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On some specific features of linear discrete descriptive systems

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Discrete descriptive systems are those the dynamics of which is governed by a mixture of algebraic and differential equations. In that sense, the algebraic equations represent the constraints which must be fulfilled in every moment of the system behavior. It means that a general solution of system equations has to possess the same properties. The complex nature of discrete descriptive systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control. In that sense the question of their stability deserves great attention and is tightly connected with the questions of system solution uniqueness and existence. Moreover, the question of consistent initial conditions, time series and solution in state space and phase space also deserve a great attention. Some of these questions, which do not exist when *normal systems* are treated, will be the subject of discussion in the sequel. These specific features of discrete descriptive systems can explain some of their unusual behaviors in transient responses. Some numerical examples have been worked out to illustrate the applicability of results presented.

Key words: **linear systems, discrete descriptive systems, existence and uniqueness of solution, time series analysis, consistent initial conditions, discrete fundamental matrix.**

Introduction

DISCRETE descriptive systems are those the dynamics of which is governed by a mixture of algebraic and differential equations, which disables one to make their representation in the state space in the classical form of the vector differential state equation. As the consequence of this fact one cannot use typical tools for solving system equations as in the case when *normal systems* are treated.

In that sense, the algebraic equations represent the constraints to the solution of differential equations which they have to fulfill in any moment.

The complex nature of discrete descriptive systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control. In that sense the question of their stability deserves great attention and is tightly connected with the questions of system solution uniqueness and existence. Moreover, the question of consistent initial conditions, time series and solution in state space and phase space based on discrete fundamental matrices also deserve a great attention. Some of these questions do not exist when *normal systems* are treated.

The survey of updated results for generalized state space systems and a broad bibliography can be found in *Bajić*(1992) [1], *Campbell* (1980, 1982) [2,3], *Lewis* (1986, 1987) [16,17], *Debeljković et al.* (1996.a, 1996.b, 1998, 2004.a, 2004.b) [9,10,11,12,13] and in two special issues of the journal *Circuits, Systems and Signal Procesing* (1986, 1989) [5,6].

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Mathematical description of discrete descriptive systems in the state space

The general description of this class of systems in the state space is given by the following equation*)

$$\mathbf{f}\,(k, \mathrm{x}(k+1), \mathrm{x}(k),\ldots,\mathrm{x}(0),\ \mathrm{x_i}(k), \mathrm{u}(k), \mathrm{u}(k-1),\ldots, \mathrm{u}(0)) = 0 \quad (1)$$

or by

$$\mathbf{f}_k\,(\mathrm{x}\,(k+1), \mathrm{x}(k),\ldots, \mathrm{x}(0),\ \ \mathrm{x_i}(k) \\ \mathrm{u}(k), \mathrm{u}(k-1),\ldots,\mathrm{u}(0)) = 0 \quad (2)$$

$$\mathbf{x}_i(k) = \mathbf{g}_k\,(k, \mathrm{x}(k+1), \mathrm{x}(k),\ldots, \mathrm{x}(0) \\ \mathrm{u}(k), \mathrm{u}(k-1), \ldots, \mathrm{u}(0)) = 0 \quad (3)$$

where, in general, the vector functions $\mathbf{f}_k\,(\ )$ and $\mathbf{g}_k\,(\ )$, are such that

$$\boldsymbol{f}_k : \mathrm{R}\times\mathrm{R}^n\times\mathrm{R}^m \rightarrow \mathrm{R}^n$$

$$\boldsymbol{g}_k : \mathrm{R}\times\mathrm{R}^n\times\mathrm{R}^m \rightarrow \mathrm{R}^p$$

where $\mathbf{x}(k) = \mathbf{x}(kT)$ is the state vector, $\mathbf{u}(k)$ is the control vector, $\mathbf{x}_i(k)$ is the output vector, T is the period, and k is the moment of sampling.

One of possible canonical forms of the system under consideration, when the functions $\mathbf{f}_k\,(\ )$ and $\mathbf{g}_k\,(\ )$ obey linear features, is

* Usual notations are presented in **Appendix A**

$$E(k+1)\mathbf{x}(k+1) = A(k)\mathbf{x}(k) + B(k)\mathbf{u}(k) \quad (4)$$

$$\mathbf{x}_i(k) = C(k)\mathbf{x}(k) + D(k)\mathbf{u}(k) \quad (5)$$

$$E\mathbf{x}(0) = E\mathbf{x}_0 \,,\ k = 0, 1, 2, \ldots N - 1 \quad (6)$$

and corresponds to the non-stationary, non-autonomous discrete descriptor system.

Eq.(4) represents a vector state equation and eq.(5) *is the actual output vector equation of dynamical discrete descriptor systems.*

The time varying matrices $A(k)$, $B(k)$, $C(k)$, $E(k+1)$ are of appropriate dimensions with the invariant rank matrix $E(k+1)$ necessarily singular.

A particular formulation of a set of dynamic relations is provided by a set of equations of the following form, *Luenberger* (1977, 1978) [18,19]

$$E_{k+1}\mathbf{x}(k+1) = A_k\mathbf{x}(k) + \mathbf{u}(k) \quad (7)$$

$$\mathbf{x}_i(k) = C_k\mathbf{x}(k)\,, \quad k = 0, 1, 2, \ldots, N-1 \quad (8)$$

which enables one to present them in the block matrix form as

$$\begin{bmatrix} -A_0 & E_1 & & & & \\ 0 & -A_1 & E_2 & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & E_{N-1} & 0 \\ & & & 0 & -A_{N-1} & E_N \end{bmatrix} x \quad (9)$$

$$\mathbf{x}\begin{bmatrix} \mathbf{x}(0) \\ \mathbf{x}(1) \\ \vdots \\ \vdots \\ \mathbf{x}(N-1) \\ \mathbf{x}(N) \end{bmatrix} = \begin{bmatrix} \mathbf{u}(0) \\ \mathbf{u}(1) \\ \vdots \\ \vdots \\ \mathbf{u}(N-1) \end{bmatrix}$$

The block matrix form, with each block being $n \times n$, explicitly displays the fact that the set of dynamic equations can be regarded as one (*large*) system of linear equations.

In a particular case, which is most treated in the literature, matrices in the state equation are usually defined over the field of real numbers so that the vector functions $\mathbf{f}_k(\ )$ and $\mathbf{g}_k(\ )$ are linear. The simplest state space description (*matrix description*) of this class of the systems is given with

$$E\mathbf{x}(k+1) = A\mathbf{x}(k) + B\mathbf{u}(k) \quad (10)$$

$$\mathbf{x}_i(k) = C\mathbf{x}(k) \quad k = 0, 1, 2, \ldots \text{ N - 1} \quad (11)$$

This description will be treated in this paper in the sequel.

A specific feature of this class of systems is the possibility to represent them in the form of *finite time series*, *Dai* (1989) [7,8] with the time - invariant matrices of appropriate dimensions and with the matrix E necessarily singular and with rank defect**). In that case, the finite time series of input variables $\mathbf{u}(0)$, $\mathbf{u}(1)$, $\mathbf{u}(2)$, $\mathbf{u}(L)$, determine the states $\mathbf{x}(0\ )$, $\mathbf{x}(1)$, $\mathbf{x}(L)$ of the system given by eq.(10 - 11) which are completely defined and satisfy the following equation

$$\begin{bmatrix} -A & E & & & \\ & -A & E & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & -A & E \end{bmatrix}\begin{bmatrix} \mathbf{x}(0) \\ \mathbf{x}(1) \\ \vdots \\ \vdots \\ \mathbf{x}(L) \end{bmatrix} = \begin{bmatrix} B\mathbf{u}(0) \\ B\mathbf{u}(1) \\ \vdots \\ \vdots \\ B\mathbf{u}(L) \end{bmatrix} \quad (12)$$

It should be noted that the block matrix (12) has dimensions $nL \times n(L+1)$, which means that for the given finite time series of input variables there are n independent solutions, if they exist at all.

If there exists a condition or a relation, such that different solutions are determined by this relation at least in one point, then such a realation is called the *complete condition*.

Luenberger (1977) [18] has shown that only with regular linear discrete descriptor systems the *complete condition* can be chosen from sequences $\mathbf{x}(0)$, $\mathbf{x}(1)$, $\mathbf{x}(L)$, in such a wey that any state $\mathbf{x}(k)$, $0 \le k \le L$, is *uniquely* determined by this condition and input variables $\mathbf{u}(0)$, $\mathbf{u}(1)$, $\mathbf{u}(L)$.

The nature and specific features of this class of systems, which are not of particular interest for these investigations, can be found in *Debeljković et. al* (1998, 2004.b) [11,13] as well as some of their clasifications and particularities.

Solvability of linear discrete descriptive systems

The basic questions of singular system solvability are due to *Godbout* and *Jordan* (1975) and successfully solved in mathematical sense by *Campbell et. al* (1976).

Based on the descriptor discrete time model, *Luenberger* (1977, 1978) [18,19] has generated a very well known "shuffle" algorithm as a new test for investigating system equation solvability. Moreover, he gave and excellent explanation of this concept establishing its natural connection with the system conditionability as a dual concept.

For the necessities of these exposures we shall consider the system given by eq. (9).

In eq.(9) there are (N+1) unknown vectors $\mathbf{x}(k)$, $\mathbf{x}(k) \in \mathrm{R}^n$, but there are only n matrix equations (each of which is n – dimensional).

There is, therefore, an excess of unknown vectors over equations – or in terms of scalar quantities, an excess of n unknown to equations. Under standard nondegeneracy conditions, one expects that the system given by eq.(7) posseses not one but a family o n linearly independent solutions. This is formalized by the notation of solvability introduced below. Moreover, this fact makes it possible to consider such systems as systems consisting of two parts: one slow (differential system equations) and fast one (algebraic), what can ensure the recursive computation of system solution under the known input sequence***) $\mathbf{u}(k)$.

Let us denote the coefficient matrix of (9) with $F(0, N)$.

It can be regarded as an $N \times (N+1)$ block matrix or in ordinary terms as an $nN \times n(N+1)$ matrix.

The *block matrix* $F(0, N)$ is usually called the *coefficient matrix.* Now we can give the following definition, *Luenberger* (1977) [18] .

**Rank of matrix E is equal to q, $q<n$ and terminal point satisfies: $k=L\geq n$

***The systems that satisfy this condition should be called *regular systems.*

Definition 1. The linear dynamical discrete descriptive system, given by eq.(7), is said to be *s o l v a b l e* if its coefficient matrix $F(0, N)$ is of full rank.

The matrix $G(0,N)$ (expressed in the block form)

$$G(0,N)=\begin{bmatrix} E_1 & & & & \\ -A_1 & E_2 & & & \\ & -A_2 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & E_{N-1} \\ & & & & -A_{N-1} \end{bmatrix} \tag{13}$$

can be added to the set of eq.(7).

The matrix $G(0,N)$ is the submatrix of the matrix $F(0,N)$, obtained by eliminating the first n and the last n columns. It is referred to as the *condition matrix*.

Another test of solvability can be given in the following manner:

Definition 2. The linear dynamical discrete descriptive system, eq.(7), is said to be c o n d i t i o n a b l e if the matrix G(0, N) is of full rank.

Remark 1. Both *Definitions* can be applied to the time invariant linear discrete descriptor system given by eq.(10) without any limitations.

It is obvious that *solvability* and *conditionability* are *dual concepts*.

The most important result from the previous section is given in the following Theorem, *Luenberger* (1977) [18]:

Theorem 1. System given by eq.(10) is *solvable* if and only if

$$\det(A-zE)\neq 0 \tag{14}$$

It is obvious that if the condition of Theorem 1 is satisfied, the matrix pair *(E,A)* is said to be *regular*.

If the determinant of the matrix pair *(E,A)* is identically equal to zero, then the matrix pair (or the discrete descriptor system) is *irregular*.

Such systems may be without solutions, solutions may be nonunique. In the last case there may be finite and infinite numbers of solutions.

In order to put discussions into a rigorous mathematical form, some important results will be presented, in the sequel, *Campbell* (1980) [2].

Definition 3. Let the matrices $E,A\in C^{n\times n}$ and $k_0\in \mathcal{K}$. $\mathbf{x}(k_0)=\mathbf{x}\in \mathrm{R}^n$ is called *the vector of initial consistent conditions* associated with the moment k_0 if eq. (10) *has at least one solution.*

Definition 4. Eq.(10) is *tractable* if it *has a unique solution* for any consistent initial vector.

Definition 5. If linear homogenous eq.(10) is *tractable* at least in one discrete moment $k_0 \in \mathcal{K}$ then it is *tractable* in every discrete moment $k \in \mathcal{K}$. Therefore, is simply said that it is *tractable.*

Theorem 2. For the given matrices $E,A\in \mathrm{C}^{n\times n}$, the homogenous algebraic – difference equation is *tractable* if and only if there exists the scalar $z\in \mathrm{C}$, such that there exists the matrix $(z\quad E+A)^{-1}$.

When the systems of high order are treated, checking system solvability in the before mentioned way may be an extremely difficult task. To find such a complex number z which would guarantee that the inverse matrix $(zE-A)^{-1}$ exists is sometimes an impossible task.

Therefore, some authors were looking for some other approaches to solve this significant problem. Since this system feature is based on the system matrices *(E, A)* only, the following *Theorem* has a complete analogy with the similar result deduced for *continuous linear singular systems.*

Theorem 3. The following statements are equivalent, *Yip, Sincovec* (1981)

a) (A,E) is solvable if $\det\left(zE-A\right)\neq 0$

b) If X_0 is the null space of the matrix A and

$X_i=\left\{\mathbf{x}(t):A\mathbf{x}(t)\in EX_{i-1}\right\}$

then $\aleph(E)\cap X_i=\left\{\mathbf{0}\right\},\ \forall i=0,\ 1,\ 2,\ ...$

c) If $Y_0=\aleph(A^T)$ i

$X_i=\left\{\mathbf{x}(t):A\mathbf{x}(t)\in EX_{i-1}\right\}$

then $\aleph(E^T)\cap Y_i=\left\{\mathbf{0}\right\},\ \forall i=0,\ 1,\ 2,\ ...$

d) Matrix

$$G(n)=\left.\begin{bmatrix} E & 0 & \cdots & 0 \\ A & E & \cdots & 0 \\ 0 & A & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & E \\ 0 & & \cdots & A \end{bmatrix}\right\}n+1$$

has full colomn rank for $n=1,\ 2,\ ...$

e) Matrix

$$F(n)=\underbrace{\begin{bmatrix} E & A & 0 & \cdots & 0 \\ 0 & E & A & \cdots & 0 \\ 0 & 0 & E & \cdots & 0 \\ \vdots & & & & \vdots \\ \vdots & & & & E \\ 0 & & \cdots & & A \end{bmatrix}}_{n+1}$$

has full row rank for $n=1,\ 2,\ ...$

A particular approach to this matter is given by *Luenberger* (1978) [19].

This section describes the basic *shuffle* algorithm used to check *solvability* of the system under consideration.

Solvability is the property of the matrices E and A only while the matrix B plays no role in the simplified version of algorithm. The algorithm works by modifying an n x $2n$ array.

Begin with the array

$$E\quad A$$

If the E matrix is nonsingular, the procedure terminates – the system under consideration is *solvable*.

Otherwise, row operations on the whole array are performed, until it is brought to the new form

$$\begin{matrix} T & A_1 \\ 0 & A_2 \end{matrix}$$

where the matrix T is of full rank, e.g. the matrix T poseses n columns but less then n rows.

The matrices A_1 and A_2 are parts of the second side array after the row operations.

The matrix A_1 is the same size as the matrix T.

Next step is to bring the array into form

$$\begin{matrix} T & A_1 \\ 0 & A_2 \end{matrix}$$

This elementary operation (*interchange*) is called "*shuffle*".

If the $n \times n$ matrix on the left side of the array is nonsingular, the procedure terminates – the system is solvable.

The algorithm continues in this fashion, performing row operations in order to create null rows on the left side and then shuffling the coresponding rows from the right side to the left.

The algorithm terminates in one of two ways: (1) a point where the left half becomes nonsingular is reached, in which case the system is sovable, or (2) a point where there is zero row all the way across the array is reached, in which case the system is not sovable. The algorithm always terminates, one way or another, in most n steps.

A numerical example is presented to illustrate the method proposed****.

The general type of this algorithm, which includes the case when the system under consideration operates in forced regime, can be found in *Luenberger* (1978) [19] or in *Debeljković et. al* (1998) [11].

Preposition 1. If the matrix pair $(\lambda \quad E + A)$ is *regular* then

$$\aleph(E) \cap \aleph(A) = \{\mathbf{0}\} \tag{15}$$

Campbell et.al (1976).

It should be noted that this condition can not guarantee the *regularity* of the matrix pair $(\lambda \quad E + A)$ for some $\lambda \in C$.

Preposition 2. If the matrix pair $(\lambda \quad E + A)$ is *regular*, $\lambda \in C$, then

$$W_k \cap \aleph(E) = \{\mathbf{0}\} \tag{16}$$

Debeljković, Owens (1985) [21], where W_k denotes the subspace of consistent of initial conditions*****.

Here, as it is the case in *Preposition* 1, the *vice versa* need not be fulfilled.

Finally, for the linear discrete descriptive system, given in its normal canonical form

$$\mathbf{x}_1(k+1) = A_1\mathbf{x}_1(k) + A_2\mathbf{x}_2(k) \tag{17a}$$

$$\mathbf{0} = A_3\mathbf{x}_1(k) + A_4\mathbf{x}_2(k) \tag{17b}$$

where $\mathbf{x}_1(k)$ and $\mathbf{x}_2(k)$ are the state covectors, the matrices A_i, $i = 1,\ldots,4$ are defined over the field of real numbers having dimensions $n_1 \times n_1$, $n_1 \times n_2$, $n_2 \times n_1$, and $n_2 \times n_2$, respectively, *solvability condition* yields to

$$\det(sI_{n_1} - A)\det\left\{-A_4 - A_3(sI_{n_1} - A)^{-1}A_2\right\} \neq 0 \tag{18a}$$

or

$$(-1)^{n_2} \det A_4 \det\left\{(sI_{n_1} - A_1) + A_2A_4^{-1}A_3\right\} \neq 0 \tag{18b}$$

under the assumption that the matrix A_4 is invertibile.

Consistent initial conditions of linear discrete descriptor systems

In the discrete case, the concept of smoothness is almost meaningless, but the idea of consistent initial conditions that generate solution sequences $(\mathbf{x}(k) : k \geq 0)$ has a physical meaning. These initial conditions should be called consistent initial conditions. It is obvious that this problem is more complex here, in the case of discrete systems, than when the continuous systems are treated. This problem will be discussed in the sequel.

Let us consider the linear *regular* discrete descriptor system, given by its state space repesentation, eq. (10).

There are a few ways to compute the subspace of initial conditions for the linear discrete descriptor system.

Namely, from condition

$$(I - \hat{E}\hat{E}^D)\mathbf{x}_0 = \mathbf{0} \tag{19}$$

that is equivalent to

$$W_q = \aleph(I - \hat{E}\hat{E}^D) \tag{20}$$

one can determine all vectors $\mathbf{x}_0$ which span the subspace W_q. The matrix $\hat{E}$ is defined with

$$\hat{E} = (\lambda E - A)^{-1}E \tag{21}$$

where the index "D" denotes Drazin's inversion of any matrix******.

A geometric approach can be used for determining the subspace of initial consistent conditions.

The fundamental geometric tool in the characterization of the subspace of consistent initial conditions is the subsopace sequence $\bar{W}_j\,(j \geq 0)$ which can be formed in the following way

$$\bar{W}_0 = R^n \tag{22}$$

$$\bar{W}_{j+1} = (A - \lambda \quad E)^{-1}E\bar{W}_j, \quad \lambda \in C \tag{23}$$

then

$$\bar{W}_j = W_j, \quad j \geq 0 \tag{24}$$

$W_{j=q}$ being the subspace of consistent initial conditions of the linear discrete descriptive system under consideration.

Lemma 1. The subspace sequence $\{W_0,\ W_1,\ W_2,\ \ldots\}$is formed so that

$$W_0 \supset W_1 \supset W_2 \supset W_3 \supset \tag{25}$$

Moreover

$$\aleph(A) \subset W_j, \quad \forall j \geq 0 \tag{26}$$

**** See **Appendix B.**

***** See the next section

****** See **Appendix A**

and there exists an integer $k \geq 0$, such that

$$W_{k+1} = W_k \tag{27}$$

and hence

$$W_{k+j} = W_k, \ \forall j \geq 1 \tag{28}$$

If q^* is the smallest integer with this property, then

$$W_q \cap \aleph(E) = \{\mathbf{0}\}, \quad q \geq q^* \tag{29}$$

provided that $(\lambda E - A)$ is invertible for some scalar $\lambda \in R$.

For the sake of brevity, the proof is omitted here and can be found in *Owens, Debeljković* (1985) [21].

Theorem 4. Under the conditions of *Lemma* 1, $\mathbf{x}_0$ is a consistent initial condition for the autonomous system, given by eq.(10) if and only if $\mathbf{x}_0 \in W_{q^*}$. Moreover, $\mathbf{x}_0$ generates a discrete solution sequence $(\mathbf{x}(k) : k \geq 0)$, such that $\mathbf{x}(k) \in W_{q^*}$, za $\forall k \geq 0$.

Proof.

Necessity. To prove necessity, let $(\mathbf{x}(k) : k \geq 0)$ be a solution sequence and let $j \geq 1$ be arbitrary.

Clearly

$$E\mathbf{x}(j) = A\mathbf{x}(j-1) \tag{30}$$

with

$$\mathbf{x}(j) \in W_0 \tag{31}$$

and hence

$$\mathbf{x}(j-1) \in W_1 \tag{32}$$

If

$$\mathbf{x}(j-l) \in W_1 \tag{33}$$

then

$$E\mathbf{x}(j-l) = A\mathbf{x}(j-l-1) \tag{34}$$

so that

$$\mathbf{x}(j-l-1) \in A^{-1}EW_l = W_{l+1} \tag{35}$$

and induction proves that

$$\mathbf{x}(0) \in W_j \tag{36}$$

But *j* is arbitrary, so that

$$\mathbf{x}(0) \in W_{q^*} \tag{37}$$

for all $j \geq 0$.

Sufficiency. To prove *sufficiency* let us adopt the assumption $\mathbf{x}(0) = \mathbf{x}_0 \in W_{q^*}$, and note that

$$A\overline{W}_{q^*} \in E\overline{W}_{q^*} \tag{38}$$

If $\overline{W}$ is a basis matrix for W_{q^*}, one can write

$$A\overline{W} = E\overline{W}\Lambda \tag{39}$$

for a square matrix Λ, of dimensions equal to the dimension of the subspace W_{q^*}.

Now let us write $\mathbf{x}(0) : \mathbf{x}_0 = \overline{W} \ \ \mathbf{z}_0$ and solve equation

$$\mathbf{z}(k+1) = \Lambda \ \ \mathbf{z}(k), \qquad \mathbf{z}(0) = \mathbf{z}_0 \tag{40}$$

The vector function

$$\mathbf{x}(k) = \overline{W} \ \ \mathbf{z}(k) \in W_{q^*}, \qquad k \geq 0 \tag{41}$$

is real, analytic and satisfies the initial condition $\mathbf{x}(0) = \mathbf{x}_0$.

It is in fact the uniqe solution of eq.(10), since

$$\begin{aligned} E\mathbf{x}(k+1) - A\mathbf{x}(k) &= E\overline{W} \ \ \mathbf{z}(k+1) - A\overline{W} \ \ \mathbf{z}(k) \\ &= E\overline{W}(\mathbf{z}(k+1) - \Lambda \ \ \ \mathbf{z}(k)) = \mathbf{0} \end{aligned} \tag{42}$$

which ends the *proof of sufficiency*.

Linear discrete descriptor system state space response

Free operating regime

Theorem 5. Let eq. (10) be tractable.

Then the general solution of the autonomous system, given by eq. (10), with B = 0 is determined by

$$\mathbf{x}(k) = \begin{cases} \hat{E}\hat{E}^D \mathbf{x}_0, & \text{if } k = 0 \\ (\hat{E}^D \hat{A})^k \mathbf{x}_0, & \text{if } \ k \geq 1 \end{cases} \tag{43}$$

where

$$\begin{aligned} \hat{E} = (zE - A)^{-1}E, \qquad \hat{A} = (zE - A)^{-1}A \\ \exists z \ni \det(zE - A) \neq 0 \end{aligned} \tag{44}$$

The vector $\mathbf{x}_0 \in \mathrm{R}^n$ is a vector of initial consistent conditions for the given homogenous equation if and only if it satisfies

$$\mathbf{x}_0 = \hat{E}\hat{E}^D \mathbf{x}_0 \tag{45}$$

or, in equivalent notation

$$\mathbf{x}_0 \in \Re(\hat{E}^p) = \Re(\hat{E}\hat{E}^D) \tag{46}$$

so the solution of the autonomous eq.(10), incorporating the before mentioned vector of initial consistent condition, is given with

$$\mathbf{x}(k) = (\hat{E}^D \hat{A})^k \hat{E}\hat{E}^D \mathbf{x}(0)', \ \ \forall k \geq 1 \tag{47}$$

Proof. Rigorous proof of this *Theorem* needs basic recapitulation of some previous results.

Using eq.(44), the basic system is transformed to

$$\hat{E}\mathbf{x}(k+1) = A\mathbf{x}(k) \tag{48}$$

and using the linear nonsingular transformation of the state vector

$$\mathbf{x}(k) = T\mathbf{y}(k), \quad \det T \neq 0 \tag{49}$$

eq.(49) is reduced to

$$T^{-1}\hat{E}T\mathbf{y}(k+1) = T^{-1}\hat{A}T\mathbf{y}(k) \tag{50}$$

or to

$$\begin{bmatrix} \hat{Q}_0 & 0 \\ 0 & \hat{N} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1(k+1) \\ \mathbf{y}_2(k+1) \end{bmatrix} = \begin{bmatrix} z\hat{Q}_0 - I & 0 \\ 0 & z\hat{N} - I \end{bmatrix} \begin{bmatrix} \mathbf{y}_1(k) \\ \mathbf{y}_2(k) \end{bmatrix} \quad (51)$$

since it is obvious that

$$T^{-1}\hat{E}T = \begin{bmatrix} \hat{Q}_0 & 0 \\ 0 & \hat{N} \end{bmatrix}, \ \hat{E} = T \begin{bmatrix} \hat{Q}_0 & 0 \\ 0 & \hat{N} \end{bmatrix} T^{-1} \quad (52)$$

$$\det Q_0 \neq 0, \ N^{\nu} = 0, \ \nu = Ind(N) \quad (53)$$

$$\hat{A} = z\hat{E} - I \quad (54)$$

$$T^{-1}\hat{A}T = T^{-1}(z\hat{E} - I)T = \begin{bmatrix} z\hat{Q}_0 - I & 0 \\ 0 & z\hat{N} - I \end{bmatrix} \quad (55)$$

$$\hat{A} = (z\hat{E} - I) = T \begin{bmatrix} z\hat{Q}_0 - I & 0 \\ 0 & z\hat{N} - I \end{bmatrix} T^{-1} \quad (56)$$

as well as facts that

$$\hat{E} = T \begin{bmatrix} \hat{Q}_0 & 0 \\ 0 & N \end{bmatrix} T^{-1} \ \Rightarrow \quad \hat{E}^D = E^T \begin{bmatrix} \hat{Q}_0^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1} \quad (57)$$

The solution of eq. (51) is given by

$$\mathbf{y}_1(k) = \left(\hat{Q}_0^{-1}(z\hat{Q}_0 - I)\right)^k \mathbf{d}_1 \quad (58)$$

$$\mathbf{y}_2(k) = 0, \quad \mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} = \text{const.} \quad (59)$$

Eqs.(58) and (59) can be shown as a matrix representation, so after returning to the primary state variable, one can get

$$\begin{aligned} \mathbf{x}(k) = T\mathbf{y}(k) &= T \begin{bmatrix} \left(\hat{Q}_0^{-1}(z\hat{Q}_0 - I)\right)^k & 0 \\ 0 & 0 \end{bmatrix} \mathbf{d} \\ &= T \begin{bmatrix} \left(\hat{Q}_0^{-1}T^{-1}T(z\hat{Q}_0 - I)\right)^k & 0 \\ 0 & 0 \end{bmatrix} T^{-1}T \cdot \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}T \cdot \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} \\ &= (\hat{E}^D\hat{A})^k \cdot T\mathbf{d} \end{aligned} \quad (60)$$

Eqs.(55) and (57) have been used in forming the last expression.

Since the vector **d,** in eqs.(58) and (59), has been chosen arbitrarily, the vector **d** should be chosen from the subspace of initial consistent condition for the final solution.

Then

$$\mathbf{x}_0 = T\mathbf{d} = T \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} = \hat{E}\hat{E}^D\mathbf{x}_0 \quad (61)$$

which together with eq. (60) finally leads to eq.(43).

Forced operating regime

Let us consider the linear discrete descriptor system, operating in forced regime

$$E\mathbf{x}(k+1) = A\mathbf{x}(k) + \mathbf{u}(k) \quad (62)$$

The absence of the matrix B is not crucial for this discussion.

Let us also introduce the following notations

$$\hat{\mathbf{u}}(k) = (zE - A)^{-1}\mathbf{u}(k), \ p = Ind(E) \quad (63)$$

Theorem 6. Let us suppose that eq.(63) is *tractable*.

The solution of eq.(62), for $k \geq 1$, is given by

$$\begin{aligned} \mathbf{x}(k) &= \mathbf{x}_{\text{hom}}(k) + \mathbf{x}_{part}(k) \\ &= (\hat{E}^D\hat{A})^k \hat{E}\hat{E}^D\mathbf{x}_0 \\ &+ \hat{E}^D \sum_{i=0}^{k-1} (\hat{E}^D\hat{A})^{k-i-1}\hat{\mathbf{u}}(k) \\ &- (I - \hat{E}\hat{E}^D)\sum_{i=0}^{p-1} (\hat{E}\hat{A}^D)^i \hat{A}^D\hat{\mathbf{u}}(k+i) \end{aligned} \quad (64)$$

It can be shown, that this solution is independent of choice z.

Let

$$\hat{\mathbf{w}} = -\left(I - \hat{E}\hat{E}^D\right)\sum_{i=0}^{p-1}\left(\hat{E}\hat{A}^D\right)^i \hat{A}^D\hat{\mathbf{u}}(i) \quad (65)$$

The vector of initial state is consistent, if and only if

$$\mathbf{x}_0 \in \left[\hat{\mathbf{w}} \ + \ \Re(\hat{E}^k)\right] \quad (66)$$

As seen, the vector initial consistent conditions need not be the same for the system operating in free and forced regime. We have the same situation when the continuous singular systems are treated.

Proof. It is enough to show that particular solution has the same form as it is given by eq.(64).

So, let us suppose that

$$\mathbf{x}_1(k) = \hat{E}^D \sum_{i=0}^{k-1}\left(\hat{E}^D\hat{A}\right)^{k-i-1}\hat{\mathbf{u}}(k) \quad (67)$$

$$\mathbf{x}_2(k) = -\left(I - \hat{E}\hat{E}^D\right)\sum_{i=0}^{p-1}\left(\hat{E}\hat{A}^D\right)^i \hat{A}^D\hat{\mathbf{u}}(k+i) \quad (68)$$

We need to show that

$$\hat{E}\mathbf{x}_1(k+1) = \hat{A}\mathbf{x}_1(k) + \hat{E}\hat{E}^D\hat{\mathbf{u}}(k) \quad (69)$$

$$\hat{E}\mathbf{x}_2(k+1) = \hat{A}\mathbf{x}_2(k) + (I - \hat{E}\hat{E}^D)\hat{\mathbf{u}}(k) \quad (70)$$

In the case of eq.(69) one can start from expression

$$\begin{aligned} \hat{E}\mathbf{x}_1(k+1) &= \hat{E}\left(\hat{E}^D\sum_{i=0}^{k}\left(\hat{E}^D\hat{A}\right)^{k-i}\hat{\mathbf{u}}(i)\right) = \\ &= \hat{E}\hat{E}^D\left(-\sum_{i=0}^{k}\left(\hat{E}^D\hat{A}\right)^{k-i-1}\left(\hat{E}^D\hat{A}\right)\hat{\mathbf{u}}(i)\right) = \\ &= \hat{E}\hat{E}^D\left(-\sum_{i=0}^{k-1}\left(\hat{E}^D\hat{A}\right)^{k-i-1}\left(\hat{E}^D\hat{A}\right)\hat{\mathbf{u}}(i) + \left(\hat{E}^D\hat{A}\right)^{-1}\left(\hat{E}^D\hat{A}\right)\hat{\mathbf{u}}(k)\right) \\ &= \hat{E}\hat{E}^D\left(\hat{E}^D\hat{A}\right)\sum_{i=0}^{k-1}\left(\hat{E}^D\hat{A}\right)^{k-i-1}\hat{\mathbf{u}}(i) + \hat{E}\hat{E}^D\hat{\mathbf{u}}(k) = \\ &= \hat{E}\hat{E}^D\hat{A}\mathbf{x}_1(k) + \hat{E}\hat{E}^D\hat{\mathbf{u}}(k) = \\ &= \hat{A}\mathbf{x}_1(k) + \hat{E}\hat{E}^D\hat{\mathbf{u}}(k) \end{aligned} \quad (71)$$

which ends the proof in the first case.

In the case of eq.(70) one can start from expression

$$\hat{E}\mathbf{x}_2(k+1) = -(I-\hat{E}\hat{E}^D)\sum_{i=0}^{p-1}(\hat{E}\hat{A}^D)^i\hat{E}\hat{A}^D\hat{\mathbf{u}}(k+1+i)$$
$$= -(I-\hat{E}\hat{E}^D)\hat{A}\hat{A}^D\sum_{i=0}^{p-1}(\hat{E}\hat{A}^D)^{i-1}(\hat{E}\hat{A}^D)\hat{\mathbf{u}}(k+i)$$
$$= -(I-\hat{E}\hat{E}^D)\hat{A}\sum_{i=0}^{p-1}(\hat{E}\hat{A}^D)^i\hat{A}^D\hat{\mathbf{u}}(k+i)$$
$$= -(I-\hat{E}\hat{E}^D)\hat{A}\left(\sum_{i=0}^{p-1}(\hat{E}\hat{A}^D)^i\hat{A}^D\hat{\mathbf{u}}(k+i)-(\hat{E}\hat{A}^D)^0\hat{A}^D\hat{\mathbf{u}}(k)\right)$$
$$= \hat{A}\mathbf{x}_2(k)+(I-\hat{E}\hat{E}^D)\hat{A}\hat{A}^D\hat{\mathbf{u}}(k)$$
$$= \hat{A}\mathbf{x}_2(k)+(I-\hat{E}\hat{E}^D)\hat{\mathbf{u}}(k) \tag{72}$$

It is interesting to note that the solution, eq.(64), for $\mathbf{x}(k)$ *depends not only* on the (n+1) input vectors $\hat{\mathbf{u}}(0)$, $\hat{\mathbf{u}}(1)$, ... $\hat{\mathbf{u}}(k)$ *but also on* (p - 1) *future* vectors $\hat{\mathbf{u}}(k+1)$, $\hat{\mathbf{u}}(k+2)$, ... $\hat{\mathbf{u}}(k+p-1)$, which shows possible prediction effects in system dynamical behavior and introduce a need for solvability and causality discussion in the light of system physical realization*******.

Linear discrete descriptive transfer function matrix

Let us consider the linear discrete descriptive system given by eq.(10) and (11).

Applying Z - transformation to the before mentioned system, one can get

$$(zE-A)\mathbf{X}(z) = zE\mathbf{X}(0)+B\mathbf{U}(z) \tag{73}$$

$$\mathbf{X}_i(z) = C\mathbf{X}(z) \tag{74}$$

where $\mathbf{X}(z)$, $\mathbf{U}(z)$ i $\mathbf{X}_i(z)$ are corresponding Laplace transforms.

Under the assumption that system given by eqs.(10) and (11) is *regular*, from eqs.(73) and (74), under the null conditions, one can get

$$W(z) = C\left(zE-A\right)^{-1}B = C\frac{adj\left(zE-A\right)}{\det\left(zE-A\right)}B \tag{75}$$

the linear discrete descriptive *transfer function matrix,* with associate characteristic equation, as follows

$$f_E(z) = \det(zE-A) \tag{76}$$

It is well known that *irregular discrete descriptor systems* do not posses the transfer matrix function, but this still does not mean that they do not have dynamical behavior.

Then, this behavior is described in the form of input – output relations

$$R(z)\mathbf{X}(z) = Q(z)\mathbf{U}(z) \tag{77}$$

$R(z)$ and $Q(z)$ being polynomials over the complex numbers.

More facts concerning this class of system can be found in the papers of *Dziurla*, *Newcomb* (1987) [14] and *Dai* (1989a) [7].

It is well known that the transfer matrix function for the *linear discrete descriptor system* is not, in general, strictly proper.

In general case, the transfer matrix function can be represented with two addends. The first addend is usually proper and the second one corresponds to a polynomial in z.

On the other side, it is very well known from the general control theory that for the particular choice of the transfer matrix function there are numerous state space mathematical model representations, so that all quadrells $(\tilde{E},\tilde{A},\tilde{B},\tilde{C})$ are connected with the basic system representation through the nonsingular transformation matrix T in the following manner

$$\begin{aligned}\tilde{E} &= TET^{-1} & \tilde{B} &= TB\\ \tilde{A} &= TAT^{-1} & \tilde{C} &= CT^{-1}\end{aligned} \tag{78}$$

Moreover, they have the same matrix transfer function.

It is clear that for particular quadrell $(\tilde{E},\tilde{A},\tilde{B},\tilde{C})$ there is a unique transfer function matrix.

Some features of the transfer matrix function and some important questions concerning the dynamics of linear discrete descriptor systems will be discussed in the sequel.

The regular matrix pair theory shows that there are always two nonsingular matrices U and V, such that

$$\mathbf{K} = U(zE-A)\ \ V = \begin{bmatrix} zI_r-\overline{A} & 0\\ 0 & I_{n-r}-zN\end{bmatrix} \tag{79}$$

where N is a nilpotent matrix with *the nilpotency index*:

$$\begin{aligned}\nu &= Ind\quad N\\ r &= degree\quad \det(zE-A)\end{aligned} \tag{80}$$

r denotes the degree of the system characteristic polynomial.

Moreover, the matrix N possesses a special Jordan structure with all null elements on the first diagonal.

The matrix $\mathbf{K}$ is known as a *Kronecker matrix pair* (E,A) *form*.

On the other side, system given by eqs.(10) and (11) can be rewritten as

$$\begin{bmatrix} zE-A & -B\\ C & 0\end{bmatrix}\begin{bmatrix}\mathbf{X}(z)\\ \mathbf{U}(z)\end{bmatrix} = \begin{bmatrix} zE\mathbf{X}(z)\\ \mathbf{X}_i(z)\end{bmatrix} \tag{81}$$

with the *coefficient matrix* known under the name *system matrix*.

System, given by eq. (81) is *strictly system equivalent* to the system having the following system matrix

$$\begin{aligned}&\begin{bmatrix} U & 0\\ 0 & I\end{bmatrix}\begin{bmatrix} zE-A & -B\\ C & 0\end{bmatrix}\begin{bmatrix} V & 0\\ 0 & I\end{bmatrix} =\\ &= \begin{bmatrix} U(zE-A)V & -UB\\ CV & 0\end{bmatrix}\end{aligned} \tag{82}$$

Substituting eq.(79) into eq.(82) and using the following transformation

*******This cannot be found in continuous case.

$$\bar{\mathbf{X}}(z)=\begin{bmatrix}\mathbf{X}_1(z)\\ \mathbf{X}_2(z)\end{bmatrix}=V^{-1}\quad \mathbf{X}(z) \tag{83}$$

one can get' limitedly, a system equivalent to the system given by eq. (81)

$$\begin{bmatrix} zI_r-\bar{A} & 0 & -B_1\\ 0 & I_{n-r}-zN & -B_2\\ C_1 & C_2 & 0\end{bmatrix}\begin{bmatrix}\mathbf{X}_1(z)\\ \mathbf{X}_2(z)\\ \mathbf{U}(z)\end{bmatrix}= =\begin{bmatrix} z\mathbf{X}_1(z)\\ -zN\ \ \mathbf{X}_2(z)\\ \mathbf{X}_i(z)\end{bmatrix} \tag{84}$$

where

$$\bar{C}=CV=\begin{bmatrix}C_1 & C_2\end{bmatrix}\ ,\qquad \bar{B}=UB=\begin{bmatrix}B_1\\ B_2\end{bmatrix} \tag{85}$$

If one applies inverse Z transformation to eq.(82), the new result follows

$$\mathbf{x}_1(k+1)=\bar{\mathrm{A}}\mathbf{x}_1(k)+B_1\mathbf{u}(k) \tag{86}$$

$$N\mathbf{x}_2(k+1)=I\mathbf{x}_2(k)+B_2\mathbf{u}(k) \tag{87}$$

$$\mathbf{x}_i(k)=\begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}\mathbf{x}_1(k)\\ \mathbf{x}_2(k)\end{bmatrix} \tag{88}$$

The system given by eq.(86) corresponds to the *strictly proper part* of the transfer matrix function and is of the form, *Christodolou, Mertzios* (1985) [4]:

$$\bar{W}(z)=C\left(zI_{n_1}-\bar{A}\right)^{-1}B_1 \tag{89}$$

and the system given by eq.(87) corresponds to the polynomial part of the form

$$P(z)=C_2\left(zN-I_{n_2}\right)^{-1}B_2 \tag{90}$$

so the transfer matrix function can be represented in the following form

$$W(z)=\bar{W}(z)+P(z) \tag{91}$$

It is obvious that $\bar{W}(z)$ corresponds to the *slow* part of the system and the polynomial $P(z)$ to the *fast* part of the system under consideration, *Debeljković et. al* (1996.a, 1998, 2004.a, 2004.b) [9,11,12,13].

Let us remember that the transfer matrix function is *strictly proper* if the following condition is satisfied

$$\lim_{s\to\infty}W(s)\to 0 \tag{92}$$

Practical computation of the transfer matrix function is not based on using eq.(75).

The computational procedures are based on the series expansion of the resolvent matrix $(zE-A)^{-1}$.

For example the very well known *Sourian-Frame-Faddev algoritam* can be used.

Linear discrete descriptive fundamental matrix

A dynamical analysis of normal systems given in their classical representations (state and output equation) can be performed in the *free operating regime* if the system matrix A is known.

On the other side, it is very well known that equivalent analysis cannot be performed for discrete descriptive systems, since the system matrices E and A have to be subjected to some complex numerical operations such as finding *Drazin inverse* or *transforming to the adequate*, *Weierstras form*, or some other approaches which certainly can lead to the forms that are more applicable for different points of view and other dynamical analysis necessities.

However, it has been shown recently that some aspects of dynamic analysis of linear discrete descriptor systems can be performed using the basic system matrices E and A if one can define *the system fundamental matrix*.

Let us consider the *regular* linear discrete descriptive system, given by eqs.(10) and (11).

The discrete time interval is such that $k\in(0,N)$, and $\mathbf{u}(k)\neq 0$

$$\forall k=0,\ 1,\ \ldots,\ N-1$$

Laurent expansion for the regular resolvent matrix formed of the matrix pair (E, A), *Rose* (1978) [22], *about infinity*, is given by

$$(zE-A)^{-1}=z^{-1}\sum_{i=-\mu}^{\infty}\varphi_i z^{-i} \tag{93}$$

μ being the *nilpotency index* of the resolvent matrix $(zE-A)^{-1}$, and sequence φ_i which should be determined, is known under the expression "*forward fundamental matrix*".

Laurent expansion of resolvent matrix about zero vicinity is:

$$(zE-A)^{-1}=\sum_{i=-\rho}^{\infty}\psi_{-i}z^{i} \tag{94}$$

ρ being the *nilpotency index* of the resolvent matrix $(zE-A)^{-1}$, and sequence ψ_i, which is known, is known under the expression "*backward fundamental matrix*".

When "normal" systems are treated we have $E = I$ so $\varphi_i=0$ *for* $i<0$, *and* $\varphi_i=A^i$ *for* $i\geq 0$. If $E=I$ and $\det A\neq 0$ then $\psi_i=0$ *for* $i>0$, *and* $\psi_i=-A^{-i}$ *for* $i\leq 0$.

The relative fundamental matrix of the linear discrete descriptor system which, having in mind its nature and structure, may be called the *fundamental sequence* is very important from the dynamic analyzing point of view.

Some of these questions are of particular significance:

- Determination of the system state space rsponse
- Determination of the resolvent matrix
- Determination of controllable and observable forms of corresponding matrices
- Determination of the semi-state transition matrix
- Determination of Hankel`s matrix, Markov`s paremeters and Tschirnhausn`s polynomials.

For more informations one can use the original papers of *Rose* (1978) [22], *Lewis* (1986) [16], *Mertzios*, *Lewis* (1989) [4] or *Debeljković et. al* (1998, 2004.b) [11,13].

Conclusion

Some specific features of linear discrete descriptor systems have been presented and analyzed in the light of possible dynamical treatment of such a class of systems. In that sense, some questions of existence and uniqueness are discussed throughout the concepts of solvability, causality and conditionability. The initial consistent conditions that generate the state space sequence $(\mathbf{x}(k) : k \geq 0)$ are also discussed. The state space response of this system is also given both for free and forced operating regimes. The transfer matrix function and the fundamental matrix of linear discrete descriptor systems have been defined and analyzed.

A numerical example has been performed to show a detailed procedure in the investigation of these specific features of the system under consideration. The direct comparison is performed towards the *normal systems*, which do not obey in this manner.

Appendix A - Usual notations

Drazin matrix inversion

Given *n* x *n* matrix *F,* then F^D is the unique solution of the following matrix equations

$$FF^D = F^D F$$
$$F^D F F^D = F \tag{A1}$$
$$F^D F^{k+1} = F^k$$

k being the *index* of the matrix *F,* denoted with *k* = *Ind* (*F*), defined as the smallest integer such that the following condition is satisfied

$$rank \quad F^{j+1} = F^j \tag{A2}$$

$\aleph(F)$ and $\Re(F)$ denote *kernell* or the *null F space of the matrix and the range of the matrix F*, respectively, e.g.:

$$\aleph(F) = \{\mathbf{x}: F\mathbf{x} = \mathbf{0}, \forall\ \mathbf{x} \in \mathbb{R}^n\} \tag{A3}$$

$$\Re(F) = \{\mathbf{y} \in \mathbb{R}^n, \mathbf{y} = F\mathbf{x},\ \mathbf{x} \in \mathbb{R}^n\} \tag{A4}$$

with

$$\dim \aleph(F) + \dim \Re(F) = n \tag{A5}$$

Appendix B – Numerical example showing the application procedure of *shuffle algorithm*

It is necessary to test the solvability of the matrix pair (*E, A*) using the "*shuffle*" algorithm:

$$E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Starting with the *E*, *A* array below, the shuffle algorithm progresses as indicated

$$\begin{matrix} E & & A \\ \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{matrix} & \qquad & \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{matrix} \end{matrix}$$

Row operations yield

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{matrix}$$

A *"shuffle"* yields

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

More row operations yield

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 1 \end{matrix}$$

An easy way to see if the shuffle algorithm checks for solvability is to consider the determinant of $\left(sE - A\right)$.

According to the given results, solvability is equivalent to the condition that this determinant does not vanish identically.

Row operations on $\left(sE - A\right)$ mostly influence the determinant by a nonzero multiplicative constant. Thus, one may well check the determinant when *E* has a special form obtained by the first step of algorithm. The shuffle of A_2 over the other side of array is equivalent to the multiplication of the lower rows by $(-\lambda)$, and each of those multiplications multiplies the determinant by $(-\lambda)$. Thus, it is clear that the shuffle algorithm is equivalent to the transformations of the original matrix pencil to a new pencil whose determinant is original determinant multiplied by a nonzero constant and $(-\lambda^b)$ where *b* is the total number of rows shuffled. If a point where an entire row is zero is reached, the determinant is *zero*. If the point where the (new) determinant of *E* is nonsingular is reached, the determinant is then considered to be nonzero.

One of these two situations must arise within *n* steps, for every shuffle increases the degree of the determinant of the (modify) matrix pencil by one, and the maximum possible degree is *n*.

Finally, new *shuffle* yields

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

The algorithm terminates because the left side is nonsingular. Thus the system under consideration is *solvable* and the matrix pair (*E*, *A*) is *regular*.

Appendix C - Performing the dynamical analysis of linear discrete descriptor system

Let us consider the system

$$E\mathbf{x}(k+1) = A\mathbf{x}(k)$$

where

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/3 & -\sqrt{3}/3 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -\sqrt{3}/3 & \sqrt{3}/3 \\ -\sqrt{3} & 0 & 0 \\ 0 & -\sqrt{3}/3 & -\sqrt{3}/3 \end{bmatrix}$$

System is solvable, since

$$\det(zE-A) = 2z^2/3 - 2\sqrt{3}/3 \neq 0$$

Necessary matrices are given by

$$\hat{E} = \begin{bmatrix} 0 & -1/3 & 1/3 \\ -\sqrt{3}/2 & 0 & 0 \\ \sqrt{3}/2 & 0 & 0 \end{bmatrix}$$

$$\sigma\{\hat{E}\} = \{0, \quad 0.7598, \quad -0.7598\}$$

$$\hat{E}^D = \begin{bmatrix} 0 & -\sqrt{3}/3 & \sqrt{3}/3 \\ -3/2 & 0 & 0 \\ 3/2 & 0 & 0 \end{bmatrix}$$

The subspace of the consistent of initial conditions W_q can be determined in the following way.

For $z = 0$ the matrix pair (E, A) is *regular*, so it can be found that

$$\hat{E}\hat{E}^D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

and finally

$$\left(I - \hat{E}\hat{E}^D\right)\mathbf{x}_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

e.g.

$$x_2 + x_3 = 0$$

The state space response can be determined in the following way, upon *Theorem* 5. Since for $z = 0$ the matrix $(zE - A)$ possesses its inverse matrix, one can have

$$\hat{E} = (zE-A)^{-1}E, \quad \hat{A} = (zE-A)^{-1}A = -I_3$$

$$\hat{E}^D\hat{A} = -\hat{E}^D = \begin{bmatrix} 0 & \sqrt{3}/3 & -\sqrt{3}/3 \\ 3/2 & 0 & 0 \\ -3/2 & 0 & 0 \end{bmatrix}$$

so

$$\mathbf{x}(k) = (\hat{E}^D\hat{A})^k\,\hat{E}\hat{E}^D\mathbf{x}_0 = (\hat{E}^D\hat{A})^k\,\mathbf{x}_0$$

for

$$\mathbf{x}_0 \in W_q, \quad \forall k \geq 0$$

and finally

$$\mathbf{x}(k) = \begin{bmatrix} 0 & \sqrt{3}/3 & -\sqrt{3}/3 \\ 3/2 & 0 & 0 \\ -3/2 & 0 & 0 \end{bmatrix}^k \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix}, \quad k \geq 0$$

For

$$\mathbf{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ x_{03} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

the phase trajectories of the system under consideration is dispatched in Fig.C1.

In order to see the main character of the system time response more clearly, the values of state variables are connected in discrete sampling moments.

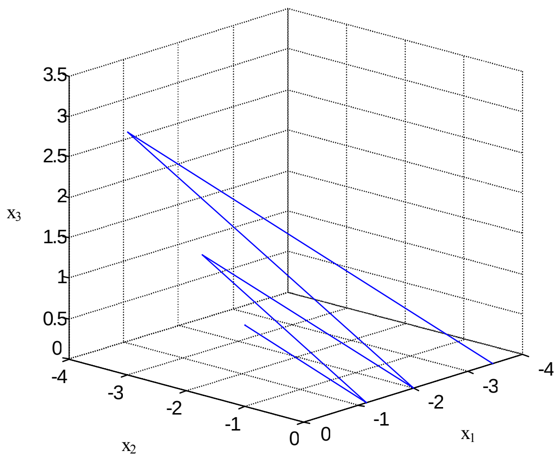

Figure C1

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O nekim specifičnim osobinama linearnih diskretnih deskriptivnih sistema

Diskretni deskriptivni sistemi predstavljeni su u matematičkom smislu kombinacijom diferencnih i algebarskih jednačina, pri čemu ove druge predstavljaju ograničenje koje opšte rešenje mora da zadovolji u svakom trenutku. Za osnovnu dinamičku analizu ove klase sistema u vremenskom domenu, potrebno je dobro poznavanje njihovih suštinskih osobina po pitanju postojanja i jedinstvenosti rešenja, konzistentnih početnih uslova i kretanja u prostoru stanja. Neka od ovih pitanja, koja sadrže niz specifičnosti koje se ne pojavljuju kod tzv. normalnih sistema, biće predmet detaljnih razmatranja u ovom radu. Nekoliko odabranih primera ilustruje prezentovane rezultate.

Ključne reči: **linearni sistemi, deskriptivni sistemi, postojanje i jedinstvenost rešenja, konzistentni početni uslovi, fundamentalna matrica deskriptivnog sistema.**

Sur quelques caractéristiques spécifiques des systèmes linéaires discrets et descriptifs

Les systèmes linéaires discrets et descriptifs sont présentés, mathématiquement, comme la combinaison des équations algébriques et celles de différence. Les équations algébriques sont la contrainte pour la solution génerale qui doit la satisfaire à chaque moment. Pour l'analyse dynamique fondamentale de cette classe de systèmes dans l'intervalle de temps fini, il est nécessaire de bien connaître leurs caractéristiques concernant l'existence et la singularité de la solution, les conditions initialles consistantes et le mouvement dans l'espace d'état. Quelques unes de cettes questions, avec leurs caractéristiques spécifiques qui n'existent pas chez les systèmes dits normaux, sont traitées en détail. Les résultats sont illustrés par quelques exemples numériques choisis.

Mots-clés: **systèmes linéaires, systèmes descriptifs, existence et singularité de la solution, conditions initialles consistantes, matrice fondamentale du système descriptif.**