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Optimal Estimation of the Roll Rate of Anti-tank Missile Part II: Kalman Filter with Polynomial Kinematic Model

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An efficient method of estimation and prediction of the roll rate of an anti-tank missile using Kalman filter with kinematic polynomial model is presented when the roll angle is measured by free gyro and measured data corrupted by noise. The optimal value of the process noise is determined to achieve minimal dispersion of the estimate of the roll angle and roll rate. The formulae for estimation are derived by using kinematic polynomial model and kinematic polynomial model with constant coefficients. By exact numerical simulation the efficiency of the models is tested and function of standard deviation of the missile roll rate with respect to normalized process noise is determined. Comparison of the applied Kalman filter with kinematic polynomial model and Kalman filter with exact model is done. Criteria for the choice of method of estimation and process noise is given to achieve minimal dispersion of the estimate of the roll angle and roll rate.

Key words: **anti-tank missile, missile roll, roll rate, angular velocity, Kalman filter, stochastic process, free gyro, numerical simulation.**

Glossary of Symbols

$\mathbf{A}$ –matrix of the dynamic coefficients of the missile in rolling motion;
$\mathbf{B}$ –control matrix;
$\mathbf{F}(t,t_0)$ –transition matrix from t_0 to t, or fundamental matrix;
$\mathbf{F}_k$ –discrete transition matrix;
l_0 –specific (reduced) active rolling moment, $l_0 = 1/J_x \times L(\alpha=\beta=p=q=r=0)$, $\left[\mathrm{rad/s^2}\right]$;
l_p –specific (reduced) damping rolling moment, $l_p = 1/J_x \times \partial L/\partial p$, $[\mathrm{rad/s}]$;
l_{dist} –specific (reduced) disturbing rolling moment, $\left[\mathrm{rad/s^2}\right]$;
$\mathbf{H}$ –measurement matrix;
$\mathbf{K}_k$ –discrete matrix of the Kalman filter gain;
$\mathbf{M}_k$ –discrete covariance matrix representing errors of the states before estimates;
$\mathbf{n}_k$ –discrete matrix of the measurement noise;
n_g –number of measured angles of the roll position for one revolution of the missile (gyroscope characteristic), $[\mathrm{rad},°]$;
$\mathbf{P}_k$ –discrete covariance matrix representing errors of the states after estimates;
p,q,r –projections of the missile angular rate on the body fixed axis system, $[\mathrm{rad/s}]$;
$\mathbf{Q}_k$ –discrete covariance matrix of the process noise;
$\bar{q}^{(3)}$ –power spectral density of the process noise of the derivative of angular acceleration, $\left[\mathrm{rad^2/s^5}\right]$;
$\mathbf{S}(t)$ –continual matrix of the process noise;
S_Φ –power spectral density of the measurement noise of the missile angle of roll, $\left[\mathrm{mrad^2/(1/s)}\right]$;
$\mathbf{R}_k$ –discrete covariance matrix of the measurement noise;
T_k, T_0 –sampling interval of the roll angle at time t_k and time $t=0$, $[\mathrm{s}]$;
t –time, $[\mathrm{s}]$;
$\bar{\mathbf{w}}$ –process noise matrix;
$\mathbf{w}_k$ –discrete process noise matrix
$\mathbf{x}$ –space state matrix (column);
$\mathbf{x}_k$ –discrete space state matrix (column);
$\tilde{\mathbf{x}}_k$ –discrete estimate of the space state matrix based on the mathematical model (column);
$\hat{\mathbf{x}}_k$ –discrete estimate of the space state matrix based on Kalman filter (column);
$\mathbf{x}_k^*$ –discrete measured space state matrix (column);
$\mathbf{x}_0$ –initial discrete space state matrix (column);
Φ –missile roll angle, $[\mathrm{rad},°]$;
$\Delta\Phi_g$ –sampling interval of the missile roll angle (gyroscope characteristic), $[\mathrm{rad},°]$;
σ_p –standard deviation of the roll rate $[\mathrm{rad/s}]$;
σ_Φ –standard deviation of the roll angle (gyroscope characteristic), $[\mathrm{rad},°]$;

Lower indexes
0 –initial state;
k –values at instant t_k;

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Upper indexes

T –matrix transposition;

$\cdot$ –time derivative;

$\wedge$ –estimated value based on Kalman filter;

$\sim$ –estimated value based on the mathematical model;

$*$ –measured quantity, reduced – non-dimensional quantity;

Introduction

IN reports [1], [2] it was explained that one of the basic factors influencing the precession of guided anti-tank missiles with one channel pulse width modulation control system, with aerodynamic fin or thrust vector, is the accuracy of prediction of the time of actuation of the control device from the equilibrium position. For this reason, an accurate estimation and prediction of the roll angle and roll angular rate is necessary. Angular rate p, is changed during the flight.

The roll angle is measured by free gyroscope built into the missile. Roll angle sensor is the encoder, which gives n_g information about the roll angle per revolution. Accordingly, after each angle $k \times \Delta\Phi_g$, time t_k is measured, where $\Delta\Phi_g = 2\pi/n_g = \text{const}$. [2]. Therefore, the corresponding sampling time is $T_k = t_k - t_{k-1}$, and it is the function of time of flight because the angular rate is not constant. For the purpose of this work the initial value $n_g = 8$ is assumed for the analysis, so that $\Delta\Phi_g = \pi/8$, [2]. Therefore, every T_k seconds Φ_k is measured with error n_k, and discrete measurement equation is

$$\Phi_k^* = \Phi_k + n_k \tag{1}$$

where Φ_k^* is the measured value of Φ_k. It is assumed that n_k is Gaussian white noise.

Upon the introduction of the column matrix of the measurement values

$$\mathbf{x}_k^* = \begin{bmatrix} \Phi_k^* & 0 & 0 \end{bmatrix}^T \tag{2}$$

and column matrix of measurement noise

$$\mathbf{n}_k = \begin{bmatrix} n_k & 0 & 0 \end{bmatrix}^T \tag{3}$$

discrete state measurement matrix is obtained

$$\mathbf{x}_k^* = \mathbf{H}\mathbf{x}_k + \mathbf{n}_k, \qquad k = 1, 2, ... \tag{4}$$

where the measurement matrix is

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{5}$$

Covariance matrix of the measurement noise is

$$\mathbf{R}_k = \mathbf{E}\left[\mathbf{n}_k \mathbf{n}_k^T\right] = \begin{bmatrix} \sigma_\Phi^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

where σ_Φ^2 – dispersion of the measured roll angle by free gyro.

Angular rate, based on the measured values of the roll angle, can be obtained using finite difference method of the first order ("two-point differencing")

$$p_k = \frac{\Phi_k - \Phi_{k-1}}{T_k} = \frac{\Delta\Phi_k}{T_k} \tag{7}$$

where: Φ_k and Φ_{k-1} – measured values of angles at moments t_k and t_{k-1}, and $T_k = t_k - t_{k-1}$ – time interval between two successive measurements of the angles Φ_k. Roll angle Φ is measured by free gyroscope with measurement error. Sampling time interval T_k is the function of the time of flight. For the purpose of this work, all the necessary data are calculated for hypothetical missile by using `SimDVPTR` program [9].

It is assumed that the measurement error is uncorrelated quantity, and that its probable density error obeys Gauss' law with null mathematical expection and standard deviation σ_Φ. It is also assumed that measurement error of the roll angle is a small value compared to the measured angle itself, so from formulae (7) error of determining angular rate and its dispersion is [1]

$$\hat{\sigma}_p^2 = 2\frac{\sigma_\Phi^2}{T_k^2} \tag{8}$$

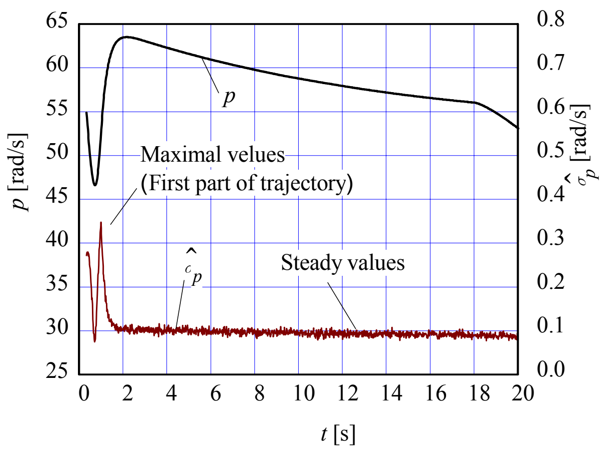

Figure 1. Estimate of the root mean square deviation of the roll angular rate $\hat{\sigma}_p$ obtained by "two-point differencing" as a function of time, $\sigma_\Phi = 0.87\ \text{mrad}$.

In report [1], estimate of the roll rate and roll angle and its standard deviation is done for Kalman filter with exact mathematical model. Optimal value of the process noise, to achieve minimal dispersion of the estimate of the roll angle and roll rate, is determined. It has been shown that there exists minimum of the estimate of standard deviation for a value of the process noise. Applying Kalman filter with exact mathematical model three times lesser estimation of standard deviation of the roll rate in the transition process (maximal values) and about five times smaller steady value compared to values obtained by numerical differentiation using first order finite difference method ("two-point differencing") is achieved.

The main disadvantage of using Kalman filter with exact mathematical model for estimating the roll angle and roll rate is necessary for storage of series of nominal values of dynamic coefficients. Therefore, the question is whether Kalman filter with polynomial kinematic model can be used for estimating roll rate and roll angle. Answer to that question is given in this report.

Differential equations of the missile rolling motion in a body fixed coordinate system are [2], [7]:

$$\begin{aligned}\dot{\Phi} &= p \\ \dot{p} &= l_0 + l_p p + \bar{w}_l\end{aligned} \tag{9}$$

where l_0 and l_p are specific (reduced) active and damping rolling moment and $\bar{w}_l = l_{\text{dist}}$ is disturbing specific moment, or process noise. It is a random function of time. For the purpose of this work, it is assumed that the disturbing moment is Gaussian white noise with null mathematical expection.

Estimation of the roll rate by Kalman filter with kinematic model of the III order

It has been explained that the angular rate is changed along the trajectory for various reasons. Due to that, Kalman filter with kinematic model of the third order is used.

Let $x_1 = \Phi$, $x_2 = \dot{\Phi} = p$ and $x_3 = \ddot{\Phi} = \dot{p}$. Then a state vector can be introduced

$$\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} \Phi & p & \dot{p} \end{bmatrix}^T \tag{10}$$

and differential equation of the states becomes

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \bar{\mathbf{w}}(t) \tag{11}$$

where : $\mathbf{A}(t)$ – matrix of dynamic coefficients

$$\mathbf{A}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{12}$$

$\bar{\mathbf{w}}$ – matrix of process noise

$$\bar{\mathbf{w}} = \begin{bmatrix} 0 & 0 & \bar{w}_l \end{bmatrix}^T \tag{13}$$

Continual covariance matrix of the process noise $\mathbf{S}(t)$ is defined by

$$\mathbf{E}\left[\bar{\mathbf{w}}(t)\bar{\mathbf{w}}^T(\tau)\right] = \mathbf{S}(t)\delta(t-\tau) \tag{14}$$

By numerical solution of the state equations in the time interval $t_0 = 0$ to t_k for initial state vector

$$\mathbf{x}(0) = \mathbf{x}_0 = \begin{bmatrix} \Phi_0 & p_0 & 0 \end{bmatrix}^T \tag{15}$$

state vector can be obtained.

It is easy to obtain discrete difference equation from (11)

$$\mathbf{x}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{w}_{k-1}, \qquad k = 1,2,... \tag{16}$$

where: $\mathbf{F}_k$ – discrete transition matrix

$$\mathbf{F}_k = \begin{bmatrix} 1 & T_k & 0.5T_k^2 \\ 0 & 1 & T_k \\ 0 & 0 & 1 \end{bmatrix} \tag{17}$$

and discrete matrix of the process noise

$$\mathbf{w}_{k-1} = \int_{t_{k-1}}^{t_k} e^{(t_k-\tau)\mathbf{A}}\bar{\mathbf{w}}(\tau)\,d\tau = \int_0^T e^{\mathbf{A}\tau}\bar{\mathbf{w}}(\tau)\,d\tau \tag{18}$$

while

$$\mathbf{Q}_k = \begin{bmatrix} T_k^5/20 & T_k^4/8 & T_k^3/6 \\ T_k^4/8 & T_k^3/6 & T_k^2/2 \\ T_k^3/6 & T_k^2/2 & T_k \end{bmatrix} \bar{q}_{\Phi}^{(3)} \tag{19}$$

Quantity $\bar{q}_{\Phi}^{(3)}$ represents power spectral density of the process noise – derivative of a specific rolling moment. Its dimension is $\dim\left[\bar{q}_{\Phi}^{(3)}\right] = \left[\mathrm{T}^{-5}\right]$. The change of angular acceleration in the period of discretization is of the order $\sqrt{\bar{q}_{\Phi}^{(3)}T_k} = \sqrt{S_l}$, where $S_l = \sigma_l^2$ variance of random component of the roll angular acceleration

The Riccati equations which define Kalman filter gains are:

$$\begin{aligned}\mathbf{M}_k &= \mathbf{F}_k\mathbf{P}_{k-1}\mathbf{F}_k^T + \mathbf{Q}_k \\ \mathbf{K}_k &= \mathbf{M}_k\mathbf{H}^T\left(\mathbf{H}\mathbf{M}_k\mathbf{H}^T + \mathbf{R}_k\right)^{-1} \\ \mathbf{P}_k &= \left(\mathbf{I} - \mathbf{K}_k\mathbf{H}\right)\mathbf{M}_k\end{aligned} \tag{20}$$

By solving these equations matrix of gains of the Kalman filter $\mathbf{K}_k$ and discrete matrix of variances of differences between the estimated and true values of states are obtained, while estimated values of the states are obtained by Kalman filter as follows

$$\hat{\mathbf{x}}_k = \tilde{\mathbf{x}}_k + \mathbf{K}_k\left(\mathbf{x}_k^* - \tilde{\mathbf{x}}_k\right) \tag{21}$$

In equation (21) $\tilde{\mathbf{x}}_k$ is discrete two-dimensional vector of the estimates of the states based on mathematical model

$$\tilde{\mathbf{x}}_k = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1} \tag{22}$$

Because only one state variable – roll angle Φ is measured, covariance matrix of the measurement noise is scalar $\mathbf{R}_k = R_k = \sigma_{\Phi}^2$, and Riccati equations can be simplified. When products of matrices in the second and third Riccati equations are expanded it can be obtained

$$\begin{aligned}\mathbf{M}\mathbf{H}^T &= \begin{bmatrix} M_{11} & M_{21} & M_{31} \end{bmatrix}^T \\ \mathbf{H}\mathbf{M} &= \begin{bmatrix} M_{11} & M_{12} & M_{13} \end{bmatrix} \\ \mathbf{H}\mathbf{M}\mathbf{H}^T &= M_{11} \\ \left(\mathbf{H}\mathbf{M}_k\mathbf{H}^T + \mathbf{R}_k\right)^{-1} &= \left(M_{11} + \sigma_{\Phi}^2\right)^{-1}\end{aligned} \tag{23}$$

In the expressions above the subscript "k" is omitted to simplify writing.

$$\mathbf{K} = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}^T \tag{24}$$

where the parameters K_{Φ}, K_p and $K_{\dot{p}}$ are:

$$K_1 = K_{\Phi} = \frac{M_{11}}{M_{11} + \sigma_{\Phi}^2} \tag{25}$$

$$K_2 = K_p = \frac{M_{12}}{M_{11} + \sigma_{\Phi}^2} \tag{26}$$

$$K_3 = K_{\dot{p}} = \frac{M_{13}}{M_{11} + \sigma_{\Phi}^2} \tag{27}$$

Coefficients M_{11}, M_{12} are M_{13} are members of the first column of the matrix $\mathbf{M}$. Further, it is

$$(\mathbf{I}-\mathbf{KH})=\begin{bmatrix}(1-K_1) & 0 & 0\\ -K_2 & 1 & 0\\ -K_3 & 0 & 1\end{bmatrix} \tag{28}$$

The third Riccati equation in expanded form is

$$\mathbf{P}=\begin{bmatrix}(1-K_1)M_{11} & (1-K_1)M_{12} & (1-K_1)M_{13}\\ -K_2M_{11}+M_{12} & -K_2M_{12}+M_{22} & -K_2M_{13}+M_{23}\\ -K_3M_{11}+M_{13} & -K_3M_{12}+M_{23} & -K_3M_{13}+M_{33}\end{bmatrix} \tag{29}$$

Note that matrices $\mathbf{M}$ and $\mathbf{P}$ are symmetrical.

In the expression for Kalman filter (21) in gain matrix exist only three gains which are different from zero on each step of correction of estimates of the states. These gains do not depend on the estimated values of states since fundamental matrix, which goes into Riccati equations (20) does not depend on the estimated values of states, but only on the dynamic coefficients and sampling time.

Estimation of the stated variables based on the mathematical model (prediction) is

$$\tilde{\mathbf{x}}_k=\mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1} \tag{30}$$

For numerical simulation, the same values of input data will be taken as in [1]: for the simulation of real values of state variables by using program SimDVPTR it will be taken $l_0(t)=\left[l_0(t)\right]_{\text{nom}}+m_l+\overline{w}_l=(1+0.05)\left[l_0(t)\right]_{\text{nom}}+\overline{w}_l$, and for variance of the random variable w_l it will be taken $S_l=(0.05\times l_0)^2$

For estimation of the state variables, nominal value of specific moment $l_0(t)$ and variance $S_l=(0.05\times l_0)^2$ will be taken.

For the initialization of Kalman filter, the initial matrix $\mathbf{P}_k$ will be obtained in the same way as it was done in [1], while for the estimation of the initial roll angle $\hat{\Phi}_k$ the measured value will be taken, for roll rate value obtained by "two-point differencing" according to expression (7) and for the determining angular acceleration "two-point differencing" the following formula will be used:

$$\dot{p}_k^*=\frac{p_k^*-p_{k-1}^*}{T_k}=\frac{\Phi_k-\Phi_{k-2}^*}{T_k^2} \tag{31}$$

Matrix $\mathbf{P}_k$ in expanded form is

$$\mathbf{P}_k=\mathbf{E}\left[(\mathbf{x}_k-\hat{\mathbf{x}}_k)(\mathbf{x}_k-\hat{\mathbf{x}}_k)^T\right]=$$

$$\mathbf{E}\begin{bmatrix}(\Phi_k-\hat{\Phi}_k)^2 & (\Phi_k-\hat{\Phi}_k)(p_k-\hat{p}_k) & (\Phi_k-\hat{\Phi}_k)(\dot{p}_k-\hat{\dot{p}}_k)\\ (\Phi_k-\hat{\Phi}_k)(p_k-\hat{p}_k) & (p_k-\hat{p}_k)^2 & (p_k-\hat{p}_k)(\dot{p}_k-\hat{\dot{p}}_k)\\ (\Phi_k-\hat{\Phi}_k)(\dot{p}_k-\hat{\dot{p}}_k) & (p_k-\hat{p}_k)(\dot{p}_k-\hat{\dot{p}}_k) & (\dot{p}_k-\hat{\dot{p}}_k)^2\end{bmatrix} \tag{32}$$

Statistics of the quantity above can easily be determined (see Appendix [1]), so the covariance matrix at initial instant of time and zero values of the process noise is

$$(\mathbf{P}_0)_{S_l=0}=\begin{bmatrix}\sigma_\Phi^2 & \sigma_\Phi^2/T_0 & 2\sigma_\Phi^2/T_0^2\\ \sigma_\Phi^2/T_0 & 2\sigma_\Phi^2/T_0^2 & 3\sigma_\Phi^2/T_0^3\\ 2\sigma_\Phi^2/T_0^2 & 3\sigma_\Phi^2/T_0^3 & 4\sigma_\Phi^2/T_0^4\end{bmatrix} \tag{33}$$

The influence of deviation of dynamic coefficients from nominal values (process noise) is expressed by matrix $\mathbf{Q}_k$ given by formula (19), while the values are taken for the initial instant of time

$$\mathbf{Q}_0=\overline{q}_\Phi^{(3)}\begin{bmatrix}T_0^5/20 & T_0^4/8 & T_0^3/6\\ T_0^4/8 & T_0^3/6 & T_0^2/2\\ T_0^3/6 & T_0^2/2 & T_0\end{bmatrix} \tag{34}$$

Eventually, covariance matrix at the initial instant of time is

$$\mathbf{P}_0=(\mathbf{P}_0)_{S_l=0}+\mathbf{Q}_0 \tag{35}$$

It is easy to show from (35), (34) and (33) that

$$\frac{(q_{ij})_0}{(p_{ij})_{S_l=0}}=O\left(T_s^4\frac{\overline{q}_\Phi^{(3)}T_s}{\sigma_\varepsilon^2}\right)=O(\lambda^2)$$

where λ is a small quantity of the first order, so the second term in expression (35) can be neglected and (35) becomes

$$\mathbf{P}_0\approx(\mathbf{P}_0)_{S_l=0}=\sigma_\Phi^2\begin{bmatrix}1 & 1/T_0 & 2/T_0^2\\ 1/T_0 & 2/T_0^2 & 3/T_0^3\\ 2/T_0^2 & 3/T_0^3 & 4/T_0^4\end{bmatrix} \tag{36}$$

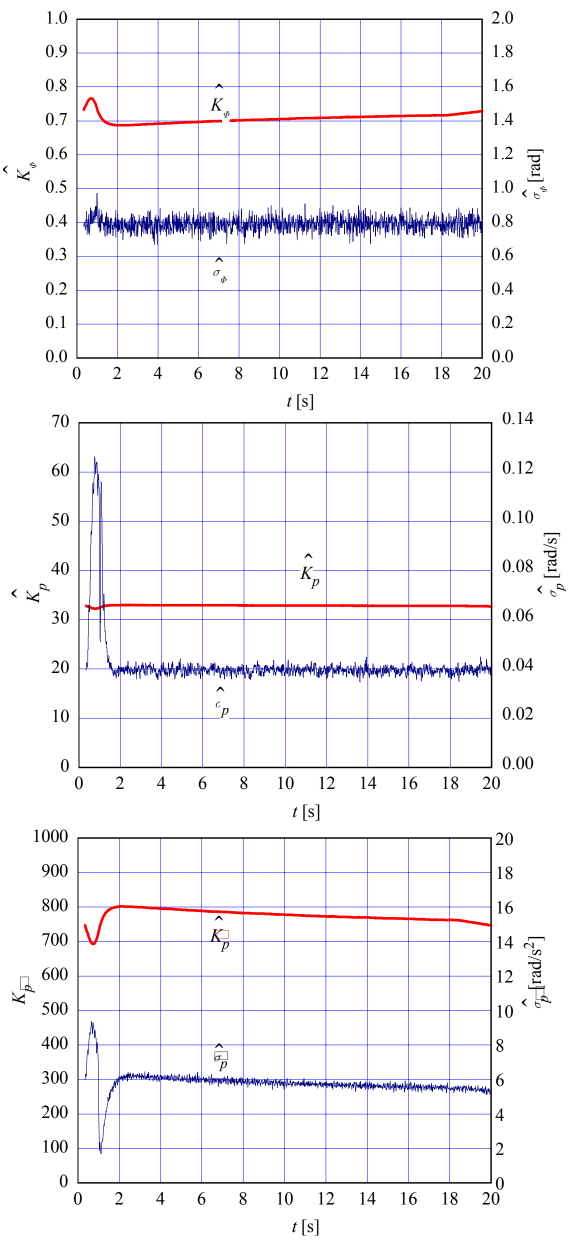

Figure 2. Estimates of the standard deviations and coefficients of Kalman filter with polynomial kinematic model of III order: $\sigma_\Phi=0.87$ mrad , $m_l=0.05\times l_0$, $S_l=(0.05\times l_0)^2$.

In Fig.2 results of simulation are shown. It can be seen that the roll angle and roll rate are well determined by filter for $t > 1.3\,\text{s}$, while for $t < 1.3\,\text{s}$, it is not so good because of the change of sign of the roll angular acceleration. However, root mean square of the estimate of the roll rate is still several times less than the estimate obtained by "two-point differencing" (Fig.1).

Gains of the filter converge to the steady values is very fast. Based on that, it can be concluded that for the filtering, constant values of gains can be taken for the whole time of flight.

Estimation of the roll rate by α-β-γ filter

The analysis undertaken in the previous two sections shows that Kalman's covariance matrix of errors and gain matrix converge to the steady values is very fast. Based on that, it can be concluded that for the estimation of state variables by Kalman filter constant values of filter gains can be taken during the flight. That enables eliminating calculation of gains and variances. Gains of the Kalman filter are then given by [5]:

$$\mathbf{K} = \left[\alpha \quad \beta/T \quad \gamma/2T^2\right]^T \tag{37}$$

where the parameters α, β and γ are:

$$\alpha = \frac{M_{11}}{M_{11} + \sigma_\Phi^2} \tag{38}$$

$$\frac{\beta}{T} = \frac{M_{12}}{M_{11} + \sigma_\Phi^2} \tag{39}$$

$$\frac{\gamma}{2T^2} = \frac{M_{13}}{M_{11} + \sigma_\Phi^2} \tag{40}$$

They can be determined analytically in function of one parameter – index of manoeuvrability of the object λ_Φ, which is proportional to the ratio of the standard deviation of the process noise $\sigma_l = \sqrt{S_l} = \sqrt{\bar{q}_\Phi^{(3)} T_k}$ and standard deviation of the measurement noise

$$\lambda_\Phi = T_k^2 \frac{\sigma_l}{\sigma_\Phi} \tag{41}$$

In the example under consideration $T_k = 0.011\,\text{s}$, $\sigma_l \approx 1.3\,\text{rad/s}^2$, $\sigma_\Phi = 0.87\,\text{mrad}$, so that $\lambda_\Phi = 0.017$. Mean steady values of filter gains obtained by numerical simulation in the preceding section by using the Kalman filter with polynomial kinematic filter are $K_1 = K_\Phi = 0.4$, $K_2 = K_p = 9.5$, $K_3 = K_{\dot{p}} = 110$. With these values of the filter gains, state values are estimated by using `SimDVPTR`. Results show that almost same values of states as those estimated by polynomial Kalman filter are obtained. By using Kalman filter with constant coefficients, the amount of necessary numerical calculations in the estimation process is reduced considerably.

Comparative analysis of the applied methods

In Table 1 values of root mean square of deviations of the estimated values of the roll angle and roll rates with respect to the true values are presented. The calculation is done by program `SimDVPTR` which uses Monte-Carlo simulation with 200 generated trajectories for testing. Results of the estimation of the roll angle and roll rates by Kalman filter with exact mathematical model is given in [1].

In all analyses, measurement noise is simulated by normal distribution of random numbers $\mathcal{N}\left(0, \sigma_\Phi^2\right)$, and process noise by normal distribution of random numbers $\mathcal{N}\left(m_l, \sigma_l^2\right)$, where $\sigma_\Phi = 0.05° = 0.87\,\text{mrad}$, $m_l = 0.05 l_0$, $\sigma_l = 0.05 l_0$. Power spectral densities which correspond to the standard deviation of the simulated values were used for estimation, that is $R = \sigma_\Phi^2$, $S_l = \sigma_l^2$.

Table 1. Estimated values for $S_l = \sigma_l^2 = (0.05 l_0)^2$

No.	Estimation method	Root mean square deviation with respect to true value			
		Roll angle [mrad]		Roll rate [rad/s]	
		Maximal value	Steady value	Maximal value	Steady value
1.	"Two-point differencing"	0.87	0.87	0.35	
2.	Kalman filter with exact model	0.94	0.77	0.120	0.069
3.	Kalman filter with kinematic polynomial model of III order	0.95	0.80	0.126	0.040
4.	$\alpha-\beta-\gamma$ filter	0.97	0.82	0.130	0.040

Analysis of the results in the Table show that the best estimation of the roll angle and roll rate is obtained by Kalman filter with polynomial kinematic model. By using Kalman filter with kinematic polynomial model of III order, root mean square deviation of the roll rate four times less than values obtained by numerical "two-point differencing" is obtained.

Applying α-β-γ filter, that is Kalman filter with constant gains, the accuracy of the estimate of the state variables is slightly decreased compared to the estimates obtained by Kalman filter with kinematic polynomial model.

Up to this point analysis was made with the same values of power spectral density of the random moment S_l as that used for simulation of the measurement roll angle Φ. As it is done in [1] the influence of S_l on the estimation of states will be analyzed.

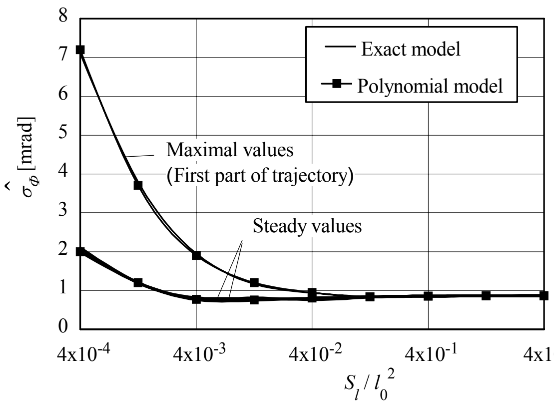

Figure 3. Comparative estimates of *rms* deviations of the roll angle in function of S_l/l_0^2 : $\sigma_\Phi = 0.87\,\text{mrad}$.

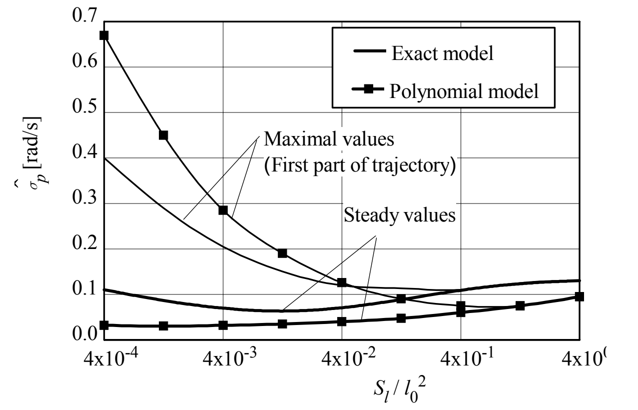

Figure 4. Comparative estimates of *rms* deviations of the roll rate in function of S_l/l_0^2 : $\sigma_{\Phi} = 0.87\,\text{mrad}$.

In Fig.3, estimate of the standard deviation of the roll angle obtained by Kalman filter with exact mathematical model and with polynomial model against reduced variance of the process noise S_l/l_0^2 is shown.

It can be seen that both models give practically the same estimates of the roll angles. With the increase of the parameter of the process noise S_l/l_0^2 maximal values of the estimates converge to steady values and for $S_l/l_0^2 > 5\times10^{-2}$ they are practically equal to the steady values. Steady values have the minimum of $\hat{\sigma}_{\Phi} \approx 0.77\,\text{mrad}$ for $S_l/l_0^2 \approx 0.02$, i.e. 11% less than the value used for simulation $(\sigma_{\Phi} \approx 0.87\,\text{mrad})$. Note that for value $S_l/l_0^2 \approx 0.02$, where the steady values are at minimum, maximal values of the estimates are twice the minimal values (Fig.3).

Standard deviation of the estimate of the roll rate by Kalman filter with exact mathematical model and with kinematic polynomial model against the reduced process noise S_l/l_0^2 is shown in Fig.4 for maximal values, which, as it was explained, arises at the beginning part of the trajectory, and for steady values, which are associated with the second part of trajectory. First, the steady values will be analyzed. It can be seen that both curves, with exact model and with kinematic model have a minimum. Estimate of the standard deviation of the roll rate for Kalman filter with exact model has a minimum of $\hat{\sigma}_p = 0.061\,\text{rad/s}$ for $S_l/l_0^2 \approx 1.2\times10^{-3}$, while estimate of the standard deviation of the roll rate for Kalman filter with polynomial model has a minimum of $\hat{\sigma}_p \approx 0.030\,\text{rad}/s$ for $S_l/l_0^2 \approx 1.5\times10^{-3}$. Therefore, applying Kalman filter with polynomial kinematic model about 50% less, standard deviation of the estimate of the roll rate can be achieved compared to the results obtained by using Kalman filter with the exact model.

Curves representing maximal values, which appear at the initial part of the trajectory (Fig.1), show that at higher values of the process noise S_l/l_0^2 polynomial Kalman filter gives lower value of the estimate of the standard deviation, while at smaller values of the process noise, better result are obtained by Kalman filter with exact mathematical model, because changes of the roll rate are greater in the first part of the trajectory.

One of the additional benefits of using Kalman filter with kinematic polynomial model is easy calculation of the roll angular acceleration, necessary for forecasting the roll angle. They can be obtained directly from Kalman filter equations (21), which in developed form are:

$$\begin{aligned}\hat{\Phi}_k &= \tilde{\Phi}_k + K_{\Phi}\left(\Phi_k^* - \tilde{\Phi}_k\right)\\ \hat{p}_k &= \tilde{p}_k + K_p\left(p_k^* - \tilde{p}_k\right)\\ \hat{\dot{p}}_k &= \tilde{\dot{p}}_k + K_{\dot{p}}\left(\dot{p}_k^* - \tilde{\dot{p}}_k\right)\end{aligned} \tag{42}$$

If Kalman filter with exact model is applied, estimate of the roll angular acceleration can be obtained by “two-point differencing” as follows

$$\hat{\dot{p}}_k = \frac{\hat{p}_k - \hat{p}_{k-1}}{\Delta t_k} = \frac{\Delta\hat{p}_k}{\Delta t_k} \tag{43}$$

Conclusion

The problem of efficient estimate of the roll angle and roll rate is considered for the anti-tank guided missile for which the angle is measured by free gyro with the presence of measurement noise. Expression for the estimate of the standard deviation of the roll rate is derived for Kalman filter with exact mathematical model, with Kalman filter with kinematic polynomial model and, in the end, with Kalman filter with constant gains in kinematic model (alpha-beta-gama filter).

Efficiency of the proposed method is analyzed by numerical simulation. Also, numerical simulation is used to obtain the estimate of the standard deviation of the roll rate with respect to the reduced value of the process noise. The optimal value of the process noise to achieve minimal dispersion of the estimate of the roll angle and roll rate is determined. It has been shown that there exists minimum of the estimate of the standard deviation for some value of the process noise for all three applied methods. Conditions for determining the method of estimation of the roll rate and process noise are determined. Applying Kalman filter, three times less estimate of standard deviation of the roll rate in the transition process (maximal values) is achieved and about ten times smaller steady value compared to the values obtained by numerical differentiation using first order finite difference method (“two-point differencing”).

Parallel analysis of the applied methods shows that the influence of the process noise on standard deviation of the roll angle and roll rate is different in the first part of the trajectory (maximal values) and in the second part (steady values). At the beginning of the flight, where the changes of angular rate are considerably greater, application of Kalman filter with an exact mathematical method gives better results compared to that obtained by kinematic polynomial method at smaller values of covariance of the process noise. This is due to the fact that exact mathematical model describes dynamics of the rolling motion at small values of the process noise better, while at higher values of the process noise the influence of mathematical model is of less importance, so both exact and kinematic polynomial model give the same results.

Application of Kalman filter with kinematic polynomial model provides 50% less estimate of standard deviation of the roll rate and roll angle compared to Kalman filter with exact model for steady values. Furthermore, by using polynomial model it is achieved that the estimation process does not depend on the unknown coefficients and real trajectory, which is not known due to the stochastic nature of disturbing forces and moments.

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Optimalna ocena ugaone brzine valjanja potivtenkovske rakete Deo II: Kalmanov filter sa polinomnim modelom

Predstavljen efikasan metod za ocenu i predikciju ugaone brzine valjanja protivoklopne rakete koristeći Klamanov filter sa polinomnim kinematičkim modelom kretanja, pri čemu se meri ugao valjanja rakete pomoću slobodnog žiroskopa, a merena veličina je opterećena šumom. Određena je optimalna vrednost procesnog šuma iz uslova da se postigne minimalna ocena disperzije ugla valjanja i ugaone brzine valjanja. Izvedene su neophodne formule za ocenu stanja pomoću kinematičkog modela i polinomnog modela sa konstantnim koeficijentima. Pomoću tačnog numeričkog simulacionog modela testirana je efikasnost metode i određena je standardna devijacija ugaone brzine valjanja u funkciji normalizovane vrednosti procesnog šuma. Dato je poređenje rezultata sa Kalmanovim filtrom sa tačnim matematičkim modelom. Dat je kriterijum za izbor metoda estimacije i procesnog šuma da se postigne minimalna ocena disperzije ugla i ugaone brzine valjanja.

Ključne reči: **protivoklopna raketa, valjanje rakete, brzina valjanja, ugaona brzina, Klamanov filtar, slučajni proces, slobodni žiroskop, numerička simulacija.**

Оптимальная оценка угловой скорости крена противотанковой ракеты - Часть II: фильтр Калмана со полиномной моделью

Здесь представлен эффективный метод для оценки и определения угловой скорости крена противотанковой ракеты при пользовании фильтра Калмана со полиномной кинематической моделью движения, при чём измеряется угол крена ракеты при помощи свободного гироскопа, а измеряемая величина усиленна шумом. Здесь определено оптимальное значение обработанного шума из условий достижения минимальной оценки рассеяния угла крена и угловой скорости крена. Выведены необходимые уравнения для оценки состояния при помощи кинематической модели и полиномной модели со постоянными коэффициентами. Эффективность метода испытывана при помощи безошибочной цифровой моделированной модели и определена стандартная девиация угловой скорости крена в функции нормализованной величины обработанного шума. Здесь тоже дано сравнение результатов со фильтром Келмана со безошибочной математической моделью. Здесь тоже приведен и критерий для выбора метода оценивания и обработанного шума из условий достижения минимальной оценки рассеяния угла крена и угловой скорости крена.

Ключевые слова: противотанковая ракета, крен ракеты, скорость крена, угловая скорость, фильтр Калмана, случайный процесс, свободный гироскоп, цифровое моделирование.

Estimation optimale de la vitesse d'angle du roulement chez le missile antichar- première partie: le filtre de Kalman avec le modèle exact

On a présenté une méthode efficace pour estimation et prédiction de la vitesse d'angle du roulement chez le missile antichar.Pour ce faire, on a utilisé le filtre de Kalman avec le modèle du mouvement polynôme cinématique en mesurant l'angle du roulement du missile à l'aide du gyroscope libre alors que la donnée mesurée est altérée par le bruit.La valeur optimale du bruit de procès est déterminée afin de réaliser l'estimation minimale de la dispersion et la vitesse d'angle du roulement.Les formules nécessaires pour estimation de l'état par les modèles cinématiques et polynômes aux coefficients constants sont dérivées.L'efficacité de la méthode a été testée par le modèle exact de simulation et on a déterminé la déviation ordinaire de la vitesse du roulement en foncion de la valeur normalisée du bruit de procès.On a donné la comparaison des résultats avec le filtre de Kalman et le modèle mathématique exact.Le critère pour le choix de la méthode d'estimation et du bruit de procès, qui permet de réaliser l'estimation minimale de la dispersion d'angle et la vitesse d'angle du roulement, est donné aussi.

Mots clés: **missile antichar, roulement du missile, vitesse du roulement, vitesse d'angle, filtre de Kalman, procès stochastique, gyroscope libre, simulation numérique.**