UDK: 681.5.037
COSATI: 12-01

Asymptotic stability of singular continuous time delayed system

Dragutin Lj. Debeljković, PhD (Eng)[1)]

This paper gives sufficient conditions for the stability of linear singular continuous delay systems of the form $E\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau)$. These new, delay–independent conditions are derived using on approach based on Lyapunov's direct method. Two different methods are applied: one based on crucial idea presented in paper by Owens, *Debeljković* (1985) and the second in an anthological paper by *Pandolfi* (1980). Numerical examples have been worked out to show the applicability of the results derived.

Key words: **continuous system, singular system, linear system, delayed system, time delay, system stability, asymptotic stability, Lyapunov's stability.**

Introduction

IT should be noticed that in some systems consider their character of dynamic and static state must be considered at the same time. Singular systems (also referred to as degenerate, descriptor, generalized, differential - algebraic systems or semi – state) are those the dynamics of which are governed by a mixture of algebraic and differential equations. Recently, many scholars have paid much attention to singular systems and obtaining numerous good results. The complex nature of singular systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control.

The problem of investigation of time delay systems has been exploited over many years. Time delay is very often encountered in various technical systems, such as electric, pneumatic and hydraulic networks, chemical processes, long transmission lines, etc. The existence of pure time lag, regardless if it is present in the control or/and that state, may cause undesirable system transient response, or even instability. Consequently, the problem of stability analysis for this class of systems has been one of the main interests for many researchers. In general, the introduction of time delay factors makes the analysis much more complicated.

It must be emphasized that there is a lot of systems that are singular and demonstrate the phenomena of time delay simultaneously; such systems are called *the singular differential systems with time delay*.

These systems have many special characters. To describe them more exactly, to design them more accurately and to control them more effectively, tremendous effort to investigate them must be made, but that is obviously very difficult work. In recent references, authors has discussed such systems and obtained certain results. But in the study of such systems, there are still many problems to be considered. When the general time delay systems are considered, in the existing stability criteria, two main approaches have been adopted.

Namely, one direction is to contrive the stability condition which does not include the information on the delay, and the other is the method which takes it into account. The former case is often called the delay - independent criteria, and generally provides simple algebraic conditions. In that sense, the question of their stability deserves great attention.

In the short overview that follows, only the results achieved in the area of Lyapunov stability of *linear, continuous singular time delay systems* (LCSTDS) will be taken into consideration. In that sense, the contributions presented in papers tackling the problem of robust stability, stabilization of this class of systems with parameter uncertainty (see the list of references) as well as other questions in connection with the stability of (LCSTDS) being necessarily transformed by Lyapunov – Krasovski functional, to the state space model in the form of differential – integral equations, *Fridman* (2001, 2002) will not be discussed.

Moreover, over the last few years, numerous papers have been published in the area of linear discrete descriptor time delay systems, but this discussion is out of the scope of this paper. The list of references provides more insight into this metter.

To the best of our knowledge, some attempts in stability investigation of (LCSTDS) was due to *Saric* (2001, 2002) where sufficent conditions for convergence of appropriate fundamental matrix were established.

Recently, in the paper of *Xu et al*. (2002) the problem of robust stability and stabilization for uncertain (LCSTDS) was addressed and necessary and sufficient conditions were obtained in terms of strict LMI. Moreover in the same paper, using suitable canonical description of (LCSTDS), a rather simple criterion for asymptotic stability testing was also proposed.

In this paper a quite different approach to this problem is presented. Namely, the result is expressed directly in terms

[1)] Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Belgrade

of matrices E, A_0 and A_1 naturally occurring in the system model and avoiding the need to introduce any canonical form into the statement of the *Theorem*.

The geometric theory of consistency leads to the natural class of positive definite quadratic forms on the subspace containing all solutions. This fact makes possible the construction of Lyapunov stability theory even for the (LCSTDS) in the sense that asymptotic stability is equivalent to the existence of symmetric, positive definite solutions to a weak form of Lyapunov matrix equation incorporating the condition which refers to the time delay term.

A definite aim of this paper is to present new results concerning asymptotic stability of a particular class of *linear continuous singular time delay systems*.

Notation and preliminaries

$\mathbb{R}$	– real vector space
$\mathbb{C}$	– complex vector space
I	– unit matrix
$F=(f_{ij})\in\mathbb{R}^{n\times n}$	– real matrix
F^T	– transpose of matrix F
$F>0$	– positive definite matrix
$F\geq 0$	– positive semi definite matrix
$\Re(F)$	– range of matrix F
$\mathbb{N}(F)$	– null space (kernel) of matrix F
$\lambda(F)$	– eigen value of matrix F
$\sigma_{(\ )}(F)$	– singular value of matrix F
$\lVert F\rVert$	– Euclidean matrix norm of $\lVert F\rVert=\sqrt{\lambda_{\max}\left(A^TA\right)}$
F^D	– dazing inverse of matrix F
$\Rightarrow$	– follows
$\mapsto$	– such that

Generally, the singular differential control systems with time delay can be written as:

$$E(t)\mathbf{x}(t)=\mathbf{f}\left(t,\mathbf{x}(t),\mathbf{x}(t-\tau),\mathbf{u}(t)\right),\quad t\geq 0$$
$$\mathbf{x}(t)=\varphi(t),\ -\tau\leq t\leq 0 \tag{1}$$

where $\mathbf{x}(t)\in\mathbb{R}^n$ is a state vector, $\mathbf{u}(t)\in\mathbb{R}^l$ is a control vector, $E(t)\in\mathbb{R}^{n\times n}$ is a singular matrix, $\varphi\in C=C\left(\left[-\tau,\ 0\right],\Re^n\right)$ is an admissible initial state functional, $C=C\left(\left[-\tau,\ 0\right],\mathbb{R}^n\right)$ is the *Banach space* of continuous functions mapping the interval $\left[-\tau,\ 0\right]$ into $\mathbb{R}^n$ with topology of uniform convergence.

Some previous results

Consider a linear continuous singular system with state delay, described by

$$E\dot{\mathbf{x}}(t)=A_0\mathbf{x}(t)+A_1\mathbf{x}(t-\tau) \tag{2a}$$

with known compatible vector valued function of initial conditions

$$\mathbf{x}(t)=\varphi(t),\quad -\tau\leq t\leq 0 \tag{2b}$$

where A_0 and A_1 are constant matrices of appropriate dimensions.

Moreover it shall be assumed that $rank\,E=r<n$.

Definition 1. The matrix pair $(E,\ A_0)$ is said to be regular if $\det(sE-A_0)$ is not identically zero, *Xu et al*. (2002).

Definition 2. The matrix pair $(E,\ A_0)$ is said to be impulse free if $\deg\left(\det(sE-A_0)\right)=rang\,E$, *Xu et al*. (2002).

The linear continuous singular time delay system (2) may have an impulsive solution, however, the regularity and the absence of impulses of the matrix pair $(E,\ A_0)$ ensure the existence and uniqueness of an impulse free solution to the system under consideration, which is defined in the following *Lemma*.

Lemma 1. Suppose that the matrix pair $(E,\ A_0)$ is *regular* and *impulsive free* and *unique* on $[0,\infty)$, *Xu et al* (2002).

Necessity for system stability investigation produces the need for establishing a proper stability definition. So, the following applies:

Definition 3.

a) *Linear continuous singular time delay system*, (2) is said to be *regular* and *impulse free* if the matrix pair $(E,\ A_0)$ is regular and impulsive free.

b) *Linear continuous singular time delay system*, (2), is said to be stable if for any $\varepsilon>0$ there exists a scalar $\delta(\varepsilon)>0$ such as that, for any compatible initial conditions $\varphi(t)$, satisfying condition: $\sup\limits_{-\tau\leq t\leq 0}\lVert\varphi(t)\rVert\leq\delta(\varepsilon)$, the solution $\mathbf{x}(t)$ of system (2) satisfies $\lVert\mathbf{x}(t)\rVert\leq\varepsilon,\ \forall t\geq 0$.

Moreover if $\lim\limits_{t\to\infty}\lVert\mathbf{x}(t)\rVert\to 0$, system is said to be *asymptotically stable*, *Xu et al* (2002).

General solution to continuous singular time delay state equitation

Campbell's (1980) approach

Eq.(2) can be assumed to be in the form:

$$\dot{\mathbf{x}}(t)+A_0\mathbf{x}(t)=A_1\mathbf{x}(t-1)+\mathbf{f}(t),\ t\geq 0. \tag{3}$$

To uniquely determine the solution of eq. (3) an arbitrary initial function $\varphi(t)=\mathbf{x}(t)=\mathbf{x}_0$ must be specified defined on $[-1,0]$, so that $\mathbf{x}(0)=\mathbf{x}_0(0^-)$.

Continuing in this manner, the given solution exists on $[0,n]$, eq.(3) has a unique solution on $[n,\ n+1]$ such as that $\mathbf{x}(n^+)=\mathbf{x}(n^-)$ and the solution exists on $[0,n+1]$. Thus for eq.(3), a unique continuous solution exists on $[-1,\infty)$ for any continuous specification of $[-1,0]$.

Eq.(2) shall be considered under the assumption that $\lambda E+A_0\Delta$ is invertible for some λ. The behaviour of eq.(2) is different from that of eq.(3). As expected, $\mathbf{x}(t)$ can no longer be taken to be arbitrary on $[-1,0]$. If in eq.(2), $\mathbf{f}=0$, the following associated homogeneous equation is obtained:

$$E\dot{\mathbf{x}}(t)+A_0\mathbf{x}(t)=A_1\mathbf{x}(t-1)\,. \tag{4}$$

Clearly all solutions of eq.(2) are of the form $\mathbf{x}_p(t)+\mathbf{x}_h(t)$ where $\mathbf{x}_p(t)$ is a solution of eq.(2) and $\mathbf{x}_h(t)$ is an arbitrary solution of eq.(4).

It shall be proven that eq.(2) has always at least one initial condition for which eq.(2) has a solution on $[-1,\infty)$. All the consistent initial conditions of eq.(4) both for $[-1,\infty)$ and $[-1,n)$ time periods shall then be characterized.

Let $\{\mathbf{x}_n(t)\}, n\geq 1$ be two sequences of infinitely differentiable functions defined on $[0,1]$. $\mathbf{x}_n(t)$ e.g. (respectively $\mathbf{f}_n(t)$) should be thought of as $\mathbf{x}(t)$ eq. (respectively $\mathbf{f}(t)$) on $[n-1,n]$.

As it will be demonstrated, infinite differentiability is a natural assumption since the existence of solutions often requires at least some components of $\mathbf{x}(t)$, $\mathbf{f}(t)$ to be infinitely differentiable at the integers.

The system eq.(2), now becomes:

$$E\dot{\mathbf{x}}_n(t)+A_0\mathbf{x}_n(t)=A_1\mathbf{x}_{n-1}(t)+\mathbf{f}_n(t)\,,\; n\geq 1, \tag{5}$$

for given $\mathbf{x}_0$.

The characterization of those $\mathbf{x}_0(t)$ such as that eq.(5) has a solution $\{\mathbf{x}_l\}_{l=0}^{r}$ such as that $\mathbf{x}_l(1)=\mathbf{x}_{l+1}(0)$ is sought.

From *Campbell* (1980), for $n\geq 1$ follows:

$$\begin{aligned}\mathbf{x}_n(t)&=e^{-\hat{E}^D\hat{A}_0t}\hat{E}^D\hat{E}\mathbf{x}_n(0)\\&+\hat{E}^De^{-\hat{E}^D\hat{A}_0t}\int_0^t e^{-\hat{E}^D\hat{A}_0\kappa}\left(A_1\mathbf{x}_{n-1}(\kappa)+\mathbf{f}_n(\kappa)\right)d\kappa\\&+(I-\hat{E}^D\hat{E})\sum_{m=0}^{k-1}\left(-\hat{E}\hat{A}_0^{\,D}\right)^m\hat{A}_0^{\,D}\left(A_1\mathbf{x}_{n-1}^{(m)}(t)+\mathbf{f}_n^{(m)}(t)\right)\end{aligned} \tag{6.a}$$

where:

$$\begin{aligned}\hat{E}&=(\lambda E+A_0)^{-1}E, & \hat{A}_0&=(\lambda E+A_0)^{-1}A_0,\\ \hat{A}_1&=(\lambda E+A_0)^{-1}A_1, & \hat{\mathbf{f}}_n(t)&=(\lambda E+A_0)^{-1}\mathbf{f}_n(t)\end{aligned} \tag{6.b}$$

and k is the index of matrix $\hat{E}$.

There is a need to manipulate this expression a lot, so let it be:

$$P=\hat{E}^D\hat{E}\,,\; Q=\hat{E}^D\hat{A}_0\,,\; H=-\hat{E}\hat{A}_0^{\,D}. \tag{7}$$

Note that P is a projection and P, Q, H all commute.

Thus we have:

$$\begin{aligned}\mathbf{x}_n(t)&=e^{-Qt}P\mathbf{x}_n(0)\\&+\hat{E}^De^{-Qt}\int_0^t e^{-Q\kappa}\left(\hat{A}_1\mathbf{x}_{n-1}(\kappa)+\mathbf{f}_n(\kappa)\right)d\kappa\\&+(I-P)\sum_{m=0}^{k-1}H^m\,\hat{A}_0^{\,D}\left(\hat{A}_1x_{n-1}^{(m)}(t)+\boldsymbol{f}_n^{(m)}(t)\right)\end{aligned} \tag{8}$$

Regardless what $\mathbf{x}_{n-1}$ is, letting $P\mathbf{x}_n(0)=P\mathbf{x}_{n-1}(1)$ makes $P\mathbf{x}$ continuous at n. The difficulty occurs with $(I-P)\mathbf{x}$ at n.

That:

$$(I-P)\mathbf{x}_1(0)=(I-P)\mathbf{x}_0(1) \tag{9}$$

gives:

$$\begin{aligned}&(I-P)\mathbf{x}_0(1)=(I-P)\times\\&\times\sum_{m=0}^{k-1}H^m\,\hat{A}_0^{\,D}\left(\hat{A}_1\mathbf{x}_{n-1}^{(m)}(0)+\hat{\mathbf{f}}_n^{(m)}(0)\right)\end{aligned} \tag{10}$$

It will be shown that the given $\mathbf{f}(t)$ and any $\{\mathbf{x}_0^{(m)}(0)\}, m\geq 0$ a solution can be obtained by specifying $\mathbf{x}_0^{(m)}(1)$.

Take eq.(9) as the definition of $(I-P)\mathbf{x}_0(1)$.

For $n\geq 1$, it follows that:

$$\begin{aligned}&(I-P)\mathbf{x}_n(t)=(I-P)\times\\&\times\sum_{m=0}^{k-1}H^m\,\hat{A}_0^{\,D}\left(\hat{A}_1\mathbf{x}_{n-1}^{(m)}(0)+\hat{\mathbf{f}}_n^{(m)}(0)\right)\end{aligned} \tag{11}$$

Thus the requirement that:

$$(I-P)\mathbf{x}_n(0)=(I-P)\mathbf{x}_{n-1}(1)\,, \tag{12}$$

for $n\geq 2$, is

$$(I-P)\sum_{m=0}^{k-1}H^m\,\hat{A}_0^{\,D}\hat{A}_1\left(\mathbf{x}_{n-1}^{(m)}(0)-\mathbf{x}_{n-2}^{(m)}(1)\right)=\boldsymbol{0}\,, \tag{13}$$

since $\hat{\mathbf{f}}_n^{(m)}(0)=\hat{\mathbf{f}}_{n-1}^{(m)}(1)$, $m\geq 0$, $n\geq 2$.

From eq.(7), for $n\geq 1$, $r\geq 1$:

$$\begin{aligned}x_n^{(r)}(t)&=(-Q)^re^{-Qt}P\mathbf{x}_n(0)\\&+\hat{E}^D(-Q)^re^{-Qt}\int_0^t e^{Q\kappa}\left(\hat{A}_1\mathbf{x}_{n-1}(\kappa)+\mathbf{f}_n(\kappa)\right)d\kappa\\&+\hat{E}^D\sum_{l=0}^{r-1}Q^{r-l-1}\left(\hat{A}_1\mathbf{x}_{n-1}^{(l)}(t)+\hat{\mathbf{f}}_n^{(l)}(t)\right)\\&+(I-P)\sum_{m=0}^{k-1}H^m\,\hat{A}_0^{\,D}\left(\hat{A}_1\mathbf{x}_{n-1}^{(m+r)}(t)+\hat{\mathbf{f}}_n^{(m+r)}(t)\right)\end{aligned} \tag{14}$$

In particular:

$$\begin{aligned}\mathbf{x}_1^{(r)}(0)&=(-Q)^rPx_1(0)\\&+\hat{E}^D\sum_{l=0}^{r-1}Q^{r-l-1}\left(\hat{A}_1\mathbf{x}_0^{(l)}(0)+\hat{\mathbf{f}}_1^{(l)}(0)\right)\\&+(I-P)\sum_{m=0}^{k-1}H^m\,\hat{A}_0^{\,D}\left(\hat{A}_1\mathbf{x}_0^{(m+r)}(0)+\hat{\mathbf{f}}_0^{(m+r)}(0)\right)\end{aligned} \tag{15}$$

Define $\mathbf{x}_0^{(r)}(1)=\mathbf{x}_1^{(r)}(0)$, where $\mathbf{x}_1^{(r)}(0)$ is given by eq.(11)

That an infinitely differentiable function on $[0,1]$ exists for arbitrary $\{\mathbf{x}^{(r)}(0)\}$, $\{\mathbf{x}^{(r)}(1)\}$ follows from *Campbell* (1980), *Lemma* 13.1.

Let $\mathbf{x}_0(t)$ be such a function.

It will be shown that is a consistent initial condition.

Given this $\mathbf{x}_0$, $\mathbf{x}_1$ will be computed and by construction

$\mathbf{x}_0(1) = \mathbf{x}_0(0)$, $m \geq 1$.

Suppose then that there exist $\mathbf{x}_0, ..., \mathbf{x}_n$ and:

$$\mathbf{x}_r^{(m)}(1) = \mathbf{x}_{r+1}^{(m)}(0), \; m \geq 0, \; r \leq n-1. \tag{16}$$

It will be shown that one can get a similar solution to $\mathbf{x}_{n+1}$ can be obtained.

By eq.(16), eq.(13) is satisfied.

The definition of $P\mathbf{x}_{n+1}$ in terms of eq.(6), the infinite differentiability of $\mathbf{f}$, and the induction hypothesis eq.(12) applied to eq.(13) means that $\mathbf{x}_n{}^{(m)}(1) = \mathbf{x}_{n+1}{}^{(m)}(0)$ so that the induction is complete.

Thus the following theorem can be proved.

Theorem 1. If $\mathbf{f}(t)$ is infinitely differentiable on $[0,\infty)$ and $\{\mathbf{x}_0{}^{(m)}(0)\}$ is an arbitrary sequence of numbers, and $\mathbf{x}_0(t)$ is any infinitely differentiable function on $[0,1]$ with these derivatives at zero such that $\mathbf{x}_0{}^{(m)}(1)$ is given by eq.(15), then eq.(2) is consistent and has an infinitely differentiable solution.

Let C be the space of $\mathbf{C}^n$ - valued infinitely differentiable function on $[0,1]$ with the family of semi-norms:

$$\rho_m(\mathbf{f}) = \sup_{0 \leq t \leq 1} \left\| \mathbf{f}^{(m)}(t) \right\|. \tag{17}$$

For any integer *n*, let C_n be those initial conditions $\mathbf{x}_0(t)$ in C for which a continuous solution to eq.(4) exists on $[0, n]$.

Theorem 2. Each C_n is a closed subspace of $C_n \supseteq C_{n+1}$.

The set of consistent initial conditions $C_\infty = \bigcap_{n=1}^{\infty} C_n$ is an infinite dimensional closed subspace of C.

Proof. Using eq.(7) and eq.(9) it can be seen that each C_n consists of those $\mathbf{x}_0 \in C$ whose derivatives at 0 and 1 satisfying *n* relationships.

For example C_1 consists of those which satisfy:

$$(I-P)\sum_{m=0}^{k-1} H^m \hat{A}_0{}^D \hat{A}_1 \mathbf{x}_0^{(m)}(0) = (I-P)\mathbf{x}_0(1), \tag{18}$$

while C_2 consists of those $\mathbf{x}_0$ which also satisfy:

$$(I-P)\sum_{m=0}^{k-1} H^m \hat{A}_0{}^D \hat{A}_1 \left(\mathbf{x}_0^{(m)}(1) + \mathbf{x}_1{}^{(m)}(0)\right) = \mathbf{0}. \tag{19}$$

That is:

$$(I-P)\times\sum_{m=0}^{k-1} H^m \hat{A}_0{}^D \hat{A}_1 \begin{pmatrix} \mathbf{x}_0{}^{(m)}(1) - (-Q)^m P\mathbf{x}_0(1) + \\ +\hat{E}^D \sum_{l=0}^{m-1} Q^{m-l-1} \hat{A}_1 \mathbf{x}_0{}^{(l)}(0) + \\ +(I-P)\sum_{l=0}^{k-1} H^l \hat{A}_0{}^D \hat{A}_1 \mathbf{x}_0{}^{(l+m)}(0) \end{pmatrix} = 0 \tag{20}$$

That C_n is closed follows from the continuity of evaluation of derivatives in C.

That C_n is infinite dimensional follows from Theorem 2.

For:

$$E\dot{\mathbf{x}}(t) + A_0\mathbf{x}(t) = \mathbf{f}(t), \tag{21}$$

the assumption that $(\lambda E + A_0)$ was invertible for some scalar λ was equivalent to consistent initial conditions uniquely determining solutions.

For the delay equation eq.(2), the situation is more complicated.

The existence of λ is equivalent to $\mathbf{x}_n$ and $\mathbf{f}_n$ uniquely determining $\mathbf{x}_{n+1}$ but that is different from $\mathbf{x}_0$ and $\mathbf{f}$ uniquely determining the $\mathbf{x}_n$.

While the infinitely differentiable initial conditions were the appropriate space for the general problem on $[0,\infty)$, if the point of interest is the existence on $[0,n)$, then only $n(k-1)$ differentiability is needed.

Also, in the context of a particular problem as low as $k-1$ times differentiability of some components of $\mathbf{f}$ and $\mathbf{x}_0$ will suffice.

Of course, in general as many as $n(k-1)$ times differentiability of C on $[0,n)$ and infinite differentiability on $[0,\infty)$ can be made.

Similarly, the complication caused by several delays is that the consistent initial conditions must satisfy derivative conditions at points other than the end points.

Wei's (2004) approach

Another approach to the general solution of singular differential systems with time delay, given in (2), is presented.

Definition 4. Let E be a square matrix, if there exists a matrix E^d satisfying:

1. $EE^d = E^d E$
2. $E^d EE^d = E^d$
3. $E^d EE^d = E^d$

E^d the Drazin inverse matrix of matrix E, is simply called D-inverse matrix.

l is the index of the matrix E, it is the smallest nonnegative integer which makes:

$$rank(E^{l+1}) = rank(E^l)$$

true.

Lemma 2. For any square matrix E, its Drazin inverse matrix E^d is existent and unique.

If the *Jordan* normalized form of E is

$$E = T\begin{pmatrix} J_1 & 0 \\ 0 & J_1 \end{pmatrix} T^{-1}, \tag{22}$$

then:

$$E^d = T\begin{pmatrix} J_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} T^{-1}. \tag{23}$$

Here J_0 is a nilpotent matrix, J_1 and T are invertible matrices.

Consider system

$$E\dot{x}(t) = Ax(t) + Bx(t-\tau)\,,\ t \ge 0\,,$$

$$x(t) = \varphi(t)\,, -\tau \le t \le 0 \tag{24}$$

and system

$$E\dot{x}(t) = Ax(t) + Bx(t-\tau) + Du(t) + f(t)\,,$$

$$t \ge 0\,,\ x(t) = 0\,,\ -\tau \le t \le 0\,. \tag{25}$$

It is not difficult to prove the following result.

Lemma 3. If $\bar{x}(t), \hat{x}(t)$ are respectively the solutions of (24) and (25), then $x(t) = \bar{x}(t) + \hat{x}(t)$ is the solution of (22).

Definition 5. Let $X(t) \in \mathbb{R}^{n\times n}$, $X(t)$ is called the first class foundation solution of singular differential systems with delay, if it satisfies the matrix equation:

$$\begin{cases} E\dot{X}(t) = AX(t) + BX(t-\tau), t \ge 0, \\ X(t) = \begin{cases} EE^d, t = 0, \\ 0, -\tau \le t \le 0, \end{cases} \end{cases} . \tag{26}$$

Definition 6. Let $Y(t) \in \mathbb{R}^{n\times n}$, $Y(t)$ is called the second class foundation solution of singular differential systems with time delay, if it satisfies matrix equations:

$$\begin{cases} E\dot{Y}(t) = AY(t) + BY(t-\tau) \\ \qquad + (I - EE^d)\delta(t),\ t \ge 0, \\ Y(t) = \begin{cases} I - EE^d, t = 0, \\ 0, -\tau \le t \le 0, \end{cases} \end{cases} \tag{27}$$

where $\delta(t)$ is a delta function, or impulse function.

Lemma 3. For a delta function $\delta(t)$, there exists

$$\int_0^t \delta(t-s) f(s) ds = f(t)\,. \tag{28}$$

Proof. Define $f * g$:

$$f * g = \int_0^t g(t-s) f(s) ds\,, \tag{29}$$

the convolution formula is known

$$L(f * g) = L(f)L(g)\,. \tag{30}$$

It also shows that $L(\delta) = 1$.

Then it follows that:

$$L\left(\int_0^t \delta(t-s) f(s) ds\right) = L(\delta(t))(L(f(t)) = L(f(t))\,, \tag{31}$$

That is:

$$\int_0^t \delta(t-s) f(s) ds = f(t)\,. \tag{32}$$

Lemma 4. For any square matrix *E*, one can have:

$$(E+I)(I - EE^d) \times (I + E(I - EE^d))^{-1} = I - EE^d\,. \tag{33}$$

Proof. Let *I* be an identity matrix with appropriate dimension, and

$$E = T\begin{pmatrix} J_1^{-1} & 0 \\ 0 & J_0 \end{pmatrix} T^{-1}\,. \tag{34}$$

from *Lemma* 4.31, it follows:

$$E^d = T\begin{pmatrix} J_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} T^{-1}\,. \tag{35}$$

For J_0 is a nilpotent matrix, $J_0 + I$ is invertible.

$$\begin{aligned} I - EE^d &= T\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} T^{-1} - T\begin{pmatrix} J_1 & 0 \\ 0 & J_0 \end{pmatrix} T^{-1} T\begin{pmatrix} J_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} T^{-1} \\ &= T\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} T^{-1} - T\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} T^{-1} = T\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} T^{-1} \end{aligned} \tag{36}$$

$$\begin{aligned} (E+I)(I - EE^d) &= T\begin{pmatrix} J_1 + I & 0 \\ 0 & J_0 + I \end{pmatrix} T^{-1} T\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} T^{-1} \\ &= T\begin{pmatrix} 0 & 0 \\ 0 & J_0 + I \end{pmatrix} T^{-1} \end{aligned} . \tag{37}$$

$$\begin{aligned} I - EE^d &= T\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} T^{-1} \\ &= T\begin{pmatrix} 0 & 0 \\ 0 & J_0 + I \end{pmatrix} T^{-1} T\begin{pmatrix} I & 0 \\ 0 & J_0 + I \end{pmatrix}^{-1} T^{-1}\,, \\ &= (E+I)(I - EE^d)\left(I + E(I - EE^d)\right)^{-1} \end{aligned} \tag{38}$$

i.e. is (38) is true.

Definition 7. If $\det(\lambda E - A) \ne 0$, the matrix pair (E, A) is called regular. If (A, E) is regular, the system (2) is called *regular*.

Remark 1. Using the standard method, it can be proven that if (E, A) is regular, systems (2), (6) and (7) *are solvable*.

Theorem 8. Suppose matrix pair (E, A) is regular, $\bar{x}(t)$ is the solution of (24), then provided that:

$$t \ge +\tau\,,$$

$$\begin{aligned} \bar{x}(t) &= \left[X(t) + Y(t)\left(I + E(I - EE^d)\right)^{-1} E\right]\varphi(0) \\ &+ \int_{-\tau}^{0} \begin{bmatrix} X(t-\theta-\tau)E^d B + \\ +Y(t-\theta-\tau)\left(I + E(I - EE^d)\right)^{-1} B \end{bmatrix} \varphi(\theta) d\theta \end{aligned} , \tag{39}$$

when:

$$0 \le t \le \tau\,,$$

$$\begin{aligned} \bar{x}(t) &= \left[X(t) + Y(t)(I + E(I - EE^d))^{-1} E\right]\varphi(0) \\ &+ \int_{-\tau}^{t-\tau} \begin{bmatrix} X(t-\theta-\tau)E^d B + \\ +Y(t-\theta-\tau)\left(I + E(I - EE^d)\right)^{-1} B \end{bmatrix} \varphi(\theta) d\theta \end{aligned} . \tag{40}$$

There $X(t)$ is the first class of foundation solution of singular differential systems with delay, $Y(t)$ is the second class of foundation solution of singular differential systems with delay.

To prove *Theorem* 8, the homogeneous differential systems with delay (24) can be partitioned into two classes of systems:

$$E\dot{x}(t) = Ax(t) + Bx(t-\tau)\,, \quad t \ge \tau,$$
$$E\dot{x}(t) = Ax(t) + EE^d B\varphi(t-\tau)\,, 0 \le t \le \tau, \tag{41}$$
$$x(t) = EE^d \varphi(t)\,, \quad -\tau \le t \le 0$$

and:

$$E\dot{x}(t) = Ax(t) + Bx(t-\tau)\,, \quad t \ge \tau,$$
$$E\dot{x}(t) = Ax(t) + (I - EE^d)B\varphi(t-\tau)\,, \; 0 \le t \le \tau, \tag{42}$$
$$x(t) = (I - EE^d)\varphi(t)\,, \quad -\tau \le t \le 0\,.$$

Lemma 5. If $x_1(t), x_2(t)$ is respectively the solution of (41) and (42), then $\overline{x}(t) = x_1(t) + x_2(t)$ is the solution of (24).

Lemma 6. Suppose matrix pair (A, E) is regular, then the solution of (41) can be written as:

$$x_1(t) = \begin{cases} X(t)\varphi(0) + \\ +\int_{-\tau}^{0} X(t-\theta-\tau)E^d B\varphi(\theta)d\theta, t \ge \tau, \\ X(t)\varphi(0) + \\ +\int_{0}^{\tau} X(t-\theta)E^d B\varphi(\theta-\tau)d\theta, 0 \le t \le \tau. \end{cases} \tag{43}$$

There $X(t)$ is the first class of foundation solution of singular differential systems with delay.

Proof. Since $X(t)$ is the first class of foundation solution of singular differential systems with delay, $X(t)$ satisfies matrix equations (26).

By taking Laplace transformation for (26), then:

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}(E\dot{X}(t))dt = E\int_0^\infty e^{-\lambda t}\dot{X}(t)dt \\ &= E\lambda\int_0^\infty e^{-\lambda t}\dot{X}(t)dt - EX(0) = \lambda ELX(t) - EEE^d\end{aligned}\,, \tag{44}$$

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}(AX(t) + BX(t-\tau))dt = \\ &= A\int_0^\infty e^{-\lambda t}X(t)dt + B\int_0^\infty e^{-\lambda t}X(t-\tau)dt \\ &= ALX(t) + B\int_{-\tau}^\infty e^{-\lambda t-\lambda\tau}X(t)dt \\ &= ALX(t) + Be^{-\lambda\tau}\int_0^\infty e^{-\lambda t}X(t)dt = (A + Be^{-\lambda\tau})LX(t)\end{aligned} \;.\tag{45}$$

$$\lambda ELX(t) - EEE^d = (A + Be^{-\lambda\tau})LX(t)\,. \tag{46}$$

Because (E, A) is regular, for λ large enough, $\lambda E - A - Be^{-\lambda\tau}$ is invertible, the following is obtained:

$$L\{X(t)\} = \left(\lambda E - A - Be^{-\lambda\tau}\right)EEE^d\,. \tag{47}$$

If $x_1(t)$ is the solution of (41), then:

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}(E\dot{x}(t))dt = E\int_0^\infty e^{-\lambda t}\dot{x}_1(t)dt = \\ &E\lambda\int_0^\infty e^{-\lambda t}x_1(t)dt - Ex_1(0) = ELx_1(t) - EEE^d\varphi(0)\end{aligned}\,, \tag{48}$$

and:

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}(E\dot{x}_1(t))dt = \int_0^\tau e^{-\lambda t}(E\dot{x}_1(t))dt + \int_\tau^\infty e^{-\lambda t}(E\dot{x}_1(t))dt \\ &= A\int_0^\tau e^{-\lambda t}(E\dot{x}_1(t))dt + \int_0^\tau EE^d Be^{-\lambda t}\varphi(t-\tau)dt \\ &+ A\int_\tau^\infty e^{-\lambda t}x_1(t)dt + B\int_\tau^\infty e^{-\lambda t}x_1(t-\tau)dt \\ &= A\int_0^\infty e^{-\lambda t}x_1(t)dt + Be^{-\lambda\tau}\int_0^\infty e^{-\lambda t}x_1(t-\tau)dt \\ &+ \int_0^\tau EE^d Be^{-\lambda t}\varphi(t-\tau)dt = AL\{x_1(t)\} + Be^{-\lambda\tau}L\{x_1(t)\} \\ &+ \int_0^\tau EE^d Be^{-\lambda t}\varphi(t-\tau)dt\end{aligned} \tag{49}$$

That is:

$$\begin{aligned}&(\lambda E - A - Be^{-\lambda\tau})^{-1}L(x_1(t)) = \\ &= EEE^d\varphi(0) + \int_0^\tau EE^d Be^{-\lambda t}\varphi(t-\tau)dt\end{aligned} \tag{50}$$

Since $E^d = EE^d E^d$, from (47) it can be seen that:

$$\begin{aligned}&L(x_1(t)) = (\lambda E - A - Be^{-\lambda\tau})^{-1}EEE^d\varphi(0) + \\ &+ \int_0^\tau (\lambda E - A - Be^{-\lambda\tau})^{-1}EEE^d E^d Be^{-\lambda t}\varphi(t-\tau)dt\,\cdot \\ &= L(X(t))\varphi(0) + \int_0^\tau L(X(t))E^d Be^{-\lambda s}\varphi(s-\tau)ds\end{aligned} \tag{51}$$

Defining auxiliary function $\wp : [-\tau, \infty) \to [0,\; 1]$:

$$\wp(t) = \begin{cases} 0, & t \ge 0, \\ 1, & t < 0, \end{cases}, \tag{52}$$

the following is obtained:

$$\begin{aligned}&\int_0^\tau L(X(t))E^d Be^{-\lambda s}\varphi(s-\tau)ds = \\ &= L(X(t))\int_0^\tau e^{-\lambda s}E^d B\varphi(t-\tau)dt \\ &= L(X(t))\int_0^\infty e^{-\lambda s}E^d B\varphi(t-\tau)\wp(t-\tau)dt \\ &= L(X(t))L(E^d B\varphi(t-\tau)\wp(t-\tau)) \\ &= L\left(\int_0^t X(t-\theta)E^d B\varphi(\theta-\tau)\,\wp(\theta-\tau)d\theta\right)\end{aligned}\;. \tag{53}$$

Then:

$$x_1(t) = X(t)\varphi(0) + \int_0^t X(t-\theta)E^d B\varphi(\theta-\tau)\omega(\theta-\tau)d\theta\,. \tag{54}$$

When: $t \ge \tau$,

$$x_1(t) = X(t)\varphi(0) + \int_{-\tau}^0 X(t-\theta-\tau)E^d B\varphi(\theta)d\theta\,, \tag{55}$$

and, when $0 \le t \le \tau$,

$$x_1(t) = X(t)\varphi(0) + \int_0^t X(t-\theta)E^d B\varphi(\theta-\tau)d\theta . \quad (56)$$

The proof of Theorem 8 is completed.

Lemma 7. Suppose matrix couple (A, E) is regular, then the solution of (42) can be written as:

$$x_2(t) = \begin{cases} Y(t)(I+E(I-EE^d))^{-1}\varphi(0) \qquad t \ge \tau, \\ +\int_{-\tau}^{0} Y(t-\theta-\tau)\left(I+E(I-EE^d\right)^{-1} B\varphi(\theta)d\theta, \\ Y(t)(I+E(I-EE^d))^{-1}\varphi(0) \qquad 0 \le t \le \tau \\ +\int_0^t Y(t-\theta)\left(I+E(I-EE^d\right)^{-1} B\varphi(\theta-\tau)d\theta, \end{cases} \quad (57)$$

There $Y(t)$ is the second class of foundation solution of singular differential systems with delay.

Proof. Since $Y(t)$ is the second class of foundation solution of singular differential systems with delay, $Y(t)$ satisfies matrix equations (27).

By taking Laplace transformation for (27), the following is obtained:

$$\begin{aligned} \int_0^\infty e^{-\lambda t}(E\dot{Y}(t))dt &= E\int_0^\infty e^{-\lambda t}\dot{Y}(t)dt \\ &= E\lambda\int_0^\infty e^{-\lambda t}Y(t)dt - EY(0) \\ &= \lambda EL(X(t)) - E(I-EE^d) \end{aligned} , \quad (58)$$

or:

$$\begin{aligned} & A\int_0^\infty e^{-\lambda t}Y(t)dt + B\int_0^\infty e^{-\lambda t-\lambda\tau}Y(t)dt + (I-EE^d) \\ &= AL\{Y(t)\} + B\int_{-\tau}^\infty e^{-\lambda t-\lambda\tau}Y(t)dt + (I-EE^d) \\ &= AL\{Y(t)\} + Be^{-\lambda\tau}\int_0^\infty e^{-\lambda t}Y(t)dt + (I-EE^d) \\ &= (A+Be^{-\lambda\tau})L\{Y(t)\}(I-EE^d) \end{aligned} \quad (59)$$

That is:

$$\lambda EL\{Y(t)\} - E(I-EE^d) = (ABe^{-\lambda\tau})L\{Y(t)\} + (I-EE^d) . \quad (60)$$

Because (E, A) is regular, for λ large enough, $\lambda E - A - Be^{-\lambda\tau}$ is invertible, then:

$$L\{Y(t)\} = (\lambda E - A - Be^{-\lambda\tau})^{-1} \times (I+E)(I-EE^d) . \quad (61)$$

If $x_2(t)$ is the solution of (10), then:

$$\begin{aligned} \int_0^\infty e^{-\lambda t}(E\dot{x}_2(t))dt &= E\int_0^\infty e^{-\lambda t}\dot{x}_2(t)dt \\ &= E\int_0^\infty e^{-\lambda t}\dot{x}_2(t)dt - Ex_2(0) \\ &= \lambda E\left(L\{x_2(t)\} - E(I-EE^d)\right)\varphi(0) \end{aligned} , \quad (62)$$

and:

$$\begin{aligned} & \int_0^\infty e^{-\lambda t}(E\dot{x}_2(t))dt = \int_0^\tau e^{-\lambda t}(E\dot{x}_2(t))dt \\ & + \int_\tau^\infty e^{-\lambda t}(E\dot{x}_2(t))dt = A\int_0^\tau e^{-\lambda t}x_2(t)dt \\ & + \int_0^\tau (I-EE^d)Be^{-\lambda t}\varphi(t-\tau)dt + A\int_\tau^\infty e^{-\lambda t}x_2(t)dt \\ & + B\int_\tau^\infty e^{-\lambda t}x_2(t-\tau)dt = A\int_0^\infty e^{-\lambda t}x_2(t)dt \\ & + Be^{-\lambda\tau}\int_0^\infty e^{-\lambda t}x_2(t)dt + \int_0^\tau (I-EE^d)Be^{-\lambda t}\varphi(t-\tau)dt \end{aligned} . \quad (63)$$

That is:

$$\begin{aligned} & (\lambda E - A - Be^{-\lambda\tau})^{-1}L\{x_2(t)\} = (I-EE^d)E\varphi(0) \\ & + \int_0^\tau (I-EE^d)Be^{-\lambda t}\varphi(t-\tau)dt \end{aligned} . \quad (64)$$

Then:

$$\begin{aligned} L\{x_2(t)\} &= (\lambda E - A - Be^{-\lambda\tau})^{-1} \times \\ & \times (I-EE^d)E\varphi(0) \\ & + \int_0^\tau \begin{matrix}(\lambda E - A - Be^{-\lambda\tau})^{-1} \times \\ \times (I-EE^d)Be^{-\lambda t}\varphi(t-\tau)dt\end{matrix} \end{aligned} . \quad (65)$$

From *Lemma* 4, follows:

$$\begin{aligned} L\{x_2(t)\} &= (\lambda E - A - Be^{-\lambda\tau})^{-1} \times \\ & \times (E+I)(I-EE^d) \times \\ & \times (I+E(I-EE^d))^{-1}E\varphi(0) + \\ & + \int_0^\tau \begin{matrix}(\lambda E - A - Be^{-\lambda\tau})^{-1}(E+I)(I-EE^d) \times \\ \times (I+E(I-EE^d))^{-1}Be^{-\lambda t}\varphi(t-\tau)dt\end{matrix} \end{aligned} . \quad (66)$$

By (61), it is obvious:

$$\begin{aligned} L\{x_2(t)\} &= L\{Y(t)\} \times \\ & \times (I+E(I-EE^d))^{-1}E\varphi(0) \\ & + \int_0^\tau \begin{matrix}L\{Y(t)\}(I+E(I-EE^d))^{-1} \times \\ \times Be^{-\lambda\theta}\varphi(\theta-\tau)d\theta\end{matrix} \end{aligned} , \quad (67)$$

By using auxiliary function $\wp : [-\tau, 0) \to [0,\ 1]$ as above, then:

$$\begin{aligned} & \int_0^\tau L\{Y(t)\}(I+E(I-EE^d))^{-1}Be^{-\lambda\theta}\varphi(\theta-\tau)d\theta \\ &= L\{Y(t)\}\int_0^\tau \begin{matrix}e^{-\lambda\theta}(I+E(I-EE^d))^{-1} \times \\ \times B\varphi(\theta-\tau)d\theta\end{matrix} \\ &= L\{Y(t)\}\int_0^\infty \begin{matrix}e^{-\lambda\theta}(I+E(I-EE^d))^{-1} \times \\ \times B\varphi(\theta-\tau)\wp(\theta-\tau)d\theta\end{matrix} \\ &= L\{Y(t)\}((I+E(I-EE^d))^{-1} \times B\varphi(t-\tau)\wp(t-\tau) \\ &= L\left\{\int_0^t \begin{matrix}Y(t-\theta)e^{-\lambda\theta}(I+E(I-EE^d))^{-1} \times \\ \times B\varphi(\theta-\tau)\wp(\theta-\tau)d\theta\end{matrix}\right\} \end{aligned} \quad (68)$$

Then:

$$\begin{aligned} x_2(t) &= Y(t)\left(I+E(I-EE^d\right)^{-1}E\varphi(0) \\ & + \int_0^t \begin{matrix}Y(t-\theta)(I+E(I-EE^d))^{-1} \times \\ \times B\varphi(\theta-\tau)\wp(\theta-\tau)d\theta\end{matrix} \end{aligned} \cdot \quad (69)$$

When $t \geq \tau$:

$$x_2(t) = Y(t)(I + E(I - EE^d))^{-1} E\varphi(0) + \int_{-\tau}^{0} Y(t-\theta-\tau)(I + E(I - EE^d))^{-1} \times B\varphi(\theta)d\theta . \quad (70)$$

when $0 \leq t \leq \tau$,

$$x_2(t) = Y(t)\left(I + E(I - EE^d\right)^{-1} E\varphi(0) + \int_0^t Y(t-\theta)\left(I + E(I - EE^d)\right)^{-1} B\varphi(\theta-\tau)d\theta . \quad (71)$$

This completes the proof of Lemma 7.

Theorem 9. Suppose matrix pair (A, E) is regular, then the solution of (24) can be written as:

$$\hat{x}(t) = \int_0^t X(t-\theta)E^d \left(Du(\theta) + f(\theta)\right)d\theta + \int_0^t Y(t-\theta)\left(I + E(I - EE^d\right)^{-1} \times \left(Du(\theta) + f(\theta)\right)d\theta . \quad (72)$$

There $X(t)$ is the first class of foundation solution of singular differential systems with delay, $Y(t)$ is the second class of foundation solution of singular differential systems with delay.

From *Lemma* 3, *Theorem* 8 and *Theorem* 9, the following is obtained:

Theorem 10. Suppose matrix pair (A, E) is regular, $x(t)$ is the solution of (25), when:

$$t \geq \tau,$$

$$x(t) = \left(X(t) + Y(t)(I + E\left(I - EE^d\right)^{-1} E\right)\varphi(0) + \int_{-\tau}^{0} \begin{pmatrix} X(t-\theta-\tau)E^d B + \\ +Y(t-\theta-\tau)(I + E\left(I - EE^d\right)^{-1} B \end{pmatrix} \varphi(\theta)d\theta + \int_0^t \left(X(t-\theta)E^d + Y(t-\theta)\left(I + E(I - EE^d\right)^{-1}\right) \times \left(Du(\theta) + f(\theta)\right)d\theta \quad (73)$$

when: $0 \leq t \leq \tau$,

$$x(t) = \left(X(t) + Y(t)\left(I + E(I - EE^d\right)^{-1} E\right)\varphi(0) + \int_0^t \begin{pmatrix} X(t-\theta)E^d B + \\ +Y(t-\theta)\left(I + E(I - EE^d\right)^{-1} B \end{pmatrix} \varphi(\theta-\tau)d\theta + \int_0^t \begin{pmatrix} X(t-\theta)E^d + \\ +Y(t-\theta)(I + E(I - EE^d))^{-1} \end{pmatrix} \left(Du(\theta) + f(\theta)\right)d\theta . \quad (74)$$

There $X(t)$ is the first class of foundation solution of singular differential systems with delay, $Y(t)$ is the second class of foundation solution of singular differential systems with delay.

The main results

Consider the case when the subspace of consistent initial conditions for *singular time delay* and *singular nondelay system* coincide.

Owens-Debelković's approach

Theorem 11. Suppose that the matrix pair $\left(E, A_0\right)$ is *regular* with the system matrix A_0 being non-singular, i.e. $\det A_0 \neq 0$.

The system (2) is *asymptotically stable*, independent of delay, if there exists a positive definite matrix P, being the solution of Lyapunov's matrix equation

$$A_0^T PE + E^T PA_0 = -2\left(S + Q\right), \quad (75)$$

with matrices $Q = Q^T > 0$ and $S = S^T$, such that:

$$\mathbf{x}^T(t)\left(S + Q\right)\mathbf{x}(t) > 0, \quad \forall \mathbf{x}(t) \in W_{k^*} \setminus \{0\}, \quad (76)$$

is a positive definite quadratic form on $W_{k^*} \setminus \{0\}$, W_{k^*} being the subspace of consistent initial conditions[1], and if the following condition is satisfied:

$$\| A_1 \| < \sigma_{\min}\left(Q^{\frac{1}{2}}\right)\sigma_{\max}^{-1}\left(Q^{-\frac{1}{2}}E^T P\right), \quad (77)$$

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are maximum and minimum singular values of matrix $(\cdot)$, respectively.

Proof. Let the following function be considered:

$$V\left(\mathbf{x}(t)\right) = \mathbf{x}^T(t)E^T PE\mathbf{x}(t) + \int_{t-\tau}^{t} \mathbf{x}^T(\kappa)Q\mathbf{x}(\kappa)\,d\kappa, \quad (78)$$

Note that and *Lemma* A1[2] and *Theorem* A1 indicates that

$$V\left(\mathbf{x}(t)\right) = \mathbf{x}^T(t)E^T PE\mathbf{x}(t), \quad (79)$$

is a *positive quadratic form* on W_{k^*}, and it is obvious that all smooth solutions $\mathbf{x}(t)$ evolve in W_{k^*}, so $V\left(\mathbf{x}(t)\right)$ can be used as a *Lyapunov function* for the system under consideration, *Owens*, *Debelković* (1985).

It will be shown that the same argument can be used to declare the same property of another quadratic form present in (78).

Clearly, using the equation of motion of system, given (2), the following applies:

$$\dot{V}\left(\mathbf{x}(t)\right) = \mathbf{x}^T(t)\left(A_0^T PE + E^T PA_0 + Q\right)\mathbf{x}(t) + 2\mathbf{x}^T(t)\left(E^T PA_1\right)\mathbf{x}(t-\tau) - \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau), \quad (80)$$

and after some manipulations, yields to:

$$\dot{V}\left(\mathbf{x}(t)\right) = \mathbf{x}^T(t)\left(A_0^T PE + E^T PA_0 + 2Q + 2S\right)\mathbf{x}(t) + 2\mathbf{x}^T(t)\left(E^T PA_1\right) - \mathbf{x}^T(t)Q\mathbf{x}(t) - \mathbf{x}^T(t)S\mathbf{x}(t) - \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau) . \quad (81)$$

From (77) and the fact that the choice of matrix S, can be done, such that:

$$\mathbf{x}^T(t)S\mathbf{x}(t) \geq 0, \quad \forall \mathbf{x}(t) \in W_{k^*} \setminus \{0\}, \quad (82)$$

the following result is obtained:

[1] W_{k^*} subspace of consistent initial conditions, *Owens*, *Debeljković* (1985).

[2] See *Apendix* A.

$$\Delta\dot{V}(\mathbf{x}(t)) \le 2\mathbf{x}^T(t)\left(E^T P A_1\right)\mathbf{x}(t-\tau) -\mathbf{x}^T(t)Q\mathbf{x}(t) - \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau), \quad (83)$$

and based on the well known inequality[3]:

$$2\mathbf{x}^T(t)E^T P A_1 \mathbf{x}(t-\tau) = 2\mathbf{x}^T(t)\left(E^T P A_1 Q^{-\frac{1}{2}} Q^{\frac{1}{2}}\right)\mathbf{x}(t-\tau) \le \mathbf{x}^T(t)E^T P A_1 Q^{-1} A_1^T P E^T \mathbf{x}(t) + \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau), \quad (84)$$

and by substituting into (83), it yields:

$$\dot{V}(\mathbf{x}(t)) \le -\mathbf{x}^T(t)Q\mathbf{x}(t) + \mathbf{x}^T(t)E^T P A_1 Q^{-1} A_1^T P E\mathbf{x}(t), \quad (85)$$

or:

$$\dot{V}(\mathbf{x}(t)) \le -\mathbf{x}^T(t)Q^{1/2}\Gamma Q^{1/2}\mathbf{x}(t), \quad (86)$$

with matrix Γ defined by:

$$\Gamma = \left(I - Q^{-1/2}E^T P A_1 Q^{-1/2} Q^{-1/2} A_1^T P E Q^{-1/2}\right). \quad (87)$$

$\dot{V}(\mathbf{x}(t))$ is a negative definite form if:

$$1 - \lambda_{\max}\left(Q^{-1/2}E^T P A_1 Q^{-1/2} Q^{-1/2-1} A_1^T P E Q^{-1/2}\right) > 0, \quad (88)$$

which is satisfied if:

$$1 - \sigma_{\max}^2\left(Q^{-1/2}E^T P A_1 Q^{-1/2}\right) > 0. \quad (89)$$

Using the properties of the singular matrix values, *Amir-Moez* (1956), the condition (17) holds if:

$$1 - \sigma_{\max}^2\left(Q^{-1/2}E^T P\right)\sigma_{\max}^2\left(A_1 Q^{-1/2}\right) > 0, \quad (100)$$

which is satisfied if:

$$1 - \frac{\|A_1\|^2 \sigma_{\max}^2\left(Q^{-1/2}E^T P\right)}{\sigma_{\min}^2(\Omega)^{1/2}} > 0, \quad (101)$$

what completes the proof.

Remark 2. Equations (75-76) are modified forms taken from *Owens*, *Debelković* (1985).

Remark 3. If the system under consideration is just an ordinary time delay, e.g. $E = I$, the result is identical to that presented in *Tissir*, *Hmamed* (1996).

Remark 4. Let first the case when the time delay *is absent* be discussed.

Then the *singular* (weak) Lyapunov's matrix equation (75) is naturally the generalization of classical Lyapunov's theory.

In particular

If E is *non-singular matrix*, then the system is asymptotically stable if and only if $A = E^{-1}A_0$ Hurwitz matrix.

Equation (75) can be written in the form:

$$A^T E^T P E + E^T P E A = -Q, \quad (102)$$

with matrix Q being symmetric and a positive definite form, in the whole state space, since then

$$W_{k^*} = \Re(E^{k^*}) = \mathbb{R}^n.$$

[3] $2\mathbf{u}^T(t)\mathbf{v}(t) \le \mathbf{u}^T(t)P\mathbf{u}(t) + \mathbf{v}^T(t)P^{-1}\mathbf{v}(t),\ P > 0$

In those circumstances $E^T P E$ is a Lyapunov's function for the system.

The matrix A_0 is by necessity non-singular and hence the system has the form

$$E_0\dot{\mathbf{x}}(t) = \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0. \quad (103)$$

Then for this system to be stable (75) it must also hold and have familiar Lyapunov's structure

$$E_0^T P + P E_0 = -Q, \quad (104)$$

where Q is a symmetric matrix but it is only required to be a positive definite form on W_{k^*}.

Remark 5. There is no need for the system under consideration to posses properties given in *Definition* 2, since this is obviously guaranteed by demand that all smooth solutions $\mathbf{x}(t)$ evolve in W_{k^*}.

Example 1. Consider the linear continuous singular time delay system with matrices as follows:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0{,}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Based on the above presented procedure the following data, can easily be found, *Debeljković et. al.* (2006):

$$\hat{E} = (\lambda E + A_0)^{-1}_{|\lambda=0} \cdot E$$

$$\hat{E} = A_0^{-1}E = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \Rightarrow$$

$$\hat{E}^D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbb{N}\left(I - \hat{E}\hat{E}^D\right) = \left(I - \hat{E}\hat{E}^D\right)\mathbf{x}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\mathbf{x}_0 = 0, \quad \Rightarrow$$

$$\mathbb{N}\left(I - \hat{E}\hat{E}^D\right) = W_{k^*} = \left\{\mathbf{x} : x_1 \in \mathbb{R},\ x_2 \in \mathbb{R},\ x_2 = -x_3\right\}.$$

$$\det A_0 \ne 0, \quad \exists\lambda \mapsto \det(\lambda E - A_0) \ne 0,$$

$$rang\, E = 2, \quad \deg\det(sE - A_0) = 2.$$

It can be adopted that:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = Q^T > 0,$$

$$S=S^T=\begin{bmatrix}0&1&1\\1&0&-1\\1&-1&-1\end{bmatrix},$$

$$S+Q=\begin{bmatrix}1&1&1\\1&1&-1\\1&-1&0\end{bmatrix}\Rightarrow$$

$$\begin{aligned}\mathbf{x}^T(t)S\mathbf{x}(t)&=\begin{bmatrix}x_1&x_2&x_3\end{bmatrix}\begin{bmatrix}0&1&1\\1&0&-1\\1&-1&-1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\\&=\left(2x_1x_2+2x_1x_3-2x_2x_3-x_3^2\right)_{x_2=-x_3}\\&=\left(2x_1\left(x_2+x_3\right)+2x_3^2-x_3^2\right)_{x_2=-x_3}\\&=x_3^2>0,\ \forall\mathbf{x}(t)\in W_{k^*}\setminus\{0\}\end{aligned}.$$

$$\begin{aligned}\mathbf{x}^T(t)Q\mathbf{x}(t)&=\begin{bmatrix}x_1&x_2&x_3\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\\&=\left(x_1^2+x_2^2+x_3^2\right)_{x_2=-x_3}\\&=x_1^2+2x_2^2>0,\quad\forall\mathbf{x}(t)\in W_{k^*}\setminus\{0\}\end{aligned}$$

Moreover:

$$\mathbf{x}^T(t)Q\mathbf{x}(t)=x_1^2+x_2^2+x_3^2>0,\quad\forall\mathbf{x}(t)\in W\ .$$

$$\det Q=1\neq 0\ ,$$

and:

$$\begin{aligned}\mathbf{x}^T(t)\left(S+Q\right)\mathbf{x}(t)&=\begin{bmatrix}x_1&x_2&x_3\end{bmatrix}\begin{bmatrix}1&1&1\\1&1&-1\\1&-1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\\&=\left(x_1^2+x_2^2+2\left(x_2+x_3\right)+2x_2^2\right)_{x_2=-x_3}\\&=x_1^2+3x_2^2>0,\quad\forall\mathbf{x}(t)\in W_{k^*}\setminus\{0\}\end{aligned}$$

Also it can be computed:

$$\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}p_{11}&p_{12}&p_{13}\\p_{12}&p_{22}&p_{23}\\p_{13}&p_{23}&p_{33}\end{bmatrix}\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&0\end{bmatrix}+$$

$$+\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&0\end{bmatrix}\begin{bmatrix}p_{11}&p_{12}&p_{13}\\p_{12}&p_{22}&p_{23}\\p_{13}&p_{23}&p_{33}\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix}=-2\begin{bmatrix}1&1&1\\1&1&-1\\1&-1&0\end{bmatrix}$$

$$\begin{aligned}&P=\begin{bmatrix}1&0&2\\0&3&-2\\2&-2&p_{33}\end{bmatrix},\Rightarrow\qquad\Delta_3\left(p_{33}\right)>0,\Rightarrow\\&p_{33}>\frac{16}{3}\quad\Rightarrow\quad P=P^T>0\end{aligned}.$$

Generally it can be adopted:

$$P=\begin{bmatrix}1&0&2\\0&3&-2\\2&-2&6\end{bmatrix}=P^T>0\,.$$

Finally, condition (76) has to be checked:

$$\left\|A_1\right\|=0{,}10\qquad\sigma\left\{Q\right\}=\left\{1,1,1\right\},$$

$$Q^{\frac{1}{2}}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad Q^{-\frac{1}{2}}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},$$

$$Q^{-\frac{1}{2}}E^TP=\begin{bmatrix}-1&-\frac{1}{2}&1\\-\frac{1}{2}&-3&2\\0&0&0\end{bmatrix}$$

$$\sigma_{\min}\left(Q^{\frac{1}{2}}\right)=1\ ,\quad\sigma_{\max}=\left(Q^{-\frac{1}{2}}E^TP\right)=3.82\ ,$$

$$0.10=\|A_1\|<\frac{\sigma_{\min}\left(Q^{\frac{1}{2}}\right)}{\sigma_{\max}\left(Q^{-\frac{1}{2}}E^TP\right)}<0.26\ ,$$

so, the system under consideration is asymptotically stable. For the sake of further investigation let, for the previous case, the situation when $E=I$ be adopted.

Then it can be calculated:

$$\sigma\left\{A_0\right\}=\left\{\lambda_1,\lambda_2,\lambda_3\right\}=\left\{-1,\ -1,\ -1\right\},\quad\lambda_{\max}=-1.$$

$$\left\|A_0\right\|=\sigma_{\max}\left(A_0\right)=\sqrt{\lambda_{\max}\left(A_0^TA_0\right)}\,,$$

$$A_0+A_0^T=\begin{bmatrix}-2&0&0\\0&-2&-1\\0&-1&-2\end{bmatrix},$$

$$\lambda_i\left(A_0+A_0^T\right)=\left\{\lambda_1,\ \lambda_2,\ \lambda_3\right\}=\left\{-1,\ -2,\ -3\right\},$$

$$\mu\left(A_0\right)=\frac{1}{2}\lambda_{\max}\left(A_0+A_0^T\right)=-\frac{1}{2}.$$

Following *Mori et al.* (1981):

$$\mu\left(A_0\right)+\left\|A_1\right\|<0,\ \mapsto-0.5+0.10=-0.4<0.$$

the asymptotic stability of the *non-delay* system under consideration is confirmed.

Moreover:

$$\begin{aligned}\mathbf{x}^T(t)E^TPE\mathbf{x}(t)&=\begin{bmatrix}x_1&x_2&x_3\end{bmatrix}\begin{bmatrix}1&0&0\\0&3&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\\&=\left(x_1^2+3x_2^2\right)_{x_2=-x_3}>0,\quad\forall\mathbf{x}(t)\in W_{k^*}\setminus\{0\}\end{aligned}$$

so $V\left(\mathbf{x}(t)\right)$ can be used as a *Lyapunov's function* for the system (2).

Pandolffi approach

Our result is stated as follows:

Theorem 12. Suppose that the system matrix A_0 is non-singular, i.e. $\det A_0\neq 0$.

Then system (2) with the known compatible vector valued function of initial conditions can be considered and it can be assumed that $rank\,E_0=r<n$.

Matrix E_0 is defined in the following way $E_0 = A_0^{-1}E$

The system (2) is *asymptotically stable*, independent of delay, if

$$\| A_1\| < \sigma_{\min}\left(Q^{1/2}\right)\sigma_{\max}^{-1}\left(Q^{-1/2}E_0^T P\right), \qquad (105)$$

and if there exists:

(i) $(n\times n)$ matrix P, being the solution of Lyapunov's matrix:

$$E_0^T P + PE_0 = -2I_\Omega, \qquad (106)$$

with the following properties:

a) $P = P^T$ (107a)

b) $P\mathbf{q}(t) = 0, \quad \mathbf{q}(t)\in\Lambda$ (107b)

c) $\mathbf{q}^T(t)P\mathbf{q}(t) > 0, \quad \mathbf{q}(t)\neq 0, \quad \mathbf{q}(t)\in\Omega$, (107c)

where:

$$\Omega = \mathbb{N}\left(I - EE^D\right), \qquad (108)$$

$$\Lambda = \mathbb{N}\left(EE^D\right), \qquad (109)$$

with matrix I_Ω representing generalized operator on $\mathbb{R}^n$ and identity matrix on subspace Ω and zero operator on subspace Λ and matrix Q being any positive definite matrix.

Moreover matrix P is a symmetric and positive definite form on the subspace of consistent initial conditions.

Here $\sigma_{\max}(\cdot)$ and $\sigma_{\min}(\cdot)$ are maximum and minimum singular values of matrix $(\cdot)$, respectively.

Proof. If (106) has the solution P with the properties (107), then matrix E_0 cannot have eigenvalues with positive real parts, *Pandolfi* (1980). Hence, the system (2) *without delay* is stable.

Let matrix P be defined in the following way

$$\mathbf{q}^T(t)P\mathbf{q}(t) = \int_0^\infty \left(\left\| e^{E_0 t}E\,\mathbf{q}(t)\right\|\right)^2 dt. \qquad (110)$$

The integral equal zero if $\mathbf{q}(t)\in\Lambda$, and is finite if $\mathbf{q}(t)\in\Omega$.

Then it is clear that P is the solution of (106) with properties (107), *Pandolfi* (1980).

Remark 6. Equations (106 - 107) are taken from *Pandolfi* in a modified form (1980).

Remark 7 It is obvious that Ω corresponds the subspace of consistent initial conditions, *Campbell* (1980) or *Owens*, *Debeljković* (1985) there denoted with W_{k^*}.

Remark 8. So the stability of (2) *without delay* is proven, *Pandolfi* (1980).

In the sequel the rest of the proofs are presented, establishing conditions under which (LCSTDS) will be asymptotically stable.

Let the function be considered:

$$V\left(\mathbf{x}(t)\right) = \mathbf{x}^T(t)E^T PE\mathbf{x}(t) + \int_{t-\tau}^{t}\mathbf{x}^T(\kappa)Q\mathbf{x}(\kappa)\,d\kappa, \qquad (111)$$

Note that result presented in *Owens*, *Debeljković* (1985), indicates that

$$V\left(\mathbf{x}(t)\right) = \mathbf{x}^T(t)E^T PE\mathbf{x}(t), \qquad (112)$$

is a *positive quadratic form* on $\Omega = W_{k^*}$, and it is obvious that all smooth solutions $\mathbf{x}(t)$ are involved in W_{k^*}, so $V\left(\mathbf{x}(t)\right)$ can be used as a *Lyapunov's function* for the system under consideration, *Owens*, *Debeljković* (1985).

It will be shown that the same argument can be used to declare the same property of another quadratic form present in (111).

Clearly, using the equation of motion of (2), it can be stated that:

$$\begin{aligned}\dot V\left(\mathbf{x}(t)\right) = &\ \mathbf{x}^T(t)\left(E_0^T P + PE_0 - 2I_\Omega\right)\mathbf{x}(t)\\ &+ 2\mathbf{x}^T(t)\left(E_0 PA_1^T\right)\mathbf{x}(t-\tau)\\ &- \mathbf{x}^T(t)Q\mathbf{x}(t)\mathbf{x}^T - 2\mathbf{x}^T(t)S\mathbf{x}(t)\\ &- \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau)\end{aligned} \qquad (113)$$

where matrix I_Ω is defined by:

$$I_\Omega = Q + S, \qquad (114)$$

with the symmetric matrix $S = S^T$, with the following property:

$$\mathbf{x}^T(t)S\mathbf{x}(t)\geq 0, \ \forall\mathbf{x}(t)\in\Omega. \qquad (115)$$

and after some adjustemnts, following the ideas presented in *Tissir*, *Hmamed* (1996), it yields to:

$$\begin{aligned}\dot V\left(\mathbf{x}(t)\right) = &\ 2\mathbf{x}^T(t)\left(E_0^T PA_1\right)\mathbf{x}(t-\tau)\\ &-\mathbf{x}^T(t)Q\mathbf{x}(t) - \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau)\end{aligned}, \qquad (116)$$

and based on the well known inequality:

$$\begin{aligned}&2\mathbf{x}^T(t)E_0^T PA_1\mathbf{x}(t-\tau) = 2\mathbf{x}^T(t)\left(E_0^T PA_1Q^{-1/2}Q^{1/2}\right)\mathbf{x}(t-\tau)\\ &\leq \mathbf{x}^T(t)E_0^T PA_1Q^{-1}A_1^T PE_0^T\mathbf{x}(t) + \mathbf{x}^T(t-\tau)Q\mathbf{x}(t-\tau)\end{aligned}, \qquad (117)$$

and by substituting into (116), it yields:

$$\dot V\left(\mathbf{x}(t)\right)\leq -\mathbf{x}^T(t)Q\mathbf{x}(t) + \mathbf{x}^T(t)E_0^T PA_1Q^{-1}A_1^T PE_0\mathbf{x}(t), \qquad (118)$$

or:

$$\dot V\left(\mathbf{x}(t)\right)\leq -\mathbf{x}^T(t)Q^{1/2}\ \Psi\, Q^{1/2}\mathbf{x}(t), \qquad (119)$$

with matrix Ψ defined by:

$$\Psi = \left(I - Q^{-1/2}E_0^T PA_1Q^{-1/2}Q^{-1/2}A_1^T PE_0Q^{-1/2}\right). \qquad (120)$$

$\dot V\left(\mathbf{x}(t)\right)$ is a negative definite form if:

$$1-\lambda_{\max}\left(Q^{-1/2}E_0^T PA_1Q^{-1/2}Q^{-1/2}A_1^T PE_0Q^{-1/2}\right) > 0, \qquad (121)$$

which is satisfied if:

$$1-\sigma_{\max}^{2}\left(Q^{-1/2}E_0^T P A_1 Q^{-1/2}\right)>0 . \qquad (122)$$

Using the properties of the singular matrix values, *Amir-Moez* (1956), the condition (122) holds if:

$$1-\sigma_{\max}^{2}\left(Q^{-1/2}E_0^T P\right)\sigma_{\max}^{2}\left(A_1 Q^{-1/2}\right)>0 , \qquad (123)$$

which is satisfied if:

$$1-\frac{\left\| A_1\right\|^2 \sigma_{\max}^{2}\left(Q^{-1/2}E_0^T P\right)}{\sigma_{\min}^{2}\left(Q^{1/2}\right)}>0 , \qquad (124)$$

that completes the proof.

In the sequel, there is an example to show the effectiveness of the proposed method.

Example 2. Consider the linear continuous singular time delay system with matrices as follows:

$$E_0=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 1 & 0\end{bmatrix},\quad A_0=I,\quad A_1=\begin{bmatrix}0{,}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Based on the given data, it can be calculated:

$$E_0^D=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 1 & 0\end{bmatrix},\quad E_0E_0^D=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -1 & 0\end{bmatrix},$$

$$I-E_0E_0^D=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}.$$

$$\mathbb{N}\left(I-E_0E_0^D\right)=\left(I-E_0E_0^D\right)\mathbf{x}_0=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & 1\end{bmatrix}\mathbf{x}_0=0,\Rightarrow$$

$$\mathbb{N}\left(I-\hat{E}\hat{E}^D\right)=\Omega=\left\{\mathbf{x}: x_1\in\mathbb{R},\ x_2\in\mathbb{R},\ x_2=-x_3\right\}.$$

$$\mathbb{N}\left(E_0E_0^D\right)=\left(E_0E_0^D\right)\mathbf{x}_0=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & -1 & 0\end{bmatrix}\mathbf{x}_0=0,\ \Rightarrow$$

$$\mathbb{N}\left(E_0E_0^D\right)=\Lambda=\left\{\mathbf{x}: x_1=0,\ x_2=0,\ x_3\in\mathbb{R}^n, x_3=1\right\}$$

$$\det A_0\neq 0,\quad \exists\lambda \ni \det(\lambda E-A_0)\neq 0,$$
$$rang\ E=2,\ \deg\det(sE-A_0)=2.$$

It can be adopted:

$$Q=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 1\\ 0 & 1 & 1\end{bmatrix}=Q^T>0,\quad S=S^T=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & -1\end{bmatrix},$$

$$\Rightarrow\ I_\Omega=S+Q=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 0\end{bmatrix}\ \Rightarrow$$

$$\mathbf{x}^T(t)S\mathbf{x}(t)=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=$$

$$=\left(-2x_2x_3-x_3^2\right)_{x_2=-x_3}=2x_3^2-x_3^2=x_3^2>0,\ \forall\mathbf{x}(t)\in\Omega$$

$$\begin{aligned}\mathbf{x}^T(t)Q\mathbf{x}(t)&=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 1\\ 0 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\\ &=\left(2x_1^2+2x_2^2+2x_2x_3+x_3^2\right)\\ &=\left(2x_1^2+x_2^2+x_2^2+2x_2x_3+x_3^2\right),\\ &=\left(2x_1^2+x_2^2+\left(x_2+x_3\right)^2\right)_{x_2=-x_3}\\ &=2x_1^2+x_2^2>0,\quad \forall\mathbf{x}(t)\in\Omega\end{aligned}$$

$$\det Q=2\neq 0 ,$$

$$\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{12} & p_{22} & p_{23}\\ p_{13} & p_{23} & p_{33}\end{bmatrix}+$$
$$+\begin{bmatrix}p_{11} & p_{12} & p_{13}\\ p_{12} & p_{22} & p_{23}\\ p_{13} & p_{23} & p_{33}\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}-2 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & 0\end{bmatrix},$$

with solution:

$$P=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix},$$

In the sequel, the properties of matrix P are checked.

a) $P=P^T$.

b)
$$\begin{aligned}P\mathbf{q}(t)&=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\mathbf{q}(t)\\ &=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}=0,\quad \forall\mathbf{q}(t)\in\Lambda\end{aligned}.$$

c)
$$\begin{aligned}\mathbf{q}^T(t)P\mathbf{q}(t)&=\mathbf{q}^T(t)\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\mathbf{q}(t)\\ &=\begin{bmatrix}0 & q_2(t) & -q_3(t)\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\ q_2(t)\\ -q_3(t)\end{bmatrix}\\ &=q_2^2(t)>0,\quad \mathbf{q}(t)\neq 0,\quad \forall\mathbf{q}(t)\in\Omega\end{aligned}$$

Moreover there is a need to check (5).

Based on:

$$Q=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 1\\ 0 & 1 & 1\end{bmatrix}\ \wedge\ A_1=\begin{bmatrix}0{,}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

simple calculations yield to:

$$\|A_1\| = 0.10 \quad \sigma\{Q\} = \{\, 2.62 \quad 2.00 \quad 0.38 \,\},$$

$$Q^{1/2} = \begin{bmatrix} 1.41 & 0 & 0 \\ 0 & 1.34 & 0.45 \\ 0 & 0.45 & 0.90 \end{bmatrix}, \quad Q^{-1/2} = \begin{bmatrix} 0.71 & 0 & 0 \\ 0 & 0.90 & -0.45 \\ 0 & -0.45 & 1.34 \end{bmatrix}$$

$$Q^{-1/2} E_0^T P = \begin{bmatrix} 0.71 & 0 & 0 \\ 0 & -0.90 & 0 \\ 0 & 0.45 & 0 \end{bmatrix},$$

$$\sigma_{\min}\left(Q^{1/2}\right) = 0.62 \;\wedge\; \sigma_{\max} = \left(Q^{-1/2} E_0^T P\right) = 1.00 .$$

$$0{,}10 = \| A_1 \| < \frac{\sigma_{\min}(Q)^{1/2}}{\sigma_{\max}(Q^{-1/2} E_0^T P)} < 0.26 ,$$

so, the system under consideration is asymptotically stable.

Moreover:

$$\mathbf{x}^T(t) E^T P E \mathbf{x}(t) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \left(x_1^2 + 3x_2^2\right)_{x_2 = -x_3} > 0, \quad \forall \mathbf{x}(t) \in W_{k^*} \setminus \{0\}$$

so $V(\mathbf{x}(t))$ can be used as a *Lyapunov's function* for the system (2).

Conclusion

Quite new, sufficient delay–independent criteria for asymptotic stability of (LCSTDS) are presented. In some sense this result may be treated as the further extension of results derived in *Debeljković et. al* (2006).

In comparison with some other papers on this matter, there is neither the need for linear transformations of the basic system, nor need of solving the systems of high order linear matrix in the qualities. State space solutions are given in two different ways as useful tool for checking the presented results.

Numerical examples are presented to show the applicability of the derived results.

Appendix A

The fundamental geometric tool in the characterization of the subspace of consistent initial conditions is the subspace sequence for *linear singular system without delay*

$$W_0 = \mathbb{R}^n , \tag{A1}$$

$$\vdots$$

$$W_{j+1} = A_0^{-1}(E W_j), \;\; j \ge 0 , \tag{A2}$$

where $A_0^{-1}(\cdot)$ denotes the inverse image of $(\cdot)$ under the operator A_0.

Lemma A1. The subsequence $\{W_0, W_1, W_2, ...\}$ is nested in the sense that:

$$W_0 \supset W_1 \supset W_2 \supset W_3 \supset ... \tag{A3}$$

Moreover:

$$\mathbb{N}(A) \subset W_j, \; \forall j \ge 0, \tag{A4}$$

and there exists an integer $k \ge 0$, such as that:

$$W_{k+1} = W_k. \tag{A5}$$

Then it is obvious that:

$$W_{k+j} = W_k, \;\; \forall j \ge 1. \tag{A6}$$

If k^* is the smallest such integer with this property, then:

$$W_k \cap \mathbb{N}(E) = \{0\}, \;\; k \ge k^* , \tag{A7}$$

provided that $\left(\lambda E - A_0\right)$ is invertible for some $\lambda \in \mathbb{R}$.

Theorem A.1. Under the conditions of *Lemma A*1, $\mathbf{x}_0$ is a consistent initial condition for the system under consideration if and only if $\mathbf{x}_0 \in W_{k^*}$.

Moreover $\mathbf{x}_0$ generates a unique solution $\mathbf{x}_0 \in W_k^*$, $t \ge 0$, that is really analytic on $\{t : t \ge 0\}$.

Proof. See *Owens*, *Debeljković* (1985).

References

[1] AMIR-MOEZ,A.: *Extreme Properties of a Hermitian Transformations and Singular Values of Sum and Product of Linear Transformations*, Duke Math J., 23 (1956), pp.463–476.

[2] CAMPBELL,S.L.: *Singular systems of differential equation*, Pitman, London, 1980.

[3] CAMPBELL,S.L.: *Singular Systems of Differential Equations II*, Pitman, Marshfield, Mass., 1982.

[4] DEBELJKOVIĆ,D.LJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B.: *Application of singular system theory in chemical engineering : Analysis of process dynamics*, Internat. Congress of Chemical and Process Eng., CHISA 96, Prague (Czech Rep.), 25.-30. August, 1996.

[5] DEBELJKOVIĆ,D.LJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B.: *Continuous Singular Control Systems*, GIP Kultura, Belgrade, 2005.a.

[6] DEBELJKOVIĆ,D.LJ., MILINKOVIC,S.A., JOVANOVIC,M.B., JACIC,Lj.A.: *Discrete Singular Control Systems*, GIP Kultura, Belgrade, 2005.b.

[7] DEBELJKOVIĆ,D.LJ., STOJANOVIĆ,S.B., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B.: *Singular Time Delayed System Stability Theory in the sense of Lyapunov: New Results*, Nonlinear Analysis - 2006 (accepted).

[8] MORI,T., FUKUMA,N., KUWAHARA,M.: *Simple Stability Criteria for Single and Composite Linear Systems with Time Delays*, Int. J. Control 34 (6) (1981) pp.1175–1184.

[9] OWENS,H.D., DEBELJKOVIĆ,D.LJ.: *Consistency and Liapunov Stability of Linear Descriptor Systems: A Geometric Analysis*, IMA Journal of Mathematical Control and Information, (2), (1985), pp.139-151.

[10] PANDOLFI L.: *Controlabillity and Stabilization for Linear System of Algebraic and Differential Equations*, Jota 30 (4) (1980), pp.601–620.

[11] SARIĆ,B.: *On fundamental matrix of linear time - invariant time - delayed singular systems*, SACTA, (2001), Vol.4, No.1, pp.11-25.

[12] SARIĆ,B.: *Dynamical Behavior Analysis of Linear Implicit-Singular Control Systems with delayed State on finite and Infinite Time Interval*, Ph.D., Dept. of Control Eng., Faculty of Mech. Eng., Univ. of Belgrade, July 2002a.

[13] SARIĆ,B.: *Coditions for convergence of the fundamental matrix of linear time – invariant time - delayed singular systems*, Int. J. of Math. and Math. Science, (2002b),Vol.31, No.8, pp.463–475.

[14] TISSIR,E., HMAMED,A.: *Further Results on Stability of* $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{x}(t-\tau)$, Automatica, 32 (12) (1996), pp.1723–1726.

[15] XU,S., YANG,C.: *An Algebraic Approach to the Robust Stability Analysis and Robust Stabilization of Uncertain Singular Systems*, Int. J. System Science, (2000.a),Vol.31, pp.55–61.

[16] XU,S., YANG,C.: *H_∞ State Feedback Control for Discrete Singular Systems*, IEEE Trans. Automat. Control AC-45 (6) (2000.b), pp.1405–1409.

[17] XU,S., YANG,C., NIU,Y., LAM,J.: *Robust Stabilization for Uncertain Discrete Singular Systems*, Automatica, (2001.a),Vol.37, pp.769–774.

[18] XU,S., LAM,J., YANG,C., NIU,Y.: *Quadratic Stability and Stabilization of Uncertain Linear Discrete – time Systems with State Delay*, Systems Control Let. (43), (2001.b), pp.77–84.

[19] XU,S., DOOREN,P.V., STEFAN,R., LAM,J.: “Robust Stability and Stabilization for Singular Systems with State Delay and Parameter Uncertainty”, I E E E Trans. Automat. Control AC-47 (7) (2002), pp.1122–1128.

[20] XU,S., LAM,J., YANG,C.: *Robust H_∞ Control for Discrete Singular Systems with State Delay and Parameter Uncertainty*, Dynamics Continuous, Discrete Impulsive Systems, (2004).

Received: 15.01.2006.

Asimptotska stabilnost singularnih kontinualnih sistema sa čistim vremenskim kašnjenjem

U ovom radu izvedeni su dovoljni uslovi posebne klase linearnih kontinualnih singularnih sistema sa čistim vremenskim kašnjenjem, čija se reprezentacija u prostoru stanja može predstaviti sledećom vektorskom diferencijalnom jednačinom stanja $E\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau)$. Ovi novi uslovi, koji ne uključuju iznos čisto vremenskog kašnjenja, u eksplicitan kriterijum dobijeni su standardnim prilazom koji počiva na Ljapunovskoj stabilnosti. U tom smislu korišćena su dva, ranije razvijena prilaza data u radovima Owens, Debeljković-a (1985) i Pandolfi-a (1980).

Ključne reči: **kontinualni system, singularni system, linearni system, sistem sa kašnjenjem, vremensko kašnjenje, stabilnost sistema, asimptotska stabilnost, ljapunovska stabilnost.**

Устойчивость асимптоты сингулярных непрерывных систем со чистой временной задержкой

В настоящей работе выведены довольные условия особого класса линейных непрерывных сингулярных систем со чистой временной задержкой, чью презентацию состояния в просторе возможно представить следующим векториальным дифференциальным уравнением состояния $E\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau)$. Эти новые условия, которые не включают сумму чистой временной задержки в явный критерий, получены стандартным подходом, обоснованым на устойчивости Ляпунова. В этом смысле пользованы два, раньше развиты подходы, представлены в работах Оввенса, Дебельковича (1985.) и Пандолфия (1980.). Очевидными численными примерами показано применение выведеных результатов.

Ключевые слова: непрерывная система, сингулярная система, линейная система, система со чистой временной задержкой, временная задержка, устойчивость системы, устойчивость асимптоты, устойчивость Ляпунова.

La stabilité asymptotique des systèmes singuliers continus à délai temporel pur

Dans ce travail on donne les conditions suffisantes de classe particulière des systèmes linéaires continus et singuliers à délai temporel pur dont la représantation dans l’espace d’état peut être présentée par une équation vectoriele différentielle d’état $E\dot{\mathbf{x}}(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau)$. Ces nouvelles conditions, qui ne comprennent pas le total du délai temporel pur dans le critère explicite, ont été obtenues par approche standardisée, basée sur la stabilité de Lyapunov. A cet effet on a utilisé deux approches développées auparavent dans les travaux d’Owens, Debeljakovic (1985.) et Pandolfi (1980.). L’applicabilité des résultats obtenus est démontrée au moyen des exemples numériques.

***Mots clés*: système continu, système singulier, système linéaire, système à délai, délai temporel, stabilité du système, stabilité asymptotique, stabilité de Lyapunov.**