UDK: 681.5,519.876
COSATI: 14-02

Synthesis of Generalized Linear Singular System Using Global Proportional-Differential Feedback

Ivan M. Buzurović, PhD (Eng) [1)]
Dragutin Lj. Debeljković, PhD (Eng)[1)]

In this paper, the use of differential and proportional state feedback, as a means for gaining regulability and controllability of linear singular systems was investigated. Two approaches were analysed: first which solves the problem using geometric approach, and second based on classic automatic control theory. There are numerous examples which illustrate efficiency of the given procedures. All results could be applied to regular – non-singular systems, bearing in mind the geometric approach. Appropriate transformations for designing state feedback with standard controllers are given as well.

Key words: **singular regulation, singular systems, system synthesis, system regulation, linear system, proportional differential regulation.**

Introduction

SINGULAR systems appear naturally in many engineering disciplines and problems, like electric, electronic and magnetic circuits, in flight dynamic and robotic problems, in large energetic systems and feedback systems. They appear in control and optimization problems, just as in modelling process for some nontechnical disciplines.

The question is: why do singular systems appear in the mathematical modelling process? Which are the reasons for the expansion of this relatively young part of control theory in the recent times? How is it that this kind of systems did not appear in earlier researches? Answer to these questions is quite simple. Singularity in mathematical models was avoided using appropriate assumptions with substantial foundations. Because of that, mathematical model based on the adopted system model has had lower proximity level with real physical process. For more accurate presentation of the physical system, singular system theory was developed, a theory which supports singularities and treats them in an adequate manner.

To be precise in the following presentation, it is necessary to determine singular system class and input signal class which will be subject of this research. In that sense, only continuous, time-invariant, stationary singular systems with or without input signals are considered.

Point of interest is the system described by equation:

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)\,, \tag{1}$$

with singular matrix E. System given in equation (1) could be presented with matrix triple (E, A, B), so if matrix E is singular it means that system (1) is singular, as well. Otherwise it is said that the system is regular. Very often there is a need for a unique theory which can solve the problems of both, regular and singular systems simultaneously. One of those results could be obtained using singular perturbations. In that process, singular system is approximated with idealized regular system in some parts. In that way, boundary regular system model is brought closer. Exactly in these cases there is a strong need for the development of various syntheses methods for developing controllers placed in global state feedback. These methods have to be effective in designing any kind of generalized automatic control system.

Singular system described in the regular way is considered, eq. (1), where $\mathbf{x}(t)$ is a generalized n-state vector, of $n\times 1$ dimension and $\mathbf{u}(t)$ is an input vector, of $m\times 1$ dimension. Assuming that the considered system is *regular*; it means that det $(sE - A) \neq 0$.

It is well know that the state of the system (1) is completely determined for every $t \geq 0$ using $\boldsymbol{x}(0-)$ and $\boldsymbol{u}(t)$. However, due to the existence of endless physical frequent modes, $\boldsymbol{x}(t)$ could appear as mode with *impulse* behaviour. This usually happens in cases when $\boldsymbol{u}(t)$ or derivatives of $\boldsymbol{u}(t)$ are *discontinue*. This, of course, has negative consequence because it limits the class of expected input functions and system becomes extremely sensitive to noise. In those cases the structure of the system needs to be modified to gain equalized system trajectory. In this paper the possibility to apply state feedback in order to exchange *endless physical frequent modes* with ***final frequent modes*** was investigated.

The application of proportional state feedback in normal systems is well known in control theory and could be found in many books and papers. It is also known that eigenvalues of normal system (poles) could be transferred to arbitrary

1) Faculty of Mechanical Engineering, Kraljice Marije 16, 11120 Belgrade

final places using state feedback with *determinate amplification* if and only if the system is *infinite controllable* and each system frequency could be arbitrarily distributed and if and only if the system is completely controllable.

Using differential state feedback was not a popular method for normal systems. It is quite adequate because differential state feedback is less effective than proportional state feedback. However, in case of singular systems, which are very noise sensitive, it is possible that differential state feedback reduces noise sensitivity. Moreover, differential state feedback could be used for singular system *dynamical rank* changing.

Some of the existing methods for finding solutions of this type for singular systems are based on Weierstrass's decomposition to subsystems with infinite and noninfinite frequencies. Because the dimension of system with infinite frequencies is equal to rank of matrix pencil *det* $(sE-A)$, it is necessary to use numerical approach. However, rank of matrix pencil *det* $(sE-A)$ is not constant at open surroundings of singular systems. That is why the Weierstrass's decomposition method is very often limited on account of the given reasons.

Geometric approach in solving proportional-differential feedback synthesis problem

In order to solve this kind of problem subspaces family $\{\Sigma_\theta\}$ is defined, where each of subspaces is isomorphous in relation to regular system space of values, with characteristic that this subspace family is related to all subspaces of the generalized system. A set of regular systems, ones which do not have infinite eigenvalues, is denoted with Σ_0. Because every family subset Σ_θ is opened, generalized system given is inner point of each subset to which it belongs. Therefore, if system (*E, A, B*) belongs to Σ_θ, control design technique applied to Σ_θ could be used not only for (*E, A, B*) systems, but for every system which belongs to open surroundings of the defined system (*E, A, B*).

The next element in the discussed procedure is a group symmetrical to $\{\Sigma_\theta\}$ set. It consists of transformation group $\{R_\phi\}$ with the characteristic that R_ϕ isomorphly transforms Σ_θ to $\Sigma_{\theta=\phi}$.

The third element of the theory allows that feedback could be applied to every subset Σ_θ with the characteristic that it exactly corresponds to that subset. Feedback control vector is in the form of:

$$\mathbf{u}(t)=F\left(\mathbf{x}(t)\cos\theta-\dot{\mathbf{x}}(t)\sin\theta\right)+\mathbf{v}(t)\,. \tag{2}$$

Parameter θ denotes constant value in the feedback. If $\theta=0$, then feedback has a static state character. Therefore, it can be concluded that the following analysis will include most common and practically possible cases.

Singular System Transformation

Let $\hat{\Sigma}(n,m)$ denote space of all matrix triples (*E, A, B*) $\in R^{nxn}$ x R^{nxn} x R^{nxm}. Let $\Sigma(n,m)$ denote subspace of $\hat{\Sigma}(n,m)$ defined by the following relation:

$$\Sigma(n,m)=\left\{(E,A,B)\in\hat{\Sigma}(n,m):\det(sE-A)\neq 0\right\}. \tag{3}$$

On condition that polynomial *det* $(sE-A)$ is not equal to zero, this guarantees uniqueness of the system solution, given in equation (1). In references, systems belonging to subspace $\Sigma(n,m)$ are called *regular*.

For every $\theta\in R$ $\hat{\Sigma}_\theta(n,m)$ is defined as a subset of $\Sigma(n,m)$ given with next equation:

$$\begin{aligned}&\Sigma_\theta(n,m)=\\&=\left\{(E,A,B)\in\Sigma(n,m):\det(\cos\theta E-\sin\theta A)\neq 0\right\}\end{aligned} \tag{4}$$

***Note* 1.** If $\theta=0$, $\Sigma_0(n,m)$ consits of triple (*E, A, B*) for which *E* is non-singular, the system is regular. Therefore regular systems build one open subspace of closeness $\{\Sigma_\theta(n,m)\}$.

Further more, a group of symmetric copy functions $\{\Sigma_\theta(n,m):\theta\in[0,\pi)\}$ - transformation which copy all subspaces one into others is defined.

For every $\Phi\in R$, the defined transformation is

$$R_\Phi:\hat{\Sigma}(n,m)\mapsto\hat{\Sigma}(n,m)$$

with the following relation:

$$R_\Phi(E,A,B)\cong(\cos\Phi E+\sin\Phi A)-\sin\Phi E+\cos\Phi A,B\,. \tag{5}$$

If $(\hat{E},\hat{A},B)\cong R_\Phi(E,A,B)$, then:

$$\begin{bmatrix}\hat{E}\\ \hat{A}\end{bmatrix}=\begin{bmatrix}\cos\Phi I & \sin\Phi I\\ -\sin\Phi I & \cos\Phi I\end{bmatrix}\cdot\begin{bmatrix}E\\ A\end{bmatrix}, \tag{6}$$

$(\hat{E},\hat{A})$ are got from (E,A), using rotation for Φ angle.

Further investigations show how positions of eigenvalues are changed when the system is in rotation. Expanded complex plane $\boldsymbol{CU}(\infty)$ and complex projected space $\boldsymbol{CP}(1)$ are defined. Also, relation on $\boldsymbol{C}^2$ is defined by relation $(s_1,s_2)\sim(\hat{s}_1,\hat{s}_2)$ if and only if there is no null complex number λ which fulfils equation $(\hat{s}_1,\hat{s}_2)=\lambda(s_1,s_2)$. If $(s_1,s_2)\in\boldsymbol{C}^2$, with $[(s_1,s_2)]$ the corresponding element from $\boldsymbol{CP}(1)$ is denoted, i.e. equivalency classes which contain (s_1,s_2). (s_1,s_2) are called *homogenous coordinates* of $[(s_1,s_2)]$.

Let (*E, A, B*) be a regular system, (*E, A, B*) $\in\Sigma(n,m)$. It is said that $[(s_1,s_2)]\in\boldsymbol{CP}(1)$ is eigenvalue of the system (*E, A, B*) if and only if *det* $(s_1E-s_2A)=0$.

It can be seen that if $[(s_1,s_2)]$ = $[(\hat{s}_1,\hat{s}_2)]$], then *det* $(s_1E-s_2A)=0$ if and only if $\det(\hat{s}_1E-\hat{s}_2A)=0$. Only if these conditions are fulfilled, eigenvalues of the system are well defined.

***Note* 2.** If $[(s_1,s_2)]$ was the same as expanded complex number $\alpha\cong[(s_1,s_2)]$, it could be concluded that α is the system eigenvalue, gained from equation *det*

$(s_1E - s_2A) = 0$. If $s_2 \neq 0$, than α is a finite complex number and it denotes system eigenvalue if and only if *det* $(\alpha E - A) = 0$. This coincides with the standard definition of generalized linear system eigenvalue. If $s_2 = 0$, it is then $\alpha = \infty$ system eigenvalue if and only if det $E = 0$. Manifold of $\alpha = \infty$ is equal to subsystem dimension with infinite frequency in Weierstrass's decomposition and does not depend on Joradn's structure.

Recent description of infinite eigenvalues is characteristic, because there is no difference between dynamics of impulse modes and nondynamic restrictions. It is more common to define the value of manifold $\alpha = \infty$ as a number of independent impulse modes i.e. as a degree of det $(sE - A)$ or as *rang E*. According to definition, the system has full number of eigenvalues (finite and infinite) equal to the rang of the matrix *E*. Total number of system eigenvalues is invariant to system rotation, as previously mentioned. Since *rang E* is not a customary definition of eigenvalues it does not support this infinite values characteristic.

***Note* 3.** It directly follows from definition of $\sum_\theta(n,m)$ that system (*E, A, B*) belongs to $\sum_\theta(n,m)$ if and only if there are no eigenvalues in point $[(\cos\theta, \sin\theta)]$. Then $\sum_0(n,m)$, as a set of regular systems consists of systems which do not have eigenvalues at infinity.

The following result shows that eigenvalues of a generalized linear system are moved in case of system rotation. If system rotates for an angle Φ, homogenous coordinates of eigenvalues rotate for angle - Φ.

Theorem 1. Let $(E,A,B) \in \sum(n,m)$ and let $(\hat{E},\hat{A},B) \cong R_\Phi(E,A,B)$. Then $[(s_1,s_2)]$ is eigenvalue of the system (E,A,B) if and only if $[(\hat{s}_1,\hat{s}_2)]$ is eigenvalue of the system $(\hat{E},\hat{A},B)$, where:

$$\begin{bmatrix}\hat{s}_1\\ \hat{s}_2\end{bmatrix} \cong \begin{bmatrix}\cos\Phi & -\sin\Phi\\ \sin\Phi & \cos\Phi\end{bmatrix}\cdot\begin{bmatrix}s_1\\ s_2\end{bmatrix}. \qquad (7)$$

Equivalent to this, expanded complex number α is eigenvalue of the system (E,A,B) if and only if expanded complex number $\hat{\alpha}$ is eigenvalue of $(\hat{E},\hat{A},B)$, where:

$$\hat{\alpha} \cong \frac{(\cos\Phi)\alpha - \sin\Phi}{(\sin\Phi)\alpha + \cos\Phi}. \qquad (8)$$

Controllability and Observability Analysis

Let (E,A,B) be regular system and let R (E,A,B) be *controllable subspace.* $\mathrm{R}(E,A,B)$ consists of the states in R^n which are reachable at positive time from onset state $\boldsymbol{x}(0-) \cong 0$. And if $R(E,A,B) = \mathrm{R}^n$, then (E,A,B) are called *controllable system*. If P is linear transformation on R^n an S is subspace in R^n, $\langle P \mid S\rangle$ denote subspace $S + P(S) + ... + P^{n-1}(S)$, i.e. least P invariant subspace which contains S.

***Lemma* 1.** If $(E,A,B) \in \sum(n,m)$ and α is a real number which satisfies *det* $(\alpha E - A) \neq 0$, then:

$$\mathrm{R}(E,A,B) = \left\langle (\alpha E - A)^{-1} E \mid R\left(\alpha E - A^{-1}\right)B\right\rangle \qquad (9)$$

Theorem 2. The generalized system (E,A,B) is controllable if and only if regular system $\left(I,(\alpha E - A)^{-1}E,(\alpha E - A)^{-1}B\right)$ is controllable.

Given result is explained using *Lemma* 1.

***Lemma* 2.** Let $(E,A,B) \in \sum_\theta(n,m)$ and let $(\hat{E},\hat{A},B) \cong R_\theta(E,A,B)$. Then:

$$\mathrm{R}(E,A,B) = \mathrm{R}(\hat{E},\hat{A},B). \qquad (10)$$

Theorem 3. $(E,A,B) \in \sum_\theta(n,m)$ is controllable if and only is the regular system $R_\theta(E,A,B)$ is controllable.

***Corollary* 1.** Let $(E,A,B) \in \sum_\theta(n,m)$ and let:

$$\begin{aligned}&(\hat{E},\hat{A},B) \cong R_{-\theta}(E,A,B)\\ &= (\cos\theta E - \sin\theta A, \sin\theta E + \cos\theta A, B)\end{aligned}.$$

Regular normal system $(\hat{E},\hat{A},B)$ is controllable if and only if:

$$rang\begin{vmatrix}\hat{E}^{-1}B,\left(\hat{E}^{-1}\hat{A}\right)\hat{E}^{-1}B,...,\\ \left(\hat{E}^{-1}\hat{A}\right)^{n-1}\hat{E}^{-1}B\end{vmatrix} = n \qquad (11)$$

which follows from *Theorem* 3, so generalized linear system (E,A,B) is controllable if and only if condition (11) is fulfilled. If $\theta = 0$, than equation (11) is led to controllability condition of normal systems.

In the continuation, the state equation of the system given with expression (1) along with linear output equation is considered:

$$\boldsymbol{y}(t) = C\boldsymbol{x}(t), \qquad (12)$$

where C is real $p \times x$ matrix. System given by equations (1) and (17) could be written using symbol (E,A,B,C). Instead of $\hat{\sum}(n,m)$, $\sum(n,m)$ and $\sum_\theta(n,m)$ groups $\hat{\Gamma}(n,m,p)$, $\Gamma(n,m,p)$ and $\Gamma_\theta(n,m,p)$ were defined. Let $\hat{\Gamma}(n,m,p)$ space of all matrix groups (E,A,B,C) $\in \mathrm{R}^{nxn}\times\mathrm{R}^{nxn}\times\mathrm{R}^{nxm}\times\mathrm{R}^{pxn}$ be denoted. Let $\Gamma(n,m,p)$ denote open subspace of $\hat{\Gamma}(n,m,p)$ defined by:

$$\begin{aligned}&\Gamma(n,m,p) \cong\\ &\left\{(E,A,B,C) \in \hat{\Gamma}(n,m,p) : \det(sE - A) \neq 0\right\}\end{aligned} \qquad (13)$$

Let $\Gamma_\theta(n,m,p)$ be a subspace of $\Gamma(n,m,p)$ given with:

$$\Gamma_\theta(n,m,p) \cong \begin{Bmatrix}(E,A,B,C) \in \Gamma(n,m,p):\\ \det(\cos\theta E - \sin\theta A) \neq 0\end{Bmatrix} \qquad (14)$$

Accepting the notation given, symbol R_Φ is adopted as a sign for function onto $\hat{\Gamma}(n,m,p)$ given as follows:

$$R_{\Phi}(E,A,B,C) \cong (\cos\Phi E + \sin\Phi A, -\sin\Phi E + \cos\Phi A, B, C) \tag{15}$$

Further, some relations between system rotation and its observability are examined. Let (E,A,B,C) denote a regular system and let K (E,A,B,C) denote nonobservable subspace. K (E,A,B,C) consists of some states x_0 with the characteristic that if $\boldsymbol{x}(0-)=\mathbf{x}_0$, than $\boldsymbol{y}(0-)=0$, then the system response is identically equal to zero on $[0,\infty]$. (E,A,B,C) is observable if K $(E,A,B,C)=0$, or equivalently, if $\boldsymbol{y}(0-)$ together with input and output on $[0,\infty]$ it presents sufficient condition for determination of $\boldsymbol{x}(0-)$.

***Lemma* 3**. If $(E,A,B,C)\in\Gamma(n,m,p)$ and if α denotes a real number which fulfils condition det $(\alpha E-A)\neq 0$, then it is:

$$(E,A,B,C)=\bigcap_{i=0}^{n-1} N\left\{C\left[(\alpha E-A)^{-1}E\right]^i\right\} \tag{16}$$

Theorem 4. Generalized system (E,A,B,C) is observable if and only if regular system $\left(I,(\alpha E-A)^{-1}E,(\alpha E-A)^{-1}B,C\right)$ is observable.

Using *Lemma 3*, it is possible to prove the following result.

***Lemma* 4.** Let $(E,A,B,C)\in\Gamma_{\theta}(n,m,p)$ and let $\left(\hat{E},\hat{A},B,C\right)\cong R_{-\theta}(E,A,B,C)$. Then the following applies:

$$\text{K }(E,A,B,C) = \text{K }\left(\hat{E},\hat{A},B,C\right). \tag{17}$$

Proof. If *sin* $\theta=0$, the equation is trivial, so it could be assumed that $\sin\theta\neq 0$. Because $\left(\hat{E},\hat{A},B,C\right)$ is normal system, then:

$$\text{K}\left(\hat{E},\hat{A},B,C\right)=\bigcap_{i=0}^{n-1} N\left[C\left(\hat{E}^{-1}\hat{A}\right)^i\right], \tag{18}$$

where $\hat{E}^{-1}\hat{A}$ is the biggest invariant subspace in $N(C)$. Let $\alpha\cong\cos\theta/\sin\theta$. Because of $(E,A,B,C)\in\Gamma_{\theta}(n,m,p)$, det$(\alpha E\text{-}A)\neq 0$, and according to *Lemma 3* it is:

$$\text{K}(E,A,B,C)=\bigcap_{i=0}^{n-1} N\left\{C\left[(\alpha E-A)^{-1}E\right]^i\right\}. \tag{19}$$

biggest $(\alpha E\text{-}A)^{-1}E$ invariant subspace included in $N(C)$. From the proof of *Lemma* 2 it is known that $\hat{E}^{-1}\hat{A}$ and $(\alpha E-A)^{-1}E$ have the same invariant subspaces. From the equation (17) and (18) follows that K (E,A,B,C) =system (E,A,B). θ is chosen to fulfil equation $(E,A,B)\in\sum_{\theta}(n,m)$. Using rotation of the system for angle $-\theta$ normal system $(E_0,A_0,B_0)\cong R_{-\theta}(E,A,B)\in\sum_0(n,m)$ is obtained.

1. Rotation for obtaining the requested performances. First of all, it is necessary to define the required characteristics of a normal system $(\hat{E}_0,\hat{A}_0,B)\in\sum_0(n,m)$ which a system needs to have in order for the system $(\hat{E},\hat{A},B)\cong R_{\theta}(\hat{E}_0,\hat{A}_0,B)\in\sum_{\theta}(n,m)$ to fulfil the required performances in the synthesis procedure.
2. Solving synthesis problem for normal system (E_0,A_0,B). Amplification matrix F is chosen first, for the normal system with closed feedback $(\hat{E}_0,\hat{A}_0,B)$, obtained from (E_0,A_0,B) from state feedback and control algorithm $\boldsymbol{u}=F\boldsymbol{x}+\boldsymbol{v}$, to satisfy the prescribed performance from step 2. Please note that $(\hat{E}_0,\hat{A}_0,B)=g_0(F)$ (E_0,A_0,B).
3. Last synthesis step
   PD control law is applied:

$$\boldsymbol{u}(t)=F\left(\boldsymbol{x}(t)\cos\theta-\dot{\boldsymbol{x}}(t)\sin\theta\right)+\boldsymbol{v}(t), \tag{21}$$

For original generalized system (E,A,B) using amplification F, defined at step 3.

***Note* 4.** It is clear that the solution of the problem given demands two results. It is necessary to solve synthesis feedback problem for normal system, step 3, and also to know how certain system characteristics behave when the system is in rotation, for e.g. where the poles of the system are, step 2.

***Note* 5.** Because the given regular system (E,A,B) belongs to $\sum_{\theta}(n,m)$ for each $\theta\in[0,\pi]$ it is possible to treat θ as synthesis parameter, which needs to be determined.

***Note* 6.** According to *Note* 4, it can be concluded that there are some other analogous procedures for determining the control with output feedback. In that case, synthesis problem for output feedback system needs to be solved.

***Example* 1**. Considering normal system $(E,A,\boldsymbol{b},\boldsymbol{c},\boldsymbol{d})$, where:

$$E=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ A=\begin{bmatrix}0&1\\0&0\end{bmatrix},\ \boldsymbol{b}=\begin{bmatrix}0\\1\end{bmatrix},$$

$$\boldsymbol{d}=\begin{bmatrix}1\\2\end{bmatrix},\ \boldsymbol{c}=\begin{bmatrix}1&-1\end{bmatrix}.$$

Because $R(\boldsymbol{d})$ is not a part of $EN(\boldsymbol{c})$, it is clear that the state feedback does not solve the problem. Considering θ as a parameter which needs to be determined, a necessary condition for problem solving is:

$$R(\boldsymbol{d})\subset(\cos\theta E-\sin\theta A)N(\boldsymbol{c}). \tag{22}$$

It is obvious that the previous relation is fulfilled if and only if $tg\theta=1/2$. Using the stated fact, $F=\left[\delta\sqrt{5}\ -\delta\right]$ is obtained, where δ could be chosen arbitrarily. For every δ, except for $\delta=2\sqrt{5}$, the resulting system is regular.

***Example* 2**. The system $(E,A,\boldsymbol{b})$ is considered, with matrices given:

$$E=\begin{bmatrix}0&1\\0&0\end{bmatrix},\ A=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ \boldsymbol{b}=\begin{bmatrix}0\\1\end{bmatrix}.$$

$(E,A,\boldsymbol{b})\in\sum_\theta(n,m)$ or every not null θ. Then:

$$E_0=\begin{bmatrix}-\sin\theta&\cos\theta\\0&-\sin\theta\end{bmatrix},$$

$$A_0=\begin{bmatrix}\cos\theta&\sin\theta\\0&\cos\theta\end{bmatrix}.$$

Ordinary normal system, associate system given $(E_0,A_0,\boldsymbol{b})$ is $(E_0^{-1}A_0,\ E_0^{-1}\boldsymbol{b})$, where is:

$$E_0^{-1}A_0=\begin{bmatrix}-ctg\theta&-\csc^2\theta\\0&-ctg\theta\end{bmatrix},$$

$$E_0^{-1}\boldsymbol{b}=\begin{bmatrix}-ctg\theta\csc\theta\\-\csc\theta\end{bmatrix}.$$

If θ is approximately zero, then system $(E_0,A_0,\boldsymbol{b})$ is almost singular, and the characteristics of $(E_0^{-1}A_0,\ E_0^{-1}\boldsymbol{b})$ approach eternity.

In order to escape computing problems connected with solving synthesis problems for almost singular regular systems, it is advisable to choose θ in the way that the system (E,A,B) is inside $\sum_\theta(n,m)$. In other words, system eigenvalues in open circuit would not have to be near $ctg\theta$. Different problems appear if the desired eigenvalues are near $ctg\theta$ point. In that case state amplification, defined with matrix F in step 3, has to be chosen in that way in order for the closed normal system $g_0(F)$ $(E_0,A_0,\boldsymbol{b})$ to have eigenvalues close to infinite.

***Example* 3.** In this example $(E,A,\boldsymbol{b})$ are chosen in the same way as in *Example 2*. Presuming that the desirable eigenvalues of the closed system are in point -1 with multiplication 2, in resulting PD feedback with the control law given:

$$\boldsymbol{u}(t)=F\left(\boldsymbol{x}(t)\cos\theta-\dot{\boldsymbol{x}}(t)\sin\theta\right)+\boldsymbol{v}(t), \tag{23}$$

elements of the matrix are:

$$f_1=\frac{-\sin\theta}{1+\sin 2\theta},\ f_2=\frac{-2\sin\theta-\cos\theta}{1+\sin 2\theta} \tag{24}$$

If $\theta\to3\pi/4(ctg\theta\to-1)$ elements of the matrix F tend to infinity. This possible problem could be avoided choosing θ so $ctg\theta$ is not close to any desired eigenvalue.

Classical approach to solving synthesis problem for proportional-differential feedback

In this part of the paper, *regulation* problems of linear singular system considering geometric approach are given. Methodology given in this part is based on classical approach for this kind of problems, which in form of necessary and sufficient conditions gives the possibility for introducing proportional-differential feedback for regularization of the given singular system. Base investigation, which is the outset for further results, could be found in *Mukandan, Dayawansa* (1984).

It is important to point out that it is assumed that matrices *E*, *B*, *A*, are defined in the field of real numbers, even if most problems of this kind could be solved for matrices with complex numbers. Controllers have constant amplification, unless stated otherwise.

All matrix forms of $(sE-A)$ type, $(sE-A-BF)$, $(sE+sBF_1-A-BF_2)$ from this point on are considered to be non-singular.

Regularization of Singular System

The importance of achieving system regularity is illustrated by the fact that a regular system can be transformed into normal system and after this treated in the customary way. Classic automatic control theory could be applied in case of a system prepared in this way. PDF has the following form

$$\boldsymbol{u}(t)=-F_1\dot{\boldsymbol{x}}(t)+F_2\boldsymbol{x}(t)+\boldsymbol{v}(t). \tag{25}$$

In that case, system with feedback has the form given by equation (26):

$$(E+BF)_1\dot{\boldsymbol{x}}(t)=(A+BF_2)\boldsymbol{x}(t)+B\boldsymbol{v}(t). \tag{26}$$

***Definiton* 1**. System with closed feedback given in eq. (26) could be regular if using PDF where there is a matrix F_1 with the characteristic that the matrix pencil $(E+BF_1)$ is non-singular.

Theorem 6. System could be transformed into a regular one using PDF if and only if matrix $[E\ B]$ has full rang.

Proof. Without loosing overall concept, it could be assumed that B is of the form: $[0\ I_m]^T$.

Existence of a F_1 matrix with the characteristic that $(E+BF_1)$ is non-singular, provided that first $(m-n)$ rows of matrix E are linearly independent. It follows that matrix E has full row rang. On the other hand, if $[E\ B]$ is of full row rang, first $(n-m)$ rows of matrix *E* are linearly independent. According to that, it is possible to choose matrix F_1, as matrix pencil $(E+BF_1)$ to be non-singular. Let implications of *Assumption* 1 in the sense of system controllability characteristics be considered.

For example, considering the system given by eq. (1), it is obvious that the system is not infinity controllable, *Verghese* (1978), *Verghese et.al* (1981). Considering the transformed form of that system, where:

$$E=\begin{bmatrix}E_1\\0\end{bmatrix},\ A=\begin{bmatrix}A_1\\A_2\end{bmatrix},\ B=\begin{bmatrix}B_1\\B_2\end{bmatrix}, \tag{27}$$

with full row rang matrix E_1. Division is done in the way that the number of rows in matrices E_1, A_1 and B_1 are equal. Since system (1) is not infinity controllable, then matrix:

$$\begin{bmatrix}E_1&0\\A_2&B_2\end{bmatrix}, \tag{28}$$

does not have full row rang.

Keeping in mind that matrix B_2 can not have full row rang it follows that matrix:

$$\begin{bmatrix} E_1 & 0 \\ 0 & B_2 \end{bmatrix}, \tag{29}$$

and matrix $\begin{bmatrix} E & B \end{bmatrix}$ will not have full row rang.

This result can be proclaimed using the following assumption.

Theorem 7. The necessary condition, in order to gain system regularity of the system given by eq. (1), is that the system has to be infinitely controllable.

Proof. If system, given by eq. (1) is not infinitely controllable, then $\begin{bmatrix} E & B \end{bmatrix}$ will not have full row rang. Therefore, according to *Assumption* 1, the system is not regulable and can not be regularized.

System synthesis using state feedback – modal control of linear singular systems

Considering infinitely controllable linear singular system, it is possible to transform the initial equations into forms where variables are divided into *nondynamic* and *dynamic*, *Verghese* (1978), *Verghese et al* (1981). This could be done using appropriate linear non-singular transformation and multiplication of matrices E, A and B from the left side with adequate non-singular matrix.

Kronecker's form of singular matrix pencil should be considered especially:

$$(sE - A) = \begin{bmatrix} sI - \tilde{J} & 0 & 0 \\ 0 & I - s\tilde{J} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{matrix} \} \; r \\ \} \, n-r \\ \end{matrix} = \begin{bmatrix} s\bar{E} - \bar{A} & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \} \; r \\ \} \, n-r \end{matrix} \tag{30},$$

where $\tilde{J}$ consists of several nilpotent Jordan's blocks, each with rang two or more.

Partitioning of matrix B and vector $\mathbf{x}(t)$ is done according to the following rule:

$$B = \begin{bmatrix} \bar{B} \\ B^0 \end{bmatrix} \begin{matrix} \} \; r \\ \} \, n-r \end{matrix}, \quad \mathbf{x} = \begin{bmatrix} \bar{x} \\ x^0 \end{bmatrix} \begin{matrix} \} \; r \\ \} \, n-r \end{matrix}. \tag{31}$$

Let *J* be a matrix in Jordan's form with special Jordan's blocks $\hat{J}_1, \hat{J}_2, ..., \hat{J}_k$ from *J*. Let rows of matrix *B*, which are correspond the rows on the last position of $\hat{J}_1$, $\hat{J}_2, ..., \hat{J}_k$, written as $\hat{b}_1, \hat{b}_2, ..., \hat{b}_k$, respectively be considered. Infinitely controllable implies that rows $\hat{b}_1, \hat{b}_2, ..., \hat{b}_k$ are linearly independent. According to that, it could be stated that matrix is:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & \bar{B} \\ 0 & -\hat{J} & 0 \end{bmatrix}, \tag{32}$$

with full rang. Likewise, according to definition of *Verghese* (1978), *Verghese et al* (1981) zeroes of $(sE - A)$ are in the same time zeroes of the matrix pencil $(s\bar{E} - \bar{A})$.

In the presupposition, some lemmas, used for further investigations are given. It should be noted that all matrix forms of type $(sE - A)$, $(sE - A - BF)$, $(sE + sBF_1 - A - BF_2)$ in the following part, are regular (non-singular).

***Lemma* 5**. Let matrix pencil be considered:

$$(sE - A) = \begin{bmatrix} sE - A_1 \\ sE - A_2 \end{bmatrix}, \tag{33}$$

where pencil $(sE - A)$, has full rang. Any null of matrix $(sE_1 - A_1)$, will be a null of matrix $(sE - A)$, .

***Lemma* 6.** Considering matrix pencil further provides:

$$(s\tilde{E} - \tilde{A}) = [sE - AB]. \tag{34}$$

Matrix B has full column rang. It is assumed that matrix $(sE - A)$, is non-singular and that there are no infinite nulls (zeroes).

In that case, a matrix F exists, with the characteristic that $[sE - (A + BF)]$ has arbitrarily given finite zeroes, if and only if matrix $(s\tilde{E} - \tilde{A})$ has no zeroes. Further more, the total number of zeroes of $[sE - A - BF]$ is independent of F.

In the following lines control applied to linear singular system using proportional feedback will be reviewed. The possibility for translating finite and infinite frequency of poles onto arbitrarily chosen location for singular linear system is considered. In that sense, it is very important to illustrate significance controllability, complete controllability and infinite controllability.

Theorem 8. Infinite zeroes of the matrix pencil $(sE - A)$, for system given with eq. (1) could be translated to any arbitrary location if and only if the system is of infinite controllability. Further more, generalised dynamic series of the system (1) *(ti. total number of independent frequent modes)* is invariant of feedback matrix.

Taking into account that the total number of zeroes stays the same when proportional feedback is applied, synthesis system matrix $[sE - A - BF]$ has no zeroes at infinity. According to that, space of zeroes of the matrix pencil $[sE - A - BF]$ is equal to the space of finite zeroes of the matrix pencil $[sE - A]$ and zeroes of matrix $[I - sJ_2 + B_2(F_{21} + F_{22})]$. Second space of zeroes on the right side could be chosen arbitrarily, to correspond infinite zeroes of the matrix pencil $[sE - A]$ and the numbers of that zeroes is equal.

Further theorem expansion is by natural isomorphism of controllability and possibility to regulate eigenvalues of normal (classic) systems defined by their own matrix models in case linear singular systems are given with state vector differential equation and output equation.

***Therem* 9**. Finite and infinite zeroes of the matrix pencil $[sE - A]$ for system given in eq. (1) could be translated onto arbitrary locations using state proportional feedback if and only if the system (1) is completely controllable, which implies that all finite and infinite modes are controllable individually.

Modal Control of Singular Systems using proportional and differential feedback

In previous considerations it was found that the generalised dynamic rank of feedback system is invariant of feedback matrix, when the system is controlled using only proportional feedback. In every case, the number of generalized state values which are a part of singular system dynamic is greater then its generalized dynamical system rang *Verghese* (1978), *Verghese et.al* (1981). The influence of proportional and differential feedback for generalized dynamical rang of the system are investigated here. Control law is given in the standard way, eq. (25).

Theorem 10. Matrices F_1 and F_2, with the characteristic that a closed singular system $\left(sE + sBF_1 - A - BF_2\right)$ has no infinite zeroes, exist if and only if matrix pencil $[sE - A \quad sB]$ has no infinite zeroes. Further, matrix F_1 can be chosen in the way that the dynamical rang of feedback system could be any value between:

$$\{rang[E \quad B] - rang(B)\} \text{ and } rang(B). \tag{35}$$

Hence, it can be concluded that the infinite controllability is necessary and sufficient regularity condition.

Theorem 11. Matrices F_1 and F_2 with the characteristic that $\left(sE + sBF_1 - A - BF_2\right)$ has arbitrarily given zeroes exist if and only if singular system (1) is controllable and it $[sE - A \quad sB]$ has no infinite zeroes.

Control of Regulable Systems using feedback

From this point on, the control of regulable systems using proportional state feedback is investigated.

The main idea of this approach is based on the fact that if a system is regulable, using non-singular linear transformation, the system could be transformed into a form typical for normal (classical) systems in state space.

Theorem 12. It is assumed that the system given by eq. (1) is *regulable*. PDF state feedback could be used for assigning eigenvalues (real or complex) to system with feedback, if and only if all frequent system modes, eq. (1), are controllable.

It is obvious from this assumption (theorem) that assigning eigenvalues to regulable systems is exclusively connected with the characteristics of the matrix pencil $(sE_1 - A_1)$ and that the assortment of matrix F_1 does not affect the procedure (except for amplification assignment for proportional controller in state feedback) unless feedback system, after control introduction, is *regular*.

Theorem 13. It is supposed that the basic system is *regulable*. Also, that matrices $\hat{F}_1$ and $\hat{F}_2$ are chosen in the way that $(E + B\hat{F}_1)$ and $(E + B\hat{F}_2)$ are non-singular matrices. Control law is given by equation:

$$\boldsymbol{u}(t) = -\hat{F}_1\dot{\boldsymbol{x}}(t) + \hat{F}_2\boldsymbol{x}(t) + \boldsymbol{v}(t). \tag{36}$$

There exists a matrix $\tilde{F}_2$, such that the eigenvalues of the system structure with feedback is isomorphic to system eigenstructure with feedback and the control law given

$$\boldsymbol{u}(t) = -\hat{F}_1\dot{\boldsymbol{x}}(t) + \tilde{F}_2\boldsymbol{x}(t) + \boldsymbol{v}(t). \tag{37}$$

Firstly, from the stated assumption nothing could be learned about the controllability of the system finite modes, eq. (1).

Secondly, if the system is regulable, it could be easily seen that almost every matrix F_1, with appropriate dimension, could serve regulability of closed feedback system and the control law given

$$\boldsymbol{u}(t) = F_1\dot{\boldsymbol{x}}(t) + \boldsymbol{v}(t). \tag{39}$$

Numeric and illustrative examples

Basic goal of the things displayed here is to show that many of fundamental results obtained using feedback for *normal systems*, could be extended to *regulable systems*, through the procedure called *regulability* (transforming regulable systems to regular systems) and performed using differential state feedback.

***Example* 4.** In the following part there is an illustration of application of the PDF feedback for controlling linear singular systems.

$$\begin{bmatrix} C_1 & C_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/r \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) \tag{40}$$

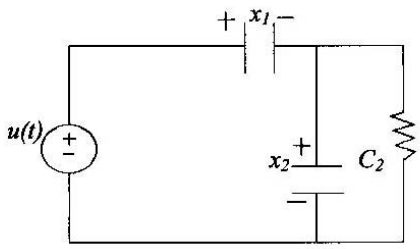

Figure 1 $x_1(t)$ is a voltage through condenser C_1, $x_2(t)$ is a voltage through condenser C_2. Input value is voltage $u(t)$

It is obvious that a large variety of differential feedback laws could be chosen for regulability electric circuit given, for this is obviously a linear singular system, eq. (40).

Let the feedback control law, which will serve for regulability system, be given in the form:

$$u(t) = -R_1C_1\dot{x}(t) + v(t) \tag{41}$$

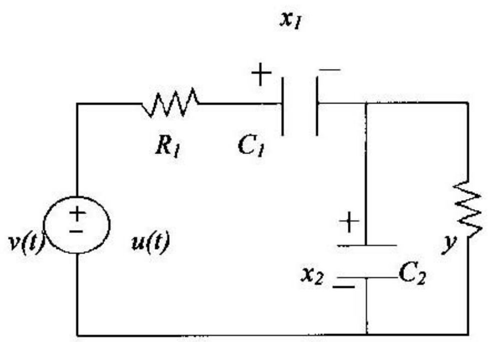

Figure 2. $v(t)$ is a new input value

Modified state equation now has a different form:

$$\begin{bmatrix} C_1 & -C_2 \\ R_1C_1 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/r \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}v(t) \tag{42}$$

therefore, for every $R_1 \neq 0$ the system is *regular*, e.g. matrix E is not singular any more. Physical interpretation of the new system is given in Fig.3. It is clear that further proceedings connected to the system obtained in this way reduced to well known control processes of non-singular e.g. *normal* systems.

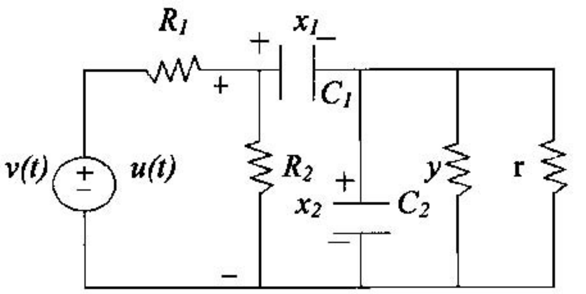

Figure 3. $u = v - R_1C_1x_1(t) - R_1/R_2(x_1 + x_2)$

Keeping in mind that system, presented by eq. (42), is closely *controllable* (all finite and infinite modes are controllable), it is valuable to notice that, in Fig.4, there are different possibilities for proportional feedback effects on dynamical behaviour of the whole system.

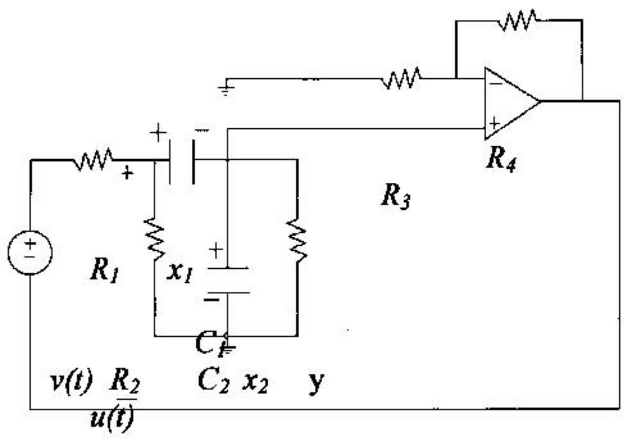

Figure 4. $u = v - R_1C_1\dot{x}_1 - \frac{R_1}{R_2}(x_1 + x_2) + x_2\frac{R_3 + R_4}{R_3}$

Conclusion

In the first part of this paper synthesis proceeding of linear singular system using geometric approach is given. Attention was especially paid to controllability and observability concepts. According to that, it is shown that in the case of regular system matrix *E*, eq.(1), conditions are reduced to the well known classical linear system results.

Proportional and differential state feedback is investigated as a method for control of *singular systems*. It is shown that controllability of infinite frequency modes plays a basic role in poles (zeroes) translation process for regularity of singular systems. It is further shown that in translation process of infinite poles nondynamic variables do not play any role.

The relation between structure determination of regulable systems and regular system itself is explicitly given.

Results obtained are followed by appropriate numerical examples, which clearly show applied methodology and practical usage of feedback controllers for regularity procedure and/or poles adjustment of linear singular automatic control systems. These results could be successfully used for synthesis of optimal singular systems and/or their estimation.

Appendix A – Some basic facts from Singular System Theory

According to what has previously been said, the system described by equations:

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)\,, \qquad \text{(A1)}$$

$$x_i(t) = Cx(t)\,, \qquad \text{(A2)}$$

where matrix E is singular matrix, is called singular systems. In this case, considered values arethe following: vector value $\mathbf{x}(t)$, which is called system state, $\mathbf{u}(t)$ is input vector or vector of input values, where $\mathbf{x}_i(t)$ is the output vector or vector of output values. E and A are, in this case, obligatory square matrixes of appropriate arbitrarily chosen matrices of appropriate dimensions.

Consistency of Primordial Conditions

For distinction of classical method of differential equation solving, given by Coushe's problem, there are limits applied on singular system primordial conditions. The basic reason for this is the existence of algebraic equations which mathematically imply the impossibility of accepting all primordial conditions. The primordial conditions which are acceptable, in the sense of generation smooth and not impulse solutions, are called *consistent primordial conditions*.

Let primordial value of the state vector be observed:

$$\mathbf{x}(0) = \mathbf{x}_0\,, \qquad \text{(A3)}$$

which, beside states from eq. (A1–A2), defines primordial (initial) condition problem. Under the assumption that state vector of initial conditions satisfies the consistency conditions, there are unique solutions and smooth solutions. For classic (normal) systems uniqueness of solutions is guarantied. Generally speaking, in case of singular systems uniqueness of solutions can not be guarantied. By the following example the previous statement will be illustrated most adequately.

***Example A*1.** The following singular system is considered:

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} t \\ 2t \end{bmatrix}. \qquad \text{(A4)}$$

It is clear that the value for $\dot{x}_1$ to fulfil equations $\dot{x}_1 = x_2 + t$ and $\dot{x}_1 = x_2 + 2t$ for $t \neq 0$ could not be found.

***Example A*2**.

The singular system given is consideed:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \text{(A5)}$$

where x_1 depends on x_2, whence x_2 could be arbitrarily chosen.

It is clear from previous two examples that the solutions of the system given are not unique, meaning that, generally speaking, uniqueness could not be guarantied. Therefore, a question logically imposes itself: which are the conditions necessary to fulfil in order to obtain unique solutions for the system given by equations (A1-A3)? To get the answer to this question it is necessary to define the notion of *consistence beginning vector*.

***Definition A*1.** $\mathbf{x}_0$ is consistent initial vector of the system given by eq. (A1-A2) if there is at least one solution whose initial condition satisfies the condition $\mathbf{x}(0)=\mathbf{x}_0$.

Theorem A1. Singular system, given by equations (A1-A2) with consistent initial vector, eq.(A3) has unique solutions if and only if complex scalar s such as $(sE-A)^{-1}$ exists.

A complete proof of this theorem will not be given, rather only a sketch which is based on the fact that if the former inversion exists, solution could be found using Laplace transformation.

Even after finding conditions that ensure solution uniqueness, the question about solutions remains open. There are four basic ways for solving linear singular system with associate consistent initial conditions. They are:

- Reduction of the system order, until reaching derivation state vector, and after that, derivation of the system of lower rank. Proceeding is well known as diminishing of system rang (dimension).
- Solution in time domain using Drazin inverse.
- Approximation of singular system with sequences which are not singular. In this case solutions are given as border occurrence of system singularity.
- Solutions obtained using Laplace transformation.

Reduction of system order

Let singular system given by equations (A1-A2) be considered. Suppose that system could be transformed and shown as matrix block, as it is given:

$$\begin{bmatrix} E_1 \\ O \end{bmatrix}\dot{\mathbf{x}}(t)=\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\mathbf{x}(t)+\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\mathbf{u}(t)\,, \tag{A6}$$

where n is dimension of the system, O is null matrix, which is obtained as sub-matrix of E_1, with characteristic that it is a singular matrix of the biggest rang. If the expression which comprises linearly independent A_2, than r state could be eliminated using the equation:

$$A_2\mathbf{x}(t)+B_2\mathbf{u}(t)=\boldsymbol{0}\,. \tag{A7}$$

Resulting equations, after eliminations, is of the form:

$$E_3\dot{\mathbf{x}}(t)=A_3\mathbf{x}(t)+B_3\mathbf{u}(t)+B_4\dot{\mathbf{u}}(t)\,. \tag{A8}$$

Eq.(A8) still could not present a regular system. In that case the procedure is continued until non-singular system appears or trivial system solution appears ($E=0$).

To carry out the previous algorithm depending on whether equation where linearly independent A_2 exists (it means that this matrix has rang equal to the order). Next theorem shows that A_2 will be of full rang always when the system, given by eq.(A6), has unique solutions.

Theorem A2. System is given by eq.(A6) with consistent initial vector $\mathbf{x}_0$. Matrix A_2 has full rang and the system has unique solution.

Proof. It is known that a system has unique solutions if and only if there is inversion of the matrix pencil $(sE-A)$. Forming matrix pencil $(sE-A)$, together with eq.(A6), following matrix is obtained:

$$(sE-A)=\begin{bmatrix} sE_1-A_1 \\ -A_2 \end{bmatrix}\;; \tag{A9}$$

matrix is invertible only if columns of matrix A_2 are linearly independent.

Reduction of the system order and reduction of the system state always lead to solutions under conditions given. The advantage of this method is that the problem is reduced to solving normal system problem, with very good theoretical bases. Main deficiency of this approach is the fact that equations in the system description could be algebraically complicated. The next example illustrates the previous facts.

***Example A*3.** Let a singular autonomous system be described by the following equation:

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{A10}$$

System given by eq. (A10) is obviously singular. Using the method given, reducing the system state the following is obtained:

$$\dot{x}_1=x_1\,, \tag{A11}$$

where x_2 is equal to zero. Initial condition is consistent for $x_2(0)=0$.

Reducing the system order, it is not possible to obtain nonconsistent initial conditions. However, it is worth mentioning that nonconsistent initial conditions are responsible for impulse behaviour of the system.

Solutions in time domain

General solution of a singular system in time domain is given by the following expression:

$$\begin{aligned}\mathbf{x}(t)&=e^{-\hat{E}^D\hat{A}(t-t_0)\hat{E}\hat{E}^D}\mathbf{x}_0+\\&+e^{-\hat{E}^D\hat{A}t}\int_{t_0}^{t}e^{\hat{E}^D\hat{A}\tau}\hat{E}^D B\hat{\mathbf{u}}(\tau)d\tau+\\&+(I-\hat{E}\hat{E}^D)\sum_{i=0}^{k-1}(-1)^i(\hat{E}\hat{A}^D)^i\hat{A}^D\hat{B}\mathbf{u}^{(i)}(t)\end{aligned} \tag{A12}$$

where:

$$\begin{aligned}\hat{E}&=(sE-A)^{-1}E\\\hat{A}&=(sE-A)^{-1}A\,,\\\hat{B}&=(sE-A)^{-1}B\end{aligned} \tag{A13}$$

Index “ D ” denotes Drazin’s inverse, and (i) denotes i – th derivation of time. I is unit matrix with appropriate order, $\mathbf{x}_0=\mathbf{x}(t_0)$ is consistent initial vector, while s denotes the complex variable. Solution does not depend on the value s, so eq.(A12) is correct for any value of s.

General solution given by eq.(A12) is valid only for consistent initial vectors, while for nonconsistent initial vectors equation does not give valuable solutions. Some methods, besides the consistent ones, deal with nonconsistent initial conditions.

Sequential approximation of linear singular systems

Singular system could be approximated by sequences and behave as normal system in those parts. Basic idea behind this procedure is that the singular matrix has to be close to regular one. Proximity could be determined as a very small value which almost does not influence the dynamical behaviour of the system. Following that, the system response could be treated as boundary value of normal system. The next example will illustrate the stated.

***Example A*3.** Infinitesimal value ε is added to elements of the singular system matrix eq.(A10), so the new system has the form:

$$\begin{bmatrix} 1 & 1 \\ 0 & \varepsilon \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \qquad (A14)$$

Motion of the singular system is determined by the equation:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t} & 1/(1+\varepsilon)\left(e^{-t} - (1/\varepsilon)e^{-(t/\varepsilon)}\right) \\ 0 & (1/\varepsilon)e^{-(t/\varepsilon)} \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \quad (A15)$$

Applying limiting value operation to expression (A15), provided that $\varepsilon \to 0$, the solution of the singular system is obtained:

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-t} & e^{-t} - \delta(t) \\ 0 & \delta(t) \end{bmatrix}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}, \qquad (A16)$$

where $\delta(t)$ is the well known Dirac function.

For consistent initial condition $x_2(0) = 0$, the solution is the same as in the method of reducing the system order. When initial conditions are not consistent, impulses appear in system solution. This statement is a general characteristic of the singular system and the following state is formulated.

***State A*1**. Impulse function and their derivatives appear in the solution as a result of nonconsistent initial conditions.

Solutions of singular systems using Laplace transformations

Solutions of singular systems for initial condition given could be obtained using Laplace's transformation. Let singular system, eq.(A1–A2) be considered.

Applying transformation it follows:

$$\mathbf{X}(s) = \mathbf{L}^{-1}\left[(sE-A)^{-1}\right]E\mathbf{X}(0) + \mathbf{L}^{-1}\left[(sE-A)^{-1}B\mathbf{U}(s)\right]. \quad (A17)$$

This solution exist only if $(sE-A)^{-1}$ exists, and it is a condition for the existence of a unique solution. This procedure could be applied to systems with nonconsistent initial conditions. According to this expression it is possible to determine transfer function of singular systems. It could be in polynomial form, depending on the appearance of impulses and their derivations in time domain. Basic form of the transfer function is:

$$W(s) = C(sE-A)^{-1}B \ . \qquad (A18)$$

References

[1] BERNHARD, P., "*On Singular Implicit Linear Dynamical Systems*", SIAM J. Control and Optimiz.20 (5), (1982), pp.612-633.

[2] BUZUROVIĆ, M.I., *Geometrijski prilaz u proučavanju linearnih singularnih sistema*, Magistarski rad, Mašinski fakultet, Beograd, 2000.

[3] CAMPBELL, S. L., *Singular Systems of Differential Equations*, Pitman, Marshfield, Mass., 1980.

[4] CAMPBELL, S. L., *Singular Systems of Differential Equations II*, Pitman, Marshfield, Mass., 1982.

[5] *Circuits, Systems and Signal Processing*, Special Issue on Semistate Systems, 5 (1) (1986).

[6] *Circuits, Systems and Signal Processing*, Special Issue: Recent Advances in Singular Systems, 8 (3) (1989).

[7] DEBELJKOVIĆ, Lj. D., MILINKOVIĆ, S. A. JOVANOVIĆ, M. B. *Kontinualni singularni sistemi automatskog upravljanja*, GIP Kultura, Beograd, 1996.

[8] DEBELJKOVIĆ, Lj. D., MILINKOVIĆ, S. A. JOVANOVIĆ, M .B. JACIĆ,LJ.A. *Diskretni singularni sistemi automatskog upravljanja*, GIP Kultura, Beograd, 1998.

[9] GANTMACHER, R. F., *The Theory of Matrices* , Chelsea Publishing Company, New York, 1977,Vol.I, Vol.II

[10] MALABRE, M., "*A Structural Approach for Linear Singular Systems*", Proc. Int. Symp. on Sing. Syst., Atlanta, GA (1987.a), pp.34–37

[11] MALABRE, M., "*More Geometry about Singular Systems*", IEEE Proc. on CDC, Los Angeles, CA (1987.b), pp.1138–1139

[12] MALABRE, M., "*Geometric Algorithm and Structural Invariants for Linear Singular Systems*", Proc. 12th IMACS World Congr., Paris (1988), pp.181–183

[13] MALABRE, M., "*On Infinite Zeros for Generalized Linear Systems*", Proc. MTNS, Amsterdam, 1 (1989.a), pp.271–278

[14] MALABRE, M., "*Generalized Linear Systems: Geometric and Structural Approaches*", *Linear Algebra Applic*., (1989.b), pp.591–621

[15] MUKUNDAN, R., DAYAWANSA, W., "F*eedback Control of Singular Systems–Proportional and Derivative Feedback of the State*", *Int. J. Syst. Sci*., 14 (1983), pp.615–632

[16] OZCALDRIAN, K., *Control of Descriptor Systems*, Ph.D.Thesis, School of Elec. Eng., Georgia Inst. Techn., Atlanta, GA, USA, 1985.

[17] OZCALDRIAN, K., "*A Geometric Characterization of the Reachable and the Controllable Subspaces of Descriptor Systems*", *Circ.* Syst. Sig. Proc., 5 (1) (1986), pp.37–48

[18] OZCALDRIAN, K., "*A Note on the Assignment of Initial and Final Manifolds for Descriptor System*s", Proc. Int. Symp. Sing. Syst., Atlanta, GA, I (1987.a) pp.38–39

[19] OZCALDRIAN, K., "*Geometric Notes on Descriptor Systems*", IEEE Proc. on CDC, Los Angeles, CA (1987.b), pp.1134–1137

[20] OZCALDRIAN, K., LEWIS, F .L. "*A Result on the Placement of Infinite Eigenvalues in Descriptor Systems*", Proc. ACC, San Diego, CA (1984), pp.366–371

[21] OZCALDRIAN,K., LEWIS,F.L., "*A Brief Survey of Geometric Theory for Singular Systems*", Proc. ICIAM, Paris (1987.a)., pp.163-165

[22] OZCALDRIAN, K., LEWIS, F. L., "*A Geometric Approach to Eigenstructure Assignment for Singular Systems*", IEEE Trans. Automat. Cont., AC-32 (7) (1987.b), pp.629–632

[23] SYRMOS, V. L., LEWIS, F. L., "*A Geometric Approach to Proportional-Plus-Derivative Feedback Using Quotient and Partitioned Subspaces*", Automatica, 27 (2) (1991.a), pp.349–370

[24] SYRMOS, V. L., LEWIS, F. L., "*On the Application of the Generalized Sylvester Equations to Feedback Design Techniques in Generalized Systems*", Proc. ECC 91, Grenoble, France (1991.b), pp.394–399

[25] VERGHESE, G., P. VAN DOOREN, T. KAILATH, "*Properties of the System Matrix of a Generalized State-Space System*", Int. J. Control, 30 (2) (1979), pp.235–243

[26] VERGHESE,G. C., KAILATH,T., "*Impulsive Behavior in Dynamical Systems; Structure and Significance*", Proc. MTNS, Delft (1979.a), pp.162–168

[27] VERGHESE, G. C., KAILATH, T., "*Eigenvectors Chains for Finite and Infinite Zeros of Rational Matrices*", IEEE Proc. CDC, Ft. Lauderdale, FL (1979.b), pp.31–32

[28] VERGHESE,G.C., KAILATH, T., "*Rational Matrix Structure*", IEEE Trans. Automat. Cont., AC-26 (2) (1981), pp.434–439

[29] VERGHESE ,G. C., LEVY, B. C., T. KAILATH, "*A Generalized State-Space for Singular Systems*", IEEE Trans. Automat. Cont., AC-26 (4) (1981), pp.811–831

[30] VERGHESE, G .C., "*Further Notes on Singular Descriptions*", Proc. JAAC, Charlottesville, VA (1981) TA-4B.

Received: 30.08.2003.

Sinteza generalisanog linearnog singularnog sistema proporcionalno diferencijalnom globalnom povratnom spregom

U ovom radu se razmatra upotreba proporcionalne i diferencijalne povratne sprege po stanju, kao mogućnost za postizanje regularnosti i upravljanja linearnim singularnim sistemima. U tom smislu analizirana su dva pristupa, prvi koji problem posmatra sa geometrijskog aspekta i drugi koji se ovom problematikom bavi sa kalasičnog stanovišta. Brojnim primerima ilustrovana je efikasnost izloženih procedura. Izloženi rezultati mogu se primeniti i na normalne – nesingularne sisteme imajući u vidu geometrijski prilaz problemu. U tom cilju date su i odgovarajuće transformacije nakon kojih je moguće projektovati povratnu spregu, sa odgovarajućim uskladnikom, standardnim metodama.

Ključne reči: **singularno upravljanje, singularni sistem, sinteza sistema, upravljanje sistemom, linearni sistem, proporcionalno diferencijalno usklađivanje.**

Синтез универсальной линейной сингулярной системы пропорционально дифференциальной общей обратной связью

В этой работе рассматривается употребление пропорциональной и дифференциальной обратной связи по состоянию, как возможности добиваться регулярности и управления линейными сингулярными системами. В этом смысле анализированы два подхода, первый рассматривающий проблему, со геометрической позиции и второй, который занимается этой проблемой со классической точки зрения. Многочисленными примерами представлена эффективность приведенных техник. Приведенные результаты могут примениться и для нормальных - несингулярных систем имея в виду геометрический подход к этой проблеме. Со такой целью даны и соответствующие трансформации, после которых возможно проектировать обратную связь, со соответствующим согласователем, и то стандартными методами.

Ключевые слова: сингулярное управление, сингулярная система, синтез систем, управление системой, линейная система, пропорциональное дифференциальное регулирование.

La synthèse du système linéaire singulier généralisé au moyen des réactions globales proportionnellement différentielles

Dans ce travail nous avons examiné l'emploi des réactions proportionnelles et différentielles de l'état comme une possibilité d'obtenir la régularité et la commande des systèmes linéaires singuliers.A cet effet nous avons analysé deux approches:la première considère le problème du point de vue géométrique, alors que la seconde traite ce problème de manière classique. L'efficacité des procédés exposés est illustrée par de nombreux exemples.Ayant en vue l'approche géométrique du problème, les résultats présentés peuvent s'appliquer aux systèmes normaux non-singuliers. Dans ce but sont données les transformations appropriées, après lesquelles il est possible de projeter des réactions avec un controleur convenable au moyen des métodes ordinaires.

Mots clés: **commande singulière, système syngulier, synthèse du système, commande du système, système linéaire, régulation différentielle proportionnelle.**