

Transfer function matrix and fundamental matrix of linear singular-descriptive systems

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The transfer matrix function has a particular significance in dynamical analysis of real *linear singular multivariable feedback control systems* from the stand point of input – output realations, under the zero conditions. Natural connection with fequency domain is obvious, necessary and therefore present in numerous methodes and approaches. Several algorithms which allow the computation of transfer function matrix of linear regular singular systems from the state space description without inverting a polynomial matrix are presented. An allternative closed-form expression for transfer function matrix in terms of matrix pencil $(sE - A)$ is also given. Some of the approaches presented are direct extensions of Leverrier's algorithm and some are its modifications.

A several numerical examples have been worked out to illustrate the methods presented.

Fundamental matrix has a particular significance in dynamical analysis of real *linear discrete descriptive multivariable feedback control systems*. This paper shows that the forward and backward fundamental matrix sequence of regular discrete descriptor system can be effciently used for computational purposes for finding its state space transient response. Moreover, for such methods there is no need to use Drazin or some other pseudo inversion procedures and Laurent series expansion is enough for these purposes.

Key words: linear systems, singular systems, descriptive systems, transfer function matrix, fundamental matrix.

Symbols:

A	– system matrix
B	– control matrix
C	– output matrix
D	– matrix
E	– singular system matrix
$f(s)$	– elements of matrix transfer function
I	– identity matrix
J	– Jordan block matrix
N	– nilpotent matrix
$q(s)$	– polynomial
R	– coefficient matrix
$R(s)$	– polynomial matrices
s	– complex variable
t	– time
$\mathbf{u}(t)$	– control vector
w	– complex variable
$W(s)$	– transfer matrix function of singular system
$\mathbf{x}(t)$	– state vector
$\mathbf{x}_i(t)$	– output vector

z	– complex variable
$\delta(k)$	– Dirack function
μ	– complex variable
ν	– index of nilpotency
ρ	– index of nilpotency
ϕ	– fundamental matrix
ψ	– fundamentalna matrix
Ψ	– transition matrix
det	– determinant
tr	– trase of matrix

Introduction

SINGULAR systems are those the dynamics of which are governed by a mixture of algebraic and differential equations. In that sense the algebraic equations represent the constraints to the solution of the differential part.

These systems are also known as descriptor and semi-state and arise naturally as a linear approximation of system models, or linear system models in many applications such as electrical networks, *aircraft dynamics*, neutral delay systems, chemical, thermal and diffusion processes, large-scale systems, interconnected systems, economics, optimization problems, feedback systems, robotics, biology, *etc.*

The complex nature of generalized state space systems

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causes many difficulties in analytical and numerical treatment that do not appear when system are considered in their normal form. In this sense questions of existence, solvability, uniqueness, and smoothness which must be solved in satisfactory manner are present. A short and concise, acceptable and understandable explanation of all these questions may be found in the papers of *Debeljkovic et. al* (2004, 2005) and *Lazarevic et al.* (2001).

The survey of updated results for generalized state space systems and a broad bibliography can be found in *Bajic* (1992), *Campbell* (1980, 1982), *Lewis* (1986.a, 1987), *Debeljkovic et al.* (1996.a, 1996.b, 1998, 2004, 2005) and in the two special issues of the journal *Circuits, Systems and Signal Processing* (1986, 1989).

Consider linear singular system represented, by:

$$\begin{aligned} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t), & \mathbf{x}(t_0) &= \mathbf{x}_0 \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{aligned}, \quad (1)$$

and:

$$\begin{aligned} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B\mathbf{u}(t), & \mathbf{x}(t_0) &= \mathbf{x}_0 \\ \mathbf{y}(t) &= C\mathbf{x}(t) \end{aligned}, \quad (2)$$

with the matrix E possibly singular, where $\mathbf{x}(t) \in \mathfrak{R}^n$ is a generalized state-space vector, $\mathbf{u}(t) \in \mathfrak{R}^m$ is a control variable, and $\mathbf{y}(t) \in \mathfrak{R}^p$. Matrices $E, A \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times m}$ and $C \in \mathfrak{R}^{p \times n}$ are of the appropriate dimensions and are defined by the field of real numbers¹.

The system given by eq.(1) is operating in a free and the one given by eq.(2) is operating in a forced regime, i.e. some external force is applied on it. It should be stressed that, in general case, the initial conditions for an autonomous and a system operating in the forced regime need not be the same.

System models of this form have some important advantages in comparison with models in the normal form, e.g. when $E = I$ and an appropriate discussion can be found in *Bajic* (1992.a) and *Debeljkovic et al.* (1996, 1996a, 1998, 2004, 2005), *Debeljkovic* (2004).

Eq.(1.1 – 1.2) arise naturally in the process of modeling various physical systems, when the equations are written in the sparse form.

They have some important advantages in comparison with the models in the normal form, *Bajic* (1992.a):

- the models preserve the sparsity of system matrices
- there is a tight relation between the system physical variables and the variables in these models,
- the structure of the physical system is well reflected in the models,
- there is a great simplicity in derivation of the eq.(1.1 – 1.2) and in this connection there is no need to eliminate the unwanted (redundant) variables, as it is not necessary to build models in the traditional form.

Models of the mentioned forms, can be found in many different fields.

The Laplace transform of system, given by (2), under the zero conditions, results in the following generalized transfer function matrix:

$$W(s) = C(sE - A)^{-1} B = C \frac{\text{adj}(sE - A)}{\det(sE - A)} B, \quad (3)$$

transfer matrix of singular system, with characteristic equation, of the form:

$$f_E(s) = \det(sE - A). \quad (4)$$

It can be shown that the transfer function of linear singular systems, in certain circumstances, can not be found.

This problem is completely determined by question of possible solvability of singular system, see *Appendix B*.

For (3), it is obvious, that only regular singular systems², can have such description.

If singular system have no transfer function, i.e. it is irregular, it may still have a general description pairing, *Dziurla, Newcomb* (1987), that is a description of the form:

$$R(s)\mathbf{Y}(s) = Q(s)\mathbf{U}(s), \quad (5)$$

where $\mathbf{Y}(s)$ and $\mathbf{U}(s)$ are Laplace transforms of the output and input, respectively. Since, irregular systems may have many or no solutions at all, the question arises as to whether we would meet them in the practice.

The mentioned reference shows that we indeed meet them, at least when we idealize certain systems.

Other aspects, concerning solvability and state structure for irregular singular systems can be found in *Dai* (1989.a).

A practical and compact procedure for obtaining transfer function of linear singular systems, especially for high order systems, is not based on eq.(3), but some specific procedures based on a finite – series expansion for $(sE - A)^{-1}$.

It is obvious that this procedure can only be applied to the class of regular singular systems, so the right choice of $\lambda \in \mathfrak{R}$, such as the regularity of conditions, (B.1), is satisfied.

Matrix function computation, based on eq.(3), can be performed with a lot of difficulties since for this procedure the inversion of proposed matrix is needed. The computation is then extremely complicated with possibilities of unstable converging procedures.

Having in mind such circumstances, usually some other approaches are used in order to overcome the difficulties mentioned above.

Some numerical methods for practical computation of matrix transfer functions of linear singular systems

Method - Paraskevopoulos, Hristodoulou, Boglu

Consider the linear singular system, given by (1-2).

To compute the inverse of matrix $(sE - A)$ the following technique will be used. Find a μ so that matrix pencil $(\mu E + A)$ is regular. It should be noted that $(\mu E + A)$ is polynomial in μ of degree at most n .

First, a μ such that $(\mu E + A)$ is invertible, a number μ which is not the root of the polynomial $\det(\mu E + A)$ must be found. This problem is simple.

¹ Basic notations are given in Appendix A

² See, eq. (B.1).

The following steps are used:

$$\begin{aligned} (sE - A)^{-1} &= (sE + \mu E - \mu E - A)^{-1} \\ &= \{(s + \mu)E - (\mu E + A)\}^{-1} \\ &= \{(s + \mu)\hat{E} - I\}^{-1} (\mu E + A)^{-1} \end{aligned} \quad (6)$$

where:

$$\hat{E} = (\mu E + A)^{-1} E. \quad (7)$$

The term $(\mu E + A)^{-1}$ can easily be evaluated using a computer, since for constant μ , $(\mu E + A)$ is given known constant matrix of appropriate dimension.

Next the *Sourian Frame-Faddev-algorithm* will be used to compute the term $\{(s + \mu)\hat{E} - I\}^{-1}$.

The following change of variable is introduced:

$$s + \mu = w = 1/z. \quad (8)$$

Then:

$$\begin{aligned} \{(s + \mu)\hat{E} - I\}^{-1} &= -z(zI - \hat{E})^{-1} \\ &= -z \frac{B_0 + B_1 z + \dots + B_{n-1} z^{n-1}}{a_0 + a_1 z + \dots + a_{n-1} z + z^n} \\ &= -\frac{B_{n-1} + B_{n-2}(s + \mu) + \dots + B_0(s + \mu)^{n-1}}{1 + a_{n-1}(s + \mu) + \dots + a_0(s + \mu)^n}, \end{aligned} \quad (9)$$

where the terms B_i and a_i , $i = 0, 1, n - 1$, are obtained from *Sourian Frame-Faddev-ovog* algorithm:

$$\begin{aligned} a_{n-1} &= -\text{tr}(\hat{E}B_{n-1}) & B_{n-1} &= I_n \\ a_{n-2} &= -\frac{1}{2}\text{tr}(\hat{E}B_{n-2}) & B_{n-2} &= a_{n-1}I + \hat{E}B_{n-1} \\ &\vdots & B_{n-3} &= a_{n-2}I + \hat{E}B_{n-2}, \\ a_1 &= -\frac{1}{n-1}\text{tr}(\hat{E}B_1) & &\vdots \\ a_0 &= -\frac{1}{n}\text{tr}(\hat{E}B_0) & B_0 &= a_1I + \hat{E}B_1 \end{aligned} \quad (10)$$

By using formula (9) in (6) the general result follows:

Theorem 1. The inverse of matrix $(sE - A)$, where $\det E = 0$, is given by the following formula:

$$\begin{aligned} (sE - A)^{-1} &= \\ &= \frac{E_0 + E_1(s + \mu) + \dots + E_{n-1}(s + \mu)^{n-1}}{1 + C_1(s + \mu) + \dots + C_{n-1}(s + \mu)^{n-1} + C_n(s + \mu)^n} \end{aligned} \quad (11)$$

where:

$$\begin{aligned} E_i &= -B_{n-1-i}(\mu E + A)^{-1}, \quad i = 1, 2, 3, n-1, \\ C_i &= a_{n-1}, \quad i = 1, 2, 3, \dots, n, \end{aligned} \quad (12)$$

and B_i and a_i have been taken from (10).

μ is any constant such that $\det(\mu E + A) \neq 0$.

Note that the formula (11) is independent of possible choice of μ .

Also, it is possible that some of terms B_i and C_i are equal zero.

Thus matrix in (11) is not necessarily *strictly proper*^{#3}.

Numerical example 1. System under consideration is given by:

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\sigma\{A\} = \{1, 0\}, \quad \sigma\{E\} = \{1, 0\}$$

$$(sE - A)^{-1} = \{(s + \mu)\hat{E} - I\}^{-1} (\mu E + A)^{-1},$$

$$\mu = -1, \quad \hat{E} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad a_1 = -\text{tr}(\hat{E}B_1) - \text{tr}(\hat{E}) = 1,$$

$$B_0 = a_1I + \hat{E}B_1 = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

$$a_0 = -\frac{1}{2}\text{tr}\left\{\begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}\right\} = 0$$

$$E_i = -B_{n-1-i}(\mu E + A)^{-1}, \quad i = 0, 1$$

$$E_0 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad c_1 = a_1 = 1,$$

$$E_1 = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}, \quad c_2 = a_0 = 0.$$

$$(sE - A)^{-1} = \frac{1}{s} \begin{bmatrix} 4s-3 & 1-2s \\ 3-2s & s-1 \end{bmatrix},$$

$$\begin{aligned} W(s) &= C(sE - A)^{-1}B = \frac{1}{s} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4s-3 & 1-2s \\ 3-2s & s-1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} \\ &= \frac{1}{s} \begin{bmatrix} 8s-6 & 6s-4 \\ 4s & 3s \end{bmatrix}. \end{aligned}$$

Numerical example 2. System under consideration is given by:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & -1 & -1 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 2 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix},$$

³ See Appendix B

$$\sigma\{A\} = \{2, 1, 0\}, \quad \sigma\{E\} = \{1, 0, 1\}$$

$$\mu = -1, \quad \hat{E} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -\frac{1}{2} \end{bmatrix}.$$

$$B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad a_2 = -\text{tr}(\hat{E}B_2) = \frac{3}{2},$$

$$B_1 = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & \frac{3}{2} \end{bmatrix} + \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -\frac{1}{2} \end{bmatrix} \\ = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$a_1 = -\frac{1}{2} \text{tr} \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 1 & 0 & 1 \end{bmatrix} \right\} = \frac{1}{2}$$

$$B_0 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$a_0 = -\frac{1}{3} \text{tr} \left\{ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \right\} = 0.$$

$$E_i = -B_{2-i}(-E + A)^{-1}, \quad i = 0, 1, 2$$

$$E_0 = -B_2(-E + A)^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ -1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix},$$

$$E_1 = -B_1(-E + A)^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{3}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

$$E_2 = -B_0(-E + A)^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$c_i = a_{3-i}, \quad i = 1, 2, 3.$$

$$(sE - A)^{-1} = \frac{1}{s^2 + s} \begin{bmatrix} s+1 & s+1 & s+1 \\ 0 & -s^2 - s & 0 \\ -2 & s-2 & s-2 \end{bmatrix}$$

$$W(s) = C(sE - A)^{-1}B = \\ = \frac{1}{s^2 + s} \begin{bmatrix} -2s^2 - s - 2 & -2s^2 + s + 3 & -2s^2 - 3 \\ -2 & 3s - 6 & s - 4 \\ -2s^2 - s - 1 & -2s^2 - 2s + 9 & -2s^2 - s + 2 \end{bmatrix}$$

Laverrier's algorithm for singular systems – Mertzios approach

The computation of $(sE - A)^{-1}$ can be carried out using Cramer's rule, which require the evaluation n^2 determinants $(n-1) \times (n-1)$ of polynomial matrices. The algorithm presented in the continuation is an extension of the Leverrier's algorithm for this class of linear singular systems.

The transfer matrix function, can be rewritten as:

$$W(s) = C (Iz - (Iz - sE + A))^{-1} B \\ = C(Iz - \hat{A})^{-1} B \quad (13)$$

where:

$$\hat{A} = Iz - Es + A, \quad (14)$$

and z is a new pseudovvariable, which does affect on the transfer matrix $W(s)$ since it can be eliminated later.

Now it is clear that the Leverrier algorithm can be to computed as the inver of the matrix $(Iz - \hat{A})$.

Therefore, the transfer matrix function $W(s)$ can be expanded:

$$W(s) = q^{-1}(s)C[z^{n-1}R_0(s) + z^{n-2}R_1(s) + \dots + \\ + zR_{n-2}(s) + R_{n-1}(s)]B \quad (15)$$

where:

$$q(s) = z^n + q_1(s)z^{n-1} + q_2(s)z^{n-2} + \dots + q_n(s) \\ = \det(Iz - \hat{A}) \quad (16)$$

and:

$$R_0(s) = I_n \quad q_1(s) = -\text{tr}[\hat{A}] \\ R_1(s) = \hat{A}R_0(s) + q_1I_n \quad q_2(s) = -\frac{1}{2}\text{tr}[\hat{A}R_1(s)] \\ R_2(s) = \hat{A}R_1(s) + q_2I_n \quad q_3(s) = -\frac{1}{3}\text{tr}[\hat{A}R_2(s)]$$

$$R_{n-1}(s) = \hat{A}R_{n-2}(s) + q_{n-1}I_n \quad q_n(s) = -\frac{1}{n}tr[\hat{A}R_{n-1}(s)] \quad (17)$$

The matrices $R_i(s)$, $i = 1, \dots, n-1$ can also be computed using the following expression:

$$R_i(s) = \hat{A}^i + q_1(s)\hat{A}^{i-1} + q_2(s)\hat{A}^{i-2} + \dots + q_i(s)I \quad (18)$$

The matrices $R_i(s)$, $i = 1, \dots, n-1$ are no longer the coefficients matrices of the power of s , but depend on the variable s itself. This can be seen from (17), since matrix \hat{A} depends on s .

Since (15) is independent of z , in the continuation, for the sake of simplicity, we shall take $z = 0$. Therefore, the relations (13 - 16) can be written as:

$$\hat{A} = -ES + A, \quad (19)$$

$$W(s) = -C\hat{A}^{-1}B = q^{-1}(s)CR_{n-1}(s)B, \quad (20)$$

$$q(s) = q_n(s). \quad (21)$$

It should be noted that we have infinite number of forms of \hat{A} , $W(s)$ and $q(s)$ depending of pseudovvariable z . It can be seen from (17) that the degree of polynomial matrix $R_i(s)$, $i = 0, 1, \dots, n-1$, and of polynomial quantity $q_i(s)$ $i = 0, 1, \dots, n$, is equal i the most. Hence, $R_i(s)$ and $q_i(s)$ can be written as:

$$R_i(s) = \sum_{k=0}^i R_{ik}s^k, \quad (22)$$

$$q_i(s) = \sum_{k=0}^i q_{ik}s^k, \quad (23)$$

where $R_{ik}(s)$ and $q_{ik}(s)$ are the constant matrix and scalar of the power s^i , respectively.

It is seen from (20) that for the computation of $W(s)$, we need only the quantities $R_{n-1}(s)$ i $q_n(s)$.

To this end the coefficient matrices $R_{n-1,k}(s)$ are given by:

$$R(s) = -\text{adj} \hat{A} = R_{n-1}(s) = \sum_{k=0}^{n-1} R_{n-1,k}s^k \quad (24)$$

will be computed recursively in terms of the coefficients matrices $R_{ik}(s)$, $i = 0, 1, \dots, n-2$.

Substituting (19) and. (22) in the recursive relations $R_i(s)$, $i = 0, 1, \dots, n-1$, in (20), we obtain the following general recursive relations by equating the coefficients of the power s in the two sides of each equation:

$$R_{i+1,k} = \begin{cases} -ER_{i,k} - q_{i+1,k}I_n & k = i+1 \\ AR_{i,k} - ER_{i,k-1} + q_{i+1,k}I_n & k = 1, \dots, i \\ AR_{i,1} + q_{i+1,k}I_n & k = 0 \end{cases} \quad (25)$$

and:

$$q_{i+1,k} = \begin{cases} \frac{1}{i+1}tr[ER_{i,k-1}] & k = i+1 \\ -\frac{1}{i+1}tr[AR_{i,k} - ER_{i,k-1}] & k = 1, \dots, i \\ -\frac{1}{1+i}tr[AR_{i,0}] & k = 0 \end{cases} \quad (26)$$

Specifically, it is shown that generally $R_{i,j}$ can be expressed in terms of $R_{i-1,j}$, and $R_{i-1,j-1}$, while $R_{i,0}$ can be expressed in terms of $R_{i-1,0}$ and $R_{i,i}$ in terms of $R_{i-1,i-1}$.

For $W(s)$ computational purposes we need only terms $R_{n-1,k}$, $k = 0, 1, \dots, n-1$, and scalars $q_{n,k}$, $k = 0, 1, \dots, n$.

The formulas (25-26) are readily reduced to the Leverrier algorithm for normal (nonsingular) if we assume that $E = I$, I being identity matrix. In that case, $R_{i,k}$ i $q_{i,k}$, for $k = 0$, are identically equal zero.

Numerical example 3. System is given with the following data:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$i = 0: R_{0,0} = I_3, \quad q_{1,1} = tr(ER_{0,0}) = 2,$$

$$R_{1,1} = -ER_{0,0} + q_{1,1}I_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$q_{1,0} = -tr(AR_{0,0}) = -3,$$

$$R_{1,0} = AR_{0,0} + q_{1,0}I_n = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -2 & 1 \\ 1 & -1 & -3 \end{bmatrix}.$$

$$i = 1: q_{2,0} = -\frac{1}{2}tr(AR_{1,0}) = 3,$$

$$q_{2,1} = -\frac{1}{2}tr(AR_{1,1} - ER_{1,0}) = -4,$$

$$q_{2,2} = \frac{1}{2}tr(ER_{1,1}) = 1,$$

$$R_{2,0} = AR_{1,0} + q_{2,0}I_3 = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ -1 & 1 & 2 \end{bmatrix},$$

$$R_{2,1} = AR_{1,1} - ER_{1,0} + q_{2,1}I_3 = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$R_{2,2} = -ER_{1,1} + q_{2,2}I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$i = 2: q_{3,0} = -\frac{1}{3}tr(AR_{2,0}) = -1,$$

$$q_{3,1} = -\frac{1}{3}tr(AR_{2,1} - ER_{2,0}) = 3,$$

$$q_{3,2} = -\frac{1}{3}tr(AR_{2,2} - ER_{2,1}) = -1,$$

$$q_{3,3} = \frac{1}{3}tr(ER_{2,2}) = 0,$$

$$R_{3,0} = AR_{2,0} + q_{3,0}I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\begin{aligned} R(s) &= \sum_{k=0}^2 R_{2,k}s^k = R_{2,0} + R_{2,1}s + R_{2,2}s^2 \\ &= \begin{bmatrix} -s+1 & -s & -1 \\ 1 & s^2-2s & s-2 \\ -1 & -s+1 & -s+2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} q(s) &= \sum_{k=0}^3 q_{3,k}s^k = q_{3,0} + q_{3,1}s + q_{3,2}s^2 + q_{3,3}s^3 = \\ &= -1 + 3s - s^2 \end{aligned}$$

$$\begin{aligned} W(s) &= \frac{1}{q(s)}cR(s)B \\ &= \frac{1}{-s^2+3s-1} \begin{bmatrix} s-2 & s^2-6 \\ -2s+1 & -3s+2 \end{bmatrix}. \end{aligned}$$

Numerical example 4. Singular system is given with the following data:

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

$$i = 0: R_{0,0} = I_2, \quad q_{1,1} = tr(ER_{0,0}) = 5,$$

$$q_{1,0} = -tr(AR_{0,0}) = -4,$$

$$R_{1,0} = AR_{0,0} + q_{1,0}I_2 = \begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix},$$

$$R_{1,1} = -ER_{0,0} + q_{1,1}I_2 = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}.$$

$$i = 1: q_{2,0} = -\frac{1}{2}tr(AR_{1,0}) = 0,$$

$$q_{2,1} = -\frac{1}{2}tr(AR_{1,1} - ER_{1,0}) = 1,$$

$$q_{2,2} = \frac{1}{2}tr(ER_{1,1}) = 0.$$

$$q(s) = \sum_{k=0}^2 q_{2,k}s^k = s$$

$$R(s) = \sum_{k=0}^1 R_{1,k}s^k = R_{1,0} + R_{1,1}s = \begin{bmatrix} 4s-3 & -2s+1 \\ -2s+3 & s-1 \end{bmatrix}$$

$$W(s) = \frac{1}{q(s)}cR(s)B = \frac{1}{s} \begin{bmatrix} 8s-6 & 6s-4 \\ 4s & 3s \end{bmatrix}.$$

Mertzios - Syrmos approach

It can be shown, that transfer matrix function of the system given by (2), may be written in the form:

$$W(s) = q^{-1}(s)C R(s)B \quad (27)$$

$$= q^{-1}(s)C R_{n-1}(s)B$$

$$= q^{-1}(s)C \begin{pmatrix} R_{n-1,n-1}s^{n-1} + R_{n-1,n-2}s^{n-2} + \dots \\ \dots \\ R_{n-1,1}s + R_{n-1,0} \end{pmatrix} B$$

where $q(s)$ is characteristic polynomial of (2), given by:

$$q(s) = q_n(s) = q_{n,n}s^n + q_{n,n-1}s^{n-1} + \dots + q_{n,1}s + q_{n,0} \quad (28)$$

The coefficient matrices $R_{n-1,j}$, $j = 0, 1, \dots, n-1$, and coefficients $q_{n,j}$, $j = 0, 1, \dots, n-1$, are given by the recursive relations:

$$R_{i+1,j} = \begin{cases} -ER_{i,j-1} + q_{i+1,j}I_n & j = i+1 \\ AR_{i,k} - ER_{i,j-1} + q_{i+1,j}I_n & j = 1, \dots, i \\ AR_{i,0} + q_{i+1,j}I_n & j = 0 \end{cases} \quad (29)$$

$$q_{i+1,j} = \begin{cases} \frac{1}{i+1}tr[ER_{i,k-1}] & j = i+1 \\ -\frac{1}{i+1}tr[AR_{i,k} - ER_{i,k-1}] & j = 1, \dots, i \\ -\frac{1}{1+i}tr[AR_{i,0}] & j = 0 \end{cases} \quad (30)$$

where $i = 0, 1, \dots, n-1$.

The initial conditions of (29) and (30) is:

$$R_{0,0} = I_n, \quad (31)$$

and final conditions are:

$$R_{n,j} = 0, \quad j = 0, 1, \dots, n. \quad (32)$$

Symbol $\langle A^k, E^l \rangle$ denotes the sum of all $\binom{k+l}{k} = \frac{(k+l)!}{k!l!}$ terms which consist of all the products of the matrices A and E , appearing k and l times, respectively, i.e.:

$$\langle A^k, E^l \rangle = A^k E^l + A^{k-1} E^l A + \dots + E^l A^k \quad (33)$$

Theorem 2. The transfer function matrix $W(s)$ may be expressed directly in terms of E, A, B, C and characteristic polynomial coefficients as follows:

$$\begin{aligned}
W(s) &= q^{-1}(s)[[C\langle A^0, E^0 \rangle B \sum_{l=0}^{n-1} q_{n-1,l} s^l + \\
&\quad + C\langle A^1, E^0 \rangle B] \\
&\quad + \sum_{l=0}^{n-2} q_{n-2,l} s^l + \dots + C\langle A^{n-1}, E^0 \rangle B] + \dots + \\
&\quad + (-1)^j [C\langle A^0, E^j \rangle \sum_{l=0}^{n-j-1} q_{n-j-1,l} s^{j+l} + C\langle A^1, E^j \rangle B \\
&\quad + \sum_{l=0}^{n-j-2} q_{n-j-2,l} s^{j+l} + \dots + C\langle A^{n-j-1}, E^j \rangle B s^j] + \dots \\
&\quad + (-1)^{n-2} [C\langle A^0, E^{n-2} \rangle B \sum_{i=0}^l q_{i,l} s^{n-2+i} + C\langle A^1, E^{n-2} \rangle B s^{n-2}] + \\
&\quad + (-1)^{n-1} C\langle A^0, E^{n-1} \rangle B s^{n-1}] = \\
&= q^{-1}(s) \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} (-1)^l C\langle A^k, E^l \rangle B [\sum_{j=0}^{n-k-l-1} q_{n-k-l-1,j} s^{j+1}]
\end{aligned}
\tag{34}$$

In (34) except for the matrices E, A, B, C and the characteristic polynomial $q(s)$ the auxiliary sequence of the coefficients $q_{k,l}$, $k, l = 0, 1, \dots, n-1$, are also involved. These coefficients may be determined either by using the recursive relations (29-30) or by the following formulas:

$$\begin{aligned}
q_{10} &= -trA \\
q_{11} &= trE \\
q_{20} &= -\frac{1}{2} tr[A(A - (trA)I)] \\
q_{21} &= -\frac{1}{2} [A[-\langle A^0, E^1 \rangle + (tr\langle A^0, E^1 \rangle)\langle A^0, E^0 \rangle] - \\
&\quad - E[\langle A^1, E^0 \rangle - (tr\langle A^1, E^0 \rangle)\langle A^0, E^0 \rangle]] \\
q_{n-1,0} &= -\frac{1}{n-1} [A[A^{n-2} - A^{n-3}(trA) - \\
&\quad - \frac{1}{2} A^{n-4} tr[A(A - (trA)I)] \\
&\quad - \frac{1}{n-2} tr[A[A^{n-3} - A^{n-4}(trA) - \frac{1}{2} An - 5tr[A(A - trI)]]]] \\
q_{n-1,l} &= -\frac{1}{n-1} tr[A[\langle A^{n-l-2}, (-E)^l \rangle - \\
&\quad - (tr\langle A^1, (-E)^0 \rangle)\langle A^{n-l-3}, (-E)^l \rangle -
\end{aligned}$$

$$\begin{aligned}
&\quad - (tr\langle A^0, (-E)^l \rangle)\langle A^{n-l-2}, (-E)^{l-1} \rangle \dots] \\
&\quad - E[\langle A^{n-l-1}, (-E)^{l-1} \rangle - (tr\langle A^1, (-E)^0 \rangle)\langle A^{n-l-2}, (-E)^{l-1} \rangle \\
&\quad - tr\langle Aa^0, (-E)^1 \rangle \langle A^{n-l-1}, (-E)^{l-2} \rangle \dots]]
\end{aligned}
\tag{35}$$

Proof. To prove (24), it is sufficient to express $R(s) = R_{n-1}(s)$, and therefore the coefficients matrices $R_{n-1,j}$, $j = 0, 1, \dots, n-1$, in terms of matrices E, A, B, C and coefficients $q_{k,l}$.

Using (29-30) recursively with the initial condition (31), the matrix $R_{n-1,j}$ may be written in the form:

$$\begin{aligned}
R_{n-1,j} &= [(-1)^j [\langle A^{n-j-1}, E^j \rangle + q_{10} \langle A^{n-j-2}, E^j \rangle + \dots \\
&\quad + q_{n-j-1,0} \langle A^0, E^j \rangle] + (-1)^{j-1} [q_{11} \langle A^{n-j-1}, E^{j-1} \rangle + \\
&\quad q_{21} \langle A^{n-j-2}, E^{j-1} \rangle + \dots + q_{n-1,j} \langle A^0, E^{j-1} \rangle] + \\
&\quad + \dots + [q_{jj} \langle A^{n-j-1}, E^0 \rangle + \\
&\quad q_{j+1,j} \langle A^{n-j-2}, E^0 \rangle + \dots + q_{n-1,j} \langle A^0, E^0 \rangle]] = \\
&= \sum_{k=0}^{n-j-1} \sum_{l=0}^j (-1)^{j-1} q_{k+l,l} \langle A^{n-j-k-1}, E^{j-l} \rangle.
\end{aligned}
\tag{36}$$

The substitution of (36) in (27) yields to the presented formula (34).

In the sequel we shall derive an alternative closed-form formula for the generalized transfer function matrix, which expressed it in terms of the generalized matrix pencil $(A - sE)$.

From the *definition*, given by (33), it results easily that:

$$(A + E)^n = \sum_{i=0}^n \langle A^i, E^{n-i} \rangle. \tag{37}$$

Therefore, it holds:

$$(A - sE)^n = \sum_{i=0}^n \langle A^i, (-sE)^{n-i} \rangle. \tag{38}$$

Using (37) and rearranging the terms on the right side of (34), we obtain the following more concise expression for the generalized transfer function matrix:

$$W(s) = q^{-1}(s)N(s), \tag{39}$$

where:

$$\begin{aligned}
N(s) &= C[(A - sE)^{n-1} + (q_{11}s + q_{10})(A - sE)^{n-2} + \\
&\quad + (q_{n-2,n-2}s^{n-2} + q_{n-2,n-3}s^{n-3} + q_{n-2,0})(A - sE) + \\
&\quad + (q_{n-1,n-1}s^{n-1} + q_{n-1,n-2}s^{n-2} + \dots + q_{n-1,0})I]B
\end{aligned}
\tag{40}$$

Numerical example 5. Let us consider, linear singular system, given by:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$\begin{aligned} W(s) &= q^{-1}(s) [C \langle A^0, E^0 \rangle B \sum_{l=0}^2 q_{2,l} s^l \\ &+ C \langle A^1, E^0 \rangle B \sum_{l=0}^1 q_{1,l} s^l + C \langle A^2, E^0 \rangle B - \\ &- [C \langle A^0, E^1 \rangle B \sum_{l=0}^1 q_{1,l} s^{l+1} + C \langle A^1, E^1 \rangle B] + \\ &+ C \langle A^0, E^2 \rangle B s^2]. \end{aligned}$$

$$q_{10} = -3, \quad q_{11} = 2, \quad q_{20} = 3, \quad q_{21} = -4.$$

$$q_{22} = 1, \quad q_{33} = 0, \quad q_{32} = -1, \quad q_{31} = 3, \quad q_{30} = -1$$

$$CB = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}, \quad CAB = \begin{bmatrix} 2 & 2 \\ 3 & -2 \end{bmatrix},$$

$$CA^2B = \begin{bmatrix} 4 & 0 \\ 4 & -7 \end{bmatrix}, \quad CEB = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix},$$

$$C(AE + EA)B = \begin{bmatrix} 3 & 1 \\ 6 & -2 \end{bmatrix},$$

$$CE^2B = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}.$$

$$W(s) = \frac{1}{-s^2 + 3s - 1} \begin{bmatrix} s-2 & s^2-6 \\ -2s+1 & -3s+2 \end{bmatrix}.$$

Numerical example 6. Let us consider, linear singular system, given by:

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}.$$

$$q_{1,0} = -trA = -4, \quad q_{1,1} = trE = 5,$$

$$q_{2,0} = -\frac{1}{2} tr[A(A - (trA)I)] = 0,$$

$$\begin{aligned} q_{2,1} &= -\frac{1}{2} tr[A[-\langle A^0, E^1 \rangle + (tr\langle A^0, E^1 \rangle)\langle A^0, E^0 \rangle] - \\ &- E[\langle A^1, E^0 \rangle - (tr\langle A^1, E^0 \rangle)\langle A^0, E^0 \rangle]] = \end{aligned}$$

$$= -\frac{1}{2} tr[A(-E + (trE)I_2)] - E[A - (trA)I_2] = 1.$$

$$q_{2,2} = -\frac{1}{2} tr[-E[\langle A^0, (-E)^1 \rangle - (tr(-E))\langle A^0, (-E)^0 \rangle]]$$

$$= -\frac{1}{2} tr[-E[-E - tr(-E)I_2]] = 0.$$

$$CB = \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}, \quad CAB = \begin{bmatrix} 2 & 0 \\ 8 & 0 \end{bmatrix},$$

$$CEB = \begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix}.$$

$$W(s) = q^{-1}(s) [CB \sum_{l=0}^1 q_{1,l} s^l + CAB - CEBs] =$$

$$= q^{-1}(s) [CB(q_{1,0} + q_{1,1}s) + CAB - CEBs] =$$

$$= \frac{1}{s} \begin{bmatrix} 8s-6 & 6s-1 \\ 4s & 3s \end{bmatrix}.$$

Mehtod - Rachid

Consider single input – single output singular system under the state space for:

$$\begin{aligned} E\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + \mathbf{b}u(t), \\ x_i(t) &= \mathbf{c}^T \mathbf{x}(t) \end{aligned} \quad (41)$$

Corresponding transfer matrix is:

$$W(s) = \mathbf{c}^T (sE - A)^{-1} \mathbf{b}. \quad (42)$$

It is assumed that A is non singular (regular) matrix. Equation (41) can be rewritten as follows:

$$\mathbf{x}(t) = D\dot{\mathbf{x}}(t) - \mathbf{d}u(t), \quad (43)$$

where:

$$D = A^{-1}E, \quad \mathbf{d} = A^{-1}\mathbf{b}. \quad (44)$$

From. (41- 43), follows:

$$\begin{aligned} x_i(t) &= \mathbf{c}^T \mathbf{x}(t) = \mathbf{c}^T (D\dot{\mathbf{x}}(t) - \mathbf{d}u(t)) \\ &= \mathbf{c}^T D(D\mathbf{x}^{(2)}(t) - \mathbf{d}\dot{u}(t)) - \mathbf{c}^T \mathbf{d}u(t) = \dots \end{aligned} \quad (45)$$

which gives at the n -th order:

$$\begin{aligned} x_i(t) &= \mathbf{c}^T D^n \mathbf{x}^{(n)}(t) - \mathbf{c}^T D^{n-1} \mathbf{d}u^{(n-1)}(t) \\ &- \dots - \mathbf{c}^T \mathbf{d}u(t) \end{aligned} \quad (46)$$

Applying this procedure to $\dot{x}_i(t) = \mathbf{c}^T \dot{\mathbf{x}}(t)$, gives:

$$\dot{x}_i(t) = \mathbf{c}^T D^{n-1} \mathbf{x}^{(n)}(t) - \dots - \mathbf{c}^T \mathbf{d}\dot{u}(t), \quad (47)$$

and similarly to all derivatives of $x_i(t)$, till the n -th one, which gives:

$$x_i^{(n)}(t) = \mathbf{c}^T \mathbf{x}^{(n)}(t). \quad (48)$$

Summarizing these relations under a matrix form, yields:

$$\begin{bmatrix} x_i \\ x_i^{(1)} \\ \vdots \\ x_i^{(n)} \end{bmatrix} = \begin{bmatrix} \mathbf{c}^T D^n \\ \mathbf{c}^T D^{n-1} \\ \vdots \\ \mathbf{c}^T \end{bmatrix} \mathbf{x}^{(n)} - \begin{bmatrix} \mathbf{c}^T \mathbf{d} & \mathbf{c}^T D \mathbf{d} & \cdots & \mathbf{c}^T D^{n-1} \mathbf{d} \\ 0 & \mathbf{c}^T \mathbf{d} & \cdots & \mathbf{c}^T D^{n-2} \mathbf{d} \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \mathbf{c}^T \mathbf{d} \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} u \\ u^{(1)} \\ \vdots \\ u^{(n-1)} \end{bmatrix} \quad (49)$$

or:

$$[Cx_i(t)]Dx_i(t) = [Cx(t)]\mathbf{x}^{(n)}(t) + [Cu(t)]Du(t), \quad (50)$$

where $[Cx_i]$ is $(n+1)$ identity matrix:

$$Dx_i^T(t) = [x_i \quad x_i^{(1)} \quad \cdots \quad x_i^{(n)}]^T,$$

$$Du^T(t) = [u \quad u^{(1)} \quad \cdots \quad u^{(n)}]^T,$$

and where $[Cx_i]$ is $(n+1) \times (n+1)$ matrix.

$[Cu]$, is also, $(n+1) \times n$ matrix

Then one has to constitute the following matrix $[Cx(t)Cx_iCu(t)]$ of order $(n+1) \times (3n+1)$. By performing Gaussian elimination we can have *zero* elements in the last row n .

If one has to extend this method to the class of multi-variable singular control systems, then having in mind the principle of superposition, the proposed method should be used for every transfer function connecting every input with every output, which is not such in general difficult task.

Numerical example 7. Consider the multivariable linear singular systems:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

This is two input and two output system.

For the sake of simplicity we shall use the following notation:

$$\mathbf{x}_i(t) = \mathbf{y}(t).$$

a) 1-st input, 1-st output:

$$\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{c}^T = [1 \quad 1 \quad 1].$$

$$\mathbf{d} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

$$D = A^{-1}E = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ -1 & 0 & 2 \end{bmatrix}.$$

$$D^2 = \begin{bmatrix} 2 & 0 & -3 \\ 3 & 0 & -5 \\ -3 & 0 & 5 \end{bmatrix}, \quad D^3 = \begin{bmatrix} 5 & 0 & -8 \\ 8 & 0 & -13 \\ -8 & 0 & 13 \end{bmatrix}$$

$$\mathbf{c}^T D^3 = [21 \quad 0 \quad -34], \quad \mathbf{c}^T \mathbf{b} = -2,$$

$$\mathbf{c}^T D^2 = [8 \quad 0 \quad -13], \quad \mathbf{c}^T D \mathbf{b} = -5,$$

$$\mathbf{c}^T D = [3 \quad 0 \quad -5], \quad \mathbf{c}^T D^2 \mathbf{b} = -13.$$

$$[Cx] = \begin{bmatrix} 21 & 0 & -34 \\ 8 & 0 & -13 \\ 3 & 0 & -5 \\ 1 & 1 & -1 \end{bmatrix},$$

$$[Cy] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad [Cu] = \begin{bmatrix} 2 & 5 & 13 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$[CxCyCu] = \cdots =$$

$$\begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 21 & -34 & 1 & 0 & 0 & 0 & 2 & 5 & 13 \\ 0 & 0 & -\frac{1}{21} & -\frac{8}{21} & 1 & 0 & 0 & -\frac{16}{21} & \frac{2}{21} & \frac{1}{21} \\ 0 & 0 & 0 & 1 & -3 & 1 & 0 & 2 & -1 & 0 \end{bmatrix}$$

$$y - 3\dot{y} + \ddot{y} = 2u - \dot{u}.$$

$$f_{11}(s) = \frac{s-2}{-s^2+3s-1}.$$

b) 1-st input, 2-nd output:

$$\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{c}^T = [1 \quad 0 \quad 1];$$

$$\mathbf{d} = A^{-1}\mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix},$$

$$D = A^{-1}E = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ -1 & 0 & 2 \end{bmatrix},$$

$$\mathbf{c}^T D^3 = [-3 \quad 0 \quad 5], \quad \mathbf{c}^T \mathbf{b} = 1,$$

$$\mathbf{c}^T D^2 = [-1 \quad 0 \quad 2], \quad \mathbf{c}^T D \mathbf{b} = 1,$$

$$\mathbf{c}^T D = [0 \quad 0 \quad 1], \quad \mathbf{c}^T D^2 \mathbf{b} = 2.$$

$$[Cx] = \begin{bmatrix} -3 & 0 & 5 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad [Cy] = I_4,$$

$$[Cu] = \begin{bmatrix} -1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$[CxCyCu] = \dots =$$

$$= \begin{bmatrix} 0 & -3 & 5 & 1 & 0 & 0 & 0 & -1 & -1 & -2 \\ 0 & -1 & 2 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -3 & 1 & 0 & -1 & 2 \end{bmatrix}.$$

Since the first column equals zero, the following submatrix is subtracted for the purposes of basic order equation reduction:

$$\begin{bmatrix} 2 & 0 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -\frac{1}{2} & 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & -3 & 1 & 0 & -1 & 2 \end{bmatrix}.$$

such that:

$$y - 3\dot{y} + \ddot{y} = 2\dot{u} - u.$$

$$f_{21}(s) = \frac{-2s+1}{-s^2+3s-1}.$$

c) 2-nd input, 1-st output:

$$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad c^T = [1 \ 1 \ -1].$$

$$\mathbf{d} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix} \quad D = A^{-1}E = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ -1 & 0 & 2 \end{bmatrix},$$

$$c^T D^3 = [21 \ 0 \ -34], \quad c^T \mathbf{b} = -6,$$

$$c^T D^2 = [8 \ 0 \ -13], \quad c^T D\mathbf{b} = -18,$$

$$c^T D = [3 \ 0 \ -5], \quad c^T D^2\mathbf{b} = -47.$$

$$[Cx] = \begin{bmatrix} 21 & 0 & -34 \\ 8 & 0 & -13 \\ 3 & 0 & -5 \\ 1 & 1 & -1 \end{bmatrix}, \quad [Cy] = I_4,$$

$$[Cu] = \begin{bmatrix} 6 & 18 & 47 \\ 0 & 6 & 18 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$[CxCyCu] = \dots = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -21 & -34 & 1 & 0 & 0 & 0 & 6 & 18 & 47 \\ 0 & 0 & -\frac{1}{21} & -\frac{8}{21} & 1 & 0 & 0 & -\frac{48}{21} & -\frac{18}{21} & \frac{2}{21} \\ 0 & 0 & 0 & 1 & -3 & 1 & 0 & 6 & 0 & -1 \end{bmatrix}$$

$$y - 3\dot{y} + \ddot{y} = 6u - \dot{u},$$

$$f_{12}(s) = \frac{s^2 - 6}{-s^2 + 3s - 1}.$$

d) 2-nd input, 2-nd output:

$$\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad c^T = [1 \ 0 \ 1],$$

$$\mathbf{d} = A^{-1}\mathbf{b} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix},$$

$$D = A^{-1}E = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ -1 & 0 & 2 \end{bmatrix},$$

$$c^T D^3 = [-3 \ 0 \ 5], \quad c^T \mathbf{b} = 2,$$

$$c^T D^2 = [-1 \ 0 \ 2], \quad c^T D\mathbf{b} = 3,$$

$$c^T D = [0 \ 0 \ 1], \quad c^T D^2\mathbf{b} = 7.$$

$$[Cx] = \begin{bmatrix} -3 & 0 & 5 \\ -1 & 0 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad [Cy] = I_4,$$

$$[Cu] = \begin{bmatrix} -2 & -3 & -7 \\ 0 & -2 & -3 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$[CxCyCu] = \dots =$$

$$= \begin{bmatrix} 0 & -3 & 5 & 1 & 0 & 0 & 0 & -2 & -3 & -7 \\ 0 & -1 & 2 & 0 & 1 & 0 & 0 & 0 & -2 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -2 & 3 \end{bmatrix}$$

$$y - 3\dot{y} + \ddot{y} = 3\dot{u} - 2u,$$

$$f_{22}(s) = \frac{-3s+2}{-s^2+3s-1}.$$

And finally:

$$W(s) = \begin{bmatrix} f_{11}(s) & f_{12}(s) \\ f_{21}(s) & f_{22}(s) \end{bmatrix}.$$

Fundamental matrix of the discrete descriptive system

In the sequel some contributions derived in the papers of Lewis (1985.a), Mertzios, Lewis (1989) and Lewis, Mertzios (1990) are presented.

For the classical state space equations an analysis may be accomplished in terms of only matrix A .

It is well known that in the singular case, eq. (1-2) an analysis in terms of E and A is not possible. Auxiliary quantities that have been used in the analysis of these systems have included the *Drazin inverse* of related matrix, the transformation to Weierstrass form and deflating special subspaces. Some other approaches, algorithmic in nature, are also important.

The sequel will show that the analysis of linear discrete descriptor systems can not be performed upon only the knowledge of basic system matrices E and A . This fact follows due to new concept introduced in Lewis, Mertzios (1990) denoted as *relative fundamental matrix*.

We are concerned with the discrete descriptive of equations:

$$E\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k), \quad (51)$$

$$k = 0, 1, 2, \dots, N-1$$

$$\mathbf{x}_i(k) = C\mathbf{x}(k), \quad (52)$$

with unusual notation and under the assumption that the discrete system under consideration is a *regular* one.

The interval of interest of index k is $k \in [0, N]$ and $u(k) \neq 0, \forall k = 0, 1, \dots, N-1$.

For regular matrix pencil (E, A) , the Laurent series expansion about infinity for the resolvent matrix exist and is given by, Rose (1978):

$$(zE - A)^{-1} = z^{-1} \sum_{i=-\nu}^{\infty} \phi_i z^{-i}, \quad (53)$$

where ν is index of nilpotence and sequence ϕ_i (which should be determined in the sequel), is known as the (*forward*) *fundamental matrix*.

The Laurent series expansion about zero, is:

$$(zE - A)^{-1} = \sum_{i=-\rho}^{\infty} \psi_i z^{-i}, \quad (54)$$

where sequence ψ_i shall be known, for the reasons to be seen, as a *backward fundamental matrix*.

In the state - space $E = I$, we have that $\phi_i = 0$ for $i < 0$, and $\phi_i = A^i$ for $i \geq 0$.

If $E = I$ and $\det A \neq 0$, then $\psi_i = 0$ for $i > 0$, and $\psi_i = A^{-i}$ for $i \leq 0$.

Relative fundamental matrix of discrete descriptor system, which according to its nature, may be called the fundamental matrix sequence, is of particular importance in the study of descriptor systems.

Some of these basic questions are:

- Determination of state space response
- Determination of resolvent matrix
- Finding expressions of controllable and observable canonical forms

- Determination of transition matrix
- Determination of Hankel's matrix, Markov's parameters and Tschirnhausen's polynomials.

In the sequel, only the two first questions will be discussed.

The fundamental sequence obeys the following properties:

Theorem 3. Let us consider regular matrix pencil and let fundamental sequence ϕ_i be defined by (53).

Then:

$$\phi_i E - \phi_{i-1} A = I \delta_{oi}, \quad (55)$$

$$E \phi_i - A \phi_{i-1} = I \delta_{oi}, \quad (56)$$

$$(\phi_o A)^i \phi_o = \phi_i, \quad i \geq 0, \quad (57)$$

$$(-\phi_{-1} E)^{-i-1} \phi_{-1} = \phi_i, \quad i < 0. \quad (58)$$

$$\phi_i = 0, \quad i < -\mu. \quad (59)$$

$$\phi_i E \phi_j = \phi_j E \phi_i, \quad (60)$$

$$\phi_i E \phi_j = \begin{cases} -\phi_{i+j}, & i < 0, j < 0 \\ \phi_{i+j}, & i \geq 0, j \geq 0, \\ 0, & \text{anywhere} \end{cases} \quad (61)$$

$$\phi_i A \phi_j = \begin{cases} -\phi_{i+j+1}, & i < 0, j < 0 \\ \phi_{i+j+1}, & i \geq 0, j \geq 0 \\ 0, & \text{anywhere} \end{cases} \quad (62)$$

For the sake of brevity *the proof* is omitted here and can be found in Lewis, Mertzios (1990).

The following exposure presents some of particular cases of (51) and (52).

These results are given in the form of the following corollary.

Corollary 1.

$$\phi_0 E \phi_i = \begin{cases} \phi_i, & i \geq 0 \\ 0, & i < 0, \end{cases} \quad (63)$$

$$-\phi_{-1} A \phi_i = \begin{cases} 0, & i \geq 0 \\ \phi_i, & i < 0, \end{cases} \quad (64)$$

$$-\phi_{-1} E \phi_i = \begin{cases} 0, & i \geq 0 \\ \phi_i, & i < 0, \end{cases} \quad (65)$$

$$\phi_0 A \phi_i = \begin{cases} \phi_{i+1}, & i \geq 0 \\ 0, & i < 0. \end{cases} \quad (66)$$

There have been several interpretations of (51).

From dynamical standpoint we may consider that the initial condition $\mathbf{x}(0)$ is given and that it is desired to determine $\mathbf{x}(k)$ in a *forward* fashion from the input sequence $\mathbf{u}(k)$ and previous values of the semistate.

A variant of this is to consider $\mathbf{x}(N)$ as given and then determine $\mathbf{x}(k)$ in a *backward* fashion from the input and

the *future* values of *semistate*.

Another interpretation, arising in economics is to consider that (51) describes a *relationship* that holds between the states and the inputs. That is, no causality is assumed. It is desired to determine, given the sequence $\mathbf{u}(k)$ and *admissible* $\mathbf{x}(0)$ and $\mathbf{x}(N)$, the semi-state for intermediate values k .

Usually this is called *symmetric solution*.

Our aim is to provide some unified treatment of these different problems. In this section we present closed-loop solutions (51) in the *forward* and *backward* case.

For the causal solution, consider that \mathbf{x}_0 is given along with the sequence $\{\mathbf{u}_k\}$.

Then formal application of the Z – transform, to (51), yields:

$$\mathbf{X}(z) = (zE - A)^{-1} zE\mathbf{x}(0) + (zE - A)^{-1} B\mathbf{U}(z). \quad (67)$$

Now, using (53) and convolution theorem, we discover that:

$$\mathbf{x}(k) = \phi_k E\mathbf{x}(0) + \sum_{i=0}^{k+v-1} \phi_{k-i-1} B\mathbf{u}(i). \quad (68)$$

This is the *forward* solution for (51).

If $E = I$ it reduces to the standard state space result.

To obtain *backward* to (51), given the final condition \mathbf{x}_N and u_{ki} , take the Z – transform of (51), to get:

$$\mathbf{X}(z) = -(zE - A)^{-1} z^{-N+1} E\mathbf{x}(N) + (zE - A)^{-1} B\mathbf{U}(z), \quad (69)$$

then apply (54) to obtain:

$$\mathbf{x}(k) = -\psi_{k-N+1} E\mathbf{x}(N) + \sum_{i=k-\psi}^{N-1} \psi_{k-i} B\mathbf{u}(i), \quad (70)$$

backward semi – state solution.

We define also the *backward semistate – transition matrix* for (51) as $-\psi_k E$.

These solutions are given in the form of fundamental matrix ϕ_i and the backward matrix ψ_i . A generalized Leverrier technique for computing ϕ_i is known, so that we may assume that these fundamental matrices are given.

Some aspects of this problem will be presented in the sequel.

It is worth examining (68) the case when the descriptor systems, given (51) is in the Weierstrass form:

$$\mathbf{x}_2(k+1) = J\mathbf{x}_1(k) + B_1\mathbf{u}(k), \quad (71)$$

$$N\mathbf{x}_2(k+1) = I\mathbf{x}_2(k) + B_2\mathbf{u}(k), \quad (72)$$

where $\mathbf{x}_1(k) \in \mathfrak{R}^{m_1}$, $\mathbf{x}_2(k) \in \mathfrak{R}^{m_2}$ J is *Jordan* matrix, and N a nilpotent *Jordan* block with $\nu = \text{Ind}(N)$.

Then, the *forward* solution of (68) becomes:

$$\mathbf{x}_1(k) = J^k \mathbf{x}_1(0) + \sum_{i=0}^{k-1} J^{k-i-1} B_1 \mathbf{u}(i), \quad (73)$$

$$\mathbf{x}_2(k) = -\sum_{i=1}^{v-1} \delta_i(k) N^i \mathbf{x}_2(0) - \sum_{i=0}^{v-1} N^i B_2 \mathbf{u}(k+i). \quad (74)$$

The first term in (74) has evidently not been displayed elsewhere, in scientific papers. It has a satisfying relationship to the well known impulsive terms in singular continuous systems due to the initial conditions. Those values of $\mathbf{x}_2(0)$ guaranteeing that (74) is equal zero for $k < 0$, are called *admissible initial conditions*.

Two additional aspects of the *forward* solution, (68), are worth noting.

First, based on (57-58), it can be said that:

$$\mathbf{x}(k) = (\phi_0 A)^k \phi_0 E\mathbf{x}(0) + \sum_{i=0}^{k-1} (\phi_0 A)^k \phi_0 B\mathbf{u}(i) + \sum_{i=k}^{k+v-1} (-\phi_{-1} E)^{-k+1} \phi_{-1} B\mathbf{u}(i), \quad k \geq 0 \quad (75)$$

making the calculation of $\mathbf{x}(k)$ easier, since it is necessary only to know ϕ_0 i ϕ_{-1} .

Theorem 4. For $k \geq 0$, (68) is equivalent to the *forward recursion*:

$$\begin{aligned} \mathbf{x}(k) &= \phi_0 A\mathbf{x}(k-1) \\ &+ \phi_{-1} B\mathbf{u}(k) \\ &+ \phi_0 B\mathbf{u}(k-1) \\ &+ \phi_{-1} E\mathbf{v}(k+1), \end{aligned} \quad (76)$$

and also to backward recursion:

$$\begin{aligned} \mathbf{x}(k) &= -\phi_{-1} E\mathbf{x}(k+1) \\ &+ \phi_{-1} B\mathbf{u}(k) + \phi_0 B\mathbf{u}(k-1) + \phi_0 A\mathbf{v}(k-1), \end{aligned} \quad (77)$$

where the intermediate input sequence $\mathbf{v}(k)$ is defined in terms of \mathbf{u}_k , by:

$$\mathbf{v}(k) = \sum_{i=0}^{k+v-1} \phi_{k-i-1} B\mathbf{u}(i). \quad (78)$$

The proof is omitted here, for the sake of brevity.

Finally, the results which follow, will lead to the expressions for forward and backward fundamental matrix sequences,

Mertzios, Lewis (1989), and Lewis, Mertzios (1990).

Theorem 5. Let (51) be regular and sequence ϕ_i is defined by (53).

Then for any $c \in \mathfrak{R}$, such that $\det(cE - A) \neq 0$:

$$\phi_i = \begin{cases} (\hat{E}^D \hat{A})^i \hat{E}^D (cE - A)^{-1} & i \geq 0 \\ (\hat{A}^D \hat{E})^{i-1} (I - \hat{E} \hat{E}^D) \hat{A}^D (cE - A)^{-1} & i < 0 \end{cases} \quad (79)$$

where:

$$\begin{aligned} \hat{E} &= (cE - A)^{-1} E, \\ \hat{A} &= (cE - A)^{-1} A. \end{aligned} \quad (80)$$

Proof. Simply compare (68) term by term to the classical solution, given for example in *Debeljković et. al* (1988).

In any case the Drazin inversion is not avoidable.

From the preceding *Theorem* the following result can be generated.

Corollary 2. Let system, given (51) is *regular*. Then for

any $c \in \mathfrak{R}$ such that $\det(cE - A) \neq 0$, follows:

$$\phi_0 = \hat{E}^D (cE - A)^{-1}, \quad (81)$$

$$\phi_{-1} = (I - \hat{E}\hat{E}^D) \hat{A}^D (cE - A)^{-1}, \quad (82)$$

$$\phi_i = (\phi_0 A)^i \phi_0, \quad i > 0, \quad (83)$$

$$\phi_i = (-\phi_{-1} E)^{-i-1} \phi_{-1}, \quad i < -1. \quad (84)$$

Proof. To obtain the first two equations, evaluate (79) for $i = 0$ and $i = -1$.

It is worth discussing further the meaning of ψ_i , e.g. *backward fundamental matrix sequence*.

Define the *backward matrix pencil* as $(zA - E)$. Its properties are exactly the same as those of $(zE - A)$ with matrices E and A interchanged.

Then, setting:

$$w = z^{-1}, \quad (85)$$

we may use (54), to write:

$$\begin{aligned} (zA - E)^{-1} &= -w(wE - A)^{-1} \\ &= -w \sum_{i=-\rho}^{\infty} \psi_{-i} w^i \\ &= -z^{-1} \sum_{i=\rho}^{\infty} \psi_i z^{-i}. \end{aligned} \quad (86)$$

By comparison of (86) to (53) it is concluded that $-\psi_i$ is *fundamental matrix sequence* of “backward” matrix pair $(zA - E)$ with ρ being nilpotence index.

Therefore, if matrices E and A interchange, *Theorem 4* and *Theorem 5* will give the properties of ψ_i .

Finally we can give the following expressions:

$$\Psi_k^\phi = \phi_k E, \quad \Psi_k^\psi = -\psi_k E, \quad (87)$$

if necessary.

These expressions are connected for so called transition matrices of the discrete descriptive linear system.

Conclusion

The first part of this paper is dedicated to the question of how it is possible to calculate the transfer function matrix for particular class of linear continuous singular, regular system. It has been shown that some numerical algorithms are very applicable for these purposes. Apart from this, some specific properties of this class of system such as the question of strict properness, makes a lot of additional problems.

This implies the variety of methods presented here in this paper.

A number of algorithms were presented, which allows the computation of transfer function matrix of a singular system from its state space description without inverting some particular matrices.

All presented methods have been followed by a suitable choice of numerical examples.

The second part of the paper presents several approaches of computational procedure for calculating the so called fundamental matrix sequences for linear regular causal discrete descriptor system, operating in free regime. Forward and backward fundamental matrices have been defined in order to calculate state space response of the systems under consideration. Some theoretical discussions have been presented to underline the applicability of the approaches presented.

Appendix A – Solvability

The singular system is *regular*, when the matrix pencil $(cE - A)$ is regular, i.e.

$$\exists c \in \mathfrak{R} : \det(cE - A) \neq 0, \quad (A.1)$$

and then solutions of (1.1) exist, they are unique and for so-called consistent initial conditions[†] generate *smooth solutions*.

Moreover, the closed form of these solution is known, *Campbell* (1980, 1982), *Dai* (1989.b).

In some circumstances, it is useful to introduce the linear nonsingular transformation of system governed by (1.1), in order to get the *first canonical form of linear singular system*, as:

$$\dot{\mathbf{x}}_1(t) = A_1 \mathbf{x}_1(t) + A_2 \mathbf{x}_2(t), \quad (A.2)$$

$$\mathbf{0} = A_3 \mathbf{x}_1(t) + A_4 \mathbf{x}_2(t), \quad (A.3)$$

The regularity condition (B.1) form, the system given by (B.2 – B.3) reduces to the following:

$$\det \begin{bmatrix} sI - A_1 & -A_2 \\ -A_3 & -A_4 \end{bmatrix} \neq 0, \quad (A.4)$$

which is equivalent to:

$$\det(sI - A_1) \det(-A_4 - A_3(sI - A_1)^{-1} A_2) \neq 0, \quad (A.5)$$

or:

$$\det A_4 \det((sI - A_1) - A_2 A_4^{-1} A_3) \neq 0, \quad (A.6)$$

Instead of (B.11), the following condition can be verified, *Campbell* (1980).

$$\mathfrak{N}(A) \cap \mathfrak{N}(E) = \{\mathbf{0}\}, \quad (A.7)$$

i.e. $\mathfrak{N}(A)$ and $\mathfrak{N}(E)$ have only the trivial intersection where $\mathfrak{N}(\cdot)$ denotes the null space or kernel of matrix (\cdot) .

Owens and *Debeljkovic* (1985) showed that (A.7) is equivalent with:

$$W_{k^*} \cap \mathfrak{N}(E) = \{\mathbf{0}\}, \quad (A.8)$$

W_{k^*} being subspace of consistent initial conditions.

It should be noted that condition (B.1) guarantees (B.8) and (B.9), but vice versa must not be true.

Alternative characterizations of regularity condition offered by other authors are also presented in current references.

[†] See, *Debeljkovic et al. (1996.a)*

Appendix B - Some specific features of transfer matrix function

Definition B1. Matrix of transfer function is *strictly proper* if the following condition is satisfied:

$$\lim_{s \rightarrow \infty} W(s) \rightarrow 0, \quad (\text{B1})$$

where 0 denotes null matrix of appropriate dimension.

Reference

- [1] BAJIC,V.B.: Lyapunov's Direct Method in The Analysis of Generalized state space systems and Networks, Shades Technical Publications, Hillcrest, Natal, RSA, 1992.a.
- [2] BAJIC,V.B., DEBELJKOVIĆ,D.LJ., GAJIC,Z., PETROVIC,B.: "Weak Domain of Attraction and Existence of Solutions Convergent to the Origin of the Phase Space of Singular Linear Systems", University of Belgrade, ETF, Series: Automatic Control, (1) (1992.b) 53–62.
- [3] CAMPBELL,S.L.: Singular Systems of Differential Equations, Pitman, Marshfield, Mass., 1980.
- [4] CAMPBELL,S.L.: Singular Systems of Differential Equations II, Pitman, Marshfield, Mass., 1982.
- [5] Circuits, Systems and Signal Processing, Special Issue on Semistate Systems, 5 (1) (1986).
- [6] Circuits, Systems and Signal Processing, Special Issue: Recent Advances in Singular Systems, 8 (3) (1989).
- [7] DAL,L.: "The Difference Between Regularity and Irregularity in Singular Systems", Circ. Syst. Sig. Proc., 8 (4) (1989.a) 435–444.
- [8] DALL,L.: Singular Control Systems, Springer Verlag, Berlin, 1989.b.
- [9] DEBELJKOVIĆ,D.LJ., MILINKOVIĆ,S.A., JOVANOVIĆ,M.B.: Application of Singular Systems Theory in Chemical Engineering, MAPRET Lecture – Monograph, 12th International Congress of Chemical and Process Engineering, CHISA 96, Praha, Czech Republic (1996.a).
- [10] DEBELJKOVIĆ,D.LJ., MILINKOVIĆ,S.A., JOVANOVIĆ M.B.: Kontinualni singularni sistemi automatskog upravljanja GIP Kultura, Beograd, 1996.b.
- [11] DEBELJKOVIĆ,D.LJ., JOVANOVIĆ M.B., MILINKOVIĆ,S.A., JACIĆ,L.J.A.: Diskretni singularni sistemi automatskog upravljanja GIP Kultura, Beograd, 1998.
- [12] DEBELJKOVIĆ,D.LJ., ZHANG, Q.L.: "Dynamic analysis of generalized nonautonomous state space systems", NTP (YU), Vol. LIII, No. 1, (2003), 30 – 39.
- [13] DEBELJKOVIĆ,D.LJ., MILINKOVIĆ,S.A., JOVANOVIĆ M.B.: Kontinualni singularni sistemi, Čigoja štampa, Beograd, 2004.
- [14] Debeljkovic, D. Lj., "Singular Control Systems", Dynamics of Continuous, Discrete and Impulsive Systems, (Canada), Vol. 11, Series A: Math. Analysis, No. 5 - 6 (2004), 691 - 706.
- [15] DEBELJKOVIĆ,D.LJ., JOVANOVIĆ M.B., MILINKOVIĆ,S.A., JACIĆ,L.J.A.: Diskretni deskriptivni sistemi, Čigoja štampa, Beograd, 2005.
- [16] DEBELJKOVIC, D.LJ., ZHANG.Q.L.: "Dynamic analysis of generalized nonautonomous state space systems", NTP (YU), Vol. LIII, No. 1, (2003), pp. 30 – 39.
- [17] DZIURLA,B., NEWCOMB,R.W.: "Nonregular Semistate Systems: Examples and Input-Output Pairing", IEEE Proc. on CDC, Los Angeles, CA (1987.b) 1125–1126.
- [18] LAZAREVIC,P.M., DEBELJKOVIC, D.Lj., JOVANOVIC,M.B., RANCIC,M.V., MULIC,V.S.: "Optimal Control of Linear Generalized state space systems with Pure Time Lag", NTP (YU), Vol. LI, No. 2, (2001) 5 – 12.
- [19] LEWIS,F.L.: "Fundamental, Reachability and Observability Matrices for Discrete Descriptor Systems", IEEE Trans. Automat. Cont., AC-30 (5) (1985) 502–505.
- [20] LEWIS,F.L.: "A Survey of Linear Singular Systems", Circ. Syst. Sig. Proc., 5 (1) (1986.a) 3–36.
- [21] LEWIS,F.L.: "Further Remarks on the Cayley-Hamilton Theorem and Laverrier's Method for the Matrix Pencil (sE–A)", IEEE Trans. Automat. Cont., AC-31 (9) (1986.b) 869–870.
- [22] LEWIS,F.L.: "Recent Work in Singular Systems", Proc. Int. Symp. on Sing. Syst., Atlanta, GA (1987) 20–24.
- [23] LUENBERGER,D.G.: "Dynamic Equations in Descriptor Form", IEEE Trans. Automat. Cont., AC-22 (3) (1977) 312–321.
- [24] LUENBERGER,D.G.: "Time-Invariant Descriptor Systems", Automatica, 14 (1978) 473–480.
- [25] LUENBERGER,D.G., ARBEL,A.: "Singular Dynamic Leontief Systems", Econometrica, (1977) 991–995.
- [26] MERTZIOS,B.G.: "Leverrier's Algorithm for Singular Systems", IEEE Trans. Automat. Cont., AC-29 (7) (1984) 652–653.
- [27] MERTZIOS,B.G.: "Recent Work in Singular Systems", Proc. Int. Symp. on Sing. Syst., Atlanta, GA (1987) 14–17.
- [28] MERTZIOS,B.G., CHRISTODOLOU,M.A.: "On the Generalized Cayley-Hamilton Theorem", IEEE Trans. Automat. Cont., AC-31 (2) (1986.b) 156–157.
- [29] MERTZIOS,B.G., LEWIS,F.L.: "Transfer Function Matrix of Singular Systems", IEEE Trans. Automat. Cont., AC-32 (9) (1987) 829–831.
- [30] MERTZIOS,B.G., LEWIS,F.L.: "An Algorithm for the Computation of the Transfer Function Matrix of Generalized Two-Dimensional Systems", Circ. Syst. Sig. Proc, 7 (4) (1988) 459–466.
- [31] OWENS,D.H., DEBELJKOVIC,D.LJ.: "Consistency and Liapunov Stability of Linear Decscriptor Systems: a Geometric Approach", IMA Journal of Math. Control and Information, (1985), No.2, 139 - 151.
- [32] MERTZIOS,B.G., LEWIS,F.L.: "Fundamental Matrix of Discrete Singular Systems", Circ. Syst. Sig. Proc., 8 (3) (1989) 341–356.
- [33] PARASKEVOPOULOS,P.N., CHRISTODOLOU,M.A., TSAKIRIS,M.A.: "Transfer Function Sensitivity Analysis of Singular Systems", Proc. MELECON 83, Athens, II (1983).
- [34] PARASKEVOPOULOS,P.N., CHRISTODOLOU,M.A., BOGLU,A.K.: "An Algorithm or the Computation of the Transfer Function Matrix for Singular Systems", Automatica, 20 (2) (1984) 259–260.
- [35] RACHID,A.: "On Singular Systems Transfer Function Determination", Control Theory and Advanced Technology, 6 (2) (1990) 215–219.
- [36] ROS,N.J.: "The Laurent Expansion of a Generalized Resolvent with some Applications", SIAM J. Math. Anal. 9 (4) (1978) 751 – 758.
- [37] THORP,J.S.: "The Singular Pencil of a Linear Dynamical Systems", Int. J. Control, 18 (3) (1973) 577–596.
- [38] YONG,N.Y., DEBELJKOVIC,D.Lj.: "Non – Lyapunov Stability of Linear Singular Systems: A Quite new Approach in Time Domain", Dynamics of Continuous, Discrete and Impulsive Systems, (Canada), Vol. 11, Series A: Math. Analysis, No. 5 - 6 (2004) 751 - 760.

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Matrica prenosnih funkcija i fundamentalna matrica linearnih singularnih sistema

Matrica prenosnih funkcija ima veliki značaj pri ispitivanju dinamičkog ponašanja savremenih sistema automatskog upravljanja sa pozicija ulazno-izlaznih relacija. Veza sa frekventnim domenom je očigledna pa uprkos činjenici da se moraju pretpostaviti nulti početni uslovi ovaj prilaz ima svoje potpuno opravdanje. Imajući u vidu da se i za posebnu klasu linearnih singularnih sistema može odrediti matrica prenosnih funkcija, bilo je od posebnog interesa pokazati kako se praktično računa u slučajevima kada je red sistema relativno visok. U radu je prezentovan određeni broj metoda koje, koristeći savremene numeričke algoritme, omogućavaju efikasno sračunavanje matrice prenosnih funkcija kako za jednostruko prenosne tako i za višestruko prenosne linearne singularne sisteme automatskog upravljanja. Sva izlaganja propraćena su pažljivo odabranim primerima koji su dovoljno rečiti da ilustruju sve prednosti predloženih postupaka.

Fundamentalna matrica ima poseban značaj kada su u pitanju linearni deskriptivni diskretni sistemi. Njeno sračunavanje ne zahteva primenu Drazinove inverzije ali zato zahteva razvijanje odgovarajućeg matičnog para u Laurentov red. Procedure ove vrste prikazane su i propraćene odgovarajućim primerima.

Ključne reči: Linerani sistemi, singularni sistemi, deskriptivni sistemi, prenosna matrica, fundamentalna matrica.

Матрица передаточных функций и фундаментальная матрица линейных сингулярно-дескриптивных систем

Матрица передаточных функций имеет большое значение при исследовании динамического поведения современных автоматических систем управления со позиции вводно-выходных релейций. Связь с частотной областью деятельности очевидна, и вопреки фактам, что нужно предположить нулевые исходные условия, этот подход является совсем обоснованным. Имея в виду, что и для особого класса линейных сингулярных систем возможно определить матрицу передаточных функций, от особого интереса было показать как практически считается в случаях когда порядок системы относительно высок. В работе приведено известное число методов, которые пользуясь современные цифровые алгоритмы, обеспечивают эффективное подсчитывание матрицы передаточных функций, как для одинарных передаточных, так и для многократных передаточных линейных сингулярных автоматических систем управления. Все объяснения проведены внимательно выбранными примерами, которые довольно иллюстрируют все преимущества предложенных поступков.

Фундаментальная матрица имеет особое значение когда речь идет о линейных дескриптивных сдержанных системах. Ее посчитывание не требует применения инверсии Дразина, но зато требует развития соответствующей основной пары в порядок Лаврента. Процедуры этой категории показаны и проведены соответствующими примерами.

Ключевые слова: линейные системы, сингулярные системы, дескриптивные системы, передаточная матрица, фундаментальная матрица.