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# Solving Two - Point Boundary Value Problem for Descriptor Systems

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**In this paper, the methods for solving the two-point boundary value problem for descriptor systems (TPBVDS) are presented. To solve this class of systems using the methods mentioned, the system of difference equations with two-point boundary condition needs to be solved. The importance of these is even more conspicuous considering the fact that these systems are often used in precise description of numerous natural phenomena.**

*Key words*: **discrete systems, singular systems, boundary conditions, two-point boundary problem, state estimation.**

## Introduction

THE topic of this paper is solving two-point boundary value problem TPBVP. In the case of continuous systems, this problem appears when attempting to estimate the states, and at the same time having unknown parameters of the process. Towards the end of the initial phase, the origin of the problem in the case of discrete singular systems which appears while solving the difference equations of states due to the systems matrices is considered. In order to solve the problem, it is necessary to know the initial conditions for the first equations subgroup, and the finite conditions for the other one. That type of descriptor systems will be called two-point boundary value descriptor systems or TPBVDS. Some areas will be followed by proper examples so that the actions can be explained in detail. It is essential to keep in mind some basic concepts related to discrete systems in order to understand the presented material more fully.

## Discrete descriptor systems

Discrete systems are generally those systems in which variables of states take their values only in specific (exactly defined) moments of time. Consequently, their mathematical models are described by difference equations. During an intensive development of computer technologies, discrete systems have bigger ingluence and importance in every theoretical and practical aspect of automatic control. This class of systems can be obtained as the result of continuous systems aproximation, mostly for the simulation on numerical computers or in the real systems in which signal sampler exists. In some other cases there are specific components or equipment which first serve one part of the system and then the other, so that the system can have a discrete nature. Same reasons lead to the existence of discrete descriptor systems.

The general description of mathematical models will be given in the following form:

$$\mathbf{f}(k,\mathbf{x}(k+1),\mathbf{x}(k),\dots\mathbf{x}(0),\mathbf{x}_i(k),\mathbf{u}(k),\mathbf{u}(k-1),\dots\mathbf{u}(0))=0, \quad (1)$$

in references, it is known as implicit discrete system, or:

$$\mathbf{f}_k(k,\mathbf{x}(k+1),\mathbf{x}(k),\dots\mathbf{x}(0),\mathbf{u}(k),\mathbf{u}(k-1),\dots\mathbf{u}(0))=0, \quad (2)$$

$$\mathbf{x}_i(k)=\mathbf{g}_k(k,\mathbf{x}(k+1),\mathbf{x}(k),\dots\mathbf{x}(0),\mathbf{u}(k),\mathbf{u}(k-1),\dots\mathbf{u}(0))=0, \quad (3)$$

where are, in general, vector functions $\mathbf{f}_k\left[(\cdot)\right]$ and $\mathbf{g}_k\left[(\cdot)\right]$, so that:

$$\begin{aligned}\mathbf{f}_k &: R\times R^n\times R^m\to R^n,\\ \mathbf{g}_k &: R\times R^n\times R^m\to R^p,\end{aligned} \quad (4)$$

where $\mathbf{x}(k)=\mathbf{x}(kT)$ is the system state vector, $\mathbf{u}(k)$ control vector, $\mathbf{x}_i(k)$ output vector, $T$ period, $k$ sampling moment.

One of the possible canonical form models given by Eqs. (2) and (3), when functions $\mathbf{f}_k\left[(\cdot)\right]$ and $\mathbf{g}_k\left[(\cdot)\right]$ are linear, is:

$$E(k+1)\mathbf{x}(k+1)=A(k)\mathbf{x}(k)+B(k)\mathbf{u}(k), \quad (5)$$

$$\mathbf{x}_i(k)=C(k)\mathbf{x}(k)+D(k)\mathbf{u}(k), \quad (6)$$

$$E\mathbf{x}(0)=E\mathbf{x}_0,\quad k=0,1,2,\dots N-1, \quad (7)$$

and belogs to an unstationary, discrete, linear, descriptor system. Eq.(5) represents the discrete vector equation of state, and eq.(6) is the output equation of the dynamical system. The following initial conditions are defined by eq.(7).

Through time, changeable matrices $A(k)$, $B(k)$, $C(k)$, $D(k)$ and $E(k+1)$ have proper dimensions, with the matrix which has a constant rank, but must be singular.

Special description of unstationary singular systems

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given by Eqs. (5) and (6), can be presented as, *Luenberger* (*1977, 1978*):

$$E_{k+1}\mathbf{x}(k+1) = A_k\mathbf{x}(k) + B_k\mathbf{u}(k), \tag{8}$$

$$\mathbf{x}_i(k) = C_k\mathbf{x}(k),\ k = 0,1,2,\ldots N-1, \tag{9}$$

that enables their presenation in a block-matrix form:

$$\begin{bmatrix} -A_0 & E_1 & & & & \\ 0 & -A_1 & E_2 & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & E_{N-1} & 0 \\ & & & 0 & -A_{N-1} & E_N \end{bmatrix} \times \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{x}(1) \\ \vdots \\ \vdots \\ \mathbf{x}(N-1) \\ \mathbf{x}(N) \end{bmatrix} = \begin{bmatrix} \mathbf{u}(0) \\ \mathbf{u}(1) \\ \vdots \\ \vdots \\ \mathbf{u}(N-1) \end{bmatrix}, \tag{10}$$

with every particular block which has a dimension $n \times n$, it is clearly shown that this system of dynamical equations can be treated like a large-scale system. Specially, the most frequently found case in references, is that the system is considered to have vector functons which are linear and matrices in the equation of state as in the output equation are constant, so that the simpliest matrical description of discrete descriptor systems in space of states can be obtained, *Dai* (*1989.b*):

$$E\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k), \tag{11}$$

$$\mathbf{x}_i(k) = C\mathbf{x}(k),\ k = 0,1,2,\ldots, \tag{12}$$

which will be the subject of research in this paper.

*Nature and characteristics of discrete descriptor systems*

Discrete descriptor systems, given by Eqs. (11) and (12), in the mathematical sense, present dynamical systems described by combinations of difference and algebraic equations, that enable their presentation in a classical form by vector difference equations of state and also by the usage of standard mathematical methods for their solving. Continuous systems, like discrete models, both have identical advantages, *Bajić* (*1992.a*).

On the other hand, this class of singular systems has many specificities of its own, which are considered through questions about the existence and uniqueness of solution, property of causality, existence of consistent initial conditions which have physical sense, possible uncharacteristic matrix of transfer functions, and with other questions especially related to the numerical solving and realization of these systems.

All of this is supported by the fact that the existence and researching of descriptor systems capture the attention of an enormous number of scientists, who actually work on system theory and controlling.

Finally, it is important to emphasize that no matter where these efforts were headed in search of its specificities, it should be admitted that present general trend of their consideration is contained in trying to derive results which emanate from general modern theory of systems. This becomes much clearer if it is realized that the basic solutions of those problems are not in the entire space of states, but in some of their subspaces.

*Two-point boundary problem for discrete descriptor system with boundary value problem (TBVPDS)*

The class of descriptor systems described by linear difference equation of state and system of boundary conditions is introduced, whereby matrices $E$ and $A$ are singular. Unlike the previous approach, this class of systems is solved considered together with boundary conditions. In this paper it is shown that it is not possible to set arbitrary initial conditions on one hand, and on the other to require that the system has unique solution and at the same time to be in an adequate form. For this reason, firstly it is required that the system be well-posed, in terms of existence and uniqueness of solution, and the procedure for translating the system to a proper form. All further derivations will be based on this assumption. In case of TPBVDS $\mathbf{x}(k), k = 0,1,\ldots N$, in order to obtain sequence of solutions $\mathbf{x}(k), k = 0,1,\ldots N$ under the effect of sequence of inputs $\mathbf{u}(k), k = 0,1,\ldots N-1$, it is necessary to solve difference equation of state with boundary conditions in two ending points of the observed interval; namely, to know the value of singular vector of state in initial and finite moments. In case of these systems, initial conditions are marked through the system of boundary values. It is not enough to know only the value of the descriptive vector of state in one moment in order to obtain a unique solution, because of the singularity of the system matrices. It is necessary to solve the two-point boundary problem (TPBVP) in these systems by using recursive methods, but here is one more problem. It is not possible to find the solution using simple recursions, because the vector of state in a specific moment of time depends on the input over the entire interval. Finally, TPBPV for descriptor systems can now be precisely defined in the following manner. The solution, for the given system of boundary values which are related to initial and finite points of the interval where the system defined is by the sequence of inputs which are known needs to be found. It is obvious that solving this problem requires induction of some other notions, where some of them are already quoted, like for e.g. well-posedness system, normalized form etc. In the following paragraph notions like: well-posedness system, standard and normalized forms will be explained.

*Well-posedness and normalized forms*

The TPBVDS considered in this paper satisfies the difference equation:

$$E\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k), \quad k = 0,\ldots,N-1, \tag{13}$$

with the two-point boundary condition:

$$V_i\mathbf{x}(0) + V_f\mathbf{x}(N) = \mathbf{v} \tag{14}$$

and output:

$$\mathbf{y}(k) = C\mathbf{x}(k), \quad k = 0,\ldots,N \tag{15}$$

where $\mathbf{x}, \mathbf{v} \in R^n$, and $\mathbf{u} \in R^m, \mathbf{y} \in R^p$.

As in *Luenberger* (*1978*), Eqs. (13) and (14) can be rewritten as a single set of equations

$$\mathbf{S}\mathbf{x} = \mathbf{B}\mathbf{u} \tag{16a}$$

$$\mathbf{x}^T = \left(\mathbf{x}^T(0),...,\mathbf{x}^T(N)\right) \tag{16b}$$

$$\mathbf{u}^T = \left(\mathbf{u}^T(0),...,\mathbf{u}^T(N-1),\mathbf{v}^T\right) \tag{16c}$$

where:

$$\mathbf{S} = \begin{bmatrix} -A & E & 0 & \dots & \dots & \dots & 0 \\ 0 & -A & E & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ 0 & \dots & \dots & \dots & 0 & -A & E \\ V_i & 0 & \dots & \dots & \dots & 0 & V_f \end{bmatrix} \tag{17a}$$

$$\mathbf{B} = diag(B,...,B,I) \tag{17b}$$

From this is immediately obvious that the well-posedness of Eqs. (13) and (14) – i.e. the existence of a unique solution $\mathbf{x}(k), k = 0,1,...,N$, for any choice of $\mathbf{v}$ and $\mathbf{u}(k), k = 0,1,...,N-1$ is equivalent to the invertibility of $\mathbf{S}$. Note that invertibility of $\mathbf{S}$ implies that the submatrix consisting of all but its last block of rows has full row rank. This in turn implies that the necessary condition for well-posedness is that $\{E, A\}$ comprise a regular pencil, i.e. that $\alpha E + \beta A$ is invertible for some and therefore for most $\alpha$ and $\beta$. Consequently throughout this paper it is assumed that this is the case.

An important aspect of regular pencils is that they can be transformed into a form that greatly simplifies the answering of a number of questions.

***Definition.*** A regular pencil $\{E, A\}$ is in standard form if for some $\alpha$ and $\beta$:

$$\alpha E + \beta A = I\ . \tag{18}$$

Note that any standard linear system (with *E=I*) is in a standard form (take $\alpha = 1$, $\beta = 0$). Furthermore, any well-posed TPBVDS can be transformed to the standard form. Specifically, find $\alpha$ and $\beta$ so that $|\alpha E + \beta A| \neq 0$ and premultiply eq.(13) by $(\alpha E + \beta A)^{-1}$. This does not change the system or the state variable $\mathbf{x}(k)$, but the new $E$ and $A$ matrices now satisfy eq.(18).

A pencil in standard form has a number of important properties, a few of which are summarized in the following result.

***Proposition.*** Suppose that $\{E, A\}$ is in the standard form. Then:

(i) $E$ and $A$ commute and thus have a common set of the generalized eigenvectors (which are referred to us generalized eigenvectors).

(ii) The pencil $\{E^k, A^k\}$ is regular for all $k > 0$.

(iii) For any $k, l > 0$, there exist coefficients $\alpha_0,...,\alpha_{n-1}$ so that

$$E^k A^l = \sum_{i=0}^{n-1} \alpha_i A^{n-i-1} E^i \tag{19}$$

***Proof.*** Suppose without loss of generality that $\alpha \neq 0$ in eq.(18). Then $E = \gamma I + \delta A$ where $\gamma = 1/\alpha$ and $\delta = -\beta/\alpha$. The commutativity of $E$ and $A$ follows immediately. The remainder of (i) follows from the fact that $E$ and $A$ can be put into Jordan form by the same similarity transformations. Indeed, the Jordan blocks must be of commensurate dimensions, i.e. no block of $E$ and $A$ can straddle rows of several blocks of the other without extending to include all of the rows of those blocks. (For e.g., two $4\times 4$ matrices in Jordan form, one with two $2\times 2$ Jordan blocks and the other with one $3\times 3$ and one $1\times 1$ Jordan block, do not commute.)

Assume that $E$ and $A$ are in Jordan form. Since $\{E, A\}$ is regular, $E$ and $A$ cannot have a zero eigenvalue associated with a common eigenvector. This in turn implies statement (ii). Finally, to prove (iii), take any $E^k A^l$ and replace $E$ by $\gamma I + \delta A$. Then, apply the usual Cayley-Hamilton theorem to all powers of $A$ higher then $n-1$. Finally, multiply each $A^k$ in the resulting expression by $I = (\alpha E + \beta A)^{n-k-1}$. Expanding yields an expression in the form of eq.(19).

Statement (iii), which states that $\{A^{n-1}, EA^{n-2},...,E^{n-1}\}$ span the same subspace as $\{A^k E^l \,|\, k,l \geq 0\}$, is a generalization of the Cayley-Hamilton theorem. Note that this statement is considerably simpler than those in *Lewis 1983 b, 1984, Mertzios i Christodolou 1986* for pencils not in standard form.

Standard form also provides the following simpler well-posedness condition.

***Theorem.*** Suppose that $\{E, A\}$ is in standard form. Then the system Eqs. (13) and (14) is well-posed if and only if:

$$V_i E^N + V_f A^N \tag{20}$$

is invertible.

***Proof.*** One method for deriving this result is to apply row elimination to solve $\mathbf{x}(0)$ and $\mathbf{x}(N)$ from eq.(16). Methods similar to this will be used in the next section in defining inward and outward processes. In this proof a different method that provides some computations that can be used immediately is applied.

To begin, let $\omega$ be any number such that:

$$\Gamma = \omega E^{N+1} - A^{N+1} \tag{21}$$

is invertible (this can always be done since $\{E^{N+1}, A^{N+1}\}$ is regular).

Then $\mathbf{S}$ can be expressed as:

$$\mathbf{S} = \mathbf{S}_1 \mathbf{S}_2 \tag{22}$$

$$\mathbf{S}_1 = \begin{bmatrix} I & 0 & \dots & \dots & 0 \\ 0 & I & \dots & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & I & 0 \\ S_{N0} & S_{N1} & \dots & S_{N,N-1} & S_{NN} \end{bmatrix} \tag{23}$$

$$S_{Nk} = \left(V_i A^{N-k} E^k + \omega V_f A^{N-k-1} E^{k+1}\right)\Gamma^{-1}, \quad k = 0,..,N-1, \tag{24a}$$

$$S_{Nk} = \left(V_i E^N + V_f A^N\right)\Gamma^{-1}, \tag{5.12b}$$

$$\mathbf{S}_2 = \begin{bmatrix} -A & E & 0 & \cdots & \cdots & 0 \\ 0 & -A & E & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & \cdots & 0 & -A & E \\ \omega E & 0 & \cdots & \cdots & 0 & -A \end{bmatrix} \quad (5.13)$$

Note that $\mathbf{S}_2$ is invertible, with:

$$\mathbf{S}_2^{-1} = \begin{bmatrix} A^N\Gamma^{-1} & EA^{N-1}\Gamma^{-1} & \cdots & E^{N-1}A\Gamma^{-1} & E^N\Gamma^{-1} \\ \omega E^N\Gamma^{-1} & A^N\Gamma^{-1} & \cdots & E^{N-2}A^2\Gamma^{-1} & E^{N-1}A\Gamma^{-1} \\ \omega E^{N-1}A\Gamma^{-1} & \omega E^N\Gamma^{-1} & \cdots & E^{N-3}A^3\Gamma^{-1} & E^{N-2}A\Gamma^{-1} \\ \vdots & \vdots & & \vdots & \vdots \\ \omega EA^{N-1}\Gamma^{-1} & \omega E^2A^{N-2}\Gamma^{-1} & \cdots & \omega E^N\Gamma^{-1} & A^N\Gamma^{-1} \end{bmatrix} \quad (26)$$

Consequently $\mathbf{S}$ is invertible if and only if $\mathbf{S}_1$ is invertible. Examining Eqs. (23) and (24) it can be seen that this is the case if and only if the matrix in eq.(20) is invertible.

***Definition.*** The system Eqs. (13) and (14) is in normalized form if $\{E, A\}$ is in standard form and if:

$$V_i E^N + V_f A^N = I \,. \quad (27)$$

This form is the counterpart of *Krener`s* (*1980, 1981, 1987*) standard form. Note that any well-posed system can be put in the normalized form by left multiplication of Eqs. (13) and (14). Specifically $\{E, A\}$ are first transform to standard form as described previously, to obtain new $E$ and $A$ matrices, and then multiply eq.(14) by $\left(V_i E^N + V_f A^N\right)^{-1}$ to obtain new $V_i$ and $V_f$ matrices satisfying eq.(27). From this point on it is assumed that Eqs. (13) and (14) is in the standard form.

Next, note that if eq.(20) is invertible, the inverse of $\mathbf{S}_1$ has the same form as eq.(23) except that the last block row of $\mathbf{S}_1^{-1}$ is:

$$\left(-S_{NN}^{-1}S_{N0,}-S_{NN}^{-1}S_{N1,},\ldots,-S_{NN}^{-1}S_{N,N-1,}S_{NN}^{-1}\right)$$

Using the expressions for $\mathbf{S}_1^{-1}$ and $\mathbf{S}_2^{-1}$ the Green`s function solution of Eqs. (13) and (14) can be written down:

$$\mathbf{x}(k) = A^k E^{N-k}\mathbf{v} + \sum_{l=0}^{N-1} G(k,l)B\mathbf{u}(l) \,, \quad (28)$$

where:

$$G(k,l) = \begin{cases} A^k\left[A - E^{N-k}\left(V_iA + \omega V_f E\right)E^k\right]E^{l-k}A^{N-l-1}\Gamma^{-1}, & l \ge k \\ E^{N-k}\left[\omega E - A^k\left(V_iA + \omega V_f E\right)A^{N-k}\right]E^l A^{k-l-1}\Gamma^{-1} & l < k \end{cases} \quad (29)$$

Here $G(k,l)$ is called the Green`s function of the TPBVDS. When $E$ and $A$ are both invertible, eq.(29) can be simplified:

$$G(k,l) = \begin{cases} A^k E^{N-k} V_f E^{l-N} A^{N-l-1}, & l \ge k \\ A^k E^{N-k} V_i E^l A^{-l-1}, & l \le k \end{cases} \quad (30)$$

For simplicity, in the rest of the paper it is assumed that $\Gamma$ is invertible for $\omega = 1$ and so eq.(29) is used. This assumption is equivalent to assuming that no $(N+1)$-th root of unity is an eigenmode of the system (where $\sigma$ is an eigenmode if $|\sigma E - A| = 0$ ). All of the results in the paper have obvious extensions to the case of an arbitrary value of $\omega$, as $\omega$ must be carried along in the various expressions.

*Example* 1

In this example the normalized form system given by following difference state equation is considered:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x}(k+1) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{u}(k)$$

with system of boundary conditions

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\mathbf{x}(0) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x}(N) = \mathbf{v} \,, \quad N = 10$$

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix},$$

$$\mathbf{x}_2 = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}, \ \ldots \ \mathbf{x}_9 = \begin{bmatrix} -9 \\ -1 \\ -1 \end{bmatrix}, \quad \mathbf{x}_{10} = \begin{bmatrix} -10 \\ -1 \\ -1 \end{bmatrix}$$

By simple replacement of the system matrices, of vector $\mathbf{v}$, with $\omega = 1$ in Eqs. (28) and (29) it is obtained that solution seems in the space of states as shown in Fig.1.

Proper time changes, for $\mathbf{v}^T = [0 \ \ -1 \ \ -1]^T$ are given in Fig.1.

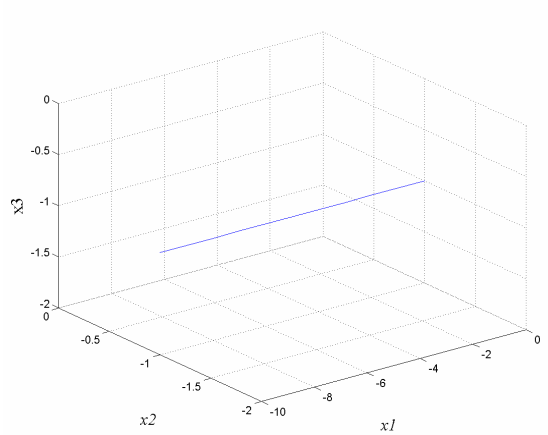


**Figure 1.**

*Inward and outward processes*

One of Krener`s most important observations was that boundary-value systems admit two notions of recursion, namely expanding inward from or outward toward the boundaries. In this section, the counterparts to these notions for TPBVDS are introduced. As it will be shown, the possible singularity of both $E$ and $A$ leads to several differences in the context.

Each of the processes associated with these recursions have interpretations as state processes: the outward process summarizes all that is required to know about the input inside any interval in order to determine $\mathbf{x}(k)$ outside the interval, while the inward process simply uses input values near the boundary to propagate the boundary condition inwards. In Krener`s context, the outward process represented a *jump*, i.e. the difference between $\mathbf{x}(k)$ at one and of any interval and the value predicted for $\mathbf{x}(k)$ at that point given $\mathbf{x}(k)$ at the other end of the interval and assuming zero input inside the interval. In the given context it cannot necessarily be predicted in either direction (because of the possible singularity of $E$ and $A$) and therefore a slightly modified definition of the outward process must be used.

$$\mathbf{z}_0(k,l) = E^{l-k}\mathbf{x}(l) - A^{l-k}\mathbf{x}(k),\ k<l \tag{31}$$

Note that this definition agrees with Krener`s if $E = I$. However, in general $z_0(k,l)$ can only be propagated outward whereas in Krener`s case the outward process could be propagated inwards as well. An explicit expression for $z_0(k,l)$, in terms of the inputs between $[k,l]$, can be obtained by premultiplying (16) by:

$$\begin{bmatrix} 0 & \dots & 0 & A^{k-l-1} & EA^{k-l-1} & \dots & E^{k-l-1} & 0 & \dots & 0 \end{bmatrix}$$

This yields:

$$z_0(k,l) = \sum_{j=k}^{l-1} E^{j-k} A^{l-j-1} Bu(j) \tag{32}$$

There are also the recursive relations:

$$z_0(k-1,l) = Ez_0(k,l) + A^{l-k}Bu(k-1) \tag{33}$$

$$z_o(k,l+1) = Az_o(k,l) + E^{l-k}Bu(l) \tag{34}$$

Furthermore, as in Krener (1987), it is straightforward to show that the four-point boundary-value system:

$$E\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k) \tag{35}$$

$$V_i\mathbf{x}(0) + V_f\mathbf{x}(N) = \mathbf{v} \tag{36}$$

$$E^{L-K}\mathbf{x}(L) - A^{L-K}\mathbf{x}(K) = \mathbf{z}_0(K,L) \tag{37}$$

has the same solution as eq.(13-14) for $k \in [0,N]\backslash[K+1,L-1]$ (i.e. over $[0,K]$ and $[L,N]$), so $\mathbf{z}_0(K,L)$ does indeed summarize all we need to know about inputs between K and L.

The inward process $\mathbf{z}_i(k,l)$ can also be defined in a manner analogous to that of *Krener* (*1987*). Unfortunately, in the present context $\mathbf{z}_i(k,l)$ is a complex function of the boundary matrices, the boundary value $\mathbf{v}$, and the inputs $\mathbf{u}(j), j \in [0,N-1]\backslash[k,l-1]$. Specifically, as will be demonstrated below, for *k<l*, $\mathbf{z}_i(k,l)$ has the form:

$$\begin{aligned}\mathbf{z}_i(k,l) &= W_i(k,l)\mathbf{x}(k) + W_f(k,l)\mathbf{x}(l)\\ &= F_{kl}\left[\mathbf{u}(0)\ \mathbf{u}(1) \dots \mathbf{u}(k-1)\ \mathbf{u}(l)\right.\\ &\qquad \left.\mathbf{u}(l+1) \dots \mathbf{u}(N-1)\ \mathbf{v}\right]\end{aligned} \tag{38}$$

and, in addition:

$$\mathbf{z}_i(0,N) = \mathbf{v}\ \ W_i(0,N) = V_i\ \ W_f(0,N) = V_f \tag{39}$$

$$z_i(k,k) = \mathbf{x}(k) = F_{kk}(\mathbf{u}(0),...,\mathbf{u}(N-1),\mathbf{v}) \tag{40}$$

where $F_{kl}$ are linear functions of their arguments. Furthermore the TPBVDS

$$E\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k) \tag{41}$$

$$W_i(K,L)\mathbf{x}(K) + W_f(K,L)\mathbf{x}(L) = \mathbf{z}_i(K,L) \tag{42}$$

has the same solution as eq. (13-14) for $k \in [K,L]$, so $\mathbf{z}_i(k,l)$ does indeed represent an inwardly-propagated boundary condition for the original system.

Let it first be indicated how Eqs. (38), (39) and (40) can be computed in a recursive manner. The basic idea here is to eliminate the values of $\mathbf{x}(k)$ near the boundary from eq.(16) in order to obtain a reduced set of equations. The resulting right-hand side will then involve the remaining $\mathbf{u}$`s and a new boundary (see eq.(16c)). Specifically, suppose the goal is to propagate one step in from the left, i.e. to compute $\mathbf{z}_i(1,N)$. Note that for $\mathbf{S}$ in eq.(5a) to be invertible it is necessary for:

$$\begin{bmatrix} -A \\ V_i \end{bmatrix}$$

to have full column rank. Consequently a block matrix $[T \quad P]$ of full row rank can be found so that:

$$\begin{bmatrix} T & P \end{bmatrix} \begin{bmatrix} -A \\ V_i \end{bmatrix} = 0 \tag{43}$$

Premultiplying eq.(16) by the matrix

$$\Omega = \begin{bmatrix} 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \\ T & 0 & \dots & 0 & P \end{bmatrix}$$

then eliminates $\mathbf{x}(0)$ and leaves the following TPBVDS on $[1,N]$:

$$E\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k) \tag{44}$$

$$TE\mathbf{x}(1) + PV_f\mathbf{x}(N) = P\mathbf{v} + TB\mathbf{u}(0) \tag{45}$$

It is easy to see that this system is well-posed, since $rank(\Omega\mathbf{S}) = rank(\Omega) = rank(\mathbf{S}) = n$, and the system is defined over an interval with one less time step. The boundary matrices in eq.(45) are not necessarily in the normalized form, so it is necessary to pre multiply eq.(45) by:

$$\Lambda = \left(TE^N + PV_fA^{N-1}\right)^{-1} \tag{46}$$

yielding:

$$W_i(1,N) = \Lambda TE\ \ W_f(1,N) = \Lambda PV_f \tag{47}$$

$$F_{1N}[\mathbf{u}(0),\mathbf{v}] = \Lambda P\mathbf{v} + \Lambda TB\mathbf{u}(0) \tag{48}$$

In a similar fashion the right boundary can be moved inward, in this case premultiplying eq.(16) by:

$$\begin{bmatrix} I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \\ 0 & 0 & \dots & S & Q \end{bmatrix} \quad (49)$$

where $[S \quad Q]$ is a full-rank solution of:

$$[S \quad Q]\begin{bmatrix} E \\ V_f \end{bmatrix} = 0 \quad (50)$$

It is also possible to obtain a direct rather than a recursive expression for the $W$ `s and at the same time to expose the relationship between the inward and outward processes that will be used later. Using the expression eq.(31) for the outward process $\mathbf{z}_0$ and eq.(16) it can be written:

$$\begin{bmatrix} -A^k & E^k & 0 & 0 \\ 0 & -A^{l-k} & E^{l-k} & 0 \\ 0 & 0 & -A^{N-1} & E^{N-1} \\ V_i & 0 & 0 & V_f \end{bmatrix} \times \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{x}(k) \\ \mathbf{x}(l) \\ \mathbf{x}(N) \end{bmatrix} = \begin{bmatrix} \mathbf{z}_0(0,k) \\ \mathbf{z}_0(k,l) \\ \mathbf{z}_0(l,N) \\ \mathbf{v} \end{bmatrix} \quad (51)$$

As was done earlier, a full-rank matrix is constructed:

$$\begin{bmatrix} T_i(k,l) & T_f(k,l) & P(k,l) \end{bmatrix}$$

so that:

$$\begin{bmatrix} T_i(k,l) & T_f(k,l) & P(k,l) \end{bmatrix} \begin{bmatrix} -A^k & 0 \\ 0 & E^{N-1} \\ V_i & V_f \end{bmatrix} \quad (52)$$

If eq.(51) is then multiplied by:

$$\Omega(k,l) = \begin{bmatrix} 0 & I & 0 & 0 \\ T_i(k,l) & 0 & T_f(k,l) & P(k,l) \end{bmatrix} \quad (53)$$

the following is obtained:

$$\begin{bmatrix} -A^{l-k} & E^{l-k} \\ T_i(k,l) & -T_f(k,l)A^{N-1} \end{bmatrix} \begin{bmatrix} x(k) \\ x(l) \end{bmatrix} = \begin{bmatrix} z_0(k,l) \\ T_i(k,l)z_0(0,k) + T_f(k,l)z_0(l,n) + P(k,l)v \end{bmatrix} \quad (54)$$

Eq.(54) is essentially the result of eliminating all variables in eq.(16) other than $\mathbf{x}(k)$ and $\mathbf{x}(l)$, by propagating outward to summarize all inputs between *k* and *l* and inward to summarize the effect of the boundary condition and inputs from 0 to *k* and 1 to *N*. Therefore letting:

$$\Lambda(k,l) = \left[T_i(k,l)E^l - T_f(k,l)A^{N-k}\right]^{-1} \quad (55)$$

it is obtained :

$$W_i(k,l) = \Lambda(k,l)T_i(k,l)E^k \quad (56)$$

$$W_f(k,l) = -\Lambda(k,l)T_f(k,l)A^{N-1} \quad (57)$$

$$\mathbf{z}_i(k,l) = \Lambda(k,l)[T_i(k,l)\mathbf{z}\ (0,K) + T_f(k,l)\mathbf{z}\ (1,N) + P(k,l)\mathbf{v}] \quad (58)$$

In continuation the way that shows how these results are used for determining the solutions of the system will be demonstrated. Three methods will be considered, where one is based on the separation of the system in two subsystems, so that one is solved by finite forward differences and the other by finite backward differences. The other two methods are based on direct usage of results given in this section. In the first method the inward and outward recursive processes are used and in the second only outward, while the inward is omitted. Each of these methods will be followed by an example as illustration of their practical usage. The solution will be given in forced working regime with step input change in order to understand these methods more fully. If all are considered in free working regime, outward process becomes equal to zero, regardless of the interval which is referred to. It should be emphasized that these methods do not require the choice of consistential initial condition, unlike the approach when Drazin`s inversion for solving is used.

## Efficient solution of TPBVDS

Unlike causal systems, the solution of a TPBVDS cannot be computed using a simple recursion since the solution $\mathbf{x}(k)$ depends on inputs over the entire interval. There are, however, several efficient methods which will be described in this section.

*Two-filter solution*

In his study, Krener derived a solution by solving his continuous-time linear system assuming a zero initial condition and then correcting for the actual boundary conditions. Since $E$ and $A$ may both be singular for a TPBVDS, the analogous procedure, first described in *Nikoukhah et al.* (*1986*), is somewhat more complex it must be identified which part of the system can be solved in the forward and backward directions.

From Kronecker`s canonical form for a regular pencil, *Van Dooren* (*1979*), non-singular matrices $T$ and $F$ can be found so that:

$$FET^{-1} = \begin{bmatrix} I & 0 \\ 0 & A_2 \end{bmatrix} \quad (59)$$

$$FAT^{-1} = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \quad (60)$$

and all of the eigenvalues of $A_1$ and $A_2$ have magnitudes no larger than 1. (The decomposition in *Van Dooren* (*1979*) splits the pencil $zE - A$ into forward dynamics corresponding to a pencil of the form $zI - \tilde{A}_1$ and backward dynamics corresponding to $Iz^{-1} - \tilde{A}_2$ where $\tilde{A}_2$ is nilpotent. The only difference in Eqs. (46) and (47) is that the unstable forward modes of $\tilde{A}_1$ have been shifted into the backward dynamics $A_2$). Define:

$$\begin{bmatrix} \mathbf{x}_1(k) \\ \mathbf{x}_2(k) \end{bmatrix} = T\mathbf{x}(k) \tag{61}$$

Then it is obtained:

$$\mathbf{x}_1(k+1) = A_1\mathbf{x}_1(k) + B_1\mathbf{u}(k) \tag{62a}$$

$$\mathbf{x}_2(k) = A_2\mathbf{x}_2(k+1) - B_2\mathbf{u}(k) \tag{62b}$$

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = FB \tag{63}$$

Note that Eqs. (62a), and (62b) are asymptotically stable recursions if $|zE-A|$ has no zeros on the unit circle. Finally, given the transformation eq.(61), the boundary condition eq.(14) takes the form:

$$[V_{1i} \vdots V_{2i}] \begin{bmatrix} \mathbf{x}_1(0) \\ \mathbf{x}_2(0) \end{bmatrix} + [V_{1f} \vdots V_{2f}] \begin{bmatrix} \mathbf{x}_1(N) \\ \mathbf{x}_2(N) \end{bmatrix} = \mathbf{v} \tag{64}$$

$$[V_{1i} \vdots V_{2i}] = V_i T^{-1}, [V_{1f} \vdots V_{2f}] = V_f T^{-1} \tag{65}$$

Employing the forward/backward representation eq.(62) of the dynamics, a general solution to Eqs.(13) and (14) is derived as follows. Let $\mathbf{x}_1^0$ denote the solution to eq.(62a) with zero initial condition, and let $\mathbf{x}_2^0$ denote the solution of eq.(62b) with zero final condition. Then:

$$\mathbf{x}_1(k) = A_1^k \mathbf{x}_1(0) + \mathbf{x}_1^0(k) \tag{66}$$

$$\mathbf{x}_2(k) = A_2^{N-k} \mathbf{x}_2(0) + \mathbf{x}_2^0(k) \tag{67}$$

Substituting Eqs. (66) and (67) into (64) and solving for $\mathbf{x}_1(0)$ and $\mathbf{x}_1(N)$ yields:

$$\begin{bmatrix} \mathbf{x}_1(0) \\ \mathbf{x}_2(N) \end{bmatrix} = H^{-1}\left\{\mathbf{v} - V_{1f}\mathbf{x}_1^0(N) - V_{2i}\mathbf{x}_2^0(0)\right\} \tag{68}$$

where:

$$\begin{aligned} H &= \left[V_{1i} + V_{1N}A_1^N \vdots V_{2i}A_2^N + V_{2f}\right] = \\ &= V_i T^{-1}\left(FET^{-1}\right)^N + V_f T^{-1}\left(FAT^{-1}\right)^N \end{aligned} \tag{69}$$

Finally, substituting eq.(68) into Eqs. (66) and (67) yields:

$$\begin{aligned} \begin{bmatrix} \mathbf{x}_1(k) \\ \mathbf{x}_2(k) \end{bmatrix} &= \begin{bmatrix} A_1^k & 0 \\ 0 & A_2^{N-k} \end{bmatrix} \times \\ &\times H^{-1}\left[\mathbf{v} - V_{1f}\mathbf{x}_1^0(N) - V_{2i}\mathbf{x}_2^0(0)\right] \begin{bmatrix} \mathbf{x}_1^0(k) \\ \mathbf{x}_1^0(k) \end{bmatrix} \end{aligned} \tag{70}$$

The solution in the original basis can then be obtained by inverting eq.(60).

Note that the transformed matrices in Eqs. (59) and (60) commute and are in fact in a form close to the normalized form. However, the full importance of transforming the system into normalized form, and in particular its implication for a generalized Cayley-Hamilton theorem and the resulting form of reachability and observability results, has not been previously recognized. Also, the algorithm just described provides an equivalent well-posedness condition, namely the invertibility of $H$ in eq.(69).

*Parallel outward-inward solution*

A second efficient algorithm can be constructed by noting that the solution $\mathbf{x}(k)$ can be recovered from the outward process $\mathbf{z}_0$ and the inward process $\mathbf{z}_i$. For simplicity, let it be assumed that $N$ is odd and that $E$ and $A$ commute (as they would if Eqs. (13) and (14) is in the normalized form). It is then possible to specify a recursive algorithm for the computation of $\mathbf{z}_0(j, N-j)$ for $j = 0,...,(N-1)/2$, starting from the initial condition at the centre of the interval (with $j = (N-1)/2$):

$$\mathbf{z}_0((N-1)/2, (N+1)/2) = B\mathbf{u}((N-1)/2) \tag{71}$$

and propagating symmetrically outward from the centre:

$$\begin{aligned} \mathbf{z}_0(j-1, N-j+1) &= EA\mathbf{z}_0(j, N-j) + \\ &+ A^{N-2j+1}B\mathbf{u}(j-1) + \\ &+ E^{N-2j+1}B\mathbf{u}(N-j) \end{aligned} \tag{72}$$

Similarly, $\mathbf{z}_i(j, N-j)$ can be computed recursively inward from the initial condition:

$$\mathbf{z}_i(0, N) = \mathbf{v} \tag{73}$$

using a recursive procedure based on that outlined (see Eqs. (43), (44), (45), (46), (47), (48), (49) and (50)).

The solution $\mathbf{x}(k)$ can then be computed as:

$$\begin{aligned} \begin{bmatrix} \mathbf{x}(j) \\ \mathbf{x}(N-j) \end{bmatrix} &= \begin{bmatrix} -A^{N-2j} & E^{N-2j} \\ W_i(j, N-j) & W_f(j, N-j) \end{bmatrix}^{-1} \times \\ &\times \begin{bmatrix} \mathbf{z}_0(j, N-j) \\ \mathbf{z}_i(j, N-j) \end{bmatrix} \end{aligned} \tag{74}$$

where the inverse on the right-hand side of eq.(74) is guaranteed to exist thanks to the well-posedness of Eqs. (13) and (14).

*Example* 2

A practical example, will illustrate how to determine the solution of TPBVDS in the force working regime, which is based on previously presented algorithm. The system given by the following state equation and set of boundary conditions will be considered.

$$E\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k), \quad k = 0,1,2,...,6$$

$$V_i\mathbf{x}(0) + V_f\mathbf{x}(N) = \mathbf{v}, \quad N = 7$$

where:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{u}(k) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad k = 0,1,...,N-1$$

$$V_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; V_f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 7 \\ 1 & 0 & 1 \end{bmatrix}, \; \mathbf{v} = \begin{bmatrix} 2 \\ 8 \\ 1 \end{bmatrix}$$

First, eq.(5.19) is calculated, which presents external recursive process, starting from the centre of the interval and computing from outward to boundaries (*k* is decreasing and *l* is increasing for one step). In this case going from $\mathbf{z}_0(3,4)$ to $\mathbf{z}_0(0,7)$, the following values are obtained:

$$\mathbf{z}_0(3,4)=\begin{bmatrix}2\\2\\1\end{bmatrix},\ \mathbf{z}_0(2,5)=\begin{bmatrix}2\\1\\1\end{bmatrix},$$

$$\mathbf{z}_0(1,6)=\begin{bmatrix}2\\-1\\1\end{bmatrix},\ \mathbf{z}_0(0,7)=\begin{bmatrix}2\\-3\\1\end{bmatrix}$$

Then eq.(32) is calculated, representing the internal recursive process, starting from the boundaries to the centre of the interval (*k* is increasing and *l* is decreasing). It is necessary to compute $\mathbf{z}_0(0,3)$, $\mathbf{z}_0(0,2)$ as well, $\mathbf{z}_0(0,1)$ and also $\mathbf{z}_0(4,7)$, $\mathbf{z}_0(5,7)$, $\mathbf{z}_0(6,7)$ which are:

$$\mathbf{z}_0(0,1)=\mathbf{z}_0(0,2)=\begin{bmatrix}2\\2\\1\end{bmatrix},\ \mathbf{z}_0(0,3)=\begin{bmatrix}2\\1\\1\end{bmatrix}$$

$$\mathbf{z}_0(4,7)=\begin{bmatrix}2\\1\\1\end{bmatrix},\ \mathbf{z}_0(5,7)=\mathbf{z}_0(6,7)=\begin{bmatrix}2\\2\\1\end{bmatrix}$$

In order to completely calculate the internal recursive process, it is necessary to determine matrices $T_i(k,l), T_f(k), P(k,l)$ using eq.(52). These matrices are as follows:

$$T_i(3.4)=\begin{bmatrix}1&0&0\\2&0&0\\3&0&0\end{bmatrix},\ T_f=\begin{bmatrix}0&2&1\\0&1&1\\0&3&3\end{bmatrix},$$

$$P=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Obviously, for any *k, l* it is necessary to determine the following matrices $T_i(k,l), T_f(k), P(k,l)$ as well.

Using eq.(55) $\Lambda(k,l)$ is computed, and then internal recursion $\mathbf{z}_i(k,l)$ using eq.(58) can easily be computed. The following results will be obtained:

$$\mathbf{z}_i(0,7)=\begin{bmatrix}2\\8\\1\end{bmatrix},\ \mathbf{z}_i(1,6)=\begin{bmatrix}2\\-8\\-1\end{bmatrix},$$

$$\mathbf{z}_i(2,5)=\begin{bmatrix}2\\-7\\-1\end{bmatrix},\ \mathbf{z}_i(3,4)=\begin{bmatrix}2\\-5\\-1\end{bmatrix}$$

Using Eqs. (56) and (57) $W_i(k,l), W_f(k,l)$ can be computed for e.g.:

$$W_i(3,4)=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix},\ W_f(3,4)=\begin{bmatrix}0&0&0\\0&1&1\\0&0&1\end{bmatrix}$$

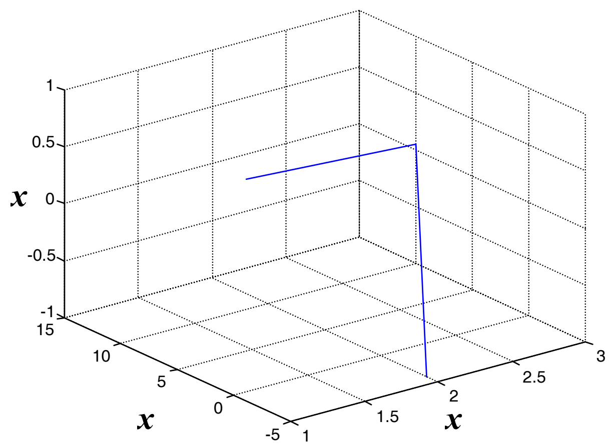


**Figure 2.**

Finally, the state variables in discrete moments of time need to be determined as this was the main aim. This solution in space of states is as shown in Fig.2.

*Serial outward-inward solution*

As a first step in this algorithm $\mathbf{z}_0(j,N-j)$ is computed outward from the interval centre as in Eqs. (71) and (72). These values, are used along with the boundary condition $\mathbf{v}$, to solve $\mathbf{x}(j)$ and $\mathbf{x}(N-j)$ recursively while propagating back towards the interval centre. To begin, note that:

$$\begin{bmatrix}\mathbf{x}(0)\\\mathbf{x}(N)\end{bmatrix}=\begin{bmatrix}-A^N & E^N\\ V_i & V_f\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{z}_0(0,N)\\\mathbf{v}\end{bmatrix} \tag{75}$$

where the inverse indicated on the right-hand side is again guaranteed to exist, thanks to well-posedness. To continue with the inward recursion, note that from eq.(31):

$$-A^{N-2j}\mathbf{x}(j)+E^{N-2j}\mathbf{x}(N-j)=\mathbf{z}_0(j,N-j) \tag{76}$$

while from eq.(13):

$$\begin{aligned}&\delta_j E\mathbf{x}(j)+A\mathbf{x}(N-j)=\\&=\delta_j A\mathbf{x}(j-1)+E\mathbf{x}(N-j+1)+\\&+\delta_j B\mathbf{u}(j-1)-B\mathbf{u}(N-j)\end{aligned} \tag{5.64}$$

for any $j\in[1,(N-1)/2]$ and any $\delta_j$. Then the recursion

$$\begin{bmatrix}\mathbf{x}(j)\\\mathbf{x}(N-j)\end{bmatrix}=\begin{bmatrix}-A^{N-2j} & E^{N-2j}\\ \delta_j E & A\end{bmatrix}^{-1}\times\begin{bmatrix}\mathbf{z}_0(j,N-j)\\ \delta_j A\mathbf{x}(j-1)+E\mathbf{x}(N-j-1)+\\ +\delta_j B\mathbf{u}(j-1)-B\mathbf{u}(N-j)\end{bmatrix} \tag{78}$$

where $\delta_j$ is chosen so that the inverse on the right-hand eq.(78) exists (for e.g., if $|zE-A|$ has no roots on the unit circle, $\delta_j$ can be considered equal to 1).

*Example* 3.

As in the previous chapter, the method for determining the solution of TPBVDS in force working regime, using the algorithm given in previous chapter will be shown. The system given by the following state equation and set of boundary conditions is considered:

$$E\mathbf{x}(k+1)=A\mathbf{x}(k)+B\mathbf{u}(k)\,,\quad k=0,1,2,...,6$$

$$V_i\mathbf{x}(0)+V_f\mathbf{x}(N)=\mathbf{v}\,,\quad N=7$$

where:

$$E=\begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix},\ A=\begin{bmatrix}0&0&0\\0&1&-1\\0&0&1\end{bmatrix},$$

$$B=\begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix},\ \mathbf{u}(k)=\begin{bmatrix}1\\1\\1\end{bmatrix},\ k=0,1,...,N-1$$

$$V_i=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ V_f=\begin{bmatrix}0&0&0\\0&1&7\\1&0&1\end{bmatrix},\ \mathbf{v}=\begin{bmatrix}1\\2\\1\end{bmatrix}$$

First, eq.(32) is computed, representing the external recursion, starting from the centre of the interval and computed outwards to the boundaries (*k* is decreasing and *l* is increasing for one step). The following results are obtained:

$$\mathbf{z}_0(3,4)=\begin{bmatrix}2\\2\\1\end{bmatrix},\ \mathbf{z}_0(2,5)=\begin{bmatrix}2\\1\\1\end{bmatrix},$$

$$\mathbf{z}_0(1,6)=\begin{bmatrix}3\\-1\\2\end{bmatrix},\ \mathbf{z}_0(0,7)=\begin{bmatrix}4\\-2\\3\end{bmatrix}$$

Using eq.(75) $\mathbf{x}(0)$ and $\mathbf{x}(7)$ are computed and their values are:

$$\mathbf{x}(0)=\begin{bmatrix}1\\-4\\-1\end{bmatrix},\ \mathbf{x}(7)=\begin{bmatrix}2\\-5\\-1\end{bmatrix}$$

Finally, when all these are determined, it only remains to determine the rest of state values from eq.(78) in discrete moments of time, as was the basic goal. In Fig.3 the solution in the space of states is shown.

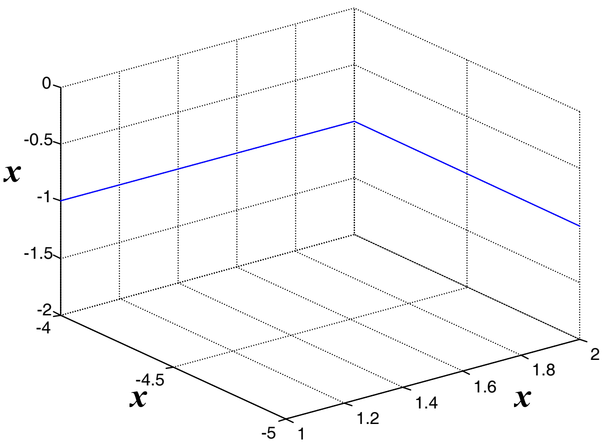


**Figure 3.**

## Shift - invariant TPBVDS

A TPBVDS is described by Eqs. (13) and (14). It is assumed that $N\ge 2n$ so that all modes can be excited and observed. In chapter 5.1 it is shown that if system Eqs. (13) and (14) is well-posed, it can be assumed, without loss of generality, that Eqs. (13) and (14) is in normalized form, i.e., that there exist scalars $\alpha$ i $\beta$ such that satisfy eq.(18) (this is referred to as the standard from for the pencil $\{E,A\}$) and in addition eq.(27). Note that eq.(18) implies that $E$ and $A$ commute and also that $\{E^k,A^k\}$ is regular for all $k\ge 0$.

As derived earlier, the map from $\{\mathbf{u}(k),\mathbf{v}\}$ to $\mathbf{x}(k)$ has the following form:

$$\mathbf{x}(k)=A^kE^{N-k}\mathbf{v}+\sum_{l=0}^{N-1}G(k,l)B\mathbf{u}(l) \tag{79}$$

where Green's function $G(k,l)$ is given by:

$$(k,l)=\\=\begin{cases}A^k\left[A-E^{N-k}\left(V_iA+\omega V_fE\right)E^k\right]E^{l-k}A^{N-l-1}\Gamma^{-1},l\ge k\\E^{N-k}\left[\omega E-A^k\left(V_iA+\omega V_fE\right)A^{N-k}\right]E^lA^{k-l-1}\Gamma^{-1},l<k\end{cases} \tag{80}$$

and where $\omega$ is any number for witch $\Gamma$ is invertible and given by eq.(21).

In the marked contrast to the case for casual systems $(E=I\,,\ V_f=0\,,\ G(k,l))$ does not, in general, depend on the difference in its arguments. Borrowing previous terminology:

**Definition 3.** The TPVDS Eqs. (13) and (14) is a displacement system if (with the usual use of notation):

$$G(k,l)=G(k-l),\ 0\le k\le N,\ 0\le j\le N-1 \tag{81}$$

With $\mathbf{v}=0$ in eq.(14) then Eqs. (13), (14) and (15) define a linear map of the form:

$$\mathbf{x}_i(k)=\sum_{l=0}^{N-1}W(k,l)\mathbf{u}(l) \tag{82}$$

$$W(k,l)=CG(k,l)B \tag{83}$$

**Definition 4.** The TPBVDS Eqs. (13) and (14) is stationary if:

$$W(k,l)=W(k-l),\ 0\le k\le N,\ 0\le l\le N-1 \tag{84}$$

**Theorem 2.** The TPBVDS Eqs. (13) and (14) is stationary if and only if:

$$O_s[V_i,E]R_s=O_s[V_i,A]R_s=0 \tag{85a}$$

$$O_s\left[V_f,E\right]R_s=O_s\left[V_f,A\right]R_s=0 \tag{85b}$$

where $[X,Y]=XY-YX$, and:

$$R_s=\left[A^{n-1}B\middle|EA^{n-2}B\middle|\ \ldots\ \middle|E^{n-1}B\right] \tag{86}$$

$$O_s=\begin{bmatrix}CA^{n-1}\\CEA^{n-2}\\\vdots\\CE^{n-1}\end{bmatrix} \tag{87}$$

**Corollary.** The TPBVDS Eqs. (13) and (14) is a displacement system if and only if:

$$[V_i, E] = [V_i, A] = 0 \tag{88a}$$

$$[V_f, E] = [V_f, A] = 0 \tag{88b}$$

The matrices $R_s$ and $O_s$ in Eqs. (86) and (87) are the strong reachability and strong observability matrices of the TPBVDS. Thus eq.(85) states that $V_i$ and $V_f$ must commute with $E$ and $A$ except for parts that are either in the left nullspace of $R_s$ or the right nullspace of $O_s$. If $R_s$ and $O_s$ are of full rank - i.e., if the TPBVDS is strongly reachable and strongly observable - $V_i$ and $V_f$ must commute with $E$ and $A$. Turning to the corollary, it can be seen that these are precisely the conditions for a TPBVDS to be displacement, so that a displacement is always stationary. Furthermore, the only way in witch a TPBVDS can be stationary without being displacement is if the system is not strongly reachable or strongly observable.

**Proof of the corollary .** It is assumed that Theorem 2 holds. From the theorem, a TPBVDS is displacement if and only if eq.(85) holds with $R_s$ and $O_s$ defined with $C = B = I$. However, thanks to the generalized Cayley-Hamilton theorem for pencils in the standard form, the matrices $\{A^k E^{n-k-1} | 0 \le k \le n-1\}$ span the same set as $\{E^k A^j | k, j \ge 0\}$. Thus $R_s$ and $O_s$ are of full rank, so that eq.(85) is equivalent to eq.(88).

**Proof of Theorem 2.** What must be shown is that eq.(85) is equivalent to:

$$W(k+1, j+1) = W(k, j) \tag{5.76}$$

for $0 \le k \le N-1, \quad 0 \le j \le N-2$. Then, using eq.(80), the commutativity of $E$ and $A$, and performing some algebra it can be found that eq.(89) is equivalent to:

$$\begin{aligned} &CA^{k+1}E^{N-k-1}\left[V_i A + \omega V_f E\right] A^{N-j-2} E^{j+1} \Gamma^{-1} B \\ &= CA^k E^{N-k}\left[V_i A + \omega V_f E\right] A^{N-j-1} E^j \Gamma^{-1} B \end{aligned} \tag{90}$$

From the Cayley-Hamilton theorem and the fact that $N \ge 2M$, it can be found that eq.(90) is equivalent to :

$$\begin{aligned} &O_s\left[V_i A + \omega V_f E\right] E \Gamma^{-1} R_s = \\ &= O_s E\left[V_i A + \omega V_f E\right] A \Gamma^{-1} R_s \end{aligned} \tag{91}$$

Define the strong reachability subspace

$$\mathbf{R}_s = \Re(R_s) \tag{92}$$

Then the generalized Cayley-Hamilton theorem implies that $\mathbf{R}_s$ is $A$ - and $E$ - and therefore also $\Gamma$ - invariant. Furthermore, for almost all $\omega$, $\Gamma$ is invertible so that the range of $\Gamma^{-1}\mathbf{R}_s$ is $\mathbf{R}_s$. Since this does not depend on $\omega$, it can be deduced that eq.(91) is equivalent to the following pair of equalities:

$$O_s [AV_i E - EV_i A] A \Gamma^{-1} R_s = 0 \tag{93}$$

$$O_s\left[AV_f E - EV_f A\right] E \Gamma^{-1} R_s = 0 \tag{94}$$

Since $\{E^N, A^N\}$ is regular, $\mathbf{R}_s = \Re\left(\left[A^N R_s \middle| E^N R_s\right]\right)$ so that eq. (93) is equivalent to:

$$O_s [AV_i E - EV_i A] A \Gamma^{-1} A^N R_s = 0 \tag{95}$$

$$O_s [AV_i E - EV_i A] E \Gamma^{-1} E^N R_s = 0 \tag{96}$$

In a similar fashion eq.(94) is equivalent to the pair of equalities:

$$O_s\left[AV_f E - EV_f A\right] E \Gamma^{-1} A^N R_s = 0 \tag{97}$$

$$O_s\left[AV_f E - EV_f A\right] E \Gamma^{-1} E^N R_s = 0 \tag{98}$$

Using the commutativity of $E$ and $A$ together with eq.(27), it can be seen that eq.(97) is equivalent to:

$$O_s [-AV_i E + EV_i A] E^{N+1} \Gamma^{-1} R_s = 0 \tag{99}$$

Using the definition of $\Gamma$, it can be seen that Eqs. (96) and (99) imply:

$$O_s [AV_i E - EV_i A] R_s = 0 \tag{100}$$

In a similar fashion Eqs. (96) and (99) can imply:

$$O_s\left[AV_f E - EV_f A\right] R_s = 0 \tag{101}$$

The $E$ - and $A$ - of $\mathbf{R}_s$ then imply that eq.(100-101), are, in fact, equivalent to Eqs. (93) and (94).

Finally, note that thanks to the commutativity of $E$ and $A$, eq.(85a) implies eq.(100) and eq.(85b) implies eq.(101). To see that the reverse of these implications holds, assume that $\alpha \ne 0$ in eq.(18) (if $\alpha = 0$, reverse the role $E$ and $A$). Then $E = \gamma I + \delta A$ with $\gamma \ne 0$. Substituting this into eq.(100) yields eq.(85a). Similarly eq.(101) implies eq.(85b).

The characterization of the displacement property in eq.(85) simplifies many computations. In particular, it is not difficult to check that Green`s function of a displacement system is

$$G(k) = \begin{cases} V_i A^{k-1} E^{N-k}, & k > 0, \\ -V_f E^{-k} A^{N+k-1}, & k \le 0. \end{cases} \tag{102}$$

Similarly, the weighting pattern of a stationary TPBVDS is given by

$$W(k) = \begin{cases} CV_i A^{k-1} E^{N-k} B, & k > 0, \\ -CV_f E^{-k} A^{N+k-1} B, & k \le 0. \end{cases} \tag{103}$$

## Inward processes, outward processes, and extendibility

Inward and outward processes play an important role in the analysis of TPBVDS. The outward process, which expands outward toward the boundaries, summarizes what is necessary to know about the input inside any interval in order to determine $\boldsymbol{x}$ outside the interval. The inward process uses input values near the boundary to propagate the boundary condition inward.

The outward process has a simple definition and characterization as shown:

$$\mathbf{z}_0(k,l) = E^{l-k}\mathbf{x}(l) - A^{l-k}\mathbf{x}(k),\ k < l \quad (104)$$

It is possible to express $\mathbf{z}_0(k,l)$ in terms of the intervening inputs:

$$z_0(k,l) = \sum_{j=k}^{l-1} E^{j-k} A^{l-j-1} Bu(j) \quad (105)$$

and to write outward recursions ($k$ decreasing and $l$ increasing). In general $\mathbf{z}_0(k,l)$ can only be propagated in an outward direction. Note also that $\mathbf{z}_0(k,l)$ does not involve the boundary matrices $V_i$ and $V_f$.

The inward process $\mathbf{z}_i(k,l), K \le L$, is a faction of the boundary value $\mathbf{v}$ and the inputs $\{\mathbf{u}(0),...,\mathbf{u}(K-1)\}$ and $\{\mathbf{u}(L),...,\mathbf{u}(N-1)\}$ so that the TPBVDS eq.(13) with boundary condition:

$$V_i(K,L)x(K) + V_f(K,L)x(L) = z_i(K,L) \quad (106)$$

yields the same solution as Eqs. (13) and (14) for $K \le k \le L$. Here $V_i(K,L)$ and $V_f(K,L)$ are assumed to be such that eq.(13), eq.(100) is in normalized form, i.e.:

$$V_i(K,L)E^{L-K} + V_f(K,L)A^{L-K} = I \quad (107)$$

Note in particular the starting values and the "final values":

$$\mathbf{z}_i(0,N) = \mathbf{v},\ V_i(0,N) = V_i,\ V_f(0,N) = V_f \quad (108)$$

$$\mathbf{z}_i(k,k) = \mathbf{x}(k), \qquad \forall k \in \mathbf{R} \quad (109)$$

For the general TPBVDS there are no simple formulas or recursions for $\mathbf{z}_i, V_i$ and $V_f$. However, the following does not hold for displacement systems:

**Proposition 2.** Assume that Eqs. (13) and (14) is a displacement system. Then for $k \le j$

$$V_i(k,l) = V_i E^{N-l+k} \quad (110)$$

$$V_f(k,l) = V_f A^{N-l+k} \quad (111)$$

$$\begin{aligned}\mathbf{z}_i(k,l) &= E^{N-l}A^k\mathbf{v} + V_iE^{N-l}\mathbf{z}_0(0,k) - V_fA^k\mathbf{z}_0(l,N)\\ &= E^{N-l}A^k\mathbf{v} + V_iE^{N-l}\sum_{j=0}^{k-1} E^j A^{k-j-1}B\mathbf{u}(j)\\ &\quad -V_fA^k\sum_{s=1}^{N-1} E^{s-1}A^{N-s-1}B\mathbf{u}(s)\end{aligned} \quad (112)$$

**Proof.** First, eq.(27) guarantees that the definitions in Eqs.(110) and (111) yield TPBVDSs in the normalized form for all $k \le l$. Eq.(112) is obtained by replacing $\mathbf{x}(k)$ and $\mathbf{x}(l)$ in:

$$\mathbf{z}_i(k,l) = V_iE^{N-l+k}\mathbf{x}(k) + V_fA^{N-l+k}\mathbf{x}(l) \quad (113)$$

by their expressions in terms of $\mathbf{v}$ and $\mathbf{u}(k)$ in eq.(79) using (102). Thus from eq.(112) $\mathbf{z}_i(k,l)$ depends only on $\boldsymbol{v}$ and the values of $\boldsymbol{u}$ of the interval $[k,l]$. Finally, to show that Eqs.(13) and (106) yields the same solution, it is noted that the pair of relations eq.(104) and eq.(113) have a simple inverse:

$$\begin{bmatrix}\mathbf{x}(k)\\ \mathbf{x}(l)\end{bmatrix} = \begin{bmatrix}-V_fA^{N-l+k} & E^{l-k}\\ V_iE^{N-l+k} & A^{l-k}\end{bmatrix}\begin{bmatrix}\mathbf{z}_o(k,l)\\ \mathbf{z}_i(k,l)\end{bmatrix} \quad (114)$$

Thus $\mathbf{x}(l)$ and $\mathbf{x}(k)$ can be obtained completely from $\mathbf{z}_i(k,l)$ as defined and the outward process, which is not changed by restricting the size of the interval. Thus the values of $\mathbf{x}(k)$ at these two points are correct, and therefore by moving in one step at a time it can be concluded that Eqs. (13) and (106), with the chosen $V_i$ and $V_f$ given by Eqs. (110) and (111) yields the correct solution.

*Example* 6.

The following system which is a displacement system given by its own state equation and system of boundary conditions is considered:

$$E\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k),\ k = 0,1,2,...,6$$

$$V_i\mathbf{x}(0) + V_f\mathbf{x}(N) = \mathbf{v},\ N = 7$$

where:

$$E = \begin{bmatrix}1&0&0\\0&0&1\\0&0&0\end{bmatrix},\ A = \begin{bmatrix}0&0&0\\0&1&-1\\0&0&1\end{bmatrix},$$

$$B = \begin{bmatrix}1&0&1\\0&1&1\\0&0&1\end{bmatrix},\ \mathbf{u} = \begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$V_i = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\ V_f = \begin{bmatrix}0&0&0\\0&1&7\\1&0&1\end{bmatrix},\ \mathbf{v} = \begin{bmatrix}2\\14\\2\end{bmatrix}$$

As in previous examples, in order to present the results in the best possible way the solution of the system, which is driven by input step function is considered. In this example, the already known algorithm which is given above will be used, with the difference, that the adequate internal recursive process will be computed in a simpler way using eq.(112).

First, eq.(105) is solved presenting external recursion, starting from the centre of the interval outward to the boundaries ($k$ is decreasing and $l$ is increasing for one step). Starting from $\mathbf{z}_0(3,4)$ to $\mathbf{z}_0(0,7)$ the following values are obtained:

$$\mathbf{z}_0(3,4) = \begin{bmatrix}1\\0\\1\end{bmatrix},\ \mathbf{z}_0(2,5) = \begin{bmatrix}1\\-1\\1\end{bmatrix},$$

$$\mathbf{z}_0(1,6) = \begin{bmatrix}1\\-3\\1\end{bmatrix},\ \mathbf{z}_0(0,7) = \begin{bmatrix}1\\-5\\1\end{bmatrix}$$

In order to compute eq.(112) which presents internal recursion from the boundaries to the centre of the interval ($k$ is increasing and $l$ is decreasing), it is also necessary to compute $\mathbf{z}_0(0,3)$, $\mathbf{z}_0(0,2)$, $\mathbf{z}_0(0,1)$ as well as $\mathbf{z}_0(4,7)$, $\mathbf{z}_0(5,7)$, $\mathbf{z}_0(6,7)$ and they are:

$$\mathbf{z}_0(0,1)=\mathbf{z}_0(0,2)=\begin{bmatrix}1\\0\\1\end{bmatrix},\ \mathbf{z}_0(0,3)=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

$$\mathbf{z}_0(4,7)=\begin{bmatrix}1\\-1\\1\end{bmatrix},\ \mathbf{z}_0(5,7)=\mathbf{z}_0(6,7)=\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Now, the internal recursive process $\mathbf{z}_i(k,l)$ using eq.(112) can easily be computed. Then the following results obtained are:

$$\mathbf{z}_i(0,7)=\begin{bmatrix}2\\14\\2\end{bmatrix},\ \mathbf{z}_i(1,6)=\begin{bmatrix}1\\-4\\-1\end{bmatrix},$$

$$\mathbf{z}_i(2,5)=\begin{bmatrix}1\\-5\\-1\end{bmatrix},\ \mathbf{z}_i(3,4)=\begin{bmatrix}1\\-3\\-1\end{bmatrix}$$

In the end, it remains only to determine the quantities of states in discrete moments, if it is necessary.

## Strong reachability and observability

In the case of standard linear systems, reachability corresponds the ability to drive the state of the system to an arbitrary value by appropriate choice of the input sequence. It is well known that if such a system is reachable it is possible to reach an arbitrary state value by proper choice of the $n$ previous input values, where $n$ is the dimension of the system. In case of a TPBVDS, however, there is a distinction between the concept of reachability by choosing the inputs in an $n$–point neighbourhood and the concept of reachability by choosing the inputs in the whole domain of definition (i.e. $[0,N]$). The first concept shall be referred to as strong reachability and the second weak reachability. These concepts correspond, respectively, to Krener`s reachability on and reachability off which he in turn defines in terms of the outward and inward processes, respectively. In the next two sections in which the corresponding observability concepts will also be analyzed the same will be done.

The first step will be an examination of reachability, and for this the following is needed.

**Definition 5.** The system Eqs. (13) and (14) is strongly reachable on $[K,L]$ if the map:

$$\{u(k)|\,k\in[K,L]\}\rightarrow z_o(K,L) \tag{115}$$

is onto. The system is strongly reachable if it is strongly reachable in some interval.

From eq.(32) it can be written:

$$\mathbf{z}_0(K,L)=R_s(L-K)\begin{bmatrix}\mathbf{u}(K)\\ \vdots\\ \mathbf{u}(L-1)\end{bmatrix} \tag{116}$$

$$R_s(j)=\left[A^{j-1}B\vdots EA^{j-2}B\vdots\cdots\vdots E^{j-1}B\right] \tag{117}$$

While waiting for the following result, define the strong reachability matrix

$$R_s=R_s(n) \tag{118}$$

and strong reachable subspace:

$$\mathbf{R}_s=\Re(R_s) \tag{119}$$

**Theorem 3.** The following statements are equivalent.

a) The system Eqs. (13) and (14) is strongly reachable.

b) The strong reachability matrix $R_s$ has full rank.

c) The matrix $[sE-tA\vdots B]$ has full rank for all $(s,t)\neq(0,0)$.

d) The state $\mathbf{x}(k)$ at any point $k\in[n,N-n]$ can be made to assume any desired value by proper choice of inputs $\mathbf{u}(j), j\in[k-n,k+n-1]$, and this can be accomplished for any choices of $V_i$ and $V_f$ for which Eqs. (13) and (14) is well-posed.

Before proving this result, let several comments be made. Note first that condition $(c)$ is one of the reachability conditions found in the descriptor references *(Lewis 1985, Yip and Sincovec 1981*). By introducing the standard form of a regular pencil it is possible to obtain a condition, namely that eq.(117) is of full rank for $j=n$, and that is far simpler than those presented previously. Note also that as for standard linear systems, condition $(b)$ implies that a system is strongly reachable if and only if it is strongly reachable over intervals of length $n$. On the other hand, in condition $(d)$ it is required that $\mathbf{x}(k)$ can be driven to an arbitrary value by applying appropriate inputs over the 2$n$-point symmetric neighbourhood of $k$. In fact, an $n$-point neighborhood of $k$ is the only thing necessary, but the extent of this interval before and after $k$ depends on the matrices $E$, $A$ and $B$ (i.e. on the causal/anticausal structure of eq.(13)). Condition $(d)$ simply uses the union of all such $n$-point intervals and therefore is appropriate for all TPBVDS. Finally, note that strong reachability does not depend on the boundary matrices $V_i$ and $V_f$ (as long as Eqs. (13) and (14) is well-posed). This can be seen directly from the definition of $\mathbf{z}_0(k,l)$ or from condition $(b)$.

**Proof.** The equivalence of $(a)$ and $(b)$ follows immediately from the generalized Cayley-Hamilton theorem (statement (iii) of the proposition (1)). As an alternative proof, note that

$$\Re[R_s(k+1)]=E\Re[R_s(k)]+A\Re[R_s(k)] \tag{120}$$

so that

$$\Re[R_s(k+2)]=\Re[R_s(k+1)]$$

if

$$\Re[R_s(k+1)]=\Re[R_s(k)].$$

Also, thanks to, eq.(18):

$$\Re[R_s(k)] \subseteq \Re[R_s(k+1)] \tag{121}$$

Simple dimension counting then shows that:

$$\Re[R_s(k)] = \Re[R_s(n)] \qquad \forall k \geq n \tag{122}$$

The equivalence of statements $(b)$ and $(c)$ is proved as follows. First assume that $\mathbf{z} \neq 0$ in eq.(18). In this case:

$$\mathbf{R}_s = rang\left[B \vdots AB \vdots \cdots \vdots A^{n-1}B\right] \tag{123}$$

(This can be verified by setting $j = n$ in eq.(116) and then by replacing $E$ by $(\gamma I + \delta A)$). Also, eq.(18) allows writing:

$$sE - tA = uI - vA \tag{124}$$

$$u = \left(\frac{s}{\alpha}\right), \qquad v = t - s\frac{\beta}{\alpha} \tag{125}$$

Note that $(u,v) = (0,0)$ if and only if $(s,t) = (0,0)$. Thus, statement $(c)$ is equivalent to the claim that $[uI - vA \vdots B]$ is of full rank for $(u,v) \neq (0,0)$. Note that this is a trivial case for $u \neq 0, v = 0$ since $[I \vdots B]$ has full rank. If $v \neq 0$, $uI - vA$ can be clearly divided by $\mathbf{v}$. Consequently, statement $(c)$ is equivalent to $[wI - A \vdots B]$ being of full rank for all $w \neq 0$. For the case in which $\alpha = 0$, it can be argued in a similar fashion by reversing the roles of $E$ and $A$. Note also that if $\alpha \neq 0$ and $\beta \neq 0$, then:

$$\begin{aligned}\mathbf{R}_s &= rang\left[B \vdots AB \vdots \cdots \vdots A^{n-1}B\right] \\ &= \left[B \vdots EB \vdots \cdots \vdots E^{n-1}B\right]\end{aligned} \tag{126}$$

Finally, consider the equivalence of statements $(b)$ and $(d)$. Because of the linearity of the system, it can be assumed that $\mathbf{v} = \mathbf{0}$ and $\mathbf{u}(j) = \mathbf{0}$ for $j \in [0, k-n-1]$ and $j \in [k+n, N]$. In this case Eqs. (28) and (29) and eq.(32) allow writing:

$$\begin{aligned}x(k) &= A^k\left[A - E^{N-k}\left(V_iA + V_fE\right)E^k\right]\Gamma^{-1}A^{N-k-n}z_o(k,k+n) \\ &+ E^{N-k}\left[E - A^k\left(V_iA + V_fE\right)A^{N-k}\right]\Gamma^{-1}E^{k-n}z_o(k-n,k)\end{aligned} \tag{127}$$

Let $\xi$ be an arbitrary vector and choose inputs $\mathbf{u}(j), j \in [k-n, k-1]$ so that $\mathbf{z}_0(k-n,k) = E^n\xi$ and $\mathbf{u}(j), j \in [k, k+n-1]$ so that $\mathbf{z}_0(k,k+n) = -A^n\xi$. With these choices which can be found since $R_s$ has full rank, eq.(127) is reduced to

$$\mathbf{x}(k) = \xi \tag{128}$$

This shows that $(a)$ implies $(d)$. To show the reverse implication, the following choice for $V_i$ and $V_f$ can be made:

$$V_i = \Delta^{-1}E \tag{129a}$$

$$V_f = \gamma\Delta^{-1}A \tag{129b}$$

$$\Delta = E^{N+1} + \gamma A^{+1} \tag{130}$$

$\gamma$ is any number that makes $\Delta$ invertible. Note that Eqs. (13) and (14) with this choice for $V_i$ and $V_f$ is in the normalized form. Let $\mathbf{v} = \mathbf{0}$ and $\mathbf{u}(j) = \mathbf{0}$ for $j \in [0, k-n-1]$ and $j \in [k+n, N]$.

Then in this case Eqs. (28) and (29) is reduced to:

$$\begin{aligned}\mathbf{x}(k) = \Delta\big[&A^{n-1}E^{N-n-1}B\mathbf{u}(k-n) + \\ &+ A^{n-2}E^{N-n}B\mathbf{u}(k-n-1) + \ldots + E^NB\mathbf{u}(k-1) \\ &+ \gamma A^NB\mathbf{u}(k) + \gamma A^{N-1}EB\mathbf{u}(k+1) + \ldots + \\ &+ \gamma E^{n-1}A^{N-n+1}B\mathbf{u}(k+n-1)\big]\end{aligned} \tag{131}$$

The range of the mapping defined in (5.118) is

$$\Delta\left[E^{N-n-1}\mathbf{R}_s + A^{N-n-1}\mathbf{R}_s\right]$$

Assuming that $(d)$ is true, all this must also belong to $R^n$. Consequently it can be concluded that $\mathbf{R}_s = R^n$ for this choice of $V_i, V_f$. Thanks, then, to the statement $(c)$ of the theorem, it can be seen that $\mathbf{R}_s = R^n$ for any $V_i, V_f$ for which the TPBVDS is well-posed, so that statement $(a)$ must also hold.

Then, the dual concept of strong observability is considered in a manner analogous to that of casual linear systems. Specifically, for such systems observability corresponds to being able to reconstruct the state at the same point in time, given the present and future observations, when all future inputs are zero. The counterpart to this in this context is the following.

**Definition 6.** The system Eqs. (13) and (14) is strongly observable on $[K, L]$ if the map:

$$\mathbf{z}_0(K,L) \to \{\mathbf{x}_i(k) | k \in [K,L]\} \tag{132}$$

defined by Eqs .(41) and (42) with $\mathbf{u} \equiv \mathbf{0}$ is one-to-one. The system is strongly observable if it is strongly observable on $[K, L]$ for all $K$, $L$ such that $L - K \geq n - 1$.

Since Eqs .(41) and (42) is in the normalized form, the Green`s function solution eq.(28) can be adopted to obtain an explicit expression for the mapping defined in eq.(132).

Specifically:

$$\begin{bmatrix}\mathbf{x}_i(k) \\ \mathbf{x}_i(k+1) \\ \vdots \\ \mathbf{x}_i(L)\end{bmatrix} = O_s(L-K)\mathbf{z}_i(K,L) \tag{133}$$

where:

$$O_s(j) = \begin{bmatrix}CE^j \\ CAE^{j-1} \\ \vdots \\ CA^j\end{bmatrix} \tag{134}$$

In analogy with the reachability results, the strong observability matrix is defined:

$$O_s = O_s(n-1) \tag{135}$$

and the strongly unobservable subspace

$$O_s = \aleph(O_s)$$

**Theorem 4.** The following statements are equivalent.

a) The system Eqs. (13) and (14) is strongly observable.

b) The strong observability matrix $O_s$ has full rank.

c) The matrix

$$\begin{bmatrix} sE - tA \\ C \end{bmatrix}$$

has full rank for all $(s,t) \neq (0,0)$.

d) The state $\mathbf{x}$ at any point $k \in [n, N-n]$ can be uniquely determined from the outputs $\mathbf{x}_i(j), j \in [k-n, k+n-1]$ and $\mathbf{u}(j), j \in [k-n, k+n-2]$. This can be accomplished for any choice of $V_i$ and $V_f$ for which Eqs. (13) and (14) is well-posed.

The proof of this theorem is analogous to that for Theorem 3. and is therefore omitted. Also, similar comments concerning these results can be made. For e.g., thanks to the Cayley-Hamilton theorem, statement *b)* is considerably simpler than expressions that have appeared previously. Also, strong observability depends only on $E, A$ and $C$ and not on the particular choice of boundary matrices $V_i$ and $V_f$.

## Weak raechability and observability

The concepts of weak reachability and observability, in contrast to strong reachability and observability, depend intimately on the particular choice of boundary matrices. The examination of these weaker concepts for TPBVDS is somewhat more complicated than in Krener`s case because of the possible singularity of $E$ and $A$.

**Definition 7.** The system Eqs. (13) and (14) is weakly reachable off $[K, L]$ if the map $F_{KL}$ defined in eq.(38), with $\mathbf{v} \equiv \mathbf{0}$, is onto. The weakly reachable subspace $\mathbf{R}_w(K,L)$ is the range of this map. The system is weakly reachable if it is weakly reachable off $[K, L]$ (i.e. if $\mathbf{R}_w(K,L) = R^n$) for all $K, L \in [n, N-n]$.

Note that the weak reachability condition is natural counterpart to the casual reachability definition in which it is required that the state can be driven to an arbitrary value from zero initial condition. Also, note the use of the wording „reachable off“, emphasizing the fact that the inputs used in this case are confined to the exterior of the interval $[K, L]$.

An important property of a casual system is that the dimension of the reachable space does not change, and in fact the reachable space itself is time-invariant. The following theorem shows that the first of these statements is also true for TPBVDS.

**Theorem 5.** The dimension of $\mathbf{R}_w(K,L)$ is constant for $K, L \in [n, N-n]$.

**Proof.** Let $K$, $L$ be any points in $[n, N-n]$. From eq.(70) (with $\mathbf{v} = \mathbf{0}$) it can be seen that:

$$\mathbf{R}_w(K,L) = \Lambda(K,L)\left[T_i(K,L)\mathbf{R}_s + T_f(K,L)\mathbf{R}\right] \quad (136)$$

Now assume that $K-1 \in [n, N-n]$ as well. It is attempted to show that:

$$\dim \mathbf{R}_w(K-1,L) = \dim \mathbf{R}_w(K,L) \quad (137)$$

To do this, $T_i(K-1,L)$ and $T_f(K-1,L)$ must first be found. In fact, what should be shown is that a possible set of choices for $T_i$, $T_f$ and $P$ is

$$T_i(K-1,L) = T_i(K,L)\tilde{A} \quad (138a)$$

$$T_f(K-1,L) = T_f(K,L) \quad (138b)$$

$$P(K-1,L) = P(K,L) \quad (138c)$$

where $\tilde{A}$ has the same eigenstructure as $A$ except that the zero eigenvalue in $A$ has been replaced by 1 in $\tilde{A}$. Without the loss of generality (since similarity transformations have no effect on the dimension of the reachability spaces), it can be assumed that $A$ is in Jordan form:

$$A = \begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix} \quad (139)$$

where $J$ is invertible and $N$ is nilpotent. In this case:

$$A = \begin{bmatrix} J & 0 \\ 0 & N+1 \end{bmatrix} \quad (140)$$

For eq.(138) to be a valid choice, two conditions must be satisfied.

First $\left[T_i(K-1,L) \quad T_f(K-1,L) \quad P(K-1,L)\right]$ must be a full rank. This is obviously the case since $\left[T_i(K,L) \quad T_f(K,L) \quad P(K,L)\right]$ is, and $\tilde{A}$ is invertible. Secondly to show that eq.(52) is satisfied with $k = K-1$ and $l = L$, i.e., it must be verified:

$$-T_i(K,L)\tilde{A}A^{K-1} + P(K,L)V_i = 0 \quad (141)$$

when it is known:

$$-T_i(K,L)A^K + P(K,L)V_i = 0 \quad (142)$$

However, since $K-1 \geq n, N^{K-1} = 0$, so that $\tilde{A}A^{K-1} = A^K$.

Consequently, it can be written:

$$\mathsf{R}_w(K-1,L) = \Lambda(K-1,L) \times \times\left[T_i(K,L)\mathsf{R}_s + T_f(K,L)\mathsf{R}_s\right] \quad (143)$$

Comparing Eqs. (136) and (143) and using the fact that $\Lambda(k,1)$ are all invertible, it can be seen that eq.(138) will hold if it can be shown that:

$$\tilde{A}\mathbf{R}_s = \mathbf{R}_s \quad (144)$$

Note first that $A\mathbf{R}_s \subseteq \mathbf{R}_s$, so that eq.(144) is clearly true if $A$ is invertible. If $A$ is singular, note that $\alpha$ cannot be zero in $\alpha E + \beta A = I$, so that $\mathbf{R}_s$ is given by eq.(123).

Then assuming that $A$ and $\tilde{A}$ are as in Eqs. (139) and (140) and using the fact that $J$ is invertible, it can be seen that eq.(144) will hold if it can be shown that:

$$\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \mathbf{R}_s \subseteq \mathbf{R}_s \tag{145}$$

If $B$ compatibly with eq.(139) is partitioned the following is obtained:

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{146}$$

$$\mathbf{R}_s = rang \begin{bmatrix} B_1 & JB_1 & \dots & \dots & \dots & \dots & J^{n-1}B_1 \\ B_2 & NB_2 & \dots & N^{n-1}B_2 & 0 & \dots & 0 \end{bmatrix} \tag{147}$$

where $\mu$ is the nilpotency degree of $N$. Let $J$ be $n_1 \times n_1$ and $N$ be $n_2 \times n_2$ (so that $n_1 + n_2 = n$ and $\mu \le n_2$). Suppose that $[\xi_1' \ \ \xi_2']' \in \mathbf{R}_s$; it should be shown that $[0 \ \ \xi_2']' \in \mathbf{R}_s$. However, if $[\xi_1' \ \ \xi_2']' \in \mathbf{R}_s$, inputs $\mathbf{u}_i, i = 0,...,\mu-1$ exist so that:

$$\xi_2 = \sum_{i=0}^{\mu-1} N^i B_2 \mathbf{u}_i \tag{148}$$

Wishing to show that this sequence with $\mathbf{u}_i, i = \mu,...,n$ can be augmented so that:

$$\sum_{i=0}^{\mu-1} J^i B_1 \mathbf{u}_i = 0 \tag{149}$$

i.e. so that:

$$\sum_{i=\mu}^{n-1} J^{i-\mu} B_1 \mathbf{u}_i = -J^{-\mu} \left( \sum_{i=0}^{\mu-1} J^i B_1 \mathbf{u}_i \right) \tag{150}$$

The right-hand side of eq.(150) is in the reachable space of $(J, B_1)$. Furthermore, since $n - 1 - \mu \ge n_1 - 1$, the left-hand side of eq.(150) can be driven to any point in the reachable space of $(J, B_1)$.

So far it has been shown that $\mathbf{R}_w(K-1, L)$ has the same dimension as $\mathbf{R}_w(K, L)$ as long as $K - 1 \ge n$. In a similar manner it can be shown that $\mathbf{R}_w(K, L+1)$ has the same dimension as well, as long as $L + 1 \le N - n$. This then completes the proof of the theorem.

Note that one immediate consequence of Theorem 5 is the following.

**Corollary.** The system Eqs. (13) and (14) is weakly reachable if it is weakly reachable off some $[K, L]$ with $K, L \in [n, N-n]$.

Hence, in order to test for weak reachability it is only necessary to examine the reachability space $\mathbf{R}_w(k,k)$ of $\mathbf{z}_i(k,k) = \mathbf{x}(k)$ for any $k \in [n, N-n]$. Note further that $\mathbf{R}_w(k,k)$ is the range space for the map from $\{\mathbf{u}(0),...,\mathbf{u}(N-1)\}$ to $\mathbf{x}(k)$ (with the boundary value set to zero); i.e. weak reachability corresponds to being able to drive $\mathbf{x}(k)$ to an arbitrary value using the entire interval of the controls. Thanks to statement *d*) of Theorem 5.3, it can be seen that weak reachability is indeed weaker than strong reachability which corresponds to being able to drive $\mathbf{x}(k)$ to an arbitrary value using only inputs within $n$ time steps of $k$.

While eq.(136) provides in principal a method for computing weakly reachable subspaces, it involves a significant amount of computation in order to determine $\Lambda(K,L), T_i(K,L)$ and $T_f(K,L)$. As the next theorem shows, there is an easier method for computing $\mathbf{R}_w(k,k)$.

**Theorem 6.** Let $k \in [n, N-n]$. Then:

$$\begin{aligned} \mathbf{R}_w(k,k) &= rang\left[ A^k E^{N-k} \left( V_i A + V_f E \right) R_s \vdots R_s \right] \\ & rang\left[ A^k E^{N-k} V_i R_s \vdots A^k E^{N-k} V_f R_s \vdots R_s \right] \end{aligned} \tag{151}$$

**Proof.** From Eqs. (28) and (29) (with $\omega = 1$ for simplicity) it follows that:

$$\begin{aligned} \mathsf{R}_w(k,k) = \Re\big[ & A^k \left( A - E^{N-k} \left( V_i A + V_f E \right) E^k \right) R_s(N-k) \vdots \\ & E^{N-k} \left( E - A^k \left( V_i A + V_f E \right) A^{N-k} \right) R_s(k) \big] \end{aligned} \tag{152}$$

That is, if $w \in \mathbf{R}_w(k,k)$, then there exist $\mathbf{x}, \mathbf{x}_i \in \mathbf{R}_s$ so that:

$$\begin{aligned} w &= A^k \left[ A - E^{N-k} \left( V_i A + V_f E \right) E^k \right] \mathbf{x} \\ &+ E^{N-k} \left[ E - A^k \left( V_i A + V_f E \right) A^{N-k} \right] \mathbf{x}_i \\ &= \left( A^{k+1} \mathbf{x} + E^{N-k+1} \mathbf{x}_i \right) - A^k E^{N-k} \times \\ &\times \left( V_i A + V_f E \right) \left[ E^k \mathbf{x} + A^{N-k} \mathbf{x}_i \right] \end{aligned} \tag{153}$$

Since $\mathbf{R}_s$ is $E$ - and $A$ -invariant, it can be seen that:

$$\mathbf{R}_w(k,k) \subseteq \Re\left[ A^k E^{N-k} \left( V_i A + V_f E \right) R_s \vdots R_s \right] \tag{154}$$

The first equality in eq.(151) will be proven if it can be shown that any $w$ in the range of $\left[ A^k E^{N-k} \left( V_i A + V_f E \right) R_s \vdots R_s \right]$ is in $\mathbf{R}_w(k,k)$. Clearly any such $w$ can be written as:

$$w = s - A^k E^{N-k} \left( V_i A + V_f E \right) t \tag{155}$$

with $s, t \in \mathbf{R}_s$. Comparing this to eq.(153) it can be seen that this will be concluded if it can be shown that there exists $\mathbf{x}, \mathbf{x}_i \in \mathbf{R}_s$ so that

$$\begin{bmatrix} A^{k+1} & E^{N-k+1} \\ E^k & A^{N-k} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_i \end{bmatrix} = \begin{bmatrix} \mathbf{s} \\ \mathbf{t} \end{bmatrix} \tag{156}$$

The matrix on the left-hand side of eq.(156) is invertible, and solving eq.(156) the following is obtained

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \Gamma^{-1} A^{N-k} & -\Gamma^{-1} E^{N-k+1} \\ -\Gamma^{-1} E^k & \Gamma^{-1} A^{k+1} \end{bmatrix} \begin{bmatrix} \mathbf{s} \\ \mathbf{t} \end{bmatrix} \tag{157}$$

where $\Gamma$ is defined in (5.9) (with $\omega = 1$). Since $\mathbf{R}_s$ is $E$ - and $A$ -invariant, it is also $\Gamma^{-1}$ invariant, so that $\mathbf{x}, \mathbf{x}_i \in \mathbf{R}_s$.

Finally it is necessary to verify the second equality in eq.(151). Since $\mathbf{R}_s$ is $E$ and $A$ invariant and $V_i E^N + V_f A^N = I$, it can be seen that:

$$\Re\left[ \left( V_i A + V_f E \right) R_s \vdots R_s \right] \subseteq \Re\left[ V_i R_s \vdots V_f R_s \right] \tag{158}$$

On the other hand:

$$\begin{aligned} \Re\left[ V_f R_s \vdots R_s \right] &= \Re\left[ V_f \left( E^{N+1} - A^{N+1} \right) R_s \vdots R_s \right] \\ &\subseteq \Re\left[ \left( V_i A + V_f E \right) E^N R_s \vdots \left( V_i E^N + V_f A^N \right) A R_s \vdots R_s \right] \\ &\subseteq \Re\left[ \left( V_i A + V_f E \right) R_s \vdots R_s \right] \end{aligned} \tag{159}$$

Similarly:

$$\Re\left[V_f R_s \vdots R_s\right] \subseteq \Re\left[\left(V_i A + V_f E\right) R_s \vdots R_s\right] \tag{160}$$

Combining Eqs. (158), (159) and (160) it can be seen that:

$$\Re\left[\left(V_i A + V_f E\right) R_s \vdots R_s\right] = \Re\left[V_i R_s \vdots V_f R_s\right] \tag{161}$$

Finally:

$$\begin{aligned} &\Re\left[A^k E^{N-k}\left(V_i A + V_f E\right) R_s \vdots R_s\right] \\ &= A^k E^{N-k} \Re\left[\left(V_i A + V_f E\right) R_s \vdots R_s\right] + \mathbf{R}_s \\ &= A^k E^{N-k} \Re\left[V_i R_s \vdots V_f R_s\right] + \mathbf{R}_s \\ &= \Re\left[A^k E^{N-k} V_i R_s \vdots A^k E^{N-k} V_f R_s \vdots R_s\right] \end{aligned} \tag{162}$$

Note from eq.(151) that $\mathbf{R}_s \subseteq R_w(k,k)$ for $k \in [n, N-n]$, consistent with the earlier statement that weak reachability is indeed a weaker condition.

Theorem 6 provides a computable weak reachability condition: it is then checked to see if either of the matrices in eq.(151) is full rank. The following results provide a simpler result of this type as no powers of $E$ or $A$ need be computed.

**Theorem 3.** The system Eqs. (13) and (14) is weakly reachable if and only if either of the matrices

$$\left[EA\left(V_i A + V_f E\right) R_s \vdots R_s\right] \tag{163a}$$

or

$$\left[EAV_i R_s \vdots EAV_f R_s \vdots R_s\right] \tag{163b}$$

has full rank.

**Proof.** To begin with, it is necessary to by show that for any subspace $\mathbf{D}$ of $R^n$:

$$E\mathbf{D} + \mathbf{R}_s = R^n \leftrightarrow E^2\mathbf{D} + \mathbf{R}_s = R^n \tag{164}$$

Let $\mathbf{S}$ be a subspace so that:

$$\mathbf{F} \oplus \mathbf{R}_s = E\mathbf{D} + \mathbf{R}_s \tag{165}$$

Then:

$$\begin{aligned} E^2\mathbf{D} + \mathbf{R}_s &= E\left(E\mathbf{D} + \mathbf{R}_s\right) + \mathbf{R}_s \\ &= E\left(\mathbf{F} \oplus \mathbf{R}_s\right) + \mathbf{R}_s = E\mathbf{F} \oplus \mathbf{R}_s \end{aligned} \tag{166}$$

Dimension counting then shows that the right-to-left implication in eq.(165) is true. Suppose that $E\mathbf{D} + \mathbf{R}_s = R^n$.

Then

$$\begin{aligned} E^2\mathbf{D} + \mathbf{R}_s &= E\left(E\mathbf{D} + \mathbf{R}_s\right) + \mathbf{R}_s \\ &= E\left(R^n\right) + \mathbf{R}_s \supseteq E\mathbf{D} \oplus \mathbf{R}_s = R^n \end{aligned} \tag{167}$$

Note that by iterating eq.(164) it can be seen that if $E^k\mathbf{D} + \mathbf{R}_s = R^n$ for some $k > 0$, it equals $R^n$ for all $k > 0$. A similar statement can be made with $E$ replaced by $A$, and combining these $E^k A^l \mathbf{D} + \mathbf{R}_s = R^n$ for some pair $k, l > 0$ if and only if $EA\mathbf{D} + \mathbf{R}_s = R^n$. The theorem then follows from the application of this result with $\mathbf{D} = \Re\left\{\left(V_i A + V_f E\right) R_s\right\}$.

A brief presentation of the corresponding concept of weak observability, and some relevant results follows:

**Definition 8.** The system Eqs. (13) and (14) is weakly observable off $[K, L]$ if the map from $\mathbf{z}_o(K, L)$ to $\{\mathbf{x}_i(j) \mid j \in [0, K] \cup [L, N]\}$, defined by eq.(15) and the four-point boundary-value problem Eqs. (35), (36) and (37) with $\mathbf{v} = \mathbf{0}, \mathbf{u} \equiv \mathbf{0}$, is one-to-one. The weakly unobservable subspace $\mathrm{O}_w(K, L)$ is the kernel of this map. The system is weakly observable if it is weakly observable off $[K, L]$ (i.e. if $\mathrm{O}_w(K, L) = \{0\}$ ) for all $K, L \in [n-1, N-n+1]$.

**Theorem 7**. The dimension of $\mathrm{O}_w(K, L)$ is constant for $K, L \in [n-1, N-n+1]$.

**Corollary.** The system Eqs. (13) and (14) is weakly observable if it is weakly observable off some $[K, L]$.

A consequence of this last result is that in order to test for weak observability it is only necessary to examine the unobservability space $\mathrm{O}_w(k, k+1)$ of $\mathbf{z}_0(k, k+1) = B\mathbf{u}(k)$. Furthermore, note that $\mathrm{O}_w(k, k+1)$ is the kernel of the mapping from $B\mathbf{u}(k)$ to the full sequence of measurements $\mathbf{x}_i(0), \ldots, \mathbf{x}_i(N)$ (with $\mathbf{v} = \mathbf{0}$ ). This is weaker than strong observability which involves the use of outputs restricted to lie within $n$ time steps of $k$.

**Theorem 8.** Let $k \in [n, N-n]$. Then:

$$\begin{aligned} \mathrm{O}_w(k,k) &= \aleph\begin{bmatrix} O_s \\ O_s\left(V_i A + V_f E\right) A^{N-k-1} E^k \end{bmatrix} = \\ &= \aleph\begin{bmatrix} O_s \\ O_s V_i A^{N-k-1} E^k \\ O_s V_f A^{N-k-1} E^k \end{bmatrix} \end{aligned} \tag{168}$$

Note that $\mathrm{O}_w(k,k) \subseteq \mathrm{O}_s$, demonstrating again that weak observability is a weaker condition.

**Theorem 9.** The system Eqs. (13) and (14) is weakly observable if and only if either of the matrices:

$$\begin{bmatrix} O_s \\ O_s\left(V_i A + V_f E\right) A^{N-k-1} E^k \end{bmatrix} \tag{169}$$

$$\begin{bmatrix} O_s \\ O_s V_i A^{N-k-1} E^k \\ O_s V_f A^{N-k-1} E^k \end{bmatrix} \tag{170}$$

has full rank.

## Conclusion

TPBV in case of singular systems exists due to the singularity of matrix $A$. Unlike causal systems, where TPBV is the consequence of applying adequate methods for system solving, here it is directly obtained from the nature of the system. For solving this problem in case of discrete singular systems, it is necessary to specify initial and ultimate conditions throughout the system of boundary values, with some limitations. Namely, boundary matrices should have such form, that it is obvious the system is in the normalized form. While stationary discrete singular systems are being solved big simplifications are being made. In fact, it is significantly simpler to calculate internal recursive process. The actual importance of these methods is illustrated by the fact that these systems can be found during the estimation of causal systems with some limitations and especially taking into consideration that most of natural phenomena can be presented better using singular systems.

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# Diskretni singularni sistemi sa dvo-tačkastim graničnim problemom

**U radu su izložene metode za rešavanje klase dvo-tačkastih singularnih sistema sa konturnim problemom. Pri rešavanju ove klase sistema pomenutim metodama potrebno je rešiti sistem diferencnih jednačina sa dvo-tačkastim graničnim uslovom. Značaj ovih metoda je utoliko veći koliko se zna da ova vrsta sistema nalazi sve veću primenu u vernom opisivanju brojnih prirodnih fenomena.**

*Ključne reči*: **diskretni sistem, singularni sistem, granični uslovi, dvo-tačkasti granični problemi, ocena stanja.**

# Дискретные сингулярные системы со предельной проблемой с двумя точками

В настоящей работе представлены методы для решения класса сингулярных систем со предельной проблемой со двумя точками. При решении этого класса систем упомянутыми методами нужно решить систему разностных уравнений со предельным условием со двумя точками. Этот метод имеет настолько большее значение насколько известно, что этот вид систем всё больше пользуется в прецизионном описании многочисленных натуральных явлений.

*Ключевые слова*: дискретная система, сингулярная система, предельные условия, предельная проблема со двумя точками, оценка состояния.

# Systèmes singuliers discrets au problème du contour limite à deux pointillés

**Dans ce papier on a exposé les méthodes servant à résoudre la classe des systèmes singuliers avec le problème du contour à deux pointillés. En résolvant cette classe de système par les méthodes citées, il faut résoudre le système des équations différentielles à condition limite aux deux pointillés. L'importance de ces méthodes est d'autant plus grande que l'on connaît que ce genre de système trouve une appliqation de plus en plus grande dans les déscriptions précises de nombreux phénomènes naturels.**

*Mots clés*: **système discret, système singulier, conditions limites, problème limite à deux pointillés, évaluation de l'état.**