UDK: 517.928
COSATI: 12-01

Stability of Descriptor Discrete Time Delayed Systems in Lyapunov sense

Dragutin Lj. Debeljković, PhD (Eng)[1)]
Sreten B. Stojanović, Phd (Eng)[2)]
Stevan A.Milinković, Phd (Eng)[3)]
Nemanja S. Višnjić, MSc (Eng)[1)]
Milomir Pješić MSc (Eng)[1)]

This paper gives sufficient conditions for the stability of linear singular continuous delay systems of the $E\mathbf{x}(k)=A_0\mathbf{x}(k)+A_1\mathbf{x}(k-1)$ form. These new, delay–independent conditions are derived using approach based on Lyapunov's direct method. Approach that has been applied is based on crucial idea presented in the paper of *Owens*, *Debeljkovic* (1985). Numerical examples have been worked out to show the applicability of the results derived.

Key words: **discrete system, descriptive system, linear system, system stability, asymptotic stability, Lyapunov method, Lyapunov stability.**

Introduction

IT should be noted that in some systems, their character of dynamic and static state must be considered simultaneously. *Singular continuous systems* (also referred to as degenerate, generalized, differential - algebraic systems or semi – state) are those the dynamics of which are governed by a mixture of *algebraic* and *differential* equations.

Descriptor discrete systems are those the dynamics of which is covered by a mixture of *algebraic* and *difference* equations.

Recently, many scholars have paid much attention to singular and descriptor systems, and in consequence obtained numerous good results. The complex nature of singular and descriptor systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control.

The problem of investigation of time delay systems has been exploited over many years. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time lag, whether it is present in the control or/and the state, may cause undesirable system transient response, or even instability. Consequently, the problem of stability analysis for this class of systems has been one of the main interests for many researchers. In general, the introduction of time delay factors makes the analysis much more complicated.

It must be emphasized that in a number of systems the phenomena of time delay and singular (descriptor) occur simultaneously.

Therefore, such systems are called *singular differential* or *descriptor difference systems with time delay*.

These systems have many special characters. To describe them more exactly, it is necessary to design them more accurately and control them more effectively; tremendous effort must be invested to investigate them, which is obviously very difficult work. In recent references authors had discussed such systems and obtained certain results. But in the study of such systems, there are still many problems to be considered. When the general time delay systems are considered, in the existing stability criteria, mainly two approaches have been adopted.

Namely, one direction is to contrive the stability condition which does not include the information on the delay, and the other is the method which takes it into account. The former case is often called the delay-independent criteria and generally provides simple algebraic conditions. In that sense the question of their stability deserves great attention.

In the short overview that follows, only the results achieved in the area of Lyapunov stability of *linear, continuous singular time delay systems* (LCSTDS) will be presented.

Moreover, in the last few years, numerous papers have been published in the area of linear discrete descriptor time delay systems, but this discussion is out of the scope of this paper. To find out more about this matter, see the list of references attached.

To the best of our knowledge, some attempts in stability investigation of (LCSTDS) were due to *Saric* (2001, 2002) where sufficent conditions for convergence of appropriate

[1)] Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Belgrade, SERBIA
[2)] Faculty of Technology, Bulevar oslobođenja 124, 18000 Leskovac, SERBIA
[3)] Faculty of Computer Science, Knez Mihajlova 6, 11000 Belgrade, SERBIA

fundamental matrix were established.

Recently, in the paper of *Xu et al*. (2002) the problem of robust stability and stabilization for uncertain (LCSTDS) was addressed and necessary and sufficient conditions were obtained in terms of strict LMI (Linear Matrix Inqualities). Moreover, in the same paper, using suitable canonical description of (LCSTDS) rather simple criteria for asymptotic stability testing was also proposed.

The same approach has been used when the linear descriptor discrete time delay systems (LDDTDS) have been treated.

In this paper, quite a different approach to this problem is presented. Namely, the result is expressed directly in terms of matrices E, A_0 and A_1 naturally occurring in the system model, avoiding the need to introduce any canonical form into the statement of the *Theorem.*

The geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all solutions. This fact makes possible the construction of Lyapunov stability theory even for the (LCSTDS) in that sense that asymptotic stability is equivalent to the existence of symmetric, positive definite solutions to a weak form of Lyapunov matrix equation incorporating condition which refer to the time delay term.

In the descriptor discrete case, the concept of smoothness has little meaning but the idea of consistent initial conditions being these initial conditions $\mathbf{x}_0$ that generate solution sequence $\left(\mathbf{x}(k): k \geq 0\right)$ has a physical meaning. The best explanation for this statement can be found in *Appendix* B, at the end of the paper.

A definite aim of this paper is to present a new results concerning asymptotic stability of a particular class of *linear descriptor discrete time delay systems*.

In order to have the best insight into these problems, the first part of the paper is devoted to the short recapitulation of some contributions in the field of *linear continuous singular systems*. Authors are deeply convinced that this approach will be of great help to the reader of the lines that follow.

Notation

$\mathbb{R}$	- Real vector space
$\mathbb{C}$	- Complex vector space
I	- Unit matrix
F	- $=(f_{ij}) \in \mathbb{R}^{n\times n}$ real matrix
F^T	- Transpose of matrix F
$F>0$	- Positive definite matrix
$F\geq 0$	- Positive semi definite matrix
$\Re(F)$	- Range of matrix F
$\mathbb{N}(F)$	- Null space (kernel) of matrix F
$\lambda(F)$	- Eigenvalue of matrix F
$\sigma_{(\,)}(F)$	- Singular value of matrix F
$\|F\|$	- Euclidean matrix norm of $\|F\| = \sqrt{\lambda_{\max}\left(A^T A\right)}$
F^D	- Drazin inverse of matrix F
$\Rightarrow$	- Follows
$\mapsto$	- Such that

Linear continuous singular systems

Generally, the linear singular continuous systems with time delay can be written as:

$$\begin{aligned} E(t)\dot{\mathbf{x}}(t) &= \mathbf{f}\left(t, \mathbf{x}(t), \mathbf{x}(t-\tau), \mathbf{u}(t)\right), \quad t \geq 0 \\ \mathbf{x}(t) &= \varphi(t), \ -\tau \leq t \leq 0 \end{aligned}, \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is a state vector, $\mathbf{u}(t) \in \mathbb{R}^l$ is a control vector, $E(t) \in \mathbb{R}^{n\times n}$ is a singular matrix, $\varphi \in C = C\left(\left[-\tau, 0\right], \Re^n\right)$ is an admissible initial state functional, $C = C\left(\left[-\tau, 0\right], \mathbb{R}^n\right)$ is the *Banach space* of continuous functions mapping the interval $\left[-\tau, 0\right]$ into $\mathbb{R}^n$ with topology of uniform convergence.

Some preliminaries

Let a *linear continuous singular system* with state delay, described by

$$E\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau), \tag{2.a}$$

be considered, with known compatible vector valued function of initial conditions

$$\mathbf{x}(t) = \varphi(t), \ -\tau \leq t \leq 0, \tag{2.b}$$

where A_0 and A_1 are constant matrices of appropriate dimensions.

Moreover, let it be assumed that $rank\, E = r < n$.

Definition 1. The matrix pair $\left(E, A_0\right)$ is said to be regular if $\det\left(sE - A_0\right)$ is not identically zero, *Xu et al.* (2002).

Definition 2. The matrix pair $\left(E, A_0\right)$ is said to be impulse free if $\deg\left(\det(sE - A_0)\right) = rang\, E$, *Xu et al.* (2002).

The linear continuous singular time delay system (2) may have an impulsive solution. However, the regularity and the absence of impulses of the matrix pair $\left(E, A_0\right)$ ensure the existence and uniqueness of an impulse free solution to the system under consideration, which is defined in the following *Lemma*.

Lemma 1. Suppose that the matrix pair $\left(E, A_0\right)$ is *regular* and *impulse free and unique* on $\left[0, \infty\right)$, *Xu et al* (2002).

The necessity for system stability investigation demands establishing a proper stability definition. Therefore:

Definition 3.

a) *Linear continuous singular time delay system* (2) is said to be *regular* and *impulse free* if the matrix pair $\left(E, A_0\right)$ is regular and imulse free.

b) *Linear continuous singular time delay system*, (2), is said to be stable if for any $\varepsilon > 0$ exists a scalar $\delta(\varepsilon) > 0$ such that, for any compatible initial condition $\varphi(t)$, satisfying condition: $\sup\limits_{-\tau\leq t\leq 0} \|\varphi(t)\| \leq \delta(\varepsilon)$, the solution $\mathbf{x}(t)$ of system (2) satisfies $\|\mathbf{x}(t)\| \leq \varepsilon, \ \forall t \geq 0$.

Moreover, if $\lim_{t\to\infty}\|\mathbf{x}(t)\| \to 0$, system is said to be *asymptotically stable*, *Xu et al* (2002).

Some previous results

Let the case when the subspace of consistent initial conditions for *singular time delay* and *singular nondelay system* coincide be considered.

Owens-Debeljkovic approach

Theorem 1 Suppose that the matrix pair (E, A_0) is *regular* with system matrix A_0 being *nonsingular* i.e. $\det A_0 \neq 0$.

The system (3.2) is *asymptotically stable*, independent of delay, if there exists a positive definite matrix P, being the solution of Lyapunov matrix equation

$$A_0^T PE + E^T PA_0 = -2(S+Q), \tag{3}$$

with matrices $Q = Q^T > 0$ and $S = S^T$, such that:

$$\mathbf{x}^T(t)(S+Q)\mathbf{x}(t) > 0, \ \forall \mathbf{x}(t) \in W_{k^*} \setminus \{0\}, \tag{4}$$

is positive definite quadratic form on $W_{k^*} \setminus \{0\}$, W_{k^*} being the *subspace of consistent initial conditions*[1], and if the following condition is satisfied:

$$\| A_1 \| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}E^T P\right), \tag{5}$$

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are maximum and minimum singular values of matrix $(\cdot)$, respectively.

Proof. For the sake of brevity, the proof is omitted here and can be found in *Debeljkovic et al.* (2007).

Pandolfi's approach

The result is stated as follows:

Theorem 2 Suppose that the system matrix A_0 is nonsingular; i.e. $\det A_0 \neq 0$.

Then the system (3.2) with known compatibile vector valued function of initial conditions can be considered assuming that $rank\, E_0 = r < n$.

Matrix E_0 is defined in the following way $E_0 = A_0^{-1}E$.

The system (3.2) is *asymptotically stable*, independent of delay, if

$$\| A_1 \| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}E_0^T P\right), \tag{6}$$

here exist:

(i) $(n\times n)$ matrix P, being the solution of Lyapunov matrix:

$$E_0^T P + PE_0 = -2I_\Omega, \tag{7}$$

with the following properties:

a) $P = P^T$ (8)

b) $P\mathbf{q}(t) = \mathbf{0}, \quad \mathbf{q}(t) \in \Lambda$ (9)

c) $\mathbf{q}^T(t)P\mathbf{q}(t) > 0, \quad \mathbf{q}(t) \neq \mathbf{0}, \quad \mathbf{q}(t) \in \Omega$, (10)

where:

$$\Omega = \mathbb{N}\left(I - EE^D\right), \tag{11}$$

$$\Lambda = \mathbb{N}\left(EE^D\right), \tag{12}$$

with matrix I_Ω reperesenting generalized operator on $\mathbb{R}^n$ and identity matrix on subspace Ω and zero operator on subspace Λ and matrix Q being any positive definite matrix.

Moreover, matrix P is symmetric and positive definite on the subspace of consistent initial conditions.

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ have the same meaning as in the previous section.

Proof. For the sake of brevity, the proof is omitted here and can be found in *Debeljkovic et al.* (2005.c, 2006.a).

Linear discrete descriptor systems

Some preliminaries

(LDDTDS) is described by

$$E\,\mathbf{x}(k+1) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1), \tag{13}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$ is a state vector.

The matrix $E \in \mathbb{R}^{n\times n}$ is a necessarily singular matrix, with property $rank\, E = r < n$ and with matrices A_0 and A_1 of appropriate dimensions.

For (LDDTDS), (13), the following definitions taken from, *Xu et al*. (2004) are presented.

Definition 4 The (LDDTDS) is said to be *regular* if $\det\left(z^2E - zA_0 - A_1\right)$, is not identically zero.

Definition 5 The (LDDTDS) is said to be *causal* if it is *regular* and

$$\deg\left(z^n \det\left(zE - A_0 - z^{-1}A_1\right)\right) = n + rang\, E.$$

Definition 6 The (LDDTDS) is said to be *stable* if it is *regular* and $\rho(E, A_0, A_1) \subset D(0,1)$, where

$$\rho(E, A_0, A_1) = \left\{z \mid \det\left(z^2E - zA_0 - A_1\right) = 0\right\}.$$

Definition 7 The (LDDTDS) is said to be *admissibleif* it is *regular*, causal *and* stable.

Lemma 1 The (LDDTDS) is admissible if there exist a matrix $Q > 0$ and an invertible symmetric matrix P such that

i) $E^T PE \geq 0$

ii) $A_0^T PA_0 - E^T PE + A_0^T PA_1\left(Q - A_1^T PA_1\right)^{-1} A_1^T PA_0 + Q < 0$

iii) $Q - A_1^T PA_1 > 0$,

Xu et al. (2004).

Proof. See, *Xu et al*. (2004).

---

[1] W_{k^*} *subspace of consistent initial conditions, Owens, Debeljković* (1985). See Appendix **A**

Main results

Theorem 3 Suppose that (LDDTDS) is *regular* and *causal* with system matrix A_0 being nonsingular, e.i. $\det A_0 \neq 0$.

Moreover, suppose matrix $\left(Q - A_1^T P A_1\right)$ is regular.

The system (4.1) is *asymptotically stable*, independent of delay, if

$$\| A_1 \| < \frac{\sigma_{\min}\left(\left(Q - A_1^T P A_1\right)^{\frac{1}{2}}\right)}{\sigma_{\max}\left(Q^{-\frac{1}{2}} A_0^T P\right)}, \tag{14}$$

and if there exist a symmetric positive definite matrix P, being the solution of discrete Lyapunov matrix equation

$$A_0^T P A_0 - E^T P E = -2\left(S + Q\right), \tag{15}$$

with matrices $Q = Q^T > 0$ and $S = S^T$, such that

$$\mathbf{x}^T(k)\left(S + Q\right)\mathbf{x}(k) > 0, \quad \forall \mathbf{x}(k) \in W_{k^*}^d \setminus \{0\}, \tag{16}$$

is positive definite quadratic form on $W_{k^*}^d \setminus \{0\}$, $W_{k^*}^d$ being the subspace of consistent initial conditions for both *time delay* and *non-time delay* discrete descriptor system .

Proof. Let the following functional be considered:

$$V\left(\mathbf{x}(k)\right) = \mathbf{x}^T(k) E^T P E \mathbf{x}(k) + \mathbf{x}^T(k-1) Q \mathbf{x}(k-1). \tag{17}$$

with matrices $P = P^T > 0$ and $Q = Q^T > 0$.

Remark 1 Equations (15 – 16) are, in modified form, taken from *Owens*, *Debeljkovic* (1985).

Note that *Lemma* B1 and *Theorem* B1 indicate that

$$V\left(\mathbf{x}(k)\right) = \mathbf{x}^T(k) E^T P E \mathbf{x}(k), \tag{18}$$

is *positive quadratic form* on $W_{k^*}^d$, and it is obvious that all solutions $\mathbf{x}(k)$ evolve in $W_{k^*}^d$, so $V\left(\mathbf{x}(k)\right)$ can be used as a *Lyapunov function* for the system under consideration, *Owens*, *Debeljkovic* (1985).

It will be shown that the same argument can be used to declare the same property of another quadratic form present in (17).

Clearly, using the equation of motion of (13):

$$\begin{aligned}
\Delta V\left(\mathbf{x}(k)\right) &= V\left(\mathbf{x}(k+1)\right) - V\left(\mathbf{x}(k)\right) \\
&= \mathbf{x}^T(k)\left(A_0^T P A_0 - E^T P E + Q\right)\mathbf{x}(k) \\
&\quad + 2\mathbf{x}^T(k)\left(A_0^T P A_1\right)\mathbf{x}(k-1) - \mathbf{x}^T(k-1) Q \mathbf{x}(k-1) \\
&= \mathbf{x}^T(k)\left(A_0^T P A_0 - E^T P E + 2Q + 2S\right)\mathbf{x}(k) \\
&\quad - \mathbf{x}^T(k) Q \mathbf{x}(k) - 2\mathbf{x}^T(k) S \mathbf{x}(k) \\
&\quad + 2\mathbf{x}^T(k)\left(A_0^T P A_1\right)\mathbf{x}(k-1) \\
&\quad - \mathbf{x}^T(k-1)\left(Q - A_1^T P A_1\right)\mathbf{x}(k-1)
\end{aligned} \tag{19}$$

From (3) and the inequality[2]:

$$\begin{aligned}
&2\mathbf{x}^T(k) A_0^T P A_1 \mathbf{x}(k-1) = \\
&= 2\mathbf{x}^T(k)\left(A_0^T P A_1\left(Q - A_1^T P A_1\right)^{-\frac{1}{2}}\left(Q - A_1^T P A_1\right)^{\frac{1}{2}}\right)\mathbf{x}(k-1) \\
&\leq \mathbf{x}^T(k) A_0^T P A_1\left(Q - A_1^T P A_1\right)^{-\frac{1}{2}}\left(Q - A_1^T P A_1\right)^{-\frac{1}{2}} A_1^T P A_0 \mathbf{x}(k) \\
&\quad + \mathbf{x}^T(k-1)\left(Q - A_1^T P A_1\right)^{\frac{1}{2}}\left(Q - A_1^T P A_1\right)^{\frac{1}{2}}\mathbf{x}(k-1)
\end{aligned} \tag{20}$$

it can be obtained

$$\begin{aligned}
&\Delta V\left(\mathbf{x}(k)\right) = -\mathbf{x}^T(k) Q \mathbf{x}(k) - \mathbf{x}^T(k) S \mathbf{x}(k) \\
&+\mathbf{x}^T(k)\left(A_0^T P A_1\left(Q - A_1^T P A_1\right)^{-\frac{1}{2}}\left(Q - A_1^T P A_1\right)^{-\frac{1}{2}} A_1^T P A_0\right)\mathbf{x}(k) \\
&\leq -\mathbf{x}^T(k) S \mathbf{x}(k) \\
&-\mathbf{x}^T(k) Q^{\frac{1}{2}}\left(I - Q^{-\frac{1}{2}} A_0^T P A_1\left(Q - A_1^T P A_1\right)^{-1} A_1^T P A_0 Q^{-\frac{1}{2}}\right) Q^{\frac{1}{2}}\mathbf{x}(k)
\end{aligned} \tag{21}$$

From the fact that the choice of matrix S, can be done, such that

$$\mathbf{x}^T(k) S \mathbf{x}(k) \geq 0, \quad \forall \mathbf{x}(k) \in W_{k^*}^d \setminus \{0\}, \tag{22}$$

and after some manipulations, (21) yields to

$$\Delta V\left(\mathbf{x}(k)\right) \leq -\mathbf{x}^T(k) Q^{\frac{1}{2}}\, \Gamma\, Q^{\frac{1}{2}}\mathbf{x}(k), \tag{23}$$

with matrix Γ defined by

$$\Gamma = \left(I - Q^{-\frac{1}{2}} A_0^T P A_1 Q^{-\frac{1}{2}} Q^{-\frac{1}{2}} A_1^T P A_0 Q^{-\frac{1}{2}}\right). \tag{24}$$

and following the procedure presented in *Tissir*, *Hmamed* (1996), $V\left(\mathbf{x}(k)\right)$ is negative definite if

$$1 - \lambda_{\max}\left(Q^{-\frac{1}{2}} A_0^T P A_1\left(Q - A_1^T P A_1\right)^{-\frac{1}{2}}\left(Q - A_1^T P A_1\right)^{-\frac{1}{2}} A_1^T P A_0 Q^{-\frac{1}{2}}\right) > 0 \tag{25}$$

is satisfied if

$$1 - \sigma_{\max}^2\left(Q^{-\frac{1}{2}} A_0^T P A_1\left(Q - A_1^T P A_1\right) Q^{-\frac{1}{2}}\right) > 0. \tag{26}$$

Using the properties of the singular matrix values, *Amir-Moez* (1956), the condition (26) holds if

$$1 - \sigma_{\max}^2\left(Q^{-\frac{1}{2}} A_0^T P\right) \times \sigma_{\max}^2\left(A_1\left(Q - A_1^T P A_1\right) Q^{-\frac{1}{2}}\right) > 0 \tag{27}$$

which is satisfied if

$$1 - \frac{\| A_1 \|^2 \sigma_{\max}^2\left(Q^{-\frac{1}{2}} A_0^T P\right)}{\sigma_{\min}^2\left(\left(Q - A_1^T P A_1\right)^{-\frac{1}{2}}\right)} > 0, \tag{28}$$

which completes proof. **Q.E.D.**

In the sequel an example is given to show the effectiveness of the proposed method.

Example 1 Consider the linear discrete descriptor time delay system with matrices as follows:

$$E = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0.10 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.10 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

---

[2] $2\mathbf{u}^T(t)\,\mathbf{v}(t) \leq \mathbf{u}^T(t) P\, \mathbf{u}(t) + \mathbf{v}^T(t) P^{-1} \mathbf{v}(t), \quad P > 0$

Based on the above presented procedure, the following data can easily be found, *Debeljkovic et. al.* (1996.b):

$$\hat{E} = \left(zE + A_0\right)^{-1}_{|z=0} \cdot E$$

$$\hat{E} = A_0^{-1}E = \begin{pmatrix} -20 & 0 & 0 \\ 20 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Rightarrow \hat{E}^D = \begin{pmatrix} -0.05 & 0 & 0 \\ 0{,}05 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\mathbb{N}\left(I - \hat{E}\hat{E}^D\right) = \left(I - \hat{E}\hat{E}^D\right)\mathbf{x}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\mathbf{x}_0 = \mathbf{0}, \Rightarrow$$

$$\mathbb{N}\left(I - \hat{E}\hat{E}^D\right) = \left\{\mathbf{x} \in \mathbb{R}^n : x_{10} + x_{20} = 0 \ \wedge \ x_{30} = 0\right\}.$$

$$\det A_0 \neq 0, \quad rang\, E = 1$$

$$\det\left(z^2E - zA_0 - A_1\right) = z\left(2z^2 - 0.01z - 1\right)\left(z - 0.10\right) \neq 0$$

$$\deg\left(z^n \det\left(zE - A_0 - z^{-1}A_1\right)\right) = n + rang\, E \Rightarrow$$

$$n = 3, \quad \mapsto \deg\left(z^4\right) = 4, \quad n + rank\, E = 3 + 1 = 4$$

It can be adopted

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Q^T > 0, \quad S = S^T = \begin{pmatrix} 0 & -2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$S + Q = \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow$$

$$\mathbf{x}^T(k)S\mathbf{x}(k) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{pmatrix} 0 & -2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \left(-2x_1x_2 - 2x_1x_2 - 3x_2^2 - 2x_3^2\right)_{\substack{x_1 = -x_2 \\ x_3 = 0}} = x_1^2 > 0, \ .$$

$$\forall \mathbf{x}(k) \in W^d_{k^*} \setminus \{0\}$$

$$\mathbf{x}^T(k)Q\mathbf{x}(k) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \left(\left(x_1 + x_2\right)^2 + x_2^2 + x_3^2\right)_{\substack{x_1 = -x_2 \\ x_3 = 0}} = x_2^2 > 0, \qquad ,$$

$$\forall \mathbf{x}(k) \in W^d_{k^*} \setminus \{0\}$$

Moreover

$$\mathbf{x}^T(k)Q\mathbf{x}(k) = \left(x_1 + x_2\right)^2 + x_2^2 + x_3^2 > 0, \quad \forall \mathbf{x}(k) \in \mathbb{R}^n$$

$$\det Q = 1 \neq 0,$$

and

$$\mathbf{x}^T(k)(S + Q)\mathbf{x}(k) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \left(x_1^2 + 2x_2^2 - x_2^2 - x_3^2\right)_{\substack{x_1 = -x_2 \\ x_3 = 0}} = x_1^2 + x_2^2 > 0,$$

$$\forall \mathbf{x}(k) \in W^d_{k^*} \setminus \{0\}$$

Also, it can be computed

$$\begin{pmatrix} 0.10 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix}\begin{pmatrix} 0.10 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} -$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$-2(S + Q) = -2\begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow$$

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix} =$$

$$= \begin{pmatrix} -3.99p_{11} + 0.2p_{12} + p_{22} & 0.1p_{12} + p_{22} & 0.1p_{13} + p_{23} \\ 0.1p_{12} + p_{22} & p_{22} & p_{23} \\ 0.1p_{13} + p_{23} & p_{23} & p_{33} \end{pmatrix}$$

with solution

$$P = \begin{pmatrix} 0.501 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix} = P^T > 0.$$

Moreover

$$\mathbf{x}^T(k)E^TPE\mathbf{x}(k) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{pmatrix} 0.501 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \left(0.501x_1^2 + 2x_2^2 + 2x_3^2\right)_{\substack{x_1 = -x_2 \\ x_3 = 0}} = \left(2.501x_1^2\right)_{\substack{x_1 = -x_2 \\ x_3 = 0}} > 0,$$

$$\forall \mathbf{x}(k) \in W^d_{k^*} \setminus \{0\}$$

so $V\left(\mathbf{x}(k)\right)$ can be used as a *Lyapunov function* for the system (13).

Finally, it is necessary to check the condition (15)

$$\left\|A_1\right\| = 0.10 \qquad \sigma\{Q\} = \{0.382, \quad 1.00, \quad 2.618\}$$

$$\Omega = Q - A_1^TPA_1 = \begin{pmatrix} 0.995 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{\min}\left(\left(Q - A_1^TPA_1\right)^{\frac{1}{2}}\right) = 0.618$$

$$\sigma_{\max} = \left(Q^{-\frac{1}{2}}A_0^TP\right) = 1.42$$

$$0{,}10 = \| A_1 \| < \frac{\sigma_{\min}\left(\left(Q - A_1^T P A_1\right)^{\frac{1}{2}}\right)}{\sigma_{\max}(Q^{-\frac{1}{2}} A_0^T P)} = 0.4361,$$

so, the system under consideration is *asymptotically stable*.

Moreover, it should be noted that there are eight different solutions for $Q^{1/2}$, $Q^{-1/2}$ and of course, for expression $Q^{-1/2}A_0^T P$.

But solutions for $\sigma_{\min}\left(Q^{1/2}\right)$ and $\sigma_{\max} = \left(Q^{-2/2}A_0^T P\right)$ *are unique* in all cases.

In a particular case, when $E = I$, the result is identical to that presented in *Debeljkovic et al.* (2005.c).

Now the general case is investigated namely if the *basic system matrix* is *singular*, e.g. $\det A_0 = 0$.

Note that this is an *imposible case* for linear continuous singular system; see, *Theorem* 1 and *Theorem* 2.

Theorem 4 Suppose that (LDDTDS) is *regular* and *causal.*

Moreover, suppose matrix $\left(Q_\lambda - A_1^T P_\lambda A_1\right)$ is regular, with $Q_\lambda = Q_\lambda^T > 0$.

The system (1) is *asymptotically stable*, independent of the delay, if:

$$\| A_1 \| < \frac{\sigma_{\min}\left(\left(Q_\lambda - A_1^T P_\lambda A_1\right)^{\frac{1}{2}}\right)}{\sigma_{\max}\left(Q_\lambda^{-\frac{1}{2}}\left(A_0 - \lambda E\right)^T P_\lambda\right)}, \tag{29}$$

and if there exists real positive scalar $\lambda^* > 0$ such that for all λ within the range $0 < |\lambda| < \lambda^*$ there exist *symmetric positive definite* matrix P_λ, being the solution of *discrete Lyapunov matrix equation*:

$$\left(A_0 - \lambda E\right)^T P_\lambda \left(A_0 - \lambda E\right) - E^T P_\lambda E = -2\left(S_\lambda + Q_\lambda\right), \tag{30}$$

with matrix $S_\lambda = S_\lambda^T$, such that:

$$\mathbf{x}^T(k)\left(S_\lambda + Q_\lambda\right)\mathbf{x}(k) > 0, \quad \forall \mathbf{x}(k) \in W_{k^*}^d \setminus \{0\}, \tag{31}$$

is *positive definite quadratic form* on $W_{k^*}^d \setminus \{0\}$, $W_{k^*}^d$ being the subspace of consistent initial conditions for both *time delay* and *non-time delay* discrete descriptor system.

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are maximum and minimum singular values of matrix $(\cdot)$, respectively.

Proof. Let the functional be considered

$$V\left(\mathbf{x}(k)\right) = \mathbf{x}^T(k)E^T P_\lambda E\mathbf{x}(k) + \mathbf{x}^T(k-1)Q_\lambda \mathbf{x}(k-1). \tag{32}$$

with matrices $P_\lambda = P_\lambda^T > 0$ and $Q_\lambda = Q_\lambda^T > 0$.

Remark 2 Equations (3 - 4) are, in modified form, taken from *Owens*, *Debeljkovic* (1985).

Note that *Lemma* B1 and *Theorem* B1 indicate that:

$$V\left(\mathbf{x}(k)\right) = \mathbf{x}^T(k)E^T P_\lambda E\mathbf{x}(k), \tag{33}$$

is *positive quadratic form* on $W_{k^*}^d$, and it is obvious that all solutions $\mathbf{x}(k)$ evolve in $W_{k^*}^d$, so $V\left(\mathbf{x}(k)\right)$ can be used as a *Lyapunov function* for the system under consideration, *Owens*, *Debeljkovic* (1985).

It will be shown that the same argument can be used to declare the same property of another quadratic form, present in (32).

Clearly, using the equation of motion of (13), it follows that

$$\begin{aligned}
&\Delta V\left(\mathbf{x}(k)\right) = V\left(\mathbf{x}(k+1)\right) - V\left(\mathbf{x}(k)\right) + \\
&\mathbf{x}^T(k)\left(\left(A_0 - \lambda E\right)^T P_\lambda \left(A_0 - \lambda E\right)\right)\mathbf{x}(k) - \\
&\mathbf{x}^T(k)\left(Q_\lambda - E^T P_\lambda E\right)\mathbf{x}(k) + \\
&2\mathbf{x}^T(k)\left(\left(A_0 - \lambda E\right)^T P_\lambda A_1\right)\mathbf{x}(k-1) - \\
&\mathbf{x}^T(k-1)\left(Q_\lambda - A_1^T P_\lambda A_1\right)\mathbf{x}(k-1) = \\
&\mathbf{x}^T(k)\left(\left(A_0 - \lambda E\right)^T P_\lambda \left(A_0 - \lambda E\right)\right)\mathbf{x}(k) + \\
&\mathbf{x}^T(k)\left(2Q_\lambda - E^T P_\lambda E + 2S_\lambda\right)\mathbf{x}(k) - \\
&\mathbf{x}^T(k)Q_\lambda \mathbf{x}(k) - 2\mathbf{x}^T(k)S_\lambda \mathbf{x}(k) + \\
&2\mathbf{x}^T(k)\left(\left(A_0 - \lambda E\right)^T P_\lambda A_1\right)\mathbf{x}(k-1) - \\
&\mathbf{x}^T(k-1)\left(Q_\lambda - A_1^T P_\lambda A_1\right)\mathbf{x}(k-1),
\end{aligned} \tag{34}$$

Using the same procedure, as in the previous case, the following is obtained:

$$\begin{aligned}
&2\mathbf{x}^T(k)\left(A_0 - \lambda E\right)^T P_\lambda A_1 \mathbf{x}(k-1) = \\
&2\mathbf{x}^T(k)\left(\left(A_0 - \lambda E\right)^T P_\lambda A_1 \left(Q_\lambda - A_1^T P A_1\right)^{-\frac{1}{2}}\left(Q_\lambda - A_1^T P_\lambda A_1\right)^{\frac{1}{2}}\right) \\
&\mathbf{x}(k-1) \le \mathbf{x}^T(k)\left(A_0 - \lambda E\right)^T P_\lambda A_1 \left(Q_\lambda - A_1^T P_\lambda A_1\right)^{-\frac{1}{2}} \\
&\left(Q_\lambda - A_1^T P_\lambda A_1\right)^{-\frac{1}{2}} A_1^T P_\lambda A_0 \mathbf{x}(k) + \mathbf{x}^T(k-1)\left(Q_\lambda - A_1^T P_\lambda A_1\right)^{\frac{1}{2}} \\
&\left(Q_\lambda - A_1^T P_\lambda A_1\right)^{\frac{1}{2}} \mathbf{x}(k-1)
\end{aligned} \tag{35}$$

so that:

$$\begin{aligned}
&\Delta V\left(\mathbf{x}(k)\right) = -\mathbf{x}^T(k)Q_\lambda \mathbf{x}(k) - \mathbf{x}^T(k)S_\lambda \mathbf{x}(k) \\
&+\mathbf{x}^T(k)\left(\left(A_0 - \lambda E\right)^T P_\lambda A_1 \left(Q_\lambda - A_1^T P_\lambda A_1\right)^{-\frac{1}{2}}\right. \\
&\left.\left(Q_\lambda - A_1^T P_\lambda A_1\right)^{-\frac{1}{2}} A_1^T P_\lambda A_0\right)(\,)\mathbf{x}(k) \le -\mathbf{x}^T(k)S_\lambda \mathbf{x}(k) - \\
&-\mathbf{x}^T(k)Q_\lambda^{\frac{1}{2}}\left(I - Q_\lambda^{-\frac{1}{2}}\left(A_0 - \lambda E\right)^T P_\lambda A_1 \left(Q - A_1^T P_\lambda A_1\right)^{-1}\right. \\
&\left.A_1^T P_\lambda \left(A_0 - \lambda E\right)Q_\lambda^{-\frac{1}{2}}\right)
\end{aligned} \tag{36}$$

From the fact that the choice of matrix S_λ, can be made, such that:

$$\mathbf{x}^T(k)S_\lambda \mathbf{x}(k) \ge 0, \qquad \forall \mathbf{x}(k) \in W_{k^*}^d \setminus \{0\}, \tag{37}$$

and after some manipulations, (36) yields to:

$$\Delta V\left(\mathbf{x}(k)\right) \le -\mathbf{x}^T(k)Q_\lambda^{\frac{1}{2}}\,\Upsilon_\lambda\, Q_\lambda^{\frac{1}{2}}\mathbf{x}(k), \tag{38}$$

with matrix Υ_λ defined by:

$$\Upsilon_\lambda = \Big(I - Q_\lambda^{-\frac{1}{2}} \left(A_0 - \lambda E \right)^T P_\lambda A_1 Q_\lambda^{-\frac{1}{2}} Q_\lambda^{-\frac{1}{2}} \\ A_1^T P_\lambda \left(A_0 - \lambda E \right) Q_\lambda^{-\frac{1}{2}} \Big). \tag{39}$$

and following the procedure presented in *Tissir*, *Hmamed* (1996), $V(\mathbf{x}(k))$ is negative definite if

$$1 - \lambda_{\max} \Big(Q_\lambda^{-\frac{1}{2}} \left(A_0 - \lambda E \right)^T P_\lambda A_1 \left(Q_\lambda - A_1^T P_\lambda A_1 \right)^{-\frac{1}{2}} \\ \left(Q_\lambda - A_1^T P_\lambda A_1 \right)^{-\frac{1}{2}} A_1^T P_\lambda \left(A_0 - \lambda E \right) Q_\lambda^{-\frac{1}{2}} \Big) > 0 \tag{40}$$

hich is satisfied if

$$1 - \sigma_{\max}^2 \left(Q_\lambda^{-\frac{1}{2}} \left(A_0 - \lambda E \right)^T P_\lambda A_1 \left(Q_\lambda - A_1^T P_\lambda A_1 \right) Q_\lambda^{-\frac{1}{2}} \right) > 0 \tag{41}$$

Using the properties of the singular matrix values, *Amir-Moez* (1956), the condition (41) holds if:

$$1 - \sigma_{\max}^2 \left(Q_\lambda^{-\frac{1}{2}} \left(A_0 - \lambda E \right)^T P_\lambda \right) \times \\ \sigma_{\max}^2 \left(A_1 \left(Q_\lambda - A_1^T P_\lambda A_1 \right) Q_\lambda^{-\frac{1}{2}} \right) > 0, \tag{42}$$

which is satisfied if:

$$1 - \frac{\| A_1 \|^2 \sigma_{\max}^2 \left(Q_\lambda^{-\frac{1}{2}} \left(A_0 - \lambda E \right)^T P_\lambda \right)}{\sigma_{\min}^2 \left(\left(Q_\lambda - A_1^T P_\lambda A_1 \right)^{-\frac{1}{2}} \right)} > 0, \tag{43}$$

which completes proof. **Q.E.D.**

In a particular case, when $E = I$ and $\det A_0 \neq 0$, the result is identical to that presented in *Debeljkovic et al.* (2005.c).

In the particular case, when $\det A_0 \neq 0$, the result is identical to that presented in *Debeljkovic et al.* (2006.b).

Example 2 Consider the *linear discrete descriptor time delay* system with matrices as follows:

$$E = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} \frac{1}{10} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In comparison with the *Example* 1, it should be underlined that here matrix A_0 is singular, e.g. $\det A_0 = 0$.

Based on the above presented procedure, the following data can easily be found, *Debeljkovic et. al.* (2006.b):

$$\hat{E} = \left(zE + A_0 \right)_{|z=1}^{-1} \cdot E .$$

$$\hat{E}_{z=1} = I \cdot E = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \Rightarrow \hat{E}^D = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\mathbb{N}\left(I - \hat{E}\hat{E}^D \right) = \left(I - \hat{E}\hat{E}^D \right) \mathbf{x}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{x}_0 = \mathbf{0}, \quad \Rightarrow$$

$$\mathbb{N}\left(I - \hat{E}\hat{E}^D \right) = \left\{ \mathbf{x} \in \mathbb{R}^n : x_{10} + x_{20} = 0 \wedge x_{30} = 0 \right\}.$$

$$\det A_0 = 0, \quad rang\, E = 1,$$

$$\det\left(z^2 E - zA_0 - A_1 \right) = -z\left(z^2 + 0.10 \right) \neq 0,$$

$$\det\left(zE - A_0 - z^{-1} A_1 \right) = -\left(z - \frac{0.10}{z} \right).$$

$$\deg\left(z^n \det\left(zE - A_0 - z^{-1} A_1 \right) \right) = n + rang\, E \Rightarrow$$

$$n = 3, \quad \mapsto \deg\left(z^4 \right) = 4, \quad n + rank\, E = 3 + 1 = 4$$

It can be adopted:

$$Q_\lambda = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Q_\lambda^T > 0, \quad S_\lambda = S_\lambda^T = \begin{pmatrix} 0 & -2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

$$S_\lambda + Q_\lambda = \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\mathbf{x}^T(k) S_\lambda \mathbf{x}(k) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{pmatrix} 0 & -2 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \\ \left(-2x_1x_2 - 2x_1x_2 - 3x_2^2 - 2x_3^2 \right)_{\substack{x_1 = -x_2 \\ x_3 = 0}} = x_1^2 > 0,$$

$$\forall \mathbf{x}(k) \in W_{k^*}^d \setminus \{0\}.$$

$$\mathbf{x}^T(k) Q_\lambda \mathbf{x}(k) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \\ \left(\left(x_1 + x_2 \right)^2 + x_2^2 + x_3^2 \right)_{\substack{x_1 = -x_2 \\ x_3 = 0}} = x_2^2 > 0,$$

$$\forall \mathbf{x}(k) \in W_{k^*}^d \setminus \{0\},$$

Moreover:

$$\mathbf{x}^T(k) Q_\lambda \mathbf{x}(k) = \left(x_1 + x_2 \right)^2 + x_2^2 + x_3^2 > 0, \quad \forall \mathbf{x}(k) \in \square^n$$

$$\det Q = 1 \neq 0,$$

and:

$$\mathbf{x}^T(k) \left(S_\lambda + Q_\lambda \right) \mathbf{x}(k) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \\ \left(x_1^2 + 2x_2^2 - x_2^2 - x_3^2 \right)_{\substack{x_1 = -x_2 \\ x_3 = 0}} = x_1^2 + x_2^2 > 0,$$

$$\forall \mathbf{x}(k) \in W_{k^*}^d \setminus \{0\}$$

Also, it can be computed:

$$\left(A_0-\lambda E\right)_{\lambda=2}^T P_\lambda \left(A_0-\lambda E\right)_{\lambda=2}=$$

$$\begin{pmatrix} 4p_{11}-12p_{12}+9p_{22} & 3p_{22}-2p_{12} & 3p_{23}-2p_{13} \\ 3p_{22}-2p_{12} & p_{22} & p_{23} \\ 3p_{23}-2p_{13} & p_{23} & p_{33} \end{pmatrix}$$

$$E^T P_\lambda E=\begin{pmatrix} p_{11}-2p_{12}+p_{22} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so finally:

$$\begin{pmatrix} 4p_{11}-14p_{12}+8p_{22} & 3p_{22}-2p_{12} & 3p_{23}-2p_{13} \\ 3p_{22}-2p_{12} & p_{22} & p_{23} \\ 3p_{23}-2p_{13} & p_{23} & p_{33} \end{pmatrix}=$$

$$-2\left(Q_\lambda+S_\lambda\right)=\begin{pmatrix} -2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

with solution:

$$P_\lambda=\begin{pmatrix} \frac{10}{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}=P_\lambda^T>0 .$$

Moreover:

$$\mathbf{x}^T(k)E^T P_\lambda E\mathbf{x}(k)=\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{pmatrix} \frac{10}{3} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=$$

$$\left(\frac{100}{9}x_1^2+2x_2^2+2x_3^2\right)_{\substack{x_1=-x_2 \\ x_3=0}}=\left(3.11x_1^2\right)_{\substack{x_1=-x_2 \\ x_3=0}}>0,$$

$$\forall\mathbf{x}(k)\in W_{k^*}^d\setminus\{0\}$$

so $V(\mathbf{x}(k))$ can be used as a *Lyapunov function* for the system (4.1).

Finally, the condition (29) has to be checked

$$\left\|A_1\right\|=0.10 \qquad \sigma\{Q_\lambda\}=\{0.382,\ 1.00,\ 2.618\}$$

$$\Omega_\lambda=Q_\lambda-A_1^T P_\lambda A_1=\begin{pmatrix} 0.997 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_{\min}\left(\left(Q_\lambda-A_1^T P_\lambda A_1\right)^{\frac{1}{2}}\right)=0.616$$

$$\sigma_{\max}=\left(Q_\lambda^{-\frac{1}{2}}A_0^T P_\lambda\right)=1.422$$

$$0{,}10=\left\|A_1\right\|<\frac{\sigma_{\min}\left(\left(Q_\lambda-A_1^T P_\lambda A_1\right)^{\frac{1}{2}}\right)}{\sigma_{\max}\left(Q_\lambda^{-\frac{1}{2}}A_0^T P_\lambda\right)\left(Q_\lambda^{-\frac{1}{2}}A_0^T P_\lambda\right)}=0.481,$$

so, the system under consideration is *asymptotically stable*.

The same conclusion can be derived directly from the locations of roots of the characteristic equation.

Conclusion

A quite new sufficiently delay–independent criteria for asymptotic stability of (LDDTDS) is presented. In a certain sense, this result may be treated as the further extension of results derived in *Debeljkovic et. al* (2006.b).

This approach gives an insight into the structure of discrete descriptor system under consideration, so it may be called: a geometric approach to the Lyapunov stability of this particular class of systems.

In that sense it seems the extension of weak Lyapunov equation (15) and (30) to discrete descriptor time delay systems.

In comparison with some other papers on this matter, there is no need for linear transformations of the basic system, nor the need to solve the systems of high order linear matrix inequalities.

Two numerical examples are presented to show the applicability of the results derived.

Appendix – A

The fundamental geometric tool in the characterization of the subspace of consistent initial conditions, for *linear singular system without delay*, is the subspace sequence

$$W_0=\mathbb{R}^n, \tag{A1}$$

$$\vdots$$

$$W_{j+1}=A_0^{-1}\left(EW_j\right),\qquad j\geq 0, \tag{A2}$$

where $A_0^{-1}(\cdot)$ denotes inverse image of $(\cdot)$ under the operator A_0.

Lemma A.1 The subsequence $\{W_0, W_1, W_2, ...\}$ is nested in the sense that:

$$W_0\supset W_1\supset W_2\supset W_3\supset .... \tag{A3}$$

Moreover:

$$\mathbb{N}(A)\subset W_j,\ \ \forall j\geq 0, \tag{A4}$$

and there exist an intager $k\geq 0$, such that:

$$W_{k+j}=W_k . \tag{A5}$$

Then it is obvious that:

$$W_{k+j}=W_k,\ \ \forall j\geq 1 . \tag{A6}$$

If k^* is the smallest such integer with this property, then:

$$W_k\cap\mathbb{N}(E)=\{\mathbf{0}\},\quad k\geq k^*, \tag{A7}$$

provided that $(\lambda E-A_0)$ is invertible for some $\lambda\in\mathbb{R}$.

Theorem A.1. Under the conditions of *Lemma* A1, $\mathbf{x}_0$ is a consistent initial condition for the system under consideration if and only if $\mathbf{x}_0\in W_{k^*}$.

Moreover $\mathbf{x}_0$ generates a unique solution $\mathbf{x}(t)\in W_{k^*}$,

$t \geq 0$, that is a real analytic one on $\{t : t \geq 0\}$

Proof. See *Owens*, *Debeljkovic* (1985).

Apendix – B

The fundamental geometric tool in the characterization of the subspace of consistent initial conditions, for *linear discrete descriptor system without delay* (13), is the subspace sequence

$$W_0^d = \mathbb{R}^n, \tag{B.1}$$

$$\vdots$$

$$W_{j+1}^d = A_0^{-1}\left(EW_j^d\right), \quad j \geq 0, \tag{B.2}$$

where $A_0^{-1}(\cdot)$ denotes inverse image of $(\cdot)$ under the operator A_0.

Lemma B.1

Lemma B.1 is identical to the *Lemma* A.1.

The only, change is $W_0 = W_0^d$, $W_1 = W_1^d$,......, $W_k = W_k^d$.

Final result is as follows:

$$W_k^d \cap \mathbb{N}(E) = \{\mathbf{0}\}, \; k \geq k^*, \tag{B.3}$$

provided that $(zE - A_0)$ is invertible for some $z \in \mathbb{C}$.

Proof. See *Owens*, *Debeljkovic* (1985).

Theorem B.1. Under the conditions of *Lemma* B1, $\mathbf{x}_0$ is a consistent initial condition for the system under consideration, e.g. *linear discrete singular system without delay* if and only if $\mathbf{x}_0 \in W_{k^*}^d$.

Moreover $\mathbf{x}_0$ generates a *discrete* solution sequence $(\mathbf{x}(k) : k \geq 0)$ such that $\mathbf{x}(k) \in W_{k^*}^d, \; \forall k \geq 0$.

Proof. See *Owens*, *Debeljkovic* (1985).

Apendix – C

Definition C.1 Let E be a square matrix, if there exists a matrix E^D satisfying:

1. $$EE^D = E^D E$$

2. $$E^D E E^D = E^D \tag{C.1}$$

3. $$E^{\wp+1} E^D = E^{\wp}$$

E^D is called the Drazin inverse matrix of matrix E and *D* simply - inverse matrix.

$\wp$ is the index of the matrix E, it is the smallest nonnegative integer which makes

$$rank\ E^{\wp+1} = rank\ E^{\wp}, \tag{C.2}$$

true.

Lemma C.1 For any square matrix E, its Drazin inverse matrix E^D is existent and unique.

If the *Jordan* normalized form of E is

$$E = T\begin{pmatrix} J & 0 \\ 0 & N \end{pmatrix} T^{-1}, \tag{C.3}$$

then:

$$E^D = T\begin{pmatrix} J^{-1} & 0 \\ 0 & 0 \end{pmatrix} T^{-1}. \tag{C.4}$$

Here N is a nilpotent matrix, J and T are invertible matrices.

References

[1] AMIR-MOEZ,A.: *Extreme Properties of a Hermitian Transformations and Singular Values of Sum and Product of Linear Transformations*, Duke Math J., **23** (1956), 463–476.

[2] CAMPBELL, S.L.: *Singular systems of differential equation*, Pitman, London, 1980.

[3] CAMPBELL,S.L.: *Singular Systems of Differential Equations II*, Pitman, Marshfield, Mass., 1982.

[4] DEBELJKOVIĆ,D.LJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B: *Application of singular system theory in chemical engineering: Analysis of process dynamics*, Internat. Congress of Chemical and Process Eng., CHISA 96, Prague (Czech Rep.), August, 1996, 25-30.

[5] DEBELJKOVIĆ,D.LJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B.: *Continuous Singular Control Systems*, GIP Kultura, Belgrade, 2005.

[6] DEBELJKOVIĆ,D.LJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B., JACIĆ,LJ.A.: *Discrete Descriptor Control Systems,* GIP Kultura, Belgrade, 2005.b.

[7] DEBELJKOVIĆ,D.LJ., LAZAREVIĆ,M.P., STOJANOVIĆ,S.B., JOVANOVIĆ,M.B., MILINKOVIĆ,S.A.: *Discrete Time Delayed System Stability Theory in the sense of Lyapunov: New Results*, Dynamics of Continuous, Discrete and Impulsive Systems, (Canada), Vol.12, Series B: Numerical. Analysis, Vol.12.b – Supplement (2005.c), 433 - 442.

[8] DEBELJKOVIĆ,D.LJ., STOJANOVIĆ,S.B., JOVANOVIĆ,M.B., MILINKOVIĆ,S.A.: *Further Results on Descriptor Time Delayed System Stabilty Theory in the sense of Lyapunov: Pandolfi based Approach*, International Journal of Information & System Science, (Canada), Vol.2, No.1, (2006.a), 1 – 11.

[9] DEBELJKOVIĆ,D.LJ., STOJANOVIĆ,S.B., VIŠNJIĆ,N.S., MILINKOVIĆ,S.A.: *Lyapunov Stability Theory : A Quite New Approach – Discrete Descriptive Time Delayed Systems*, Proc. American Control Conference 06, June, 14 – 16, Minneapolis, Minesota, (USA), (2006.b), 5091 – 5096.

[10] DEBELJKOVIĆ,D.LJ., JACIĆ,LJ.A., VISNJIĆ,N.S., PJEŠĆIĆ,M.: "Asymptotic Stability of General Discrete Descriptive Time Delayed Systems, Proc. of 5-th IFAC WORKSHOP on Technology Transfer in Developing Countries, Izmir, Cesme (Turkey), Maz 17 – 18, 2007, (accepted).

[11] DEBELJKOVIĆ,D.LJ., STOJANOVIĆ,S.B., VISNJIĆ,N.S., MILINKOVIĆ,S.A.: *Singular Time Delayed Stability Theory: A Quite New Approach*, Proc. American Control Conference 07, July, 11 – 13, New York, (USA),(2007), (accepted).

[12] DEBELJKOVIĆ,D.LJ., STOJANOVIĆ,S.B., , MILINKOVIĆ,S.A., VISNJIĆ,N.S.: *Singular Time Delayed System Stability Theory in the sense of Lyapunov: New Results*, Nonlinear Analysis - 2007 (accepted).

[13] MORI,T., FUKUMA,N., KUWAHARA,M.: *Simple Stability Criteria for Single and Composite Linear Systems with Time Delays*, Int. J. Control Vol.34 (6) (1981) 1175–1184.

[14] OWENS,H.D., DEBELJKOVIĆ,D.LJ.: *Consistency and Liapunov Stability of Linear Descriptor Systems: A Geometric Analysis*, IMA Journal of Mathematical Control and Information, (**2**), (1985), 139-151.

[15] PANDOLFI,L.: *Controlabillity and Stabilization for Linear System of Algebraic and Differential Equations*, Jota **30** (4) (1980) 601 – 620.

[16] SARIĆ,B.: *On fundamental matrix of linear time - invariant time - delayed singular systems*, SACTA, Vol.4, No.1, (2001), 11- 25.

[17] SARIĆ,B.: *Dynamical Behavior Analysis of Linear Implicit-Singular Control Systems with delayed State on finite and Infinite Time Interval*, Ph.D., Dept. of Control Eng., Faculty of Mech. Eng., Univ. of Belgrade, July 2002.a.

[18] SARIĆ,B.: *Coditions for convergence of the fundamental matrix of linear time – invariant time - delayed singular systems*, Int. J. of Math. and Math. Science, Vol.31, No.8, (2002.b), 463 – 475.

[19] TISSIR,E., HMAMED,A.: *Further Results on Stability of* $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{x}(t-\tau)$, Automatica, Vol.32 (12) (1996), 1723–1726.

[20] XU,S., YANG,C.: *An Algebraic Approach to the Robust Stability*

Analysis and Robust Stabilization of Uncertain Singular Systems, Int. J. System Science, Vol.31, (2000.a), 55–61.

[21] XU,S., YANG,C.: H_∞ *State Feedback Control for Discrete Singular Systems*, IEEE Trans. Automat. Control AC - 45 (6) (2000.b), 1405 – 1409.

[22] XU,S., YANG,C., NIU,Y., LAM,J.: *Robust Stabilization for Uncertain Discrete Singular Systems,* Automatica, Vol.37, (2001.a), 769 – 774.

[23] XU,S., LAM,J. YANG,C.: *Quadratic Stability and Stabilization of Uncertain Linear Discrete – time Systems with State Delay*, Systems Control Lett. (43), (2001.b), 77–84.

[24] XU,S., DOOREN,P.V., STEFAN,R., LAM,J.: *Robust Stability and Stabilization for Singular Systems with State Delay and Parameter Uncertainty*, I E E E Trans. Automat. Control AC-47 (7) (2002), 1122 – 1128.

[25] XU,S., LAM,J., YANG,C.: *Robust* H_∞ *Control for Discrete Singular Systems with State Delay and Parameter Uncertainty*, Dynamics Continuous, Discrete Impulsive Systems, (2004).

Received: 15.12.2006.

Stabilnost u smislu Ljapunova diskretnih deskriptivnih sistema sa čistim vremenskim kašnjenjem

U ovom radu izvedeni su dovoljni uslovi asimptotske stabilnosti posebne klase linearnih diskretnih deskriptivnih sistema, čija se vektorska diferencijalna jednačina stanja može predstaviti u sledećem obliku: $E\mathbf{x}(k) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1)$ **a bez ikakvih ograničenja na sistemske matrice.**

Ovi novi rezultati izvedeni su na bazi Ljapunovljeve druge metode a osnovani na geometrijskom prilazu stabilnosti koji su razvile Owens, *Debeljković* (1985).
Eklatantnim primerima pokazana je primenljivost izloženih rezultata.

Ključne reči: **diskretni sistem, deskriptivni sistem, linearni sistem, stabilnost sistema, asimptotska stabilnost, metoda Ljapunova, stabilnost Ljapunova.**

Устойчивость дискретных дескриптивных со чистой временной задержкой в направлении систем Ляпунова

В настоящей работе выведены довольные условия асимптотической устойчивости особого класса линейных дискретных дескриптивных систем, чьё векториальное дифференциальное уравнение состояния возможно представить в следующей форме: $E\mathbf{x}(k) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1)$,а без любых ограничений на матрицы системы.

Эти новые результаты выведены на основе второго метода Ляпунова, а они обоснованы на геометрическом подходе устойчивости, развитом Оввенсом и Дебельковичем (1985-ого года).
Очевидными численными примерами показано применение выведеных результатов.

Ключевые слова: дискретная система, дескриптивная система, линейная система, устойчивость системы, асимптотическая устойчивость, метод Ляпунова, устойчивость Ляпунова.

La stabilité dans le sens de Lyapunov des systèmes discrets descriptifs à délai temporel pur

Dans ce papier on a donné les conditions suffisantes de la stabilité asymptotiques de classe particulière des systèmes linéaires discrets descriptifs dont l'équation différentielle vectorielle de l'état peut être représentée sous la forme suivante: $E\mathbf{x}(k) = A_0\mathbf{x}(k) + A_1\mathbf{x}(k-1)$ **et cela sans aucune limite quant aux matrices du système. Ces nouveaux résultats ont été dérivés à partir de la méthode directe de Lyapunov et sont fondés sur l'approche géometrique à la stabilité développée par Owens, Debeljković (1985). On a démontré l'applicabilité des résultats exposés par les exemples éclatants.**

Mots clés: **système discret, système descriptif, système linéaire, stabilité du système, stabilité asymptotique, méthode de Lyapunov, stabilité de Lyapunov.**