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Strapdown Attitude Algorithms using Quaternion Transition Matrix and Random Inputs

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The algorithms based on the transition matrix and quaternion concept were developed for determination of the vehicle's attitude. They are constraint-preserving integrators and overcome the difficulty of some published algorithms which need first, second and third derivatives of vehicle's angular rates. The effects of quantization of the gyroscopes impulse on the accuracy of the body attitude were studied and two types of estimators were proposed to estimate a vehicle's attitude if the integrated angular rate is corrupted with uniformly distributed random noise. A series of numerical experiments were conducted to quantify the proposed estimators for coning motion with different motion's frequencies. The proposed estimators are useful for the specified domain of the frequency of motion and sampling rate.

Key words: **flight mechanics, navigation, navigational system, inertial guidance, spatial attitude, estimators, quaternion, estimation, numerical algorithm.**

Nomenclature

$\mathbf{b}^{A_1}$ – vector $\boldsymbol{b}$ with components in A_1 frame

$\boldsymbol{\omega}_{A_1A_2}^{A_2}$ – angular rate of A_2 frame relative to A_1 frame expressed with components in A_2 frame

$\otimes$ – quaternion multiplication operator

q_0, q_1, q_2 and q_3 – four quaternion elements

$\mathbf{q}$ – column matrix of quaternion elements $\mathbf{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T$

$\mathbf{q}_3$ – column matrix of quaternion vector elements $\mathbf{q}_3 = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^T$

$\boldsymbol{\omega}_{ib}^{b}$ – angular rate of b frame relative to i frame expressed in b frame

$\boldsymbol{\varphi}$ – rotation angle vector

$\boldsymbol{\alpha}$ – vector of incremental gyroscope angles

Definition of important matrices

$(\boldsymbol{\omega}^A \times)$ – skew symmetric matrix with components of $\boldsymbol{\omega}$ in A frame

$$(\boldsymbol{\omega}^A \mathrm{x}) = \boldsymbol{\Omega}(\boldsymbol{\omega}^A) = \begin{bmatrix} 0 & -\omega_{z_A} & \omega_{y_A} \\ \omega_{z_A} & 0 & -\omega_{x_A} \\ -\omega_{y_A} & \omega_{x_A} & 0 \end{bmatrix}$$

Superscripts

~ – approximated values

^ – estimated values

Abbreviations

INS – inertial navigation system

SDINS – strap down inertial navigation system

DCM – direction cosine matrix

i – inertial reference frame

b – body reference frame

Introduction

THE primary functions executed in the INS computer are the angular rate into attitude integration function (denoted as attitude algorithms), use of the attitude data to transform measured acceleration into a suitable navigation coordinate frame where it is integrated into velocity and integration of the navigation frame velocity into position (denoted as velocity and position algorithms). Thus three integration functions are involved: attitude, velocity, and position, each of which requires high accuracy to assure negligible error compared to inertial sensors accuracy requirements.

Strapdown analysts have focused on the design of algorithms for the attitude integration function.

Wilcox in 1967 [1] developed quaternion and direction cosine matrix algorithms for a digital computer and he found that the quaternion algorithms require less computer time and give less truncation error than the corresponding direction cosine matrix algorithms when it used in appropriate algorithm.

Due to the computer throughput limitations a two speed approach was proposed by Savage in 1966 for the attitude integration function whereby the attitude updating operation is divided into two parts: a simple high speed first order algorithm portion coupled with a more complex moderate-speed higher order algorithm portion whose input was provided by the high speed algorithm.

Bortz in 1970 [2] has developed a differential equation for the orientation vector relating the body frame to a chosen frame. He found that the time derivative of this vector is the

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sum of the inertially measurable angular rate and of the inertially non-measurable non-commutatively rate (coning) vector which causes the computational problems when numerically integrated direction cosine matrix. His formulation of the orientation vector allows the coning contribution to be isolated and treated separately and advantageously.

Miller in 1983 [3] developed application to the SDINS attitude problem of the rotation vector concept. He obtained a solution for the rotation vector and updated the attitude quaternion separately.

The direction cosine matrix algorithms are used in many aerospace systems to transform vectorial quantities from one reference frame to another. Due to the computer errors and errors introduced by mathematical algorithm the orthogonality property of DCM will be destroyed and should be restored. Itzhack in 1969 [4] introduced three orthogonalization techniques to correct errors in the computed DCM and restore the orthogonality property. In [5] the same author proposed iterative optimal orthogonalization of the DCM using the directional derivative method. By applying this method the closed form solution of the problem of finding an optimal orthogonal matrix is easy derived.

An algorithm for solving the DCM by using digital differential approach was given by Bodanskiy in 1974 [6]. In order to integrate the differential equations of a rigid body angular motion given by the DCM two types of digital procedures were used. The first is digital computer of general type with constant sampling interval and variable magnitude of the increments in angular position of the rigid body. And the second is the digital differential approach when the increments of the rigid body attitude are constant and the sampling time is variable. The estimations of the errors for these two types of algorithms were developed in the case of arbitrary angular motion.

Savage in 1998 [7] used the two speed approach where an analytically exact equation is used at moderate speed to update the attitude parameters with input provided from a high speed algorithm measuring the dynamic angular rate and acceleration effects with the parameter update time interval (coning effect).

Waldmann in 2002 [8] investigated a variety of attitude determination algorithms using DCM, quaternion and rotation vector as attitude parameters to access the tradeoffs between computational complexity and accuracy when it used for terrestrial navigation. He found that in terms of navigation errors the relative quaternion parameterization is the most adequate for pure inertial terrestrial navigation under coning motion conditions. The same conclusion was obtained by Miller in 1983 [3].

Friedland in 1978 [9] presented basic equations of SDINS using quaternion for attitude determination along with the resulting equations for error analysis. His equations were mechanized in inertial reference frame which is widely used in space flight application.

Shibata in 1986 [10] introduced the SDINS errors equations based on quaternion for terrestrial navigation where the Earth rotation should be considered. These equations will contribute greatly to construction of hybrid navigation systems. The optimal control of the propagation of the quaternion errors in spacecraft navigation was given by Vathsal [11] where the of normalization of quaternion is required to meet the constraint of unity condition. His algorithm can be implemented without increasing very much the computer loading.

Because the closed from solution of the quaternion propagation differential equation is unattainable this differential equation should be integrated numerically. One of the methods is by using the transition matrix approach given by Chelnokov in 1977 [12].

Mayo in 1979 [13] developed the transition matrix for the calculation of the relative quaternion between body and a rotating reference frame (non-inertial frame). This matrix is function of the angular rate of the reference frame given with components in same system, and the absolute angular rate presented with components in the body reference frame. In this case it wasn't necessary to determine transformation matrix from body frame to the reference frame for the integration of the differential equations for the relative quaternion.

Chiou and Yan in 2001[14] derived generalized constraint-preserving integrators for solving quaternion equations. This family of time integrators was based on the property of the skew symmetric matrix. Their proposed forth order algorithm has a very high computational cost since it needs to compute the second derivative of the angular rate. This difficulty can be overcome using the proposed quaternion algorithm in terms of transition matrix and the given expressions of the second derivative of the angular rate in terms of gyroscope outputs.

The main aim of this paper (part of [18]) is to develop a comprehensive approach to design of the principal software algorithms utilized in modern day strapdown inertial navigation systems in the presence of noise: integration of angular rate into attitude.

Differential equation of a quaternion and its transition matrix

In order to use the quaternion attitude representation it is necessary to solve the following differential equation (Titterton, [15], page 48)

$$\dot{\mathbf{q}} = \frac{1}{2}\ \mathbf{q} \otimes \mathbf{p}_q \tag{1}$$

where $\mathbf{p}_q$ is the quaternion representation of the body angular rate $\mathbf{p} = \boldsymbol{\omega}_{ib}^b$ and it has the following form

$$\mathbf{p}_q = \begin{bmatrix} 0 & \boldsymbol{\omega}_{ib}^{b\,\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \tag{2}$$

The eq.(1) may be expressed in the following matrix form

$$\dot{\mathbf{q}} = \frac{1}{2}\ \mathbf{W}\,\mathbf{q} \tag{3}$$

where

$$\mathbf{W} = \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix} \tag{4}$$

The equation (4) may be written as

$$\mathbf{W}(\boldsymbol{\omega}) = \begin{bmatrix} 0 & \left(-\boldsymbol{\omega}_{ib}^b\right)^T \\ \boldsymbol{\omega}_{ib}^b & -\boldsymbol{\Omega}(\omega_{ib}^b) \end{bmatrix} \tag{5}$$

with the skew symmetric matrix $\boldsymbol{\Omega}(\omega)$ which is 3x3 matrix, that corresponding to the vector cross product $\boldsymbol{\omega}_{ib}^b \times \mathbf{p} = \boldsymbol{\Omega}(\boldsymbol{\omega}_{ib}^b)\mathbf{p}$.

The closed form solution of eq.(3) is unattainable and it must be integrated numerically in order to update the quaternion. The signals of the angular rate taken from gyroscopes has form of increments with time step integration of Δt and it is assumed constant. The incremental gyroscope outputs for the time step size of Δt are

$$\boldsymbol{\alpha}_{i+1} = \begin{bmatrix} \alpha_{1,i+1} \\ \alpha_{2,i+1} \\ \alpha_{3,i+1} \end{bmatrix} = \int_{t_*+i\Delta t}^{t_*+(i+1)\Delta t} \boldsymbol{\omega}\, dt \tag{6}$$

where t_* is the mean point for the interval of updating the quaternion. It is necessary to find algorithm for the solution of eq.(3) with time step size equal to $H = 2s\Delta t$ where s is equal to 1,2,3, and depends on the algorithm order.

If the initial time for integration of eq.(3) is t_0 and the initial value of the quaternion is $\mathbf{q}(t_0) = \mathbf{q}_0$, then the value of quaternion at any time $\mathbf{q}(t)$ in the time interval defined by the time limits, $t_0 \le t \le t_0 + H$, can be found by the following equation

$$\mathbf{q}(t) = \mathbf{F}(t, t_0)\mathbf{q}_0 \tag{7}$$

where $\mathbf{F}(t,\ t_0)$ is the transition matrix of eq.(3). The transition matrix differential equation has a similar form of eq.(3) [17]:

$$\dot{\mathbf{F}}(t,\ t_0) = \ \frac{1}{2}\mathbf{W}\left[\boldsymbol{\omega}(t)\right]\ \mathbf{F}(t,\ t_0) \tag{8}$$

The initial transition matrix is equal to $\mathbf{F}(t_0, t_0) = \mathbf{I}_4$, where $\mathbf{I}_4$ is 4x4 identity matrix. eq.(7) represents the quaternion updating that accounts for b frame rotation relative to frame from its orientation at time t_0 to its new orientation at time t. This representation can be interpreted as the product between two quaternions $\mathbf{q}(t, t_0)$ and $\mathbf{q}_0$.

Having in mind the rule for the quaternion product given in Appendix A by eq.(A.14) the transition matrix $\mathbf{F}(t,\ t_0)$ in eq.(9) has the following form

$$\mathbf{F}(t, t_0) = \begin{bmatrix} f_0(t) & -f_1(t) & -f_2(t) & -f_3(t) \\ f_1(t) & f_0(t) & f_3(t) & -f_2(t) \\ f_2(t) & -f_3(t) & f_0(t) & f_1(t) \\ f_3(t) & f_2(t) & -f_1(t) & f_0(t) \end{bmatrix} \tag{9}$$

or, with rearrangement

$$\mathbf{F}(t, t_0) = f_0(t)\,\mathbf{I}_4 + \mathbf{W}(\mathbf{f}_3(t)) \tag{10}$$

where $\mathbf{f}_3(t) = \left[f_1(t) \quad f_2(t) \quad f_3(t)\right]^{\mathrm{T}}$ and the initial conditions are defined as

$$f_0(t_0) = 1 \qquad \mathbf{f}_3(t_0) = \mathbf{0}$$

$$f_0^{\,2}(t) + \mathbf{f}_3^{\mathrm{T}}(t)\mathbf{f}_3(t) = 1 \tag{11}$$

The last expression of the initial conditions given in eq.(11) represents the quaternion unitary condition. For notations simplicity, $f_0(t)$ and $\mathbf{f}_3(t)$ were written to represent $f_0(t, t_0)$ and $\mathbf{f}_3(t, t_0)$ respectively.

If the time is equal $t = t_0 + H$ then elements of transition matrix will be $f_0(t_0 + H)$ and $f_3(t_0 + H)$. Having in mind eq.(10), the eq.(7) can be transformed for $t = t_0 + H$ into the next equation

$$\begin{bmatrix} q_0(t_0+H) \\ \mathbf{q}_3(t_0+H) \end{bmatrix} = \Big\{ f_0(t_0+H)\mathbf{I}_4 + \\ + \begin{bmatrix} 0 & -\mathbf{f}_3^{\mathrm{T}}(t_0+H) \\ \mathbf{f}_3(t_0+H) & -\boldsymbol{\Omega}\left(f_3(t_0+H)\right) \end{bmatrix} \Big\} \begin{bmatrix} q_0(t_0) \\ \mathbf{q}_3(t_0) \end{bmatrix} \tag{12}$$

Development of transition matrix

For simplicity, a new notation will be used for the matrix part in Eqs. (3) and (8)

$$\dot{\mathbf{q}} = \mathbf{A}(t)\,\mathbf{q}(t) \tag{13}$$

$$\dot{\mathbf{F}}(t, t_0) = \mathbf{A}(t)\,\mathbf{F}(t, t_0) \tag{14}$$

where the matrix $\mathbf{A}(t)$ is given by

$$\mathbf{A}(t) = \frac{1}{2}\mathbf{W}\left[\boldsymbol{\omega}_{ib}^{b}(t)\right] \tag{15}$$

In order to find a solution for eq.(14) the method of successive approximations will be used for determination of transition matrix. If $t = t_0$ the transition matrix becomes unity matrix

$$\mathbf{F}(t_0, t_0) = \mathbf{I}_4 \tag{16}$$

The successive approximations for $\mathbf{F}_k(t, t_0)$, $k = 1, 2, 3, \ldots$, will be found from recursive relation

$$\frac{\mathrm{d}\,\mathbf{F}_k(t, t_0)}{\mathrm{d}t} = \mathbf{A}(t)\,\mathbf{F}_{k-1}(t, t_0) \tag{17}$$

where $k = 1,2,....$

By solving the first approximation $\mathbf{F}_0(t, t_0)$ is equal to unity matrix $\mathbf{I}_4$. From eq.(11) we can represent $\mathbf{F}_k(t, t_0)$ in the following form

$$\mathbf{F}_k(t, t_0) = \mathbf{I}_4 + \int_{t_0}^{t} \mathbf{A}(t)\,\mathbf{F}_{k-1}(t, t_0). \tag{18}$$

where $k = 1,2,.$

In a such way, the successive approximations are

$$\mathbf{F}_0(t, t_0) = \mathbf{I}_4$$

$$\mathbf{F}_1(t, t_0) = \mathbf{I}_4 + \int_{t_0}^{t} \mathbf{A}(t)\,\mathrm{d}t$$

$$\mathbf{F}_2(t, t_0) = \mathbf{I}_4 + \int_{t_0}^{t} \mathbf{A}(t)\left(\mathbf{I}_4 + \int_{t_0}^{t} \mathbf{A}(t)\,\mathrm{d}t\right)\mathrm{d}t = \\ = \mathbf{I}_4 + \int_{t_0}^{t} \mathbf{A}(t)\,dt + \int_{t_0}^{t} \mathbf{A}(t)\left(\int_{t_0}^{t} \mathbf{A}(t)\,dt\right)dt$$

$$\mathbf{F}_3(t, t_0) = \mathbf{I}_4 + \int_{t_0}^{t} \mathbf{A}(t)\,\mathbf{F}_2(t, t_0)\,dt \tag{19}$$

Finally, the transition matrix becomes

$$\mathbf{F}(t, t_0) = \mathbf{I}_4 + \int_{t_0}^{t} \mathbf{A}(t)\,dt + \int_{t_0}^{t} \mathbf{A}(t)\left(\int_{t_0}^{t} \mathbf{A}(t)\,dt\right)dt + \cdots \tag{20}$$

In order to find the transition matrix at the interval time $[t_0, t_0+H]$ the angular rate will be approximated by Taylor's series. The derivatives of the angular rate and the matrix $\mathbf{A}(t)$ are determined for the mean value of the interval $[t_0, t_0+H]$ which is given by

$$t_* = t_0 + s\Delta t \tag{21}$$

Having in mind that $H = 2s\Delta t$, the initial and final value of the interval $[t_0, t_0+H]$ can be given in terms of its mean value t_* as

$$\begin{aligned} t_0 &= t_* - s\Delta t \\ t_0 + H &= t_* + s\Delta t \end{aligned} \qquad 22)$$

So, if the point of developing $\boldsymbol{\omega}(t)$ into Taylor's series is the centre of the interval t_* we can obtain

$$\begin{aligned} \boldsymbol{\omega}(t) &= \sum_{m=0}^{m} \frac{1}{m!}\,\boldsymbol{\omega}_*^m (t-t_*)^m \\ \mathbf{A}(t) &= \sum_{m=0}^{m} \frac{1}{m!}\,\mathbf{A}_*^m\,(t-t_*)^m \\ \boldsymbol{\omega}_*^m &= \boldsymbol{\omega}^m\,(t_*) \\ \mathbf{A}_*^m &= \frac{1}{2}\,\mathbf{W}(\boldsymbol{\omega}_*^m) \end{aligned} \tag{23}$$

where the time t is within the limits defined as $t_* - s\Delta t \le t \le t_* + s\Delta t$ and the superscript notation m represents the order of the derivative. Substituting eq.(22) into eq .(20), the transition matrix at the final value of time interval becomes

$$\mathbf{F}(t_0+H,t_0)=\mathbf{I}_4+\int_{t_*-s\Delta t}^{t_*+s\Delta t} A(t)dt+\int_{t_*-s\Delta t}^{t_*+s\Delta t} A(t)\left(\int_{t_*-s\Delta t}^{t_*+s\Delta t} A(t)dt\right)dt+\cdots \tag{24}$$

Substituting the expressions in eq.(23) into eq.(24) the transition matrix $\mathbf{F}(t_0+H, t_0)$ can be found using the functions $f_0(t)$ and $\mathbf{f}_3(t)$ presented by eq.(9). If the order of the algorithm is first order ($s=1$) then the interval of updating quaternion is

$$H = 2\Delta t \tag{25}$$

and two incremental gyroscope outputs around the centre of the interval $[t_0, t_0+2\Delta t]$ are needed for updating. From eq.(6) the required gyroscope outputs are given as following

$$\text{for } i=-1 \quad \boldsymbol{\alpha}_0 = \int_{t_*-\Delta t}^{t_*} \boldsymbol{\omega}(t)\,dt$$

and

$$\text{for } i=0 \quad \boldsymbol{\alpha}_0 = \int_{t_*}^{t_*+\Delta t} \boldsymbol{\omega}(t)\,dt \tag{26}$$

If the algorithm is a second order ($s=2$), then the interval of the updating quaternion will be

$$H = 4\Delta t \tag{27}$$

In this case four incremental gyroscopes outputs around the centre of the interval $[t_0, t_0+4\Delta t]$ are needed to update the quaternion

$$\begin{aligned} &\text{for } i=-2 \quad \boldsymbol{\alpha}_{-1} = \int_{t_*-2\Delta t}^{t_*-\Delta t} \boldsymbol{\omega}(t)\,dt \\ &\text{for } i=-1 \quad \boldsymbol{\alpha}_0 = \int_{t_*-\Delta t}^{t_*} \boldsymbol{\omega}(t)\,dt \\ &\text{for } i=0 \quad \boldsymbol{\alpha}_1 = \int_{t_*}^{t_*+\Delta t} \boldsymbol{\omega}(t)\,dt \\ &\text{and for } i=1 \quad \boldsymbol{\alpha}_2 = \int_{t_*+\Delta t}^{t_*+2\Delta t} \boldsymbol{\omega}(t)\,dt \end{aligned} \tag{28}$$

Eqs. (26) and (28) represent the gyroscope outputs for different types of algorithms. The approach used to develop these algorithms is to obtain the values of transition matrix elements from the gyroscope outputs and using this transition matrix to perform the attitude updating by solving the quaternion multiplications.

Transition matrix components ($f_0(t)$ and $\mathbf{f}_3(t)$)

The eq.(7) with eq.(9).is the relation for the determination quaternion which transforms vector from b frame axes at time t to the i frame axes at time t_0. By using frame notation eq.(7) can be written as

$$\mathbf{q}_{b(t)}^{i(t_0)} = \mathbf{F}(t, t_0)\,\mathbf{q}_{b(0)}^{i(0)} \tag{29}$$

where

- $\mathbf{q}_{b(0)}^{i(0)}$ - The quaternion relating b frame at time t_0 to the i frame at same time t_0
- $\mathbf{q}_{b(t)}^{i(t_0)}$ - The quaternion relating b frame at time t to i frame at time t_0.
- $\mathbf{F}(t, t_0)$ - The transition matrix defined by f_0 and $\mathbf{f}_3 = [f_1 \;\; f_2 \;\; f_3]^{\mathrm{T}}$ in eq.(9)

By using the quaternion product the eq.(29) can be represented by chain rule as following

$$\mathbf{q}_{b(t)}^{i(0)} = \mathbf{q}_{b(0)}^{i(0)} \otimes \mathbf{q}_{b(t)}^{b(0)} \tag{30}$$

or, in opposite order

$$\mathbf{q}_{b(t)}^{i(0)} = \mathbf{q}_{b(t)}^{b(0)} \otimes \mathbf{q}_{b(0)}^{i(0)} \tag{31}$$

The comparison of eq.(30) with eq.(29) having in mind eq.(A.14) gives

$$\mathbf{q}_{b(t)}^{b(t_0)} = \left[f_0(t) \;\; f_1(t) \;\; f_2(t) \;\; f_3(t) \right]^{\mathrm{T}} \tag{32}$$

The quaternion $\mathbf{q}_{b(t)}^{b(t_0)}$ transforms a vector from b frame axes at time t to the b frame axes at time t_0 and can be updated for the body rotations as sensed by the strapdown gyroscopes. The quaternion in eq.(32) represents a rotation of specified magnitude of $\boldsymbol{\varphi}$ about a vector with variable direction $\boldsymbol{\varphi}$. This attitude quaternion can be given in terms of rotation vector [15]

$$\mathbf{q}_{b(t)}^{b(t_0)} = \begin{bmatrix} \cos 0.5\varphi \\ \dfrac{\sin 0.5\varphi}{\varphi}\;\boldsymbol{\varphi} \end{bmatrix} \tag{33}$$

where the rotation vector is given by

$$\boldsymbol{\varphi} = \left[\varphi_x \;\; \varphi_y \;\; \varphi_z \right]^{\mathrm{T}} \tag{34}$$

and φ - angle of finite rotation.

The vector $\boldsymbol{\varphi}$ determines the angle of rotation of body frame from its position at time t_0 to position at time t. The comparison of eq.(32) with eq .(33) under eq.(34) gives the elements of the transition matrix as following

$$\begin{aligned} f_0 &= \cos 0.5\,\varphi \\ f_1 = f_x &= \frac{\sin 0.5\varphi}{0.5\varphi}\ 0.5\varphi_x \\ f_2 = f_y &= \frac{\sin 0.5\varphi}{0.5\varphi}\ 0.5\varphi_y \\ f_3 = f_z &= \frac{\sin 0.5\varphi}{0.5\varphi}\ 0.5\varphi_z \end{aligned} \tag{35}$$

It can be shown from eq.(35) that

$$f_0 = \left(1 - \|\mathbf{f}_3\|^2\right)^{\frac{1}{2}} \tag{36}$$

where

$$\mathbf{f}_3 = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix}^{\mathrm{T}} \tag{37}$$

Algorithm for calculation of rotation vector $\boldsymbol{\varphi}$

The general expression for the rate of change of rotation vector is given in (Titterton, [15] , page 301)

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{\omega}(t) + \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\varphi}(t))\,\boldsymbol{\omega}(t) + \frac{1}{12}\boldsymbol{\Omega}(\boldsymbol{\varphi}(t))\left[\boldsymbol{\Omega}(\boldsymbol{\varphi}(t))\,\boldsymbol{\omega}(t)\right] \tag{38}$$

The triple cross product term will be assumed to be small enough to be neglected:

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{\omega}(t) + \frac{1}{2}\boldsymbol{\Omega}(\boldsymbol{\varphi}(t))\,\boldsymbol{\omega}(t) \tag{39}$$

The angular rate $\boldsymbol{\omega}(t)$ can be approximated by Taylor's series

$$\boldsymbol{\omega}(t) = \sum_m \frac{1}{m!}\,\boldsymbol{\omega}_*^{(m)}\,\left(t - t_*\right)^m \tag{40}$$

$$\boldsymbol{\omega}_*^{(m)} = \boldsymbol{\omega}^{(m)}\left(t_*\right) \tag{41}$$

where $\boldsymbol{\omega}_*^{(m)}$ is the m derivative of the angular rate for the mean time of the interval estimation.

In order to estimate the rotation vector $\boldsymbol{\varphi}$ from eq.(39), the method of successive approximations will be used. Neglecting the cross product in eq.(39) will give the first approximation as

$$\boldsymbol{\varphi} \approx \boldsymbol{\alpha}(t) = \int \boldsymbol{\omega}(t)\,dt \tag{42}$$

Or, substituting $\boldsymbol{\omega}(t)$ from eq.(40) into eq.(42), yields to

$$\boldsymbol{\alpha}(t) = \sum_m \frac{1}{(m+1)!}\,\boldsymbol{\omega}_*^{(m)}\,\left(t - t_*\right)^{m+1} \tag{43}$$

For the time t in the time interval $\left[t_0, t_0 + H\right]$ where $H = 2s\Delta t$

$$t_0 \le t \le t_0 + 2s\Delta t \tag{44}$$

$$t_* - s\Delta t \le t \le t_* + 2s\Delta t \ \text{ where } \ s = 1,2,.... \tag{45}$$

Substituting eq.(42) into eq.(39) and by integration the following results is obtained:

$$\boldsymbol{\varphi}(t_0 + H, t_0) = \boldsymbol{\alpha}(t_0 + H, t_0) + \frac{1}{2}\int_{t_0}^{t_0+H} \boldsymbol{\alpha}(t, t_0) \times \boldsymbol{\omega}(t)\,dt \tag{46}$$

where $\boldsymbol{\alpha}(t, t_0)$ is given by the following expression

$$\boldsymbol{\alpha}(t, t_0) = \sum_m \frac{1}{(m+1)!}\,\boldsymbol{\omega}_*^{(m)}\left[\,(t - t_*)^{m+1} - (t_0 - t_*)^{m+1}\right] \tag{47}$$

The first approximation eq.(42) is developed as the sum of increments in the interval $\left[t_0, t_0 + H\right]$ or $\left[t_* - s\Delta t, t_* + s\Delta t\right]$. This approximation is given by

$$\boldsymbol{\alpha}(t_0 + H, t_0) = \sum_i \boldsymbol{\alpha}_{i+1} \tag{48}$$

$$\boldsymbol{\alpha}_{i+1} = \sum_m \frac{1}{(m+1)!}\,\boldsymbol{\omega}_*^{(m)}\left[\,(i+1)^{m+1} - (i)^{m+1}\right]\Delta t^{m+1} \tag{49}$$

By substitute Eqs. (47) and (48) with eq.(49) into eq.(46) the following expression for rotation vector is obtained

$$\begin{aligned} &\boldsymbol{\varphi}(t_0 + H, t_0) = \\ &= \sum_i \sum_m \frac{1}{(m+1)!}\boldsymbol{\omega}_*^{(m)}\left[(i+1)^{m+1} - (i)^{m+1}\right]\Delta t^{m+1} + \\ &+ \frac{1}{2}\sum_i \int_{t_*-i\Delta t}^{t_*+i\Delta t} \boldsymbol{\alpha}(t, t_0) \times \boldsymbol{\omega}(t)\,dt \end{aligned} \tag{50}$$

The first order algorithm ($s = 1$, $H = 2\Delta t$)

In the case of first order algorithm the angular rate $\boldsymbol{\omega}(t)$ is for $m \le 1$ and it is given by

$$\boldsymbol{\omega}(t) = \boldsymbol{\omega}_*^{(0)} + \boldsymbol{\omega}_*^{(1)}\,(t - t_*) \tag{51}$$

where

$$\begin{aligned} \boldsymbol{\omega}_*^{(0)} &= \boldsymbol{\omega}(t_*) \\ \boldsymbol{\omega}_*^{(1)} &= \left.\frac{d\,\boldsymbol{\omega}}{dt}\right|_{t=t_*} \end{aligned} \tag{52}$$

The incremental gyroscope outputs for this algorithm are illustrated in Fig.1.

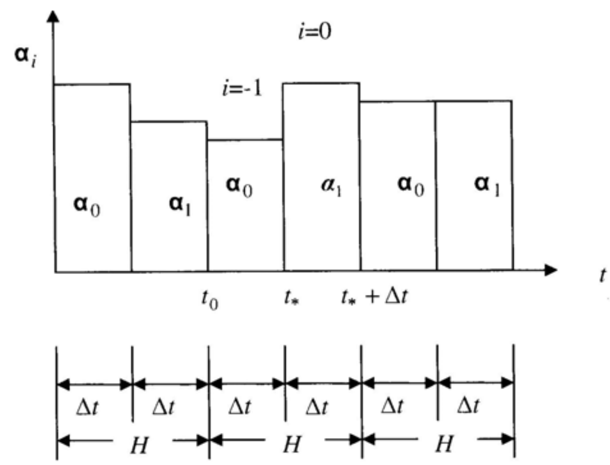

Figure 1. The incremental gyroscope outputs for the algorithm with $s = 1$

The incremental gyroscope outputs are obtained from eq.(49) as following

For $i=-1$ (before the mean time t_*) is

$$\boldsymbol{\alpha}_0 = \vec{\boldsymbol{\omega}}_*^{(0)}\ \Delta t - \frac{1}{2}\boldsymbol{\omega}_*^{(1)}\ \Delta t^2 \tag{53}$$

For $i=0$ (after the mean time t_*)

$$\boldsymbol{\alpha}_1 = \boldsymbol{\omega}_*^{(0)}\ \Delta t + \frac{1}{2}\boldsymbol{\omega}_*^{(1)}\ \Delta t^2 \tag{54}$$

From Eqs. (53) and (54) the expression for $\boldsymbol{\omega}_*^{(0)}$ and $\boldsymbol{\omega}_*^{(1)}$ can written as

$$\begin{aligned} \boldsymbol{\omega}_*^{(0)}\ \Delta t &= \frac{1}{2}(\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1) \\ \boldsymbol{\omega}_*^{(1)}\ \Delta t^2 &= (\boldsymbol{\alpha}_1 - \boldsymbol{\alpha}_0) \end{aligned} \tag{55}$$

By using (52) the expression for the rotation angle [50] becomes:

$$\boldsymbol{\varphi}(t_0+H, t_0) = \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 + \frac{1}{2}\sum_{i=-1}^{i=0}\int_{t_*-i\Delta t}^{t_*+i\Delta t} \boldsymbol{\alpha}(t,t_0)\times\boldsymbol{\omega}(t)\,dt \tag{56}$$

The solution of eq.(56) gives the components f_0 and f_x, f_y, f_z for eq.(35) that are identical to Eqs.(B.1) and (B.3).

The second order algorithm ($s=2$, $H=2\Delta t$)

The angular rate $\boldsymbol{\omega}(t)$ is given for $m\le 3$ **:**

$$\boldsymbol{\omega}(t) = \boldsymbol{\omega}_*^{(0)} + \boldsymbol{\omega}_*^{(1)}(t-t_*) + \frac{1}{2}\boldsymbol{\omega}_*^{(2)}(t-t_*)^2 + \frac{1}{6}\boldsymbol{\omega}_*^{(3)}(t-t_*)^3 \tag{57}$$

where

$$\begin{aligned} \vec{\boldsymbol{\omega}}_*^{(0)} &= \boldsymbol{\omega}(t_*) \\ \boldsymbol{\omega}_*^{(m)} &= \frac{d^m}{dt^m}[\boldsymbol{\omega}(t)]\Big|_{t=t_*} \end{aligned} \tag{58}$$

The incremental gyroscope outputs for this algorithm are illustrated in Fig.2.

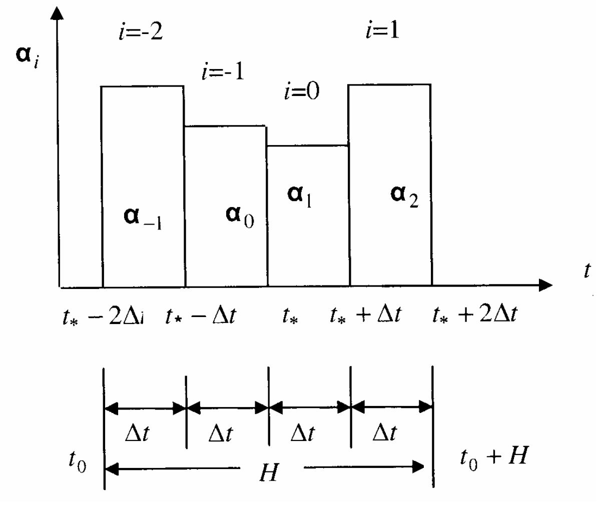

Figure 2. The incremental gyroscope outputs for the algorithm with $s=2$

The incremental gyroscope outputs are illustrated in Fig.2 and they are given as follows

$$\begin{aligned} \boldsymbol{\alpha}_{-1} &= \sum_{m=0}^{3}\frac{1}{(m+1)!}\,\boldsymbol{\omega}_*^{(m)}\ \left[(-1)^{m+1} - (-2)^{m+1}\right]\Delta t^{m+1} \\ \boldsymbol{\alpha}_0 &= \sum_{m=0}^{3}\frac{1}{(m+1)!}\,\boldsymbol{\omega}_*^{(m)}\ \left[(0)^{m+1} - (-1)^{m+1}\right]\Delta t^{m+1} \\ \boldsymbol{\alpha}_1 &= \sum_{m=0}^{3}\frac{1}{(m+1)!}\,\boldsymbol{\omega}_*^{(m)}\ \left[(1)^{m+1} - (0)^{m+1}\right]\Delta t^{m+1} \\ \boldsymbol{\alpha}_2 &= \sum_{m=0}^{3}\frac{1}{(m+1)!}\,\boldsymbol{\omega}_*^{(m)}\ \left[(2)^{m+1} - (1)^{m+1}\right]\Delta t^{m+1} \end{aligned} \tag{59}$$

The system of four equations in eq.(4.139) gives the solution for four unknowns parameters $\left(\boldsymbol{\omega}_*^{(m)}, m=0,1,2,3\right)$ in terms of $\boldsymbol{\alpha}_{-1}, \boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$. Having in mind eq.(57) with solution for eq.(59) we can determine the rotation angle by using eq.(50) and f_0 and $\mathbf{f}_3$ for eq.(35) and eventually by (B.5) and (B.6). The transition matrix is given by (10) in terms of f_0 and $\mathbf{f}_3$.

Effect of quantization in gyro impulse on the accuracy of the body attitude

The increment of the integrated body angular rate $\Delta\boldsymbol{\alpha}_l$ of the gyro sensor during the sampling time interval T_l is obtained as the sum of the impulses $\delta\boldsymbol{\alpha}_{l,i}$ of the sensor during that period. The increment of integrated body rate is given by

$$\Delta\boldsymbol{\alpha}_l = \int_{t_{l-1}}^{t_l} d\boldsymbol{\alpha} = \int_{t_{l-1}}^{t_l} \boldsymbol{\omega}_{ib}^b\, dt = \sum_{i=1}^{i_0}\delta\boldsymbol{\alpha}_{l,i} \tag{60}$$

where i_0 is the number of impulses during the interval $T_l=\Delta t$ and $\delta\boldsymbol{\alpha}_{l,i}$ is the gyroscope impulse.

During the period where the incremental measurements or impulses are taken from the sensors within the l cycle a displacement between the sum of these impulses and the true increments of the integral of the angular rate may exist. This displacement might be a source of important errors. The physical interpretation of the nature of this displacement and the impulses from the gyroscope sensor are illustrated in Fig.3.

The nature of this effect and the method of reducing the associated error will be presented in this section. The following notion will be introduced defining the period of taking the impulses from the sensors

τ_{l-1} the instant of taking the last impulse during the previous period of l cycle $T_{(l-1)}$.

τ_l the instant of taking the last impulse during the current period of l cycle (T_l)

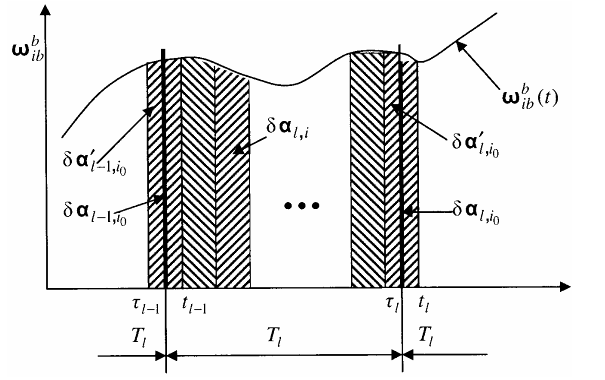

Figure 3. The impulses of gyroscope sensor during the period of l cycle

The measured increment of the angular rate $\Delta\boldsymbol{\alpha}_l$ can presented by using the following expression

$$\Delta\tilde{\boldsymbol{\alpha}}_l = \Delta\boldsymbol{\alpha}_l + \delta\boldsymbol{\alpha}'_{l-1,i_0} - \delta\boldsymbol{\alpha}'_{l,i_0} \tag{61}$$

where

$\Delta\tilde{\boldsymbol{\alpha}}_l$ - the approximated value of $\Delta\boldsymbol{\alpha}_l$.

$\Delta\boldsymbol{\alpha}_l$ - the exact value of angular rate increment.

$\delta\boldsymbol{\alpha}'_{l-1,i_0}$ - the last incremental impulse taken during the pervious l cycle.

$\delta\boldsymbol{\alpha}'_{l,i_0}$ - the last incremental impulse taken during the current l cycle.

The analysis will be done for the case of one component only for the reason of simplicity, for example $\delta\alpha'^{x}_{l-1,i_0}$ and $\delta\alpha'^{x}_{l,i_0}$ which will be denoted in the following text as $\delta\alpha'_{l-1}$ and $\delta\alpha'_l$ by omitting the superscript x and subscript i_0.

If the magnitude of the impulse is small enough, it can be considered that $\delta\alpha'_{l-1}$ and $\delta\alpha'_l$ are random variables with uniform probability density function distribution in the following interval depending on the sign of the angular rate

$$[-\sigma, 0] \quad \text{for} \quad \omega_x < 0 \tag{62}$$

or

$$[\ \sigma, 0] \quad \text{for} \quad \omega_x > 0 \tag{63}$$

The random variables $\delta\alpha'_{l-1}$ and $\delta\alpha'_l$ are uncorrelated. The error in the information on the angular rate ω_x during the T_r period can be expressed as

$$\delta\varepsilon_l = \delta\alpha'_{l-1} - \delta\alpha'_l \tag{64}$$

This error represents the error of the calculated inertial frame according to the exact position of inertial frame. Since the expectations (the mean values) for the components $\delta\alpha'_{l-1}$ and $\delta\alpha'_l$ are equal to

$$\mathrm{E}\left[\delta\alpha'_{l-1}\right] = \frac{1}{2}\sigma \,\mathrm{sign}\ \omega_x(t_{l-1}) \tag{65}$$

$$\mathrm{E}\left[\delta\alpha'_{l}\ \right] = \frac{1}{2}\sigma \,\mathrm{sign}\ \omega_x(t_{l}) \tag{66}$$

then the expectation of the x component of the error is equal to

$$\mathrm{E}\left[\delta\varepsilon_l\ \right] = \begin{cases} 0, & \omega_x(t_l)\omega_x(t_{l-1}) > 0 \\ -\sigma \,\mathrm{sign}\ \omega_x(t_l), & \omega_x(t_l)\omega_x(t_{l-1}) < 0 \end{cases} \tag{67}$$

Simulation of the increments from gyroscope sensors

The output from the gyro sensors are simulated numerically by adding the last incremental gyroscope output (impulse) taken at previous l cycle period $\delta\boldsymbol{\alpha}'_{l-1}$ to the exact value of gyroscope increment and subtracting the last incremental gyroscope output (impulse) taken at the current l cycle period $\delta\boldsymbol{\alpha}'_l$. This approach is given mathematically in eq.(61).

Since the increment $\Delta\boldsymbol{\alpha}_l$ from the gyroscope sensors are obtained by summing the gyroscope impulses during the l cycle, then $\Delta\boldsymbol{\alpha}_1$ should be an integer number of impulses. The function $\mathrm{FIX}(x)$ in MatLab was utilized to return an integer number of impulses in the quantity $\Delta\boldsymbol{\alpha}'_l = \Delta\boldsymbol{\alpha}_l + \delta\boldsymbol{\alpha}'_{l-1}$.

The Fig.4 shows the process of the numerical simulation of the measured value for the increment of the angular rate $\Delta\tilde{\boldsymbol{\alpha}}_l$.

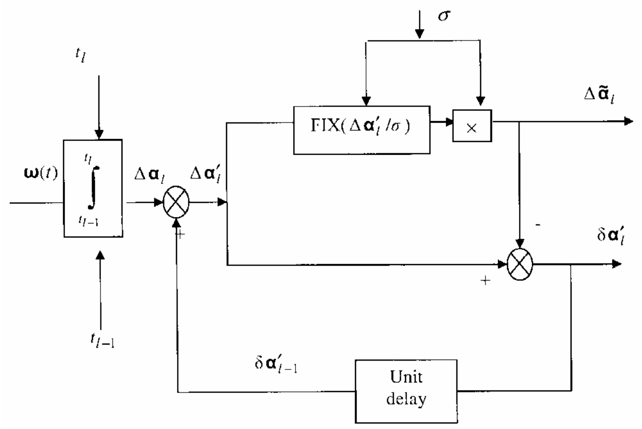

Figure 4. Numerical simulation of the measured increment angular rate $\Delta\tilde{\boldsymbol{\alpha}}_l$

As indicated in Fig.4 the approximated value of the increment of the angular rate is computed by multiplying the integer number resulted form the FIX function by the gyroscope impulse (σ). The last incremental gyroscope output obtained in current l cycle $\delta\boldsymbol{\alpha}'_l$ is simply computed as the difference between $\Delta\boldsymbol{\alpha}'_l$ and $\Delta\tilde{\boldsymbol{\alpha}}_l$. The last incremental gyroscope output for the current l cycle is assigned to be the impulse of the previous l cycle $\delta\boldsymbol{\alpha}'_{l-1}$.

Estimation of the increments from gyroscope sensors

It has been shown that the expectation of the gyroscope impulse is dependent on the sign of the angular rate. Suppose that the component of the angular rate ω_x changes its sign during the interval (t_{l-1}, t_l). There is an error due to quantization of the gyroscope increments. The calculated reference frame, i.e inertial frame, rotates relative to the exact position around the direction $\boldsymbol{e}_x$ for the angle with mathematical expectation of $-\sigma \,\mathrm{sign}\ \omega_x(t_l)$. If the change of the sign exists at other interval (t_{k-1}, t_k), the calculated inertial frame relative to the exact position around $\boldsymbol{e}_x$ for the angle with mathematical expectation of $\sigma \,\mathrm{sign}\ \omega_x(t_k)$.

Since the direction of $\boldsymbol{e}_x$ is not identical for the instant t_l and t_k, non-commutativity error appears. This error depends on the magnitude of the gyroscope impulse σ and the characteristics of the motion.

If successive change in the sign of the angular rate is existed, the non-commutative error will be accumulated for long period of time. In order to reduce the non commutative error, estimated values of the increments of integrated angular rate will be used instead of approximated values $\Delta\tilde{\boldsymbol{\alpha}}_l$. This estimated value are denoted as $\Delta\hat{\boldsymbol{\alpha}}_l$ in the following text two types of estimators will be proposed and their algorithms will be developed.

Estimator Type-1

In this type of estimation, the estimated values of the increment of the angular rate $\Delta\hat{\boldsymbol{\alpha}}_l$ is given for one component for simplifying the analysis as

$$\Delta\hat{\boldsymbol{\alpha}}_l = \Delta\tilde{\boldsymbol{\alpha}}_l + \frac{1}{2}\sigma\left(\mathrm{sign}\Delta\tilde{\boldsymbol{\alpha}}_l - \mathrm{sign}\Delta\tilde{\boldsymbol{\alpha}}_{l-1}\right) \qquad (68)$$

where

$\Delta\hat{\boldsymbol{\alpha}}_l$ - the estimated value of the current increment in the l cycle.

$\Delta\tilde{\boldsymbol{\alpha}}_l$ - the approximated value of the current increment in the l cycle.

$\Delta\tilde{\boldsymbol{\alpha}}_{l-1}$ - the approximated value of the previous increment in the l cycle.

σ - the gyroscope impulse.

The given algorithm based on eq.(68) provides reduction of the error due to quantization of the gyroscope increment. The estimator for the gyroscope increments according to this algorithm is shown in Fig.5.

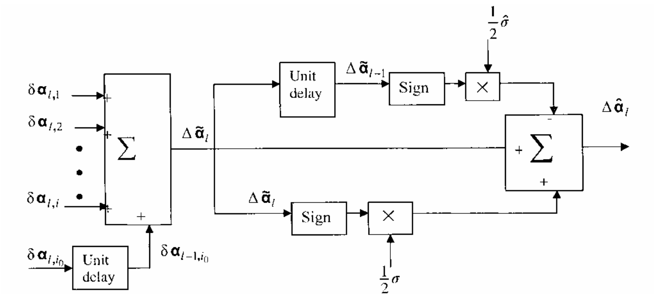

Figure 5. Estimator type-1

Estimator type-2

This type of estimator is applied on the approximated values of gyroscope increments to correct the increment components $\Delta\tilde{\boldsymbol{\alpha}}_l$ during the interval $t \in [t_{l-1}, t_l]$ by adding random variable $\mathbf{u}_l$ with uniform distribution of estimated impulse σ and the sign of the last impulse of the gyroscope rate $\delta\boldsymbol{\alpha}_{l,i_0-1}$ .This random variable are uniformly distributed within the interval defined as $[0 \quad \sigma\,\mathrm{sign}\delta\boldsymbol{\alpha}_{l,i_0-1}]$.

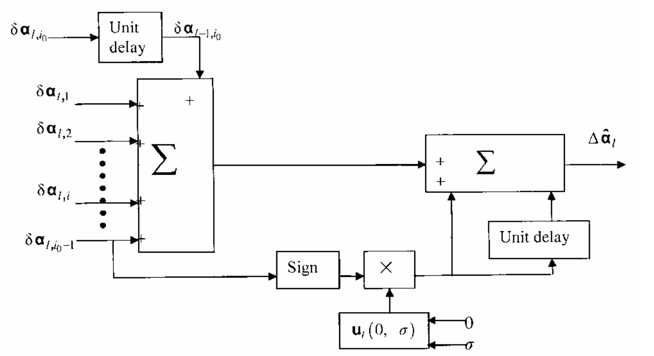

Figure 6. Estimator type 2

According to the change of the sign of the last impulse the random variable has two possible cases one positive and the other is negative. The proposed algorithm was developed based on the following expression

$$\Delta\hat{\boldsymbol{\alpha}}_l = \Delta\tilde{\boldsymbol{\alpha}}_l + \mathbf{u}_l - \mathbf{u}_{l-1} \qquad (69)$$

where $\mathbf{u}_l$ and $\mathbf{u}_{l-1}$ are uniformly distributed random variables of last impulse $\delta\boldsymbol{\alpha}_{l,i_0-1}$ for the current and the previous period of l cycle. This estimator of the second type is shown in Fig.6.

Quaternion attitude errors

The error in computed quaternion $\delta\mathbf{q}$ may be expressed in terms of the exact (true) and computed quaternion $\mathbf{q}_{b(t)}^{i(0)}$ relating b frame at time t to i frame at time $t = 0$.

$$\delta\mathbf{q} = \mathbf{q}_{i(0)}^{b(t)} \otimes \hat{\mathbf{q}}_{b(t)}^{i(0)} \qquad (70)$$

If the estimated value of the quaternion is equal to the exact value then the estimated attitude of the b frame will be identical to real position and the error quaternion will be unity quaternion. But, if an error exists in the computation the quaternion $\delta\mathbf{q}$ is no longer equal to unity quaternion. The quaternion $\delta\mathbf{q}$ relates estimated b frame at time t to the exact b frame at the same time. In general case the quaternion $\delta\mathbf{q}$ can be presented by

$$\delta\mathbf{q} = 1 + \frac{\chi_0}{2} + \frac{\boldsymbol{\chi}}{2} \qquad (71)$$

where χ_0 is a small scalar quantity and $\boldsymbol{\chi}$ is a three dimensional vector.

The drift is usually defined as the time rate of the error as follows

$$|\boldsymbol{\chi}'| \approx \frac{|\boldsymbol{\chi}|}{t - t_0} \approx \frac{|\Delta\boldsymbol{\varphi}|}{t - t_0} \qquad (72)$$

where

$$|\boldsymbol{\chi}| = 2\sqrt{(\delta q_1)^2 + (\delta q_2)^2 + (\delta q_3)^2} \qquad (73)$$

The small scalar quantity χ_0 is given by

$$\chi_0 = 2(\delta q_0 - 1) \qquad (74)$$

The components δq_0 , δq_1 , δq_2 and δq_3 are defined from the quaternion $\delta\mathbf{q}$ as

$$\delta\mathbf{q} = [\delta q_0 \quad \delta q_1 \quad \delta q_2 \quad \delta q_3]^{\mathrm{T}} \qquad (75)$$

Numerical tests and analysis

The following example will be used for studying and analyzing the attitude quaternion updating algorithms. The example is representing a coning motion and it is given in [12]. This coning motion can be described by the following body angular rate equations

$$\omega_x = a\sin\nu t, \quad \omega_y = a\cos\nu t, \quad \omega_z = \mathrm{c} = \mathrm{constant} \qquad (76)$$

where a, $\boldsymbol{\nu}$ and c are positive constants. For this numerical simulation these constants have the following values a =0.5 rad/s, $\boldsymbol{\nu}$ = 30 rad/s. and c=0.01 rad/s.

The explicit solution was obtained by solving the time rate of change of quaternion equation given in eq.(8) with the initial condition $\mathbf{q}(0)=\begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$ can be found in [12] or [16].

The effect of the quantization in the impulses from gyroscope sensors on the accuracy of the body attitude computed by the developed algorithms will be studied for different values of motion frequencies and updating time step. The obtained drift results will be compared with the results of exact quaternion solution. The numerical simulation were conducted for a motion with frequencies of f: 5, 10, 20, and 30 Hz and with sampling time steps of $\Delta t = T_l$: 0.005, 0.01, and 0.02 s and the value of gyroscope impulse was chosen to be 2.5x10^{-6}.

The drift results obtained by both first and second order algorithms are shown and discussed. The diagrams of the drift rate is given as the ratio of the real value of the drift to the reference value of $\chi'_{\mathrm{Ref}} = 1^{\circ}/\mathrm{hr} = 4.848\times 10^{-6}$ rad/s.

A comparison of results of the drift obtained using the second order algorithm for the three cases (no quantization, with quantization effect and estimation by estimator type 1) are illustrated in Fig.7.

For low frequency motion and all sampling rates good accuracy can be obtained. If the frequency of motion is equal to $f = 2{,}5\,\mathrm{Hz}$, the drift rate is increasing sixty times with double increase in time step. From the Fig.7 it can be seen that the computed drift is increased by increasing the frequency of the motion.

It is shown in Fig.7 that the quantization phenomenon affects the accuracy of the algorithm for all sampling time steps if the frequency of motion is less than 10 Hz. As the sampling time is increasing the use of the estimation will no longer improve the results especially at moderate and high motion frequencies. The similar results can be obtained by using the estimator type 2.

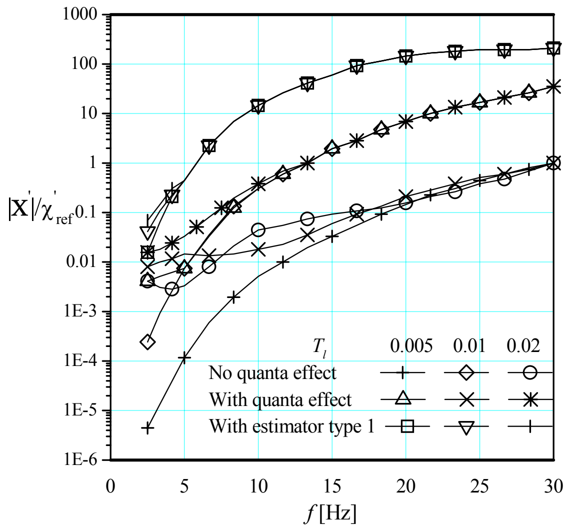

Figure 7. Effect of estimation on the drift using estimator type 1(s =2, $\dot{\chi}_{\mathrm{ref}} = 1^{\circ}/\mathrm{hr}$)

The drift results obtained by the first order algorithm are shown in Fig.8. The effect of the quantisation and estimation on the drift is less than in the case of the second order algorithm.

It is interesting to note that by using 2nd order algorithm with estimation sampled at medium sampling rate for the case of gyroscope with random inputs can produce a relatively similar accuracy of attitude computations obtained by 1st order algorithm sampled at high sampling rate.

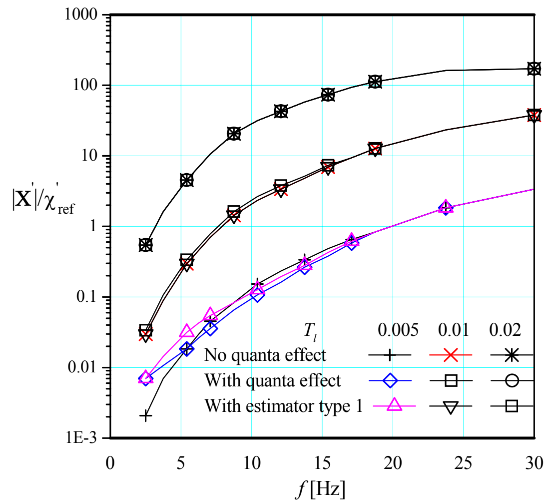

Figure 8. Effect of quantization and estimation on the drift (s=1, $\dot{\chi}_{\mathrm{ref}} = 1^{\circ}/\mathrm{hr}$)

The comparison of the computed drift by using the first and second order algorithm is shown in Fig.9 in terms of time and sampling rate for the frequency of motion $f = 5\,\mathrm{Hz}$. The first order algorithm with T_l=0.005s gives approximately the same results as the second order algorithm for T_l=0.01s.

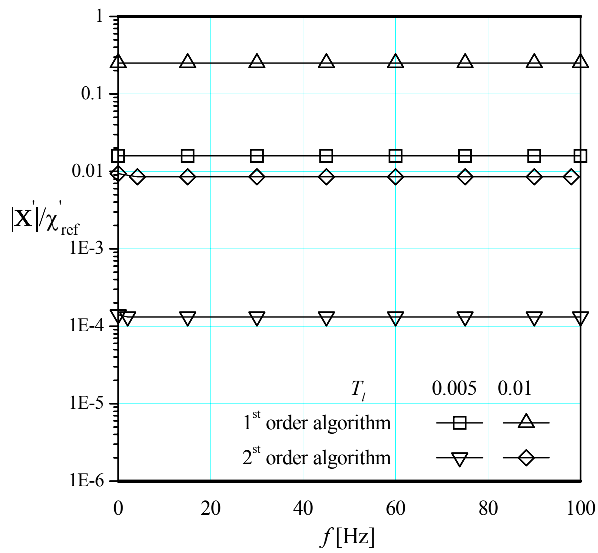

Figure 9. Comparison of the drift obtained by first and second order algorithms (f=5 Hz ,simulation time 100 s).

Conclusion

A set of algorithms based on the transition matrix and random inputs were developed. The quaternion representation was used in the development of these algorithms. These algorithms are constraint-preserving integrators, where the condition that the magnitude of the quaternion is equal to unity, is preserved. They overcome the difficulty reported in [14] where the algorithms need to compute the second derivative of angular velocity.

The effects of quantization of the gyroscopes impulse on the accuracy of the body attitude were studied. In order to estimate the increment of the gyroscope sensors, two types of estimators were proposed to estimate the integrated angular rate corrupted with uniformly distributed random impulses. A series of numerical experiments were conducted to quantify the proposed estimators with quantization errors for different motion's frequencies.

The obtained results of the 2nd order algorithm showed that the effect of quanta phenomena was especially obvious for the motion of low frequency (up to 5-10 Hz). The drift results showed that the drift was increased as the sampling time step increased. The results of the estimation of incremental gyroscope output using estimator type 1 showed that as the sampling time step reduced the accuracy of the algorithm improved (the drift was reduced) especially for low and moderate motion frequencies (2.5 to 10 Hz). The difference between two estimators for this algorithm can distinguished only at low sampling time and low frequency motion.

The 1st order algorithm results have shown that the quanta phenomena affects the accuracy only for low frequency motion at low sampling time. The estimation procedure in the case of 1st order algorithm has no effect on the improvement of the accuracy.

Appendix A: The product of two quaternions

The quaternion product may expressed in matrix form as

$$\mathbf{S}=\mathbf{q}\otimes\mathbf{p}=\begin{bmatrix}s_0\\s_1\\s_2\\s_3\end{bmatrix}=\begin{bmatrix}q_0&-q_1&-q_2&-q_3\\q_1&q_0&-q_3&q_2\\q_2&q_3&q_0&-q_1\\q_3&-q_2&q_1&q_0\end{bmatrix}\begin{bmatrix}p_0\\p_1\\p_2\\p_3\end{bmatrix} \quad (A.1)$$

where

$$\begin{aligned}\mathbf{q}&=[q_0\ \ q_1\ \ q_2\ \ q_3]^T\\\mathbf{p}&=[p_0\ \ p_1\ \ p_2\ \ p_3]^T\\\mathbf{S}&=[s_0\ \ s_1\ \ s_2\ \ s_3]^T\end{aligned} \quad (A.2)$$

Let us to define new vectors

$$\begin{aligned}\mathbf{r}_q&=[q_1\ \ q_2\ \ q_3]^{\mathrm{T}}\\\mathbf{r}_p&=[p_1\ \ p_2\ \ p_3]^{\mathrm{T}}\\\mathbf{r}_S&=[s_1\ \ s_2\ \ s_3]^{\mathrm{T}}\end{aligned} \quad (A.3)$$

The skew-symmetric matrix which can be defined as

$$\mathbf{\Omega}(\mathbf{r})=\begin{bmatrix}0&-r_3&r_2\\r_3&0&-r_1\\-r_2&r_1&0\end{bmatrix} \quad (A.4)$$

is used to obtain the vector cross-product

$$\mathbf{r}\times\mathbf{t}=\mathbf{\Omega}(\mathbf{r})\mathbf{t} \quad (A.5)$$

where

$$\begin{aligned}\mathbf{r}&=[r_1\ \ r_2\ \ r_3]^{\mathrm{T}}\\\mathbf{t}&=[t_1\ \ t_2\ \ t_3]^{\mathrm{T}}\end{aligned} \quad (A.6)$$

having in mind Eqs.(A.2), (A.3) and (A.4) the matrix equation (A.1) can be transformed :

$$\begin{aligned}\mathbf{S}=\begin{bmatrix}s_0\\\mathbf{r}_s\end{bmatrix}&=\left\{q_0\,\mathbf{I}_{4\times4}+\begin{bmatrix}0&-\mathbf{r}_q^{\mathrm{T}}\\\mathbf{r}_q&\mathbf{\Omega}(\mathbf{r}_q)\end{bmatrix}\right\}\begin{bmatrix}p_0\\\mathbf{r}_p\end{bmatrix}\\&=\begin{bmatrix}q_0p_0\\q_0\mathbf{r}_p\end{bmatrix}+\begin{bmatrix}0-\mathbf{r}_q^{\mathrm{T}}\mathbf{r}_p\\\mathbf{r}_qp_0+\mathbf{\Omega}(\mathbf{r}_q)\mathbf{r}_p\end{bmatrix}\end{aligned}$$

$$\mathbf{S}=\begin{bmatrix}s_0\\\mathbf{r}_s\end{bmatrix}=\begin{bmatrix}q_0p_0-\mathbf{r}_q^{\mathrm{T}}\mathbf{r}_p\\q_0\mathbf{r}_p+\mathbf{r}_qp_0+\mathbf{\Omega}(\mathbf{r}_q)\mathbf{r}_p\end{bmatrix} \quad (A.7)$$

or,

$$\begin{aligned}s_0&=q_0p_0-\mathbf{r}_q^{\mathrm{T}}\mathbf{r}_p\\r_s&=q_0\mathbf{r}_p+\mathbf{r}_qp_0+\mathbf{\Omega}(\mathbf{r}_q)\mathbf{r}_p\end{aligned} \quad (A.8)$$

Since;

$$\mathbf{r}_q^{\mathrm{T}}.\mathbf{r}_p=\mathbf{r}_p^{\mathrm{T}}.\mathbf{r}_q \quad (A.9)$$

and from the rule of vector cross-product

$$\mathbf{r}_q\times\mathbf{r}_p=-\mathbf{r}_p\times\mathbf{r}_q \quad (A.10)$$

the skew-symmetric matrix can follow this rule as

$$\mathbf{\Omega}(\mathbf{r}_q)\mathbf{r}_p=-\mathbf{\Omega}(\mathbf{r}_p)\mathbf{r}_q \quad (A.11)$$

Using Eqs.(A.9) and (A.11), Eq.(A.7) can be rewritten as

$$\begin{aligned}\mathbf{S}=\begin{bmatrix}s_0\\\mathbf{r}_s\end{bmatrix}&=\begin{bmatrix}p_0q_0-\mathbf{r}_p^{\mathrm{T}}\mathbf{r}_q\\q_0\mathbf{r}_p+\mathbf{r}_qp_0-\mathbf{\Omega}(\mathbf{r}_p)\mathbf{r}_q\end{bmatrix}=\\&=\left\{p_0\,\mathbf{I}_{4\times4}+\begin{bmatrix}0&-\mathbf{r}_p^{\mathrm{T}}\\\mathbf{r}_p&-\mathbf{\Omega}(\mathbf{r}_p)\end{bmatrix}\right\}\begin{bmatrix}q_0\\\mathbf{r}_q\end{bmatrix}\end{aligned} \quad (A.12)$$

Substituting $\mathbf{r}_p$ and $\mathbf{r}_q$ form Eq.(A.3) and $\mathbf{\Omega}(\mathbf{r}_p)$ from Eq.(A.4) into Eq.(A.12) gives

$$\mathbf{S}=\begin{bmatrix}p_0&-p_1&-p_2&-p_3\\p_1&p_0&p_3&-p_2\\p_2&-p_3&p_0&p_1\\p_3&p_2&-p_1&p_0\end{bmatrix}\begin{bmatrix}q_0\\q_1\\q_2\\q_3\end{bmatrix} \quad (A.13)$$

Equating (A.1) to (A.13) gives

$$\begin{aligned}\boldsymbol{S}=\begin{bmatrix}s_0\\s_1\\s_2\\s_3\end{bmatrix}&=\begin{bmatrix}q_0&-q_1&-q_2&-q_3\\q_1&q_0&-q_3&q_2\\q_2&q_3&q_0&-q_1\\q_3&-q_2&q_1&q_0\end{bmatrix}\begin{bmatrix}p_0\\p_1\\p_2\\p_3\end{bmatrix}=\\&=\begin{bmatrix}p_0&-p_1&-p_2&-p_3\\p_1&p_0&p_3&-p_2\\p_2&-p_3&p_0&p_1\\p_3&p_2&-p_1&p_0\end{bmatrix}\begin{bmatrix}q_0\\q_1\\q_2\\q_3\end{bmatrix}\end{aligned} \quad (A.14)$$

Appendix B: Transition matrix elements

First order algorithm (s=1)

In the case of first order where $s=1$ the maximum order of the quantity $\mathbf{f}_3(t)$ is 4. It can be shown [16] this quantity has the following expression

$$\mathbf{f}_3(t)=\left(\frac{1}{2}-\frac{1}{48}\|\mathbf{f}_1\|^2\right)\mathbf{f}_1+\frac{1}{3}\boldsymbol{\alpha}_0\times\boldsymbol{\alpha}_1 \quad (B.1)$$

$$\mathbf{f}_1=\boldsymbol{\alpha}_0+\boldsymbol{\alpha}_1 \quad (B.2)$$

The scalar coefficient can be found according to

$$f_0=(1-\|\mathbf{f}_3\|^2)^{0.5}=1-\frac{1}{2}\|\mathbf{f}_3\|^2-\frac{1}{8}\|\mathbf{f}_3\|^4 \quad (B.3)$$

The equation (B.3) for f_0 have the error of the order of $(\Delta t)^6$ (time step of incremental for gyroscope output). It should be noted that the condition of the magnitude of the quaternion is equal to unity (unitary condition) is included in the algorithm by Eq.(B.3). The procedure of normalization of quaternion is not required by this algorithm.

Second order algorithm (s=2)

The maximum order for this algorithm of the quantity $\mathbf{f}_3(t)$ is 6. So, the following expression were obtained

$$\mathbf{f}_1 = \boldsymbol{\alpha}_{-1} + \boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2 \quad \text{(B.4)}$$

$$\begin{aligned} \mathbf{f}_3(t) &= \left(\frac{1}{2} - \frac{1}{48}\|f_1\|^2 + \frac{1}{3840}\|\mathbf{f}_1\|^4\right)\mathbf{f}_1 + \\ &+ \left(\frac{11}{45} - \frac{1}{120}\|\mathbf{f}_1\|^2\right)(\boldsymbol{\alpha}_{-1} + \boldsymbol{\alpha}_0) \times (\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_2) + \quad \text{(B.5)} \\ &+ \frac{16}{45}\left[\boldsymbol{\alpha}_{-1} \times (\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_0 \times \boldsymbol{\alpha}_2) - \boldsymbol{\alpha}_2 \times (\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_{-1} \times \boldsymbol{\alpha}_1)\right] \end{aligned}$$

The scalar quantity f_0 can be found according to

$$f_0 = 1 - \frac{1}{2}\|\mathbf{f}_3\|^2 - \frac{1}{8}\|\mathbf{f}_3\|^4 - \frac{1}{16}\|\mathbf{f}_3\|^6 \quad \text{(B.6)}$$

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Algoritmi za određivanje ugaonog položaja besplatformnih inercijalnih navigacionih sistema primenom fundamentalne matrice kvaterniona i slučajnih ulaznih veličina

U radu su prikazani algoritmi za određivanje ugaonog položaja objekta koji se zasnivaju na primeni fundamentalne matrice kvaterniona. Spadaju u integratore koji obezbeđuju uslov jediničnog intenziteta kvaterniona i prevazilaze teškoće nekih od objavljenih algoritama kojima su potrebni prvi, drugi i treći izvod ugaone brzine objekta. Izučavaju se efekti kvantifikacije žiroskopskih impulsa na tačnost određivanja ugaonog položaja. Predložena su dva estimatora za procenu ugaonog položaja objekta ako su inkrementi ugla opterećeni slučajnim veličinama sa ravnomernom raspodelom gustine verovatnoće. Izveden je niz numeričkih eksperimenata u cilju provere opisanih estimatora za slučaj različite frekvencije konusnog kretanja. Pokazano je da su estimatori efikasni za određeni domen frekvencija kretanja i brzina odabiranja mernih signala.

Ključne reči: **mehanika leta, navigacija, navigacioni sistem, inercijalno navođenje, određivanje položaja, estimacija, estimator, kvaternion, numerički algoritam**

Algorithmes pour la détérmination de la position d'angle des systèmes de navigation inertiele sans plate-forme par l'application de la matrice fondamentale du quaternion et des vitesse d'entrée accidentelles

Ce papier représente les algorithmes de la position d'angle de l'objet et se basent sur l'application fondamentale de la matrice du quaternion. Ils font partie des intégrateurs qui assurent l'intensité unique du quaternion et surmontent, les diffucultés de certains algorithmes publiés qui nécessitent la première, la deuxième et la troisième dérivées de la vitesse angulaire d'objet . On a étudié les effets de la quantification des impulsions gyroscopiques sur la précision de détermination de la position d'angle. Deux estimateurs sont proposés pour l'estimation de la position d'angle d'objet en cas où les incréments angulaires sont chargés des valeurs accidentelles avec une répartition homogène de la densité de probabilité. On a effectué une série d'essais numériques en vue de vérifier les estimateurs décrits dans le cas de différente fréquence du mouvement conique. On a démontré que les estimateurs sont efficaces pour un domaine déterminé des fréquences du mouvement ainsi que des vitesses du choix des signaux de mesure

Mots clés: **mécanique du vol, navigation, système de navigation, guidage inertiel, précision de position, estimation, estimateur, quaternion, algorithme numérique**

Алгоритмы для определения угловых положений безплощадных инерциальных навигационных систем при помощи применения кватернарных фундаментальных пластин и произвольных входных данных

В этой работе приведены алгоритмы для определения угловых положений объекта, которые обосновываются на применении кватернарных фундаментальных пластин. Они принадлежат к интеграторам, обеспечивающим условие единичной интенсивности кватерниона и перевосходят трудности каких-то опубликованных алгоритмов, которым нужны первый, второй и третьий выводы угловых скоростей объекта. Выучиваются эффекты квантования гироимпульсов на точность определения угловых положений. Здесь предложены две оценки для оценивания угловых положений объекта, если приращение угла усилены произвольными величинами с равномерным распределением плотности вероятности. Здесь тоже сделан вывод целого ряда численных экспериментов с целью проверки описанных оценок в случае различных частот конусного движения. Здесь показано, что оценки эффективны для определённых областей частот движений и скоростей выбирания мерных сигналов.

Ключевые слова: механика полёта, навигация, навигационная система, инерциальное наведение, определение положения, расчётность, расчёт, кватернион, численный алгоритм