UDK: 623.463.2: 519.872
COSATI: 16-04, 16-01

Optimal Estimation of the Roll Rate of the Antitank Missile – Part I: Kalman Filter with Exact Model

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An efficient method of estimation and prediction of the roll rate of the antitank missile using Kalman filter with exact model of motion is presented for the case when the roll angle is measured by free gyro and measured data are corrupted by noise. First standard deviation of the roll rate is estimated by "two-point differencing" and then the formulae for estimation are derived using exact model of motion. By numerical simulation, the efficiency of the models is tested and function of standard deviation of the missile roll rate with respect to normalized process noise is determined. The optimal value of the process noise is determined to achieve minimal dispersion of the estimate of the roll angle and roll rate.

Key words: **antitank missile, missile roll, roll rate, angular velocity, Kalman filter, free gyro, stochastic process, numerical simulation.**

Glossary of Symbols

- $\mathbf{A}$ – matrix of the dynamic coefficients of the missile in rolling motion;
- $\mathbf{B}$ – control matrix;
- $\mathbf{F}(t,t_0)$ – transition matrix from t_0 to t, or fundamental matrix;
- $\mathbf{F}_k$ – discrete transition matrix;
- $\mathbf{G}_k$ – discrete control matrix;
- J_x – axial principal moment of inertia, [kgm^2];
- L – aerodynamic rolling moment, [Nm];
- l_0 – specific (reduced) active rolling moment, $l_0 = 1/J_x \times L(\alpha=\beta=p=q=r=0)$, [rad/s^2];
- l_p – specific (reduced) damping rolling moment, $l_p = 1/J_x \times \partial L/\partial p$, [rad/s];
- l_{dist} – specific (reduced) disturbing rolling moment, [rad/s^2];
- $\mathbf{H}$ – measurement matrix;
- $\mathbf{K}_k$ – discrete matrix of the Kalman filter gain;
- $\mathbf{M}_k$ – discrete covariance matrix representing errors of the previous estimates;
- $\mathbf{n}_k$ – discrete matrix of the measurement noise;
- n_g – number of the measured angles of roll position per one revolution of missile (gyroscope characteristic), [rad, $^\circ$];
- $\mathbf{P}_k$ – discrete covariance matrix representing errors of the states after estimates;
- p,q,r – projections of the missile angular rate on the body fixed axis system, [rad/s];
- $\mathbf{Q}_k$ – discrete covariance matrix of the process noise;
- $\bar{q}^{(3)}$ – power spectral density of the process noise of the derivative of the angular acceleration, [rad^2/s^5];
- $S_l = \sigma_l^2$ – variance of the random component of the rolling moment (angular acceleration), $\sqrt{S_l} = \sqrt{\bar{q}_{\Phi}^{(3)} T_k}$;
- $\mathbf{S}(t)$ – continual matrix of the process noise;
- S_{Φ} – power spectral density of the measurement noise of the missile angle of roll, [rad^2/(1/s)];
- $\mathbf{R}_k$ – discrete covariance matrix of measurement noise;
- T_k, T_0 – sampling interval of the roll angle at time t_k and at time $t=0$, [s];
- t – time, [s];
- $\mathbf{u}$ – control matrix (column);
- $\bar{\mathbf{w}}$ – process noise matrix;
- $\mathbf{w}_k$ – discrete process noise matrix
- $\mathbf{x}$ – space state matrix (column);
- $\mathbf{x}_k$ – discrete space state matrix (column);
- $\tilde{\mathbf{x}}_k$ – discrete estimate of space state matrix based on the mathematical model (column);
- $\hat{\mathbf{x}}_k$ – discrete estimate of space state matrix based on Kalman filter (column);
- $\mathbf{x}_k^*$ – discrete measured space state matrix (column);
- $\mathbf{x}_0$ – initial discrete space state matrix (column);
- Φ – missile roll angle, [rad, $^\circ$];
- $\Delta\Phi_g$ – sampling interval of the missile roll angle (gyroscope characteristic), [rad, $^\circ$];
- σ_p – standard deviation of the roll rate [rad/s];
- σ_{Φ} – standard deviation of the roll angle (gyroscope characteristic), [rad, $^\circ$];

Lower indexes

- 0 – initial state;
- k – values at instant t_k;

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Upper indexes

T	– matrix transposition;
.	– time derivative;
^	– estimated value based on Kalman filter;
~	– estimated value based on the mathematical model;
*	– measured quantity, reduced – non dimensional quantity;

Introduction

ONE of the basic factors influencing the precession of a guided antitank missile with one channel pulse modulation control system, with aerodynamic fin or thrust vector (MALJUKA, HOT, MILAN, FAGOT), is the accuracy of prediction of the time of actuation of the control device from equilibrium position. For this reason, accurate estimation and prediction of the roll angle is necessary. In order to estimate the roll angle, estimation of the roll angular rate is necessary, [1]

Angular rate p, (Fig.1), is changed during the flight for two main reasons. Firstly, the equilibrium angular rate is changed due to the change of the missile speed. Secondly, due to the missile movement around the center of mass, flight parameters are changed. In order to measure the angle of roll, the free gyroscope is built into the missile. Measurement is done discretely with some measurement error.

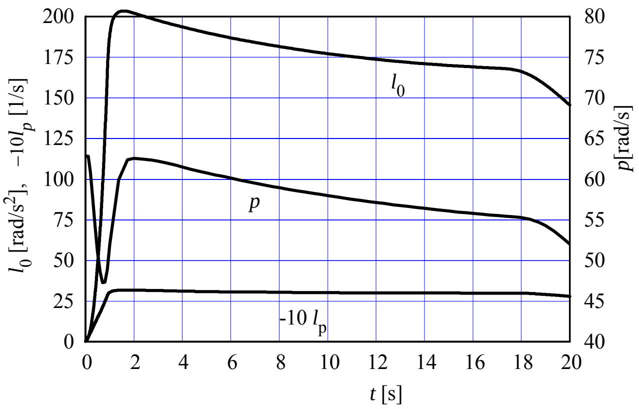

Figure 1. Roll angular rate and dynamic coefficients for a hypothetical missile [1]

Roll angle sensor is an encoder consisting of a disk fixed to outer gyroscope frame which is being rotated. On the perimeter of the disk there are n_g radial slots, and on the missile rolling body there is a photo diode and light source. The light, while passing through slots, produces current impulse based on which the time of passing the slot on the disk through referent point is determined. Accordingly, after each angle $k \times \Delta\Phi_g$, time t_k is measured, where $\Delta\Phi_g = 2\pi / n_g = \text{const}$. For the purpose of this work initial value $n_g = 8$ is assumed for the analysis, so that $\Delta\Phi_g = \pi/8$, [1].

It is known that optimal estimate of the system states can be obtained by applying Kalman filter, assuming that a necessary condition is fulfilled [2, 4, 7].

In this work, methods for estimating angular rate are considered in order to obtain a condition for which, when fulfilled, the estimates of angular rates have minimal value of standard deviations.

Estimate of the angular rate based on finite difference method of the first order

Angular rate, necessary to calculate angles and time of actuation of the control device, can be determined based on the measured values of the roll angle using finite difference method of the first order ("two-point differencing")

$$p_k = \frac{\Phi_k - \Phi_{k-1}}{T_k} = \frac{\Delta\Phi_k}{T_k} \tag{1}$$

where: Φ_k and Φ_{k-1} – measured values of angles at moment t_k and t_{k-1}, and $T_k = t_k - t_{k-1}$ – time interval between two successive measurements of the angles Φ_k. Roll angle Φ is measured by free gyroscope with a measurement error. When angular rate is constant, error of determining angular rate increases only due to measurement error of the roll angle. But, in this article, for the missile model, which is used, angular rate is variable (Figures 1 and 2), so sampling time interval T_k is a function of the time of flight. For the purpose of this work all the necessary data is calculated for hypothetical missile by using program `SimDVPTR`, [8].

Let it be assumed that the measurement error is an uncorrelated quantity, and that its probable density error obeys Gauss' law with null mathematical expectation and standard deviation σ_Φ. Further, it will be assumed that the measurement error of the roll angle is small compared to the measured angle itself, so from formulae (1) error of determining angular rate and its dispersion is

$$\sigma_p^2 = \sigma^2(p_k) = \frac{1}{\left(T_k^2\right)^2}\left[\sigma^2(\Phi_k) + \sigma^2(\Phi_{k-1})\right] = 2\frac{\sigma_\Phi^2}{T_k^2} \tag{2}$$

where σ_Φ^2 – dispersion of the measured roll angle by free gyro.

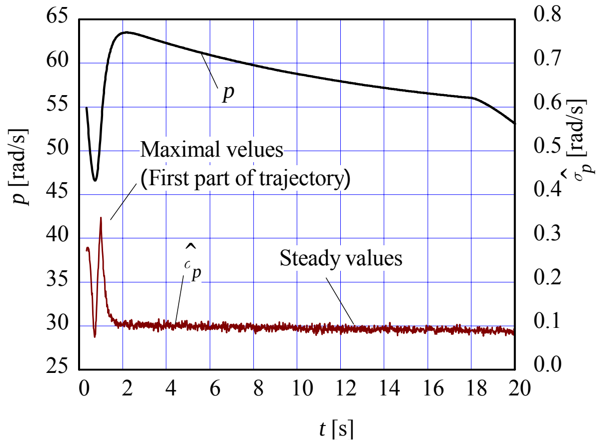

Figure 2. Estimate of the root mean square deviation of the roll angular rate $\hat{\sigma}_p$ obtained by "two-point differencing" as a function of time, $\sigma_\Phi = 0.87$ mrad .

In the preceding analysis it was assumed that dispersion of the measured angle is same in time.

The error of estimation of angular rate can be reduced by increasing the order of numerical algorithm. This analysis will not be done here. However, the conclusion of using such an approach will be given. With increasing the order of algorithm, error is decreased due to the applied method (process noise is reduced), but error due to measurement is increased.

Process noise

Differential equations of missile rolling motion in body fixed coordinate system are [1, 5, 6]:

$$\dot{p} = \frac{1}{J_x}\left(L_0 + L_{\alpha\beta}\alpha\beta + L_{\beta\delta_m}\beta\delta_m + L_p p + L_{\text{dist}}\right) \quad (3)$$

$$\dot{\Phi} = p + \left(q\sin\Phi + r\cos\Phi\right)\tan\Theta$$

where p – roll rate, r – yaw rate, q – pitch rate, Φ – roll angle, Θ – inclination angle, J_x – longitudinal principal mass moments of inertia and L_0 – aerodynamic rolling moment due to fin bank angle, L_p – derivative of rolling moment with respect to the roll rate, L_{dist} – disturbing rolling moment, α – angle of attack, β – angle of sideslip, δ_m – angle of deflection of aerodynamic control surface. Quantities L_0, L_p and L_{dist} depend on kinematic parameters α, β, δ_m and Mach number Ma.

It can be seen that equations of rolling motion (3) are nonlinear functions of the kinematic quantities of motion. They can be solved exactly only by solving numerically the complete system of equations. This is not acceptable for the estimation purpose for two reasons. Firstly, the equations are highly nonlinear and complex to be solved in real time, and secondly, disturbing moments which are stochastic functions of time always act on the missile. It is known that the disturbing moment can arise due to non steady nonlinear aerodynamic phenomena and due to the reactive moments of the rocket motor. Bearing this in mind, differential equations of the rolling motion for the estimation purposes will be written in the following form:

$$\dot{\Phi} = p \quad (4)$$

$$\ddot{\Phi} = \dot{p} = l_0(Ma) + l_p(Ma)\,p + l_{\text{dist}}(Ma,\alpha,\beta,\delta_m,...)$$

where l_0 and l_p specific (reduced) active and damping rolling moment and $l_{\text{dist}}(Ma,\alpha,\beta,\delta_m,...)$ – specific disturbing rolling moment which depends on kinematical parameters of the missile movement. It is a random function of time. Statistics of this quantity are not known, but it can be determined by detailed simulation and testing in flight. For the purpose of this work it will be assumed that the disturbing moment is Gaussian white noise with null mathematical expectation. In Fig.1 dependence of coefficients l_0 and l_p on time of flight is shown for a hypothetical missile [1].

Model equations and Kalman filter equations

Eventually, based on expression (4), equations of rolling motion (process equations) can be written in the following form

$$\dot{\Phi} = p \quad (5)$$

$$\dot{p} = l_0 + l_p p + \bar{w}_l$$

where $\bar{w}_l = l_{\text{dist}}$ is disturbing specific moment, or process noise.

If state vector is introduced

$$\mathbf{x} = \left[\Phi \quad p\right]^T, \quad (6)$$

where $x_1 = \Phi$, $x_2 = p$, equations (5) can be written in the following form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & l_p\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ l_0\end{bmatrix} + \begin{bmatrix}0\\ \bar{w}_l\end{bmatrix} \quad (7)$$

Or in matrix notation

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{u}(t) + \bar{\mathbf{w}}(t) \quad (8)$$

where: matrix of dynamic coefficient is

$$\mathbf{A} = \begin{bmatrix}0 & 1\\ 0 & l_p\end{bmatrix}, \quad (9)$$

control vector

$$\mathbf{u} = \left[0 \quad l_0\right]^T, \quad (10)$$

and process noise matrix.

$$\bar{\mathbf{w}} = \left[0 \quad \bar{w}_l\right]^T \quad (11)$$

Continual covariance matrix of the process noise $\mathbf{S}(t)$ is defined by

$$\mathbf{E}\left[\bar{\mathbf{w}}(t)\bar{\mathbf{w}}^T(\tau)\right] = \mathbf{S}(t)\delta(t-\tau) \quad (12)$$

By numerical solution of state equations (8) on the time interval $t_0 = 0$ to t_k for initial state vector

$$\mathbf{x}(0) = \mathbf{x}_0 = \left[0 \quad p_0\right]^T \quad (13)$$

Exact solution of the state vector $\mathbf{x}(t)$ can be obtained. In this work fourth order Runge-Kutta method is used while integration step is chosen from conditions to achieve the required accuracy of solution.

In order to estimate the state variables by Kalman filter in real time, it is necessary to determine discrete state equations. To achieve this, the starting point will be the general solution of the differential equation (8) given in the form:

$$\mathbf{x}(t) = \mathbf{F}(t,t_0)\mathbf{x}(t_0) + \int_{t_0}^{t}\mathbf{F}(t,\tau)\left[\mathbf{u}(\tau) + \bar{\mathbf{w}}(\tau)\right]d\tau \quad (14)$$

where is $\mathbf{x}(t_0) = \mathbf{x}_0$ initial state vector, and $\mathbf{F}(t,t_0)$ transition matrix from t_0 to t, or fundamental matrix. It is given by the following expression:

$$\mathbf{F}(t,t_0) = e^{\int_{t_0}^{t}\mathbf{A}(\tau)d\tau} \quad (15)$$

Upon applying discretization procedure [1], equation (14) becomes discrete, unlike the state equation

$$\mathbf{x}_k = \mathbf{F}_{k-1}\mathbf{x}_{k-1} + \mathbf{G}_{k-1}\mathbf{u}_{k-1} + \mathbf{w}_{k-1} \quad (16)$$

Matrices $\mathbf{F}_k$ and $\mathbf{G}_k$ are calculated assuming that matrices of dynamic coefficients l_0 and l_p do not change much in time of discretization.

Fundamental matrix $\mathbf{F}_k$ can be determined by numerical integration of matrix differential equation (8) for $\mathbf{u} = \overline{\mathbf{w}} = 0$, or by expanding matrix exponential function (15) in series and taking specific numbers of terms:

$$\mathbf{F}_k = e^{\int_0^{T_k} e^{\mathbf{A}} d\tau} \approx \left[\mathbf{F}(\tau)\right]_{\tau=T_k} = \\ = \left[\mathbf{I} + \mathbf{A}\tau + \frac{\mathbf{A}^2\tau^2}{2} + ...\right]_{\tau=T_k} = e^{\mathbf{A}(t_k)T_k} \tag{17}$$

Meanwhile, based on the assumption that on sampling interval coefficients l_0 and l_p do not change significantly, so that they can be considered constant, system of equations (7) can be integrated for $\overline{w}_l = 0$, and close form solution obtained:

$$\Phi = \Phi_0 - \frac{l_0}{l_p}t - \frac{l_0}{l_p^2}\left(1 - e^{l_p t}\right) - \frac{p_0}{l_p}\left(1 - e^{-l_p t}\right) \tag{18}$$

$$p = p_0 e^{l_p t} - \frac{l_0}{l_p}\left(1 - e^{l_p t}\right)$$

where p_0 and Φ_0 initial angular rate and initial roll angle respectively. When applied on the integration interval $t_k - t_{k-1} = T_k$ ($t = T_k$, $p_0 = p_{k-1}$, $p = p_k$, $\Phi_0 = \Phi_{k-1}$, $\Phi = \Phi_k$) expression (18) becomes

$$\Phi_k = \Phi_{k-1} - \frac{l_{0.k-1}}{l_{p.k-1}}T_k - \frac{l_{0.k-1}}{l_{p.k-1}^2}\left(1 - e^{l_{p.k-1}T_k}\right) - \\ - \frac{p_{k-1}}{l_{p.k-1}}\left(1 - e^{-l_{p.k-1}T_k}\right) \tag{19}$$

$$p_k = p_{k-1}e^{l_{p.k-1}T_k} - \frac{l_{0.k-1}}{l_{p.k-1}}\left(1 - e^{l_{p.k-1}T_k}\right)$$

From these equations it is easy to obtain matrices $\mathbf{F}_k$ and $\mathbf{G}_k$

$$\mathbf{F}_k = \begin{bmatrix} 1 & -\left(1/l_{p.k-1}\right)\left(1 - e^{-l_{p.k-1}T_k}\right) \\ 0 & e^{l_{p.k-1}T_k} \end{bmatrix} \tag{20}$$

$$\mathbf{G}_k = \begin{bmatrix} 0 & -\left[1/l_{p.k-1}\,T_k + 1/l_{p.k-1}^2\left(1 - e^{l_{p.k-1}T_k}\right)\right] \\ 0 & -\left(1/l_{p.k-1}\right)\left(1 - e^{-l_{p.k-1}T_k}\right) \end{bmatrix} \tag{21}$$

Expressions (19) are used for simulation of the roll angle and roll rate – for the calculation of "true" values. However, for the estimation of the states in real time it is better suited to use simpler approximate expressions obtained when function $e^{l_p t}$ is expanded in to series. This is justified because $e^{l_p t}$ is a small number compared to unity. If only linear terms are taken in the second row in the approximate polynomial for $\mathbf{F}_k$ and $\mathbf{G}_k$ in (20-21) and quadratic terms are neglected, matrices $\mathbf{F}_k$ and $\mathbf{G}_k$ become

$$\mathbf{F}_k \approx \begin{bmatrix} 1 & T_k + 0.5l_{p.k-1}T_k^2 \\ 0 & 1 + l_{p.k-1}T_k \end{bmatrix} \tag{22}$$

$$\mathbf{G}_k \approx \begin{bmatrix} 0 & 0.5T_k^2 \\ 0 & T_k \end{bmatrix} \tag{23}$$

With these matrices, discrete difference equation (16) becomes

$$\begin{bmatrix} \Phi_k \\ p_k \end{bmatrix} = \begin{bmatrix} 1 & T_k + 0.5l_{p.k-1}T_k^2 \\ 0 & 1 + l_{p.k-1}T_k \end{bmatrix}\begin{bmatrix} \Phi_{k-1} \\ p_{k-1} \end{bmatrix} + \\ = \begin{bmatrix} 0 & 0.5T_k^2 \\ 0 & T_k \end{bmatrix}\begin{bmatrix} 0 \\ l_{0.k-1} + w_{l.k-1} \end{bmatrix} \tag{24}$$

or in a developed form

$$\Phi_k = \Phi_{k-1} + \left(T_k + 0.5l_{p.k-1}T_k^2\right)p_{k-1} + \\ + 0.5T_k^2\left(l_{0.k-1} + w_{l.k-1}\right) \tag{25}$$

$$p_k = \left(1 + l_{p.k-1}T_k\right)p_{k-1} + T_k\left(l_{0.k-1} + w_{l.k-1}\right)$$

Discrete covariance matrix of the process noise is

$$\mathbf{Q}_k = \int_0^{T_k} \mathbf{F}(\tau)\mathbf{S}(\tau)\,\mathbf{F}^T(\tau)\,d\tau \tag{26}$$

where covariance matrix of the process noise is

$$\mathbf{S}(t) = \begin{bmatrix} 0 & 0 \\ 0 & S_l \end{bmatrix}, \tag{27}$$

and S_l – dispersion of the random component of the rolling moment.

Upon substitution of $\mathbf{F}(\tau)$ defined by equation (22) into expression (26) for $T_k = \tau$ and $\mathbf{F}^T(\tau)$, obtained by transposition of $\mathbf{F}$ and matrix of process noise defined by (27), $\mathbf{Q}_k$ becomes

$$\mathbf{Q}_k = \int_0^{T_k} \begin{bmatrix} 1 & \tau + 0.5l_p\tau^2 \\ 0 & 1 + l_p\tau \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & S_l \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \tau + 0.5l_p\tau^2 & 1 + l_p\tau \end{bmatrix}d\tau = \\ = S_l\begin{bmatrix} (1/5)l_{p.k-1}^2T_k^5 + (1/2)l_{p.k-1}T_k^4 + (1/3)T_k^3 & (1/4)l_{p.k-1}^2T_k^4 + (2/3)l_{p.k-1}T_k^3 + (1/2)T_k^2 \\ (1/4)l_{p.k-1}^2T_k^4 + (2/3)l_{p.k-1}T_k^3 + (1/2)T_k^2 & (1/3)l_{p.k-1}^2T_k^3 + l_{p.k-1}T_k^2 + T_k \end{bmatrix} \tag{28}$$

Every T_k seconds Φ_k is measured with error n_k, so discrete measurement equation is

$$\Phi_k^* = \Phi_k + n_k \tag{29}$$

where Φ_k^* is the measured value of Φ_k. It is assumed that n_k is Gaussian white noise.

Upon introduction of column matrix of measurement values

$$\mathbf{x}_k^* = \begin{bmatrix} \Phi_k^* \\ 0 \end{bmatrix} = \begin{bmatrix} \Phi_k^* & 0 \end{bmatrix}^T \tag{30}$$

and column matrix of the process noise

$$\mathbf{n}_k = \begin{bmatrix} n_k \\ 0 \end{bmatrix} = \begin{bmatrix} n_k & 0 \end{bmatrix}^T \tag{31}$$

discrete state measurement matrix is obtained

$$\mathbf{x}_k^* = \mathbf{H}\mathbf{x}_k + \mathbf{n}_k, \qquad k = 1,2,... \tag{32}$$

where the measurement matrix is

$$\mathbf{H} = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{33}$$

Covariance matrix of the measurement noise is

$$\mathbf{R}_k = \mathbf{E}\left[\mathbf{n}_k\mathbf{n}_k^T\right] = \begin{bmatrix} \sigma_\Phi^2 & 0 \\ 0 & 0 \end{bmatrix} \tag{34}$$

The Riccati equations which define Kalman filter gains are:

$$\mathbf{M}_k = \mathbf{F}_k\mathbf{P}_{k-1}\mathbf{F}_k^T + \mathbf{Q}_k$$

$$\mathbf{K}_k = \mathbf{M}_k\mathbf{H}^T\left(\mathbf{H}\mathbf{M}_k\mathbf{H}^T + \mathbf{R}_k\right) \tag{35}$$

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H})\mathbf{M}_k$$

By solving these equations the matrix of gains of the Kalman filter $\mathbf{K}_k$ and discrete matrix of the variances of differences between the estimated and true values of states are obtained, while estimated values of the states are obtained by Kalman filter as follows

$$\hat{\mathbf{x}}_k = \tilde{\mathbf{x}}_k + \mathbf{K}_k\left(\mathbf{x}_k^* - \tilde{\mathbf{x}}_k\right) \tag{36}$$

In eq. (36) $\tilde{\mathbf{x}}_k$ is a discrete two-dimensional vector of the estimates of the states based on the mathematical model

$$\tilde{\mathbf{x}}_k = \mathbf{F}_{k-1}\hat{\mathbf{x}}_{k-1} + \mathbf{G}_{k-1}\mathbf{u}_{k-1} \tag{37}$$

Since only one state variable of the roll angle Φ, is measured, covariance matrix of the measurement noise is scalar $\mathbf{R}_k = R_k = \sigma_\Phi^2$, and Riccati equations can be simplified. When products of matrices in the second and third Riccati equations are expanded, the following can be obtained

$$\mathbf{M}\mathbf{H}^T = \begin{bmatrix} M_{11} & M_{21} \end{bmatrix}^T, \ \mathbf{H}\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \end{bmatrix}, \qquad \mathbf{H}\mathbf{M}\mathbf{H}^T = M_{11} \tag{38}$$

$$\left(\mathbf{H}\mathbf{M}_k\mathbf{H}^T + \mathbf{R}_k\right)^{-1} = \left(M_{11} + \sigma_\Phi^2\right)^{-1}$$

In the expressions above, the subscript “k” is omitted to simplify writing.

Upon substitution of (38) into (35) for $\mathbf{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}^T$ it results in:

$$K_1 = K_\Phi = \frac{M_{11}}{M_{11} + \sigma_\Phi^2}, \quad K_2 = K_p = \frac{M_{21}}{M_{11} + \sigma_\Phi^2} \tag{39}$$

Coefficients M_{11} and M_{12} are members of the first column of the matrix **M**. Further, it is

$$(\mathbf{I} - \mathbf{K}\mathbf{H}) = \begin{bmatrix} (1-K_1) & 0 \\ -K_2 & 1 \end{bmatrix} \tag{40}$$

The third Riccati equation if expanded has the form:

$$\mathbf{P} = \begin{bmatrix} (1-K_1)M_{11} & (1-K_1)M_{12} \\ -K_2M_{11} + M_{12} & -K_2M_{12} + M_{22} \end{bmatrix} \tag{41}$$

Note that matrices **M** and **P** are symmetrical.

In the expression for Kalman filter (36) only two gains which are different from zero on each step of correction of estimates of the states in the gain matrix exist. These gains do not depend on the estimated values of states since fundamental matrix, which goes into Riccati equations (35) does not depend on the estimated values of states, but only on dynamic coefficients and the sampling time.

Simulation of the measured values of the roll angle

By testing gyroscopes in laboratory and by measurement in flight, characteristics of gyroscopes can be obtained. For the purpose of this work, standard deviation of measurement of the roll angle by free gyroscope will be taken $\sigma_\Phi = 0.05° = 0.87\,\text{mrad}$, while mathematical expectation is assumed to be zero. Based on this assumption, the measured noise is simulated by pseudo-normal distribution of random numbers $n_k \sim \mathcal{N}\left(0, \sigma_\Phi^2\right)$.

Based on the calculation and measurement in flight mean value of the initial angular rate and estimate of its standard deviation is determined for the missile in question: $p_0 = 62.8\,\text{rad/s}$, $\sigma_{p0} = 3\,\text{rad/s}$, [1]. Note that these values correspond to normal temperature of propellant of the rocket motor. For the initial value of the roll angle $\Phi_0 = 0$ is taken. With these values, the measured values of the roll angles during flight are simulated by using program `SimDVPTR`, [8]. Furthermore, the roll rate is simulated, as well as estimates of root mean deviation of the roll rate $\hat{\sigma}_p$ obtained by “two-point differencing” in function of time. It is shown in Fig.2 along with the simulated roll rate. For the simulation by Monte-Karlo method [3] 200 realizations of the trajectory were used. From Fig.2 it can be seen that values of $\hat{\sigma}_p$ are several times greater in the vicinity of high changes of the roll rate compared to intervals where the roll rate is changed monotonously.

Initialization of parameters of the Kalman filter

For initialization of Kalman filter, it is necessary to know initial values of variances of errors of the states. Discrete matrix of variances of the differences between the estimated and true values of states $\mathbf{P}_k$ (variance of error of estimate of the states after the correction of states) is

$$\mathbf{P}_k = \mathbf{E}\left[\left(\mathbf{x}_k - \hat{\mathbf{x}}_k\right)\left(\mathbf{x}_k - \hat{\mathbf{x}}_k\right)^T\right] = \mathbf{E}\left[\begin{array}{c|c} \left(\Phi_k - \hat{\Phi}_k\right)^2 & \left(\Phi_k - \hat{\Phi}_k\right)\left(p_k - \hat{p}_k\right) \\ \hline \left(\Phi_k - \hat{\Phi}_k\right)\left(p_k - \hat{p}_k\right) & \left(p_k - \hat{p}_k\right)^2 \end{array}\right] \tag{42}$$

Suppose that there is no process noise in the system. For determining the value of the matrix $\mathbf{P}_k$ in the first instant by using expression (42) for the estimation of the roll angle $\left(\hat{\Phi}_k\right)_{k=1}$ the measured values of the angle – Φ_k^* can be used, and for the estimation of the roll rate values which are obtained by “two-point differencing” according to expression (1). Statistics of these values are determined in [1], so covariance matrix on the first step of iteration is

$$(\mathbf{P}_0)_{S_l=0}=\begin{bmatrix}\sigma_\Phi^2 & \sigma_\Phi^2/T_0\\ \sigma_\Phi^2/T_0 & 2\sigma_\Phi^2/T_0^2\end{bmatrix} \tag{43}$$

where T_0 is sampling time for $k=0$, e.g. $T_0=t_1-t_0$. For the chosen values of the parameters $\left(\sigma_\Phi=0.05^\circ=0.87\,\text{mrad},\ T_0=0.0126\,\text{s}\right)$ the following is obtained:

$$(\mathbf{P}_0)_{S_l=0}=\sigma_\Phi^2\begin{bmatrix}1 & 1/T_0\\ 1/T_0 & 2/T_0^2\end{bmatrix}=7.5\times10^{-7}\begin{bmatrix}1 & 80\\ 80 & 12800\end{bmatrix} \tag{44}$$

When simulation of the measurement is done with nominal values of the dynamic coefficients $(S_l=0)$, with measurement noise as it is given in the previous section and initialization of matrix $\mathbf{P}_k$ done with values given by (44), coefficients of Kalman filter converge to zero very fast and the filter gives true values of the roll angle. This case is an idealization and serves only for testing purposes.

As previously said, in real flight dynamic coefficients diverge from nominal values due to various reasons. Therefore, for the initialization of matrix $\mathbf{P}_0$ it is necessary to take dispersion of the states which occurs due to the process noise. This influence is expressed through matrix $\mathbf{Q}_k$ given by formula (28) applied at initial instant of time

According to that

$$\mathbf{Q}_0=S_l\left[\begin{array}{c|c}(1/5)l_{p.0}^2T_0^5+(1/2)l_{p.0}T_0^4+(1/3)T_0^3 & (1/4)l_{p.0}^2T_0^4+(2/3)l_{p.0}T_0^3+(1/2)T_0^2\\ \hline (1/4)l_{p.0}^2T_0^4+(2/3)l_{p.0}T_0^3+(1/2)T_0^2 & (1/3)l_{p.0}^2T_0^3+l_{p.0}T_0^2+T_0\end{array}\right] \tag{45}$$

$$\mathbf{P}_0=(\mathbf{P}_0)_{S_l=0}+\mathbf{Q}_0 \tag{46}$$

For the estimation of states – roll angle and roll rate, by using Kalman filter, it is necessary to know the values of dynamic coefficients $l_0(t)$ и $l_p(t)$ (Fig.1). In real flight, dynamic coefficients always diverge from nominal, which means that they are not known on trajectory, i.e. there exists the process noise. Numerical simulation shows that deviation of $l_0(t)$ coefficient can be of the order of a few up to ten percents. Therefore, in order to estimate states by Kalman filter, nominal values of dynamic coefficients should be used, while for the simulation of the measured values of the states no nominal values of dynamic coefficient $l_0(t)$ is used

$$l_0(t)+\bar{w}_l(t) \tag{47}$$

where quantity $\bar{w}_l$ is the process noise, which is used for modeling the unknown component of the rolling moment. It is assumed that it is white noise with zero mathematical expectation.

$$E\left[\bar{w}_l(t)\bar{w}_l^T(\tau)\right]=S(t)\delta(t-\tau) \tag{48}$$

Consequently, for the simulation of real values of states by using program `SimDVPTR` value $S_l=(0.05\times l_0)^2$ is used. For the estimation of states nominal value of the specific moment $l_0(t)$ and variance $S_l=(0.05\times l_0)^2$ are used.

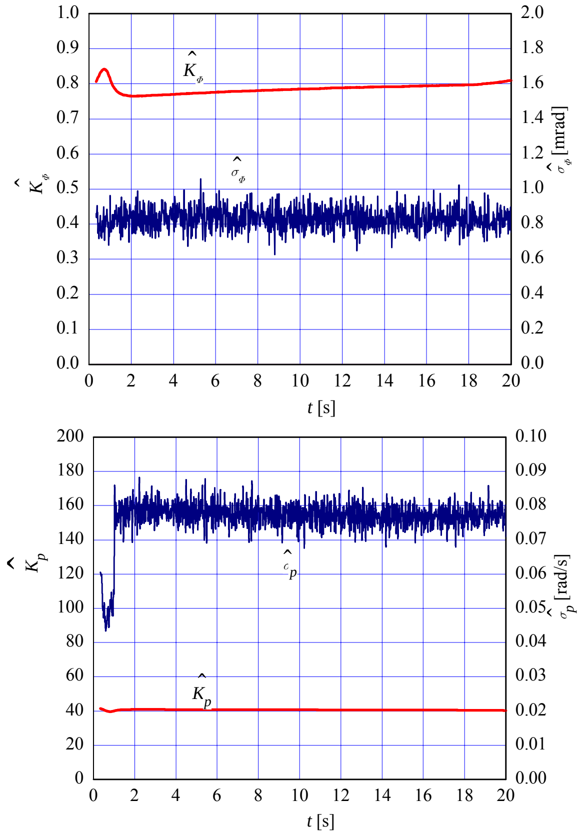

Figure 3. Estimates of standard deviations and coefficients of Kalman filter with exact model; σ_Φ= 0.87 mrad, m_l= 0, $S_l=(0.05\times l_0)^2$.

In Fig.3 results of calculation obtained by using Kalman filter with formulas derived in this section are shown. It can be seen that by using the filters, roll angle and roll rate are well estimated. This was expected because all necessary conditions concerning the measured and process noise for applying Kalman filter are fulfilled.

Deviation of the real rolling moment from nominal one in flight usually obeys Gaussian law in statistical sense. For e.g., due to deviation of the propellant temperature of the rocket motor, or temperature of air from nominal values, the speed of flight will not be equal to nominal value, but will deviate from nominal value by some deterministic value different from zero. This value can be up to 20% of the nominal value for extreme values of temperature. The question is whether in this case Kalman filter can be used for estimation of the states, because the process noise is not with null mathematical exception. Answer to that question will be given in the following analysis.

For the simulation of real values of the states by using program `SimDVPTR` it will be taken

$$l_0(t)=\left[l_0(t)\right]_{nom}+m_l+\bar{w}_l=(1+0.05)\left[l_0(t)\right]_{nom}+\bar{w}_l,$$

$$\bar{w}_l\sim\mathcal{N}(0,S_l),$$

meaning that the dynamic coefficient in flight is 5% greater than nominal. For covariance of random quantity $\bar{w}_l$ $S_l=(0.05\times l_0)^2$ will be considered.

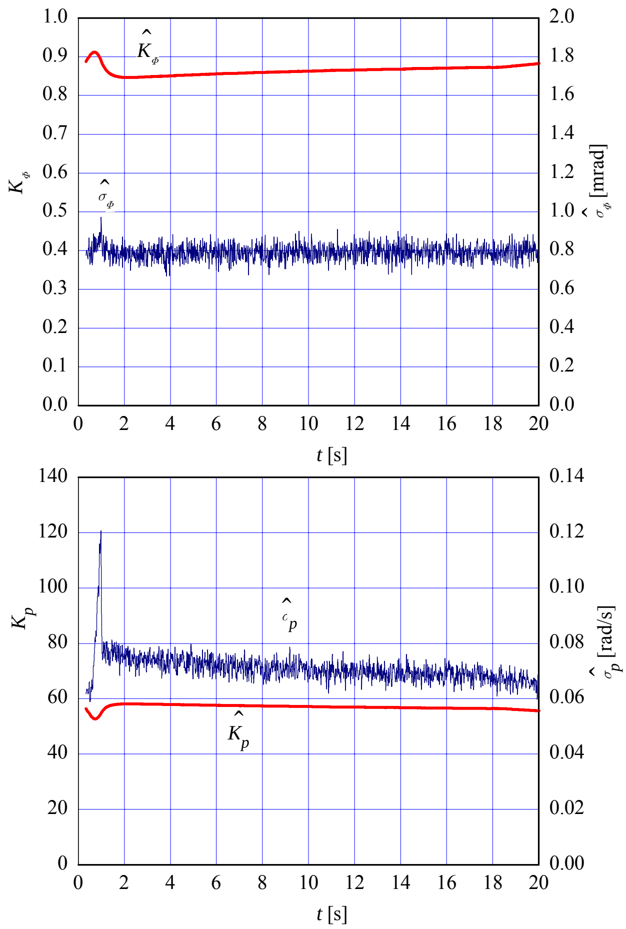

Figure 4. Estimates of the standard deviations and coefficients of the Kalman filter. σ_Φ= 0.87 mrad, m_l= 0, S_l= $(0.05 \times l_0)^2$.

For estimation of the nominal state value of specific moment $l_0(t)$ variance $S_l = (0.05 \times l_0)^2$ is taken, so the same value is used in the simulation.

In Fig.4 results of simulation are shown. It can be seen that estimates of the roll angle and roll rate are good, except in the vicinity of minimum value of $p(t)$ which corresponds the time instant $t \approx 0.4\,\mathrm{s}$. Close to this point standard deviation of the roll rate has maximal value, which is two times greater than that of the steady state value, but it is approximately equal to the third value of that, which is obtained by "two-point differencing" for the same instant of time (Fig.2). Based on that, it can be concluded that for the estimation of the roll angle and roll rate Kalman filter with exact model can be used even in case when the process noise has not got zero mean value. Filter gains and variances after the points of great changes of the roll rate converge to steady value very fast.

Analysis of the applied methods

In Table 1 values of root mean square of deviations of the estimated values of roll the angle and roll rates with respect to the true values are presented. The calculation is done by program which uses Monte-Carlo simulation with 200 generated trajectories for testing.

In the analyses measurement noise is simulated by normal distribution of random numbers $\mathcal{N}\left(0,\sigma_\Phi^2\right)$, and process noise by normal distribution of random numbers $\mathcal{N}\left(m_l,\sigma_l^2\right)$, where $\sigma_\Phi = 0.05^\circ = 0.87\,\mathrm{mrad}$, $m_l = 0.05 l_0$, $\sigma_l = 0.05 l_0$. For estimation, the power spectral densities which correspond the standard deviation of the simulated values were used; that is $R = \sigma_\Phi^2$, $S_l = \sigma_l^2$.

Table 1. Estimated values for $S_l = \sigma_l^2 = (0.05 l_0)^2$

No.	ESTIMATION METHOD	ROOT MEAN SQUARE DEVIATION WITH RESPECT TO TRUE VALUE			
		Roll angle [mrad]		Roll rate [rad/s]	
		Maximal value	Steady value	Maximal value	Steady value
1.	"Two-point differencing"	0.87	0.87	0.35	
2.	Kalman filter with exact model	0.90	0.77	0.120	0.069

Analysis of the results in the Table show that by using Kalman filter with exact model root mean square deviation of the roll rate three times less than values obtained by numerical "two-point differencing" is obtained.

Up to this point analysis was made with the same values of power spectral density of the random moment S_l to that which is used for simulation of the measurement roll angle Φ. Meanwhile, for the estimation of states variables in real time, power spectral density of the random component of the rolling moment is not known. It can be determined by simulating and analyzing of experimental values. For that, the influence of S_l on the estimation of states will be analyzed.

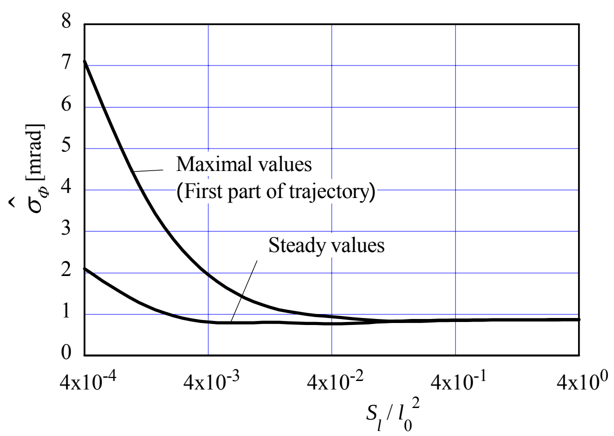

Figure 5. Estimates of rms deviations of the roll angle in function of S_l / l_0^2 : σ_Φ = 0.87 mrad .

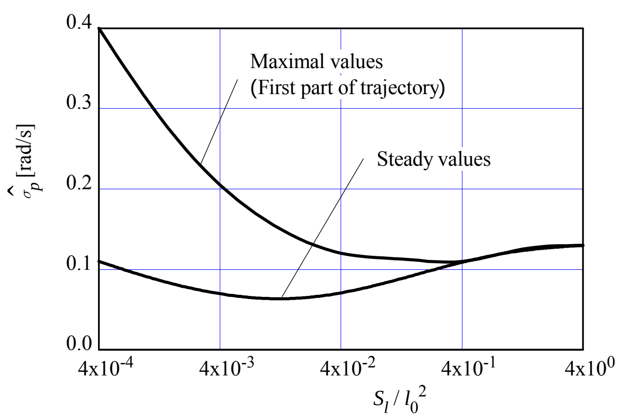

Figure 6. Estimates of rms deviations of the roll rate in function of S_l / l_0^2 : σ_Φ = 0.87 mrad .

In Fig.5 estimate of standard deviation of the roll angle obtained by Kalman filter with exact mathematical model against reduced variance of the process noise S_l/l_0^2 is shown.

With the increase of the parameter of the process noise S_l/l_0^2 maximal values of the estimates converge to steady values and for $S_l/l_0^2 > 5\times10^{-2}$ they are practically equal to the steady values. Steady values have the minimum of $\hat{\sigma}_{\Phi} \approx 0.77\,\text{mrad}$ for $S_l/l_0^2 \approx 0.02$, so, 11% less than from the value used for simulation ($\sigma_{\Phi} \approx 0.87\,\text{mrad}$). Note that for value $S_l/l_0^2 \approx 0.02$, where steady values have minimum, maximal values of the estimates are twice the minimal values (Fig.5).

Standard deviation of the estimate of the roll rate by Kalman filter with exact mathematical model with respect to the true roll rate against the reduced process noise S_l/l_0^2 is shown in Fig.6 for maximal values, which as it was explained, appears at the initial part of the trajectory, and for steady values, which are associated with the second part of the trajectory. It can be seen that curve has a minimum of $\hat{\sigma}_p = 0.061\,\text{rad/s}$ for $S_l/l_0^2 \approx 1.2\times10^{-3}$.

When the roll angular rate is determined along the trajectory, the estimate of the roll angular acceleration, which is necessary for the estimation of the roll angle, can be obtained by "two-point differencing" as follows

$$\dot{\hat{p}}_k = \frac{\hat{p}_k - \hat{p}_{k-1}}{\Delta t_k} = \frac{\Delta\hat{p}_k}{\Delta t_k} \tag{49}$$

Conclusion

The problem of efficient estimation of the roll angle and roll rate is considered for the antitank guided missile for which the angle is measured by free gyro with the presence of measurement noise. Expression for the estimation of the standard deviation of the roll rate is derived for Kalman filter with exact mathematical model and by numerical differentiation using first order finite difference method ("two-point differencing").

Efficiency of the proposed method is analyzed by numerical simulation. Numerical simulation is used to obtain the estimate of the standard deviation of the roll rate with respect to the reduced value of the process noise. In order to achieve minimal dispersion of the estimate of the roll angle and roll rate the optimal value of the process noise is determined. It has been shown that there exists minimum of the estimate of standard deviation for some value of the process noise. Three times less estimate of standard deviation of the roll rate in the transition process (maximal values) and about ten times smaller steady value compared to the values obtained by numerical differentiation using first order finite difference method ("two-point differencing") is achieved applying Kalman filter.

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Appendix

Assuming that measurement noise is with zero expected value it can be obtained:

$$\mathbf{E}\left(\Phi_k - \hat{\Phi}_k\right) = \mathbf{E}\left(\Phi_k - \Phi_k^*\right) = \mathbf{E}\left(n_k\right) = 0\,,$$

$$\mathbf{E}\left(\Phi_k - \hat{\Phi}_k\right)^2 = \mathbf{E}\left(\Phi_k - \Phi_k^*\right)^2 = \mathbf{E}\left(n_k\right)^2 = \sigma_{\Phi}^2\,,$$

$$\begin{aligned}\mathbf{E}\left[\left(\Phi_k - \Phi_k^*\right)\left(p_k - p_k^*\right)\right] &= \mathbf{E}\left[\left(\Phi_k - \Phi_k^*\right)\left(p_k - \frac{\Phi_k^* - \Phi_{k-1}^*}{T_k}\right)\right] = \\ &= \mathbf{E}\left[n_k p_k - \left(\Phi_k - \Phi_k^*\right)\frac{\Phi_k^* - \Phi_{k-1}^*}{T_k}\right] = \\ &= \frac{1}{T_k}\mathbf{E}\left[\left(\Phi_k^* - \Phi_k\right)\left(\Phi_k^* - \Phi_{k-1}^*\right)\right] = \frac{1}{T_k}\mathbf{E}\left[n_k^2\right] = \frac{\sigma_{\Phi}^2}{T_k}\end{aligned}$$

$$\mathbf{E}\left[\left(p_k - p_k^*\right)^2\right] = \mathbf{E}\left[\left(p_k - p_k^*\right)^2\right] = 2\frac{\sigma_{\Phi}^2}{T_k^2}\,,$$

$$\begin{aligned}&\mathbf{E}\left[\left(\Phi_k - \Phi_k^*\right)\left(\dot{p}_k - \dot{p}_k^*\right)\right] = \\ &= \mathbf{E}\left[\left(\Phi_k - \Phi_k^*\right)\left(\dot{p}_k - \frac{\Phi_k - \Phi_{k-2}^*}{T_k^2}\right)\right] = 2\frac{\sigma_{\Phi}^2}{T_k^2}\,,\end{aligned}$$

$$\begin{aligned}&\mathbf{E}\left[\left(p_k - p_k^*\right)\left(\dot{p}_k - \dot{p}_k^*\right)\right] = \\ &= \mathbf{E}\left[\left(p_k - \frac{\Phi_k^* - \Phi_k^*}{T_k}\right)\left(\dot{p}_k - \frac{\Phi_k^* - \Phi_{k-2}^*}{T_k^2}\right)\right] = 3\frac{\sigma_{\Phi}^2}{T_k^3}\,,\end{aligned}$$

$$\begin{aligned}\mathbf{E}\left[\left(\dot{p}_k - \dot{p}_k^*\right)^2\right] &= \mathbf{E}\left[\left(\dot{p}_k - \frac{p_k^* - p_{k-1}^*}{T_k}\right)^2\right] = \\ &= \mathbf{E}\left[\left(\dot{p}_k - \frac{\Phi_k - \Phi_{k-2}^*}{T_k^2}\right)^2\right] = 4\frac{\sigma_{\Phi}^2}{T_k^4}\end{aligned}.$$

Received: 10.11.2006.

Optimalna ocena ugaone brzine valjanja protivtenkovske rakete – Deo I: Kalmanov filter sa tačnim modelom

Predstavljen efikasan metod za ocenu i predikciju ugaone brzine valjanja protivoklopne rakete koristeći Kalmanaov filter sa tačnim modelom kretanja pri čemu se meri ugao valjanja rakete pomoću slobodnog žiroskopa, a merena veličina je opterećena šumom. Standardna devijacija ugaone brzine valjanja je prvo određena diferenciranjem kroz dve tačke, a zatim su izvedene neophodne formule za ocenu standardne devijacije pomoću Kalmanovog filtra sa tačnim modelom. Efikasnost modela je testirana numeričkom simulacijom pri čemu je određena zavisnost standardne devijacije ugaone brzine valjanja u odnosu na normalizovani procesni šum. Određena je optimalna vrednost procesnog šuma iz uslova da se postigne minimalna ocena disperzije ugla valjanja i ugaone brzine valjanja.

Ključne reči: **protivoklopna raketa, valjanje rakete, brzina valjanja, ugaona brzina, Kalmanov filtar, slobodni žiroskop, slučajni proces, numerička simulacija.**

Оптимальная оценка угловой скорости крена противотанковой ракеты - Часть I: фильтр Калмана со безошибочной моделью

Здесь представлен эффективный метод для оценки и определения угловой скорости крена противотанковой ракеты при пользовании фильтра Калмана со безошибочной моделью движения, при чём измеряется угол крена ракеты при помощи свободного гироскопа, а измеряемая величина усиленна шумом. Стандартная девиация угловой скорости крена сначала определена дифференциацией через две точки, а потом выведены необходимые уравнения для оценки стандартной девиации при помощи фильтра Калмана со безошибочной моделью. Эффективность модели испытывана цифровым моделированием, при чём определена зависимость стандартной девиации угловой скорости крена по отношению к нормализованному обработанному шуму. Здесь определено оптимальное значение обработанного шума из условий достижения минимальной оценки рассеяния угла крена и угловой скорости крена.

Ключевые слова: противотанковая ракета, крен ракеты, скорость крена, угловая скорость, фильтр Калмана, свободный гироскоп, случайный процесс, цифровое моделирование.

Estimation optimale de la vitesse d'angle de roulement chez le missile antichar -première Partie I: le filtre de Kalman avec le modèle exact

Ce papier expose une méthode pour évaluer et prédire la vitesse d'angle de roulement chez le missile antichar au moyen de filtre de Kalman avec le modèle exact du mouvement et en mesurant l'angle de roulement du missile à l'aide du gyroscope libre alors que l'unité mesurée est altérée par le bruit. La déviation ordinaire de la vitesse d'angle de roulement est déterminé d'abord par la différenciation à travers deux points; ensuite, on a dérivé les formules nécessaires pour évaluer la dérivation ordinaire au moyen du filtre de Kalman avec le modèle exact. L'efficacité du modèle a été testée par la simulation numérique en déterminant aussi la dépendance ordinaire de la déviation de vitesse d'angle de roulement par rapport au bruit normalisé de procès. On a déterminé la valeur optimale du bruit de procès pour réaliser l'estimation minimale de la dispersion de l'angle de roulement et la vitesse d'angle de roulement.

Mots clés: **missile antichar, roulement du missile, vitesse de roulement, vitesse d'angle, filtre de Kalman, gyroscope libre, procès stochastique.**