control theory.

By calculating the series of ranges of a sphere according to projective geometry theories, *Campbell* (1980) gave some results, which are useful in the robust control study.

***Theorem* 24**. The following statements are equivalent:

(i) The system (1) is stable and impulse free

(ii) There exists $rank\left(LAS\right) = n - r$ and a positive definite matrix $X > 0$ satisfying

$$\tilde{E}^T\, X^T A + A^T X\tilde{E} < 0\,, \tag{67}$$

where

$$L \in \Psi \wedge S \in \Phi\,. \tag{68}$$

(iii) $\sigma\left(E,A\right) \subseteq \Omega$,

where

$$\Phi : \left\{ L \in \mathbb{R}^{(n-r)\times n} : LE = 0,\;\; rank\, L + rank\, E = n \right\}, \tag{69}$$

$$\tilde{E} = \Upsilon^{-1} \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix} \Theta^{-1}\,, \tag{70}$$

$$\Psi : \left\{ S \in \mathbb{R}^{n\times(n-r)} : E\, S = 0,\;\; rank\, S + rank\, E = n \right\}. \tag{71}$$

$\sigma\left(E,A\right)$ is a spectrum of a generalized singular system. Ω stands for six prims, *Zhang*, *Wang*, *Cong* (2002).

Impulse free system is considered in *Theorems* 2 - 7, but there always exists an impulse on singular systems.

Therefore, there are some results analyzing the stability of (LCSS) without the consideration of impulse behaviour.

***Theorem* 25**. A regular system (1) is asymptotically stable if and only if a solution $V \geq 0$ to

$$A^T VE + E^T VA = -E^T QE\,, \tag{72}$$

exists, satisfying

$$E^T V\, E \geq 0, \quad rank\left(E^T V\, E\right) = \deg\det\left(sE - A\right), \tag{73}$$

where arbitrary $Q \geq 0$ satisfies:

$$E^T Q\, E \geq 0,\;\; rank\left(E^T Q\, E\right) = \deg\det\left(sE - A\right), \tag{74}$$

where

$$EA = AE, \quad sE - A = I\,. \tag{75}$$

Zhang et al (1998.d).

***Theorem* 26.** The regular system (1) is asymptotically stable if and only if a positive-exponent solution $V \geq 0$ satisfying

$$\left(E^h\right)^T A^T V E^{h+1} + \left(E^{h+1}\right)^T V A E^h = -\left(E^{h+1}\right)^T Q E^{h+1}, \quad (76)$$

exists, and where

$$Q > 0, \; EA = AE, \; sE - A = I, \quad (77)$$

and h is the nilpotent index of E, *Zhang et al.*(1998.b)

***Theorem* 27.** The regular system (1) is asymptotically stable if and only if a solution V to Lyapunov equation exists

$$\left(E^{h+1}\right)^T V = V^T E^{h+1} \geq 0, \quad (78.a)$$

$$A^T \left(E^h\right)^T V + V^T E^h A = -\left(E^h\right)^T Q E^T, \quad (78.b)$$

satisfying

$$rank\left(E^{h+1}\right)^T V = \deg\det(sE - A) = rank\left(E^{h+1}\right). \quad (79)$$

There exists a converse matrix T such that

$$T\begin{pmatrix}\hat{E} & \hat{A}\end{pmatrix}\begin{pmatrix}T^{-1} & 0 \\ 0 & T^{-1}\end{pmatrix} = \begin{pmatrix}\hat{E}_1 & 0 & \hat{A}_1 & 0 \\ 0 & \hat{E}_2 & 0 & \hat{A}_2\end{pmatrix}, \quad (80)$$

where $\hat{E}_1 \in \mathbb{R}^{g\times g}$, $\hat{A}_2 \in \mathbb{R}^{(n-g)\times(n-g)}$ are invertible matrices and $\hat{E}_2$ is a nilpotent matrix, *Masubuchi*, *Shimemura* (1997).

***Theorem* 28.** The regular system (1) is asymptotically stable if $\hat{E}^{-1}\hat{A}_1$ is stable, where $\hat{E} = (sE - A)^{-1} E$, $\hat{A} = (sE - A)^{-1} A$ *Yue*, *Zhang* (1998).

Theorems 9 - 11 are all for impulse systems. They will become *Theorem* 22 when the system is impulse free. Though the forms are more complicated in the *Theorems* 25 - 27, it is easy to obtain the solution when they are applied.

Considering the calculation and the numerical stability, the less the calculation of the matrix is inverse and the lower the matrix order is, the better the calculating method.

Therefore, there are both freedom and limitation in *Theorem* 28.

The stability of linear time varying continuous singular systems

Consider the linear continuous singular system represented by

$$E\dot{\mathbf{x}}(t) = A(t)\mathbf{x}(t), \; \mathbf{x}(t_0) = \mathbf{x}_0(t). \quad (81)$$

***Theorem* 29**. Suppose that the system (81) is regular and the following conditions are fulfilled at the same time:

(i) There exist constants $\gamma > 0$ and $\wp > 0$ satisfying $\|A(t)\| \leq \gamma$ and $\|A_{22}(t)\| \geq \wp$

(ii) $A_{22}(t)$ is invertible for all $t \geq 0$

(iii) There exists a positive definite matrix $P \in \mathbb{R}^{r\times r}$ and constant $\ell > 0$ such that

$$\mathbf{x}_1^T(t)\left(PW(t) + W^T(t)P\right)\mathbf{x}_1(t) \leq -\ell\left\|\mathbf{x}_1\right\|^2, \quad (82)$$

where

$$W(t) = A_{11}(t) - A_{12}(t)A_{22}^{-1}(t)A_{21}(t), \quad (83)$$

$$E = \begin{pmatrix}I_r & 0 \\ 0 & 0\end{pmatrix}, \; A(t) = \begin{pmatrix}A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t)\end{pmatrix}, \quad (84)$$

$$\begin{gathered}\mathbf{x}(t) = \left[\mathbf{x}_1^T(t) \;\; \mathbf{x}_2^T(t)\right]^T, \\ \mathbf{x}_1(t) \in \mathbb{R}^r, \quad \mathbf{x}_2(t) \in \mathbb{R}^{n-r}\end{gathered} \quad (85)$$

and E, $A(t)$ are continuous functional matrices, then the time-varying singular system (81) is stable and impulse free, *Hu*, *Sun* (2003).

Paper *Bajic*, *Milic* (1987) is focused on (LDM) with A being time varying and E being constant.

Results presented throughout *Theorems* 19 –29 are mostly taken from paper, Men *et.al* (2006).

Stability robustness considerations

Consider the singular linear systems (SLS) represented by:

$$E\dot{\mathbf{y}}(t) = A\mathbf{y}(t), \; E, A \in \mathbb{R}^{m\times n}, \; \mathbf{y}(t_0) = \mathbf{y}_0, \quad (86)$$

where $\mathbf{y}(t) \in \mathbb{R}^n$ is the phase vector (i.e. generalized state-space vector).

The matrix E, when $m = n$, is possibly singular.

When this is the case, then $rank\, E = p < n$, $nullity E = n - p = q$. If the matrix E is invertible, then (1) can be written in the normal form as:

$$\dot{\mathbf{y}}(t) = E^{-1}A\mathbf{y}(t), \quad \mathbf{y}(t_0) = \mathbf{y}_0. \quad (87)$$

Behaviour of the solution of (87) is well documented in contemporary references. However, this is not the situation for the system (86), where $m \neq n$ or when $m = n$ with $\det E = 0$.

In the control and system theory, it is of great interest to preserve various system properties under large perturbations of the system model. Such perturbations of the model may be caused by changes in the manufacturing process of the components, variations of constructive elements, or due to replacement, change of environmental conditions, etc., *Bajic* (1992). The insensitiveness of the system properties is called robustness and it is an important field of investigation. The fact is that in many practical situations the parameters of system components are not known exactly. Usually, there is only some information on the intervals to which they belong. Therefore, the robustness for any system property is an important theoretical and practical question.

In recent years, considerable attention has been given to the design of controllers for multivariable linear systems, so that certain system properties are preserved under various classes of perturbations occuring in the system. Such controllers have been termed *robust controllers*, and the resulting system is said to be robust in some context.

Patel, *Toda* (1980) first reported the robustness bounds on unstructured pertubations of linear continuous-time systems.

Yedavilli, *Liang* (1985) improved *Patel's* result for linear perturbations with the known structure and proposed a similarity transformation method to reduce robustness bounds conservatism.

Recently, *Kolla*, *Yedavalli* (1989) investigated bounds for structured and unstructured perturbations of discrete-time systems.

In this paper, the stability robustness of linear continuous singular system, in the time domain, is addressed using the Lyapunov approach. The bounds of unstructured perturbation vector function that maintain the stability of the nominal system with attractivity property of a subclass of solutions are obtained both for regular and irregular singular systems.

Zhou, *Khargonekar* (1987) considered the robust stability analysis problem by state-space methods.

They derived some lower bounds on allowable perturbations that maintain the stability of a nominally stable system with structured uncertainty. It has been shown that those bounds are less conservative than the existing ones.

Recently, *Chen*, *Han* (1994) using iterativity approach, derived new results in the same area of interest for the linear system with unstructured time-varying perturbations. In comparison with some existing methods, less conservative results have been obtained.

Consider an (SLS) given by (86).

By introducing a suitable non-singular transformation as:

$$T\mathbf{x}(t) = \mathbf{y}(t),\ T \in \mathbb{C}^{n\times n} \tag{88}$$

a broad class of SLS (86) can be transformed to the following form:

$$\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + A_2\mathbf{x}_2(t), \tag{89.a}$$

$$0 = A_3\mathbf{x}_1(t) + A_4\mathbf{x}_2(t), \tag{89.b}$$

$$ET = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},\ AT = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \tag{89.c}$$

where $\mathbf{x}(t) = \left[\mathbf{x}^T{}_1(t)\ \ \mathbf{x}^T{}_2(t)\right]^T \in \mathbb{R}^n$ is a decomposed vector, with $\mathbf{x}_1 \in \mathbb{R}^{n1}$, $\mathbf{x}_2 \in \mathbb{R}^{n2}$, and $n = n_1 + n_2$. The matrices A_i, $i = 1,...,4$, are of appropriate dimensions. Comparing (89) with (86), it is obvious that if $m = n$ the case when $\det E = 0$ is considered.

This conclusion stems from the fact that $\det(ET) = \det E \det T = 0$, and that $\det T \neq 0$.

When the matrix pencil $(sE - A)$ is regular, i.e. when:

$$\det(sE - A) \neq 0, \tag{90}$$

then solutions of (86) exist, unique for the so-called consistent initial values $\mathbf{x}_0$ of $\mathbf{x}(t)$, and moreover, the closed form of these solutions is known.

The regularity condition (90) for the system (54) reduces to following:

$$\begin{aligned}&\det(sI - A_1)\det\left(-A_4 - A_3(sI - A_1)^{-1}A_2\right) = \\ &= \det A_4 \det\left((sI - A_1) - A_2A_4^{-1}A_3\right) \neq 0\end{aligned}. \tag{91}$$

It follows from (91) that the regularity condition for (89) requires the invertibility of the matrix A_4.

It was proven in *Campbell* (1980) that $\mathbf{x}_0$ is consistent initial value that generates smooth solution if $\left(I - \hat{E}\hat{E}^D\right)\mathbf{x}_0 = 0$, where $\hat{E}^D$ is the so-called Drazin inverse of $\hat{E}$, and where $\hat{E}$ is defined by $\hat{E} = (\lambda E - A)^{-1}E$.

Let the following equation hold:

$$\mathbf{x}_2(t) = L\mathbf{x}_1(t),\ \forall t \in \mathbb{R}. \tag{92}$$

***Lemma* 6**. Let (92) hold.

Assume that the rank condition:

$$rank\begin{pmatrix} A_3 & A_4 \end{pmatrix} = rank\, A_4 = r \leq n_2, \tag{93}$$

is satisfied.

There exists a matrix L *Tseng and Kokotovic* (1988), being a solution of the matrix equation:

$$0 = A_3 + A_4L, \tag{94}$$

where 0 is a null matrix of the same dimensions as A_3.

For robustness considerations, the generalized state space systems described by the mixture of differential and algebraic equations of the form:

$$E\dot{\mathbf{y}}(t) = A\mathbf{y}(t) + \mathbf{f}_p(\mathbf{y}(t)), \tag{95}$$

are considered, with $\mathbf{f}_p(\mathbf{y}(t))$ as a perturbation factor.

Using the same non-singular linear transformation (88), the relation (95) is reduced to:

$$\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + A_2\mathbf{x}_2(t) + \mathbf{f}_1(\mathbf{x}), \tag{96.a}$$

$$0 = A_3\mathbf{x}_1(t) + A_4\mathbf{x}_2(t) + \mathbf{f}_2(\mathbf{x}), \tag{96.b}$$

with:

$$\mathbf{f}_p(\mathbf{y}(t)) = \mathbf{f}_p(T\mathbf{x}(t)) = \mathbf{f}(\mathbf{x}(t)) = \begin{pmatrix} \mathbf{f}_1(\mathbf{x}(t)) \\ \mathbf{f}_2(\mathbf{x}(t)) \end{pmatrix}. \tag{97}$$

***Assumption* 1.** Vector function $\mathbf{f}(\mathbf{x}(t))$ satisfies the following condition:

$$\mathbf{f}_2(\mathbf{x}(t)) \equiv 0,\ \mathbf{f}(\mathbf{x}(t)) = \left(\mathbf{f}_1^T(\mathbf{x}(t))\ \ \mathbf{0}^T\right)^T, \tag{98}$$

so it can be written:

$$\mathbf{f}_1(\mathbf{x}(t)) = \left(\begin{pmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{pmatrix}\right) = \mathbf{f}_1\left(\begin{pmatrix} \mathbf{x}_1(t) \\ L\mathbf{x}_1(t) \end{pmatrix}\right) = \mathbf{f}_1(\mathbf{x}_1(t)) \tag{99}$$

Djurovic et al., (1988).

The following result can be stated.

***Theorem* 30.** Suppose *Assumption* 1 and *Lemma* 6 hold and let $rank\, F < n$, where the matrix F is defined by

$$F = \begin{pmatrix} A_3 & A_4 \\ L & -I \end{pmatrix}\mathbf{x} = 0. \tag{100}$$

Then the solutions of (96), different from the null solution $\mathbf{x}(t) \equiv 0$, converge toward the origin of the phase space as time $t \to +\infty$, if

$$\frac{\|\mathbf{f}(\mathbf{x}(t))\|}{\|\mathbf{x}(t)\|} \leq \mu_1 \equiv \frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)}, \tag{101}$$

where P is unique, real, symmetric positive definite solution of the *Lyapunov matrix equation*

$$\left(A_1 + A_2 L\right)^T P + P\left(A_1 + A_2 L\right) = -2Q, \quad (102)$$

where Q is a real, symmetric positive definite matrix and $\sigma_{\min}(\cdot)$ and $\sigma_{\max}(\cdot)$ are singular values of matrix $(\cdot)$, *Debeljkovic et al.,* (1993).

To analyze the robustness of attraction property of the phase space origin, let the perturbed system (86) which for this purpose can be represented in the following form

$$E\dot{\mathbf{y}}(t) = A\mathbf{y}(t) + \mathbf{f}_p(\mathbf{y}(t)) = A\mathbf{y}(t) + G_p\mathbf{y}(t), \quad (103)$$

be considered where the factor $\mathbf{f}_p(t)$ represents the model perturbation and matrix G_p is of appropriate dimension.

To simplify the formulation of the stability robustness results (68) is first transformed to

$$\dot{\mathbf{x}}_1(t) = A_1\mathbf{x}_1(t) + A_2\mathbf{x}_2(t) + G_1(t)\mathbf{x}(t), \quad (104.a)$$

$$0 = A_3\mathbf{x}_1(t) + A_4\mathbf{x}_2(t) + G_2\mathbf{x}(t), \quad (104.b)$$

as it has been done with (86) to (89).

G_1 and G_2 are matrices of dimension $n_1 \times (n_1 + n_2)$ and $n_2 \times (n_1 + n_2)$, respectively, determined by the following expression

$$G_1 = \begin{pmatrix} G_{11} & G_{12} \end{pmatrix}, \quad G_2 = \begin{pmatrix} G_{21} & G_{22} \end{pmatrix}, \quad G_p T = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}. \quad (105)$$

Then, the following assumption can be introduced.

***Assumption* 2.** Let L be a matrix which satisfies *Lemma* 6 and let $G_2 \equiv 0$, so that:

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}\mathbf{x}(t) = \begin{pmatrix} G_{11} & G_{12} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{x}_1(t) \\ \mathbf{x}_2(t) \end{pmatrix} = \begin{pmatrix} (G_{11} + G_{12}L)\mathbf{x}_1(t) \\ 0 \end{pmatrix} = \begin{pmatrix} G_L\mathbf{x}_1(t) \\ 0 \end{pmatrix}, \quad (106)$$

Djurovic et al., (1988).

Now the results on robustness stability can be stated as follows.

***Theorem* 31.** Let the rank condition (93) and *Assumption* 2 hold. Then, the underestimate S_u of the potential domain of the attraction of the system (104) is given by:

$$S_u = \left\{\mathbf{x}(t) \in \mathbb{R} : \mathbf{x}(t) \in \aleph\left(\left(L \quad -I_{n2}\right)\right)\right\} \subseteq S, \quad (107)$$

if one of the following conditions is fulfilled:

$$\text{a) } \|G_L\|_S < \mu, \quad \text{b) } \|G_L\| < \mu, \quad \text{c) } \left|g_{Lij}\right| < \mu / n_1, \quad (108)$$

where g_{Lij} is the (i, j)-element of matrix G, and

$$\mu = \frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)}, \quad (109)$$

and where $P = P^T > 0$, is symmetric, positive definite, real matrix, being the unique solution of *Lyapunov matrix equation*

$$\left(A_1 + A_2 L\right)^T P + P\left(A_1 + A_2 L\right) = -2Q, \quad (110)$$

for any real, symmetric, positive definite matrix Q.

The set S_u contains more than one element. $\|(\cdot)\|$ and $\|(\cdot)\|_S$ denote Euclidean and spectral norm of matrix $(\cdot)$ respectively, and $\sigma_{(\cdot)}(\cdot)$ the corresponding singular value.

***Theorem* 32**. Let the rank condition (93) and *Assumption* 2 hold. Then the underestimate S_u of the potential domain of attraction of the system (104) is given by (107), if the following condition is fulfilled:

$$\left|g_{Lij}\right| = \varepsilon < \frac{1}{\sigma_{\max}\left(|P|U\right)_S} \equiv \eta, \quad (111)$$

where P satisfies the Lyapunov matrix equation given by:

$$\left(A_1 + A_2 L\right)^T P + P\left(A_1 + A_2 L\right) = -2I, \quad (112)$$

I being $n_1 \times n_1$ identity matrix with U being $n_1 \times n_1$ matrix whose entries are a unity. $((\cdot))_S$ means the symmetric part of matrix $(\cdot)$, *Djurovic et al.,* (1988).

***Theorem* 33.** Let the rank condition (93) and *Assumption* 2 hold. Moreover, let the matrix G_L be defined in the following manner:

$$G_L = \sum_{i=1}^{m} k_i G_{Li}, \quad (113)$$

where G_{Li} are constant matrices and k_i are uncertain parameters varying in some intervals around zero, i.e. $k_i \in [-\varepsilon_i, +\varepsilon_i]$.

Then the underestimate S_u of the potential domain of attraction of the system (104) is given by (107), when P satisfies the *Lyapunov matrix equation* (112), and if one of the following conditions is fulfilled

a)
$$\sum_{i=1}^{m} k_i^2 < \frac{1}{\sigma_{\max}^2(P_e)}, \quad (114)$$

or:

b)
$$\sum_{i=1}^{m} k_i\, \sigma_{\max}(P_i) < 1, \quad (115)$$

or:

c)
$$\left|k_j\right| < \frac{1}{\sigma_{\max}\left(\sum_{i=1}^{m}|P_i|\right)}, \quad j = 1, 2, \ldots, m. \quad (116)$$

where P_i and P_e are given by

$$P_i = \frac{1}{2}\left(G_{Li}^T P + PG_{Li}\right) = \left(PG_{Li}\right)_S, \quad (117)$$

and

$$P_e = \begin{pmatrix} P_1 & P_2 & \ldots & P_m \end{pmatrix}. \quad (118)$$

Moreover, S_u contains more than one element, *Djurovic et al*., (1988).

In order to illustrate the presented results, some suitable examples are given in the continuation.

***Example* 1.** Consider a singular system given by

$$\begin{pmatrix}0&1&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\dot{\mathbf{y}}(t)=\begin{pmatrix}-1&-2&0&-1\\1&-2&1&-4\\1&-1&0&1\\3&-5&2&3\end{pmatrix}\mathbf{y}(t)+ \begin{pmatrix}2k&-6k&3k&-6k\\1&0&0&1\\0&0&0&0\\0&0&0&\end{pmatrix}\mathbf{y}(t) \quad . \quad (E.1)$$

Since $\det(cE-A)\neq 0$ this is a regular singular system.

Let the behaviour of this system according to the results obtained be examined.

Using the transformation matrix

$$T=\begin{pmatrix}2&1&0&1\\1&0&0&0\\0&0&1&0\\0&1&0&0\end{pmatrix},$$

which is non-singular since $\det T=1$, the system (E.1) can be transformed to

$$\dot{\mathbf{x}}_1(t)=\begin{pmatrix}-4&-2\\0&-3\end{pmatrix}\mathbf{x}_1(t)+\begin{pmatrix}0&-1\\1&1\end{pmatrix}\mathbf{x}_2(t)+ \begin{pmatrix}-2k&-4k&3k&2k\\1&2&0&1\end{pmatrix}\mathbf{x}(t),$$

$$0=\begin{pmatrix}1&2\\1&0\end{pmatrix}\mathbf{x}_1(t)+\begin{pmatrix}0&1\\2&3\end{pmatrix}\mathbf{x}_2(t).$$

Since the rank condition (93) is satisfied, it can be found

$$L=\begin{pmatrix}1&3\\-1&2\end{pmatrix},$$

from (94), and then

$$S_u=\left\{\mathbf{x}(t)\in\mathbb{R}^4:\mathbf{x}(t)\in\aleph\left(\begin{pmatrix}1&2&-1&0\\-1&2&0&-1\end{pmatrix}\right)\right\}, \subseteq S$$

if conditions of *Theorems* 31 - 33 are satisfied.

This can be shown.

Since

$$G_L=G_{11}+G_{12}L\big|_{G_2=0}=k\begin{pmatrix}-1&1\\0&0\end{pmatrix},$$

Assumption 2 is satisfied.

For $Q=I$, from (104) it follows

$$P=\begin{pmatrix}1/3&0\\0&1/2\end{pmatrix}=P^T>0,$$

so that

$$\left|g_{Lij}\right|\leq|k|\leq\frac{\sigma_{\min}(Q)}{n_1\sigma_{\max}(P)}=\frac{1}{2\frac{1}{2}}=1.$$

The *Theorem* 32 gives a better result.

Namely

$$\left|g_{Lij}\right|<1.19,$$

since:

$$|P|=\begin{pmatrix}1/3&0\\0&1/2\end{pmatrix},\ U=\begin{pmatrix}1&1\\1&1\end{pmatrix}, \quad , \quad \sigma_{\max}\left(|P|U\right)_S=0.8416$$

$$|P|U=\begin{pmatrix}1/3&1/3\\1/2&1/2\end{pmatrix},\quad \left(|P|U\right)_S=\begin{pmatrix}1/3&5/12\\5/12&1/2\end{pmatrix}.$$

To apply *Theorem* 33, the following data has to be obtained

$$G_L=k\begin{pmatrix}-1&1\\0&0\end{pmatrix}=kG_{L1},$$

$$P_1=\left(PG_{L1}\right)_S=\begin{pmatrix}-1/3&1/6\\1/6&0\end{pmatrix}=P_e \quad , \quad \sigma_{\max}(P_e)=0.1618$$

$$k^2<\frac{1}{\sigma_{\max}^2(P_e)}\Rightarrow|k|<2.48.$$

Figures 1 and 2 represent system trajectories for possible values of an uncertain parameter k.

In the first case, Fig.1, the parameter k is chosen in such a way that the condition of the *Theorem* 32 is satisfied, so the stability robustness of attraction property of origin is proven. It can be shown that quantitative measures obtained by *Theorem* 32 are less conservative than the other two, *Zhou*, *Khargonekar,* (1987).

The second case, Fig.2, shows that the required property is not achieved, since the choice of parameter k was not adequate.

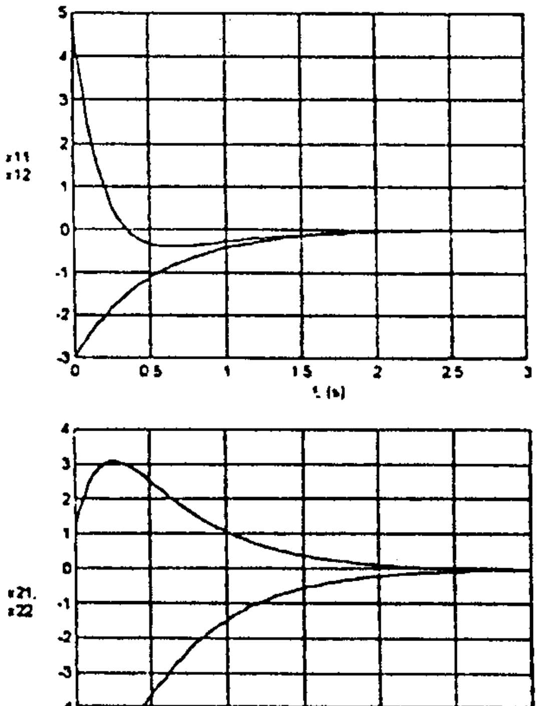


Figure 1. $k=2.45;\ x_{10}=\begin{bmatrix}5\\-3\end{bmatrix},\ x_{20}=\begin{bmatrix}-4\\1\end{bmatrix},\ x_0\in\aleph\left(\left[L-1_{n}\right]\right)$

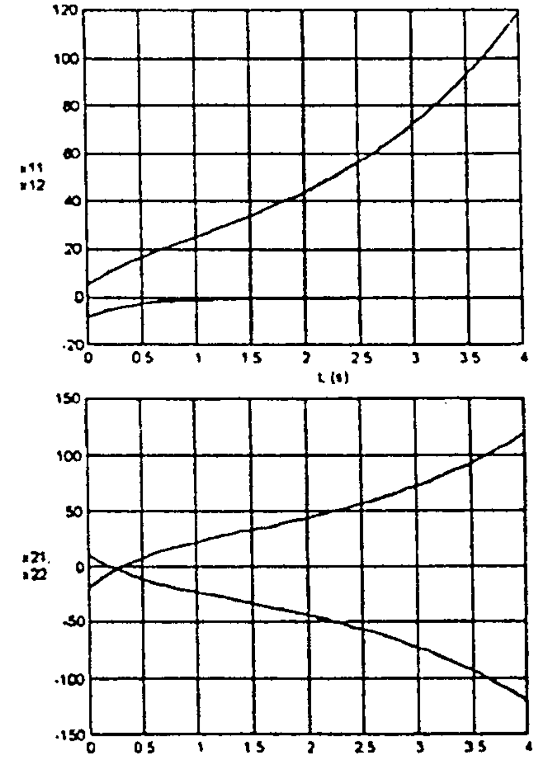


Figure 2. $k=-3.5;\ x_{10}=\begin{bmatrix}5\\-8\end{bmatrix},\ x_{20}=\begin{bmatrix}-19\\11\end{bmatrix},\ x_0\in\aleph\left(\left[L-1_{n2}\right]\right)$

***Example* 2.** Consider a singular system given by

$$\dot{\mathbf{x}}_1(t)=(-1)\mathbf{x}_1(t)+\begin{pmatrix}-1 & -3\end{pmatrix}\mathbf{x}_2(t)+G_1\mathbf{x}(t),$$

$$0=\begin{pmatrix}-1\\1\end{pmatrix}\mathbf{x}_1(t)+\begin{pmatrix}1 & 1\\-1 & -1\end{pmatrix}\mathbf{x}_2(t).$$

Since $\det(cE-A)\equiv 0$ for any c, this is an irregular singular system and solutions are not unique.

The following results can easily be obtained:

$$rank\begin{pmatrix}-1 & 1 & 1\\1 & -1 & -1\end{pmatrix}=rank\begin{pmatrix}1 & 1\\-1 & -1\end{pmatrix}=1\le 2,$$

$$L=\begin{pmatrix}a\\1-a\end{pmatrix},\quad a\in\mathbb{R},\quad G_2\equiv 0.$$

From (103), it can be obtained:

$$P=-\frac{1}{a-4},\quad a<4,$$

in order to have $P=P^T>0$.

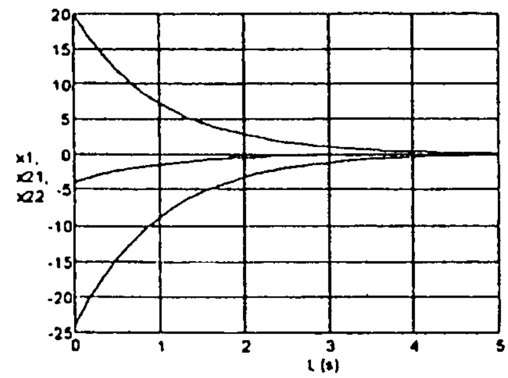
So

$$\|G_L\|\le\frac{\sigma_{\min}(Q)}{\sigma_{\max}(P)}=-(a-4), \tag{E.2}$$

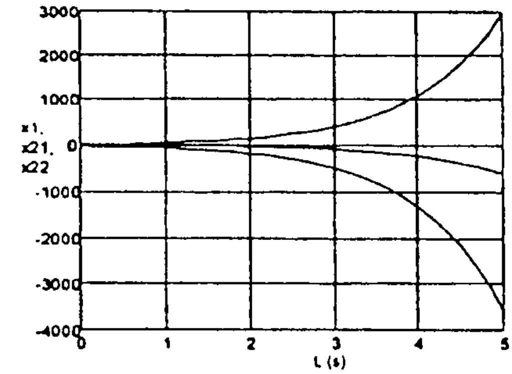
and

$$S_u=\left\{\begin{matrix}\mathbf{x}(t)\in\mathbb{R}^3:\ \mathbf{x}(t)\in\aleph\left(\begin{pmatrix}a & -1 & 0\\1-a & 0 & -1\end{pmatrix}\right),\\ a<4\end{matrix}\right\}\subseteq S$$

Two different values of the parameter a have been chosen and corresponding system responses have been depicted in Fig.3.



a) $a=-5,\ G_L=8,\ x_0\in\aleph\left(\left[L-I_{n2}\right]\right)$



b) $a=-5,\ G_L=10,\ x_0\in\left(\left[L-I_{n2}\right]\right)$

Figure 3.

In the first case, Fig.3a, condition given by (E.2) is satisfied and system has the required property.

In the second case, Fig.3b G_L is chosen to contradict (P.2) and system response diverges.

Conclusion

This survey paper is devoted to the stability of linear descriptor systems (LDS). Considering both *regular* and *irregular* continuous linear singular systems, a number of results concerning stability properties in the sense of Lyapunov were presented and relationship between them analyzed. To enssure *asymptotical stability for linear* continuous singular *systems,* it is not enough to have the eigenvalues of the matrix pair (E, A) in the left half complex plane, but also to provide an impulse-free motion of the system under consideration.

Different approaches have been shown in order to construct *Lyapunov* stability theory for a particular class of linear continuous singular systems operating in free and forced regimes.

The same results have been used to test system stabilty robustness.

Some examples have been given to show the applicability of the results derived.

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Received: 25.03.2007.

Stabilnost linearnih vremenski kontinualnih singularnih sistema u smislu Lyapunova: Pregled rezultata

U ovom radu dat je hronološki prikaz rezultata na polju ljapunovske stabilnosti posebne klase linearnih kontinualni singularnih sistema. Predmetna tematika izložena je kroz brojne definicije i odgovarajuće teoreme.
Prezentovani koncepti prošireni su na robusnost stabilnosti sistema i potkrepljeni eklatantnim primerima.

Ključne reči: **kontinualni sistem, singularni sistem, linearni sistem, stabilnost sistema, stabilnost Ljapunova, asimptotska stabilnost.**

Устойчивость линейных временных непрерывных сингулярных систем в направлении Ляпунова: Обзор и анализ результатов

В настоящей работе приведен хронологический обзор результатов в области устойчивости Ляпунова особого класса линейных непрерывных сингулярных систем. Тематика о которой идет речь растолкована через многочисленные определения и соответствующие теоремы. Показанные концепции распространены на крепкость устойчивости системы и укреплены очевидными численными примерами.

Ключевые слова: непрерывная система, сингулярная система, линейная система, устойчивость системы, устойчивость Ляпунова, асимптотическая устойчивость.

La stabilité des systèmes linéaires continus singuliers dans le sens de Lyapunov: présentation des résultats

Dans ce papier on présente chronologiquement les résultats obtenus dans le domaine de la stabilité de Lyapunov chez la classe particulière des systèmes linéaires continus singuliers. Le sujet traité est exposé au moyen de nombreuses définitions et par les théorèmes correspondantes. Les concepts présentés incluent aussi la robustesse de la stabilité du système et sont illustrés par les exemples éclatants.

Mots clés: **système continu, système singulier, système linéaire, stabilité du système, stabilité de Lyapunov, stabilité asymptotique.**