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Robustness stability analysis of linear time-invariant discrete descriptive systems

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Descriptor state space systems are those the dynamics of which is governed by a mixture of algebraic and differential equations, so it is impossible to represent them in the classical form of so-called *normal state space representations*. In that sense the algebraic equations represent the constraints to the solution of the differential part. A basic dynamic analysis of these systems means the examination of their stability in the sense of Lyapunov, as well as in the sense of finite time and practical stability. Moreover, the aspect of developing explicit upper boundaries for the perturbation of such a class of systems, so that the perturbed system remains stable, has received much attention recently and is the subject of herein discussions. This significant concept is usually denoted as the concept of robustnees.

Key words: **linear discrete descriptive systems, stability in the sense of Lyapunov, stability on finite time interval, practical stabilty, robustness.**

Introduction

DESCRIPTOR state space systems are those the dynamics of which is governed by a mixture of algebraic and differential equations, so it is impossible to represent them in the classical form of so-called *normal state space representations*. In that sense the algebraic equations represent the constraints to the solution of the differential part.

The complex nature of this class of systems causes many difficulties in their analitycal and numerical treatment that do not appear in *normal* systems. In that sense, the questions of particular importance are: the existence and uniqueness of solutions, consistent initial conditions and the determination of fundamental matrix.

The survey of updated results in this area and a broad bibliograph can be found in the following references: *Bajić* (1992) [1], *Campbell* (1980, 1982) [7,8], *Lewis* (1986, 1987), [25,26] *Debeljković at el.* (1996a, 1996b, 1998), [14,15,16] *Mertzios* (1987) [27], and in two special issues of the Journal Circuit, Systems and Signal Proceesing [9,10].

As it was underlined before, examples of descriptor systems are numerous in different technical areas. In that sense, the paper of *Stevens* (1984) [33] is dedicated to mathematical modelling and missile dynamics simulation. It has been shown that the descriptor model is the direct consequence of structural limitations and military purposes of the considered missile.

In the control and system theory, it is of great importance to preserve various system properties under large perturbations of the system model. Such perturbations of the system model may be caused by changes in the manufacturing process of components, variations of constructive elements, or changes of environmental conditions. The insensitiveness of system properties is called *robustness* and it is an important field of investigation. The fact is that in many practical situations the parameters of system components are not known exactly. Usually, we only have some information on the intervals to which they belong.

Therefore, the robustness for any system property is an important theoretical and practical question.

In recent years, a considerable attention has been focused on the design of controllers for multivariable linear systems so that certain system properties are preserved under various classes of perturbations occurring in the system. Such controllers are called *robust controllers*, and the resulting system is said to be robust in some context.

Dynamic system behavior in the presence of *small perturbations* is treated within the *sensitive theory*. The theory of robustness is related to the cases when perturbations are rather significant.

For contemporary control systems, it is of particular importance to preserve not only the stability characteristics, but also the performances such as: controllability, observability, identificabilty etc. Therefore, the robustness can be assigned to any system feature.

Robustness, besides its theoretical significance has a very impressive practical meaning, since in many cases the exact values of system parameter components are not known very well, although some boundness properties of system responses may be estimated.

Roughly speaking, some definitions of robustness are essentially based on the predefined boundaries for the perturbation of initial conditions and the allowable perturbation of the system response. In the engineering applications

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of control systems, this fact becomes very important and sometimes crucial, for the purpose of characterizing in advance, in quantitative manner, possible deviations of the system response.

Thus, the analysis of these particular boundness properties of solutions is an important step, which precedes the design of control signals in all cases.

There are significant differences in applying this concept towards single input - single output systems (SISO) in comparison with multi input - multi output systems (MIMO). More detailed information regarding this problem can be found in the cited references.

Preliminary considerations

Let us consider *linear, discrete, autonomous descriptive systems*

$$E\mathbf{y}(k+1) = A\mathbf{y}(k),\ (k = k_0, k_0+1, \ldots)\quad \mathbf{y}(k_0) = \mathbf{y}_0 \tag{1}$$

$$\mathbf{x}_1(k+1) = A_1\mathbf{x}_1(k) + A_2\mathbf{x}_2(k) \tag{2a}$$

$$\mathbf{0} = A_3\mathbf{x}_1(k) + A_4\mathbf{x}_2(k) \tag{2b}$$

and nonautonomous, given with

$$E\mathbf{x}(k+1) = A\mathbf{x}(k) + \mathbf{u}(k), \quad \mathbf{x}^*(k_0) = \mathbf{x}_0^* \tag{3}$$

defined on the discrete time interval: K=$\{k\in$N: :$k_0 \le k < k_0 + k_{kon}\}$, where quantity k_{kon} may be either a positive real number or the symbol $+\infty$, so that finite time stability and practical stability can be treated simultaneously. It is obvious that $K \in R$, and $k_0 < k_{kon}$.

In eq.(1), $\mathbf{y}(k)\in\mathbf{R}^n$ is the descriptor state vector with matrices E, $A\in\mathbf{R}^{n\times n}$, the matrix E being singular obligatorily.

In eq.(2a-2b), $\mathbf{x}_1(k)\in\mathbf{R}^{n_1}$ and $\mathbf{x}_2(k)\in\mathbf{R}^{n_2}$ are co-vectors and matrices A_i, i=1,…,4 are defined over the field of real numbers, having the following dimensions: $n_1\times n_1$, $n_1\times n_2$, $n_2\times n_1$, and $n_2\times n_2$, respectively.

As the system under consideration is time invariant, it is sufficient to consider solutions $\mathbf{x}$ only as the functions of current discrete moment k and the initial state vector $\mathbf{y}_0$ or $\mathbf{x}_0$ in the moment k_0, which is completely fixed, so one can write $\mathbf{x}(k,\mathbf{x}_0)$ or in the shortened notation $\mathbf{x}(k)$. Moreover, $\mathbf{y}_0$ are $\mathbf{x}_0$ vectors belonging to the subspace of consistent initial conditions, denoted by W_q, *Debeljković et al.* (1998) [16], which generate discrete solution sequences ($\mathbf{y}(k)$ or $\mathbf{x}(k)$: :$k\ge 0$).

Let us denote the set of consistent initial conditions of eqs.(2a–2b) by φ_I . Also, let us consider the manifold $\mathsf{M}\in\mathbf{R}^n$ determined by eq.(2b) as

$$\mathsf{M} = \{\mathbf{x}(k)\in\mathbf{R}^n : \mathbf{0} = A_3\mathbf{x}_1(k) + A_4\mathbf{x}_2(k)\} \tag{4}$$

For the system given by eqs.(2a–2b), the set φ_I of consistent initial conditions is equal to the manifold M, but in general case one can write $\varphi_I \subseteq \mathsf{M}$, or in other words a vector of consistent initial conditions $\mathbf{x}_0 = [\,\mathbf{x}_{10}^T\ \ \mathbf{x}_{20}^T\,]^T$ has to satisfy

$$A_3\mathbf{x}_{10} + A_4\mathbf{x}_{20} = \mathbf{0} \tag{5}$$

or in equivalent notation

$$\mathbf{x}_0 \in \varphi_I \subseteq \mathsf{M} \equiv \aleph([A_3\ \ A_4]) \tag{6}$$

But if it can be proved that the *rank condition*, given with

$$\operatorname{rank}\,[A_3\ \ A_4] = \operatorname{rank} A_4 \tag{7}$$

is satisfied, then it is obvious that, *Bajić* (1995) [13], $\varphi_I =$ $=\mathsf{M} = \aleph([A_3\ \ A_4])$ and the determination of φ_I requires no additional computation except those necessary to convert eq.(1) into the canonical form, given in eq.(2).

The time invariant sets, $S_{(\cdot)}$ used as the bounds of system trajectories, are assumed to be *open, connected and bounded*.

In general, one may write

$$S_\rho = \left\{\mathbf{x}(k)\in\mathbf{R}^n : \|\,\mathbf{x}(k)\,\|_G^2 < \rho,\ \ \forall\mathbf{x}(k)\in W_q \setminus \{\mathbf{0}\}\right\} \tag{8}$$

or

$$S_G(\rho) = \{\mathbf{x}(k)\in\mathbf{R}^n : \|\,\mathbf{x}(k)\,\|_G^2 < \rho\} \tag{9}$$

or

$$S_l(\rho) = \{\mathbf{x}(k)\in\mathbf{R}^n : \|\,\mathbf{x}_l(k)\,\|^2 < \rho\}\ ,\ \ l = 1,2... \tag{10}$$

where G is a real, symmetric semidefinite matrix and W_q the subset of consistent initial conditions.

A survey of basic results

The results derived in the context of stability robustness are usually based on some previous, basic results in the area of system stabilization problems. Therefore, in the sequal, some of these basic results are presented to make some new results more clear and comprehensive whether we treat regular or irregular *discrete descriptive systems.* It means it is not necessary that the following equation is valid

$$\det(zE-A)\neq 0, \quad z\in C \tag{11}$$

for the system given with eq.(1).

Thus, the *regularity condition* for eq.(3) is given with

$$\det\left(cI - A_1\right)\det\left(-A_4 - A_3\left(cI - A_1\right)^{-1}A_2\right)\neq 0 \tag{12}$$

or with

$$\det A_4 \det\left(\left(cI - A_1\right) - A_2A_4^{-1}A_3\right)\neq 0 \tag{13}$$

on condition that matrix A_4 *is nonsingular.*

In the first part of the paper, the conditions which guarantee the feature of *attraction property for all or only one subset of the systems solutions,* are presented. The analysis of the attraction property of the phase origin for discrete descriptive systems is an important step preceding the design of control. The attraction property of the origin is examined through the asymptotic stability analysis [31], where the necessary and sufficient conditions for the asymptotic stability of discrete descriptor systems were established. However, the feasible method for testing these conditions has not been derived yet.

Moreover, since only the *regular discrete descriptive systems were treated,* [31] it represents serious limitations.

In this paper we examine the existence problem of solutions convergent to the origin of the system phase space for irregular discrete descriptor systems. By a suitable nonsingular linear transformation, the original system is transformed into a convenient canonical form. This form of equations enables development and easy application of

Lyapunov's direct method for the intended existence analysis of the subclasses of solutions. In this case, when the existence of such solutions is established, the estimation of *weak domain attraction of the origin* is obtained on the basis of symmetric, positively definite solutions of the reduced order *discrete Lyapunov matrix equation.* The estimated *weak domain of attraction* consists of points of the phase space which generate at *least one solution covergent to the origin.*

Since the linear transformation, given by eqs.(1) to (3), is nonsingular, the convergence of solutions $\mathbf{y}(k)$ of eq.(1) and $\mathbf{x}(k)$ of eq.(2) is an equivalent problem.

The potential (weak) domain of attraction, for the null solution of eq.(2), $\mathbf{x}(k,0)=\mathbf{0}$ $(k\in \mathsf{K})$, is defined by

$$\begin{aligned}\mathbf{D} \triangleq \{\, & \mathbf{x}_0 \in \varphi_I : \exists(\mathbf{x}(k) : k = 0,1,2,...) \\ & \textit{which satisfies}\,(2a-2b) \\ & \ni \mathbf{x}(0) = \mathbf{x}_0\,,\ \lim_{t\to\infty} \| \mathbf{x}(t,x_0) \| \to \mathbf{0} \}\end{aligned} \tag{14}$$

Debeljković et al. (1998.a) [14].

We use term *weak* because the solutions of eq.(2) need not be unique and thus for $\mathbf{x}_0\in\mathbf{D}$ there also may exist solutions nonconvergent to the origin. In our case $\mathbf{D}=\mathsf{M}=\varphi_I$ and we may think of the *weak domain* of attraction as of the weak global domain.

This fact forced us to estimate the set $\mathbf{D}_e$ of the set $\mathbf{D}(\mathbf{D}_e\subseteq\mathbf{D})$.

We will use Lyapunov's direct method to obtain the estimation $\mathbf{D}_e$ of the set $\mathbf{D}$. Our development will not require the regularity condition, eq.(11), neither eqs.(12-13), so the treatment of both regular and irregular discrete descriptor systems will be methodologically unique.

Let us first assume that the rank condition, eq.(7), is fulfilled, so the immediate consequence that $\varphi_I = \aleph([A_3\ A_4])$ is completely acceptable for the system given by eqs.(2a–2b).

Then, there is the matrix L being any solution of the matrix equation

$$0=A_3+A_4L \tag{15}$$

where 0 is the null matrix of the same dimensions as matrix A_3.

On the basis of eqs.(7) and (15) it becomes evident that whenever the solution $\mathbf{x}(k)$ fulfills

$$\mathbf{x}_2(k) = L\mathbf{x}_1(k),\ k\in \mathrm{K} \tag{16}$$

it has also to fulfill eq.(2).

Having in mind that the *rank condition* is satisfied, eq.(7), it follows directly that $\aleph([L–I_{n_2}])\subseteq \aleph([A_3\ A_4])$. To prove this fact, let us adopt an arbitrary $\mathbf{x}^*$ such that $\mathbf{x}^*\in\aleph([L–I_{n_2}])$, i.e. $\mathbf{x}_2^*=L\mathbf{x}_1^*$, where L is the matrix being any solution of eq.(15). Multiplying equation (15), from the right side, by vector $\mathbf{x}_1^*$, using also eq.(16), it is easy to show that

$$\mathbf{0} = A_3\mathbf{x}_1^* + A_4L\mathbf{x}_1^* = A_3\mathbf{x}_1^* + A_4\mathbf{x}_2^* \tag{17a}$$

which proves the fact that: $\mathbf{x}^*\in\aleph([A_3\ \ A_4])$.

So it follows

$$\aleph([L\ –I_{n_2}])\subseteq\aleph([A_3\ \ A_4]) \tag{17b}$$

Consequently, those solutions of the system given with eqs.(2a–2b) which satisfy eq.(7), have also to satisfy the constraints imposed by eq.(2b).

For all solutions of the system given with eqs.(2a–2b), for which eq.(16) is valid, the following conclusions are extremely important:

1. The solutions of eqs.(2a–2b) have to belong to the set

$$\mathbf{x}(k,\mathbf{x}_0)\in\aleph([L\ \ -I_{n_2}]) \tag{18}$$

2. If the *rank condition,* eq.(7), is satisfied, and there exist solutions $\mathbf{x}(k)$ of the system given by eqs.(2a–2b), for which it is not dificult to show that possesses the *attraction property* to the origin of phase space, and we may say that they form the so-called *potential* (*weak*) *domain of attraction*, emanating from it and given with

$$\mathbf{D}_e=\aleph([L\ \ -I_{n_2}])\}\subseteq\mathbf{D} \tag{19}$$

For the system given with eqs.(2a–2b), the *Lyapunov–like function* can be selected as

$$V(\mathbf{x}(k)) = \mathbf{x}_1^T(k)H\mathbf{x}_1(k) \tag{20}$$

where H is a real, symmetric, positive definite matrix.

The forward difference of $V(\cdot)$ along the trajectories of the system, eqs.(2a–2b), is given with

$$\begin{aligned}\Delta V(\mathbf{x}(k)) &= V(\mathbf{x}(k+1)) - V(\mathbf{x}(k)) \\ &= \mathbf{x}_1^T(k+1)H\mathbf{x}_1(k+1) - \mathbf{x}_1^T(k)H\mathbf{x}_1(k)\end{aligned} \tag{21}$$

and by using eqs.(2a) and (16), one can get

$$\begin{aligned}\Delta V(\mathbf{x}(k)) &= \mathbf{x}_1^T(k)\big((A_1 + A_2L)^T H(A_1 + A_2) - H\big)\mathbf{x}_1(k) \\ &= -\mathbf{x}_1^T(k)Z\mathbf{x}_1(k)\end{aligned} \tag{22}$$

where

$$Z = -(A_1 + A_2L)^T H(A_1 + A_2L) + H \tag{23}$$

is a real, symmetric matrix. Since H is the symmetric, positive definite matrix, then $V(\cdot)$ is a positive definite function with respect to the co-vector $\mathbf{x}_1(k)$. Thus, if Z is the positive definite matrix, then $V(\mathbf{x}(k))$ will approach the origin of phase space, when $k\to\infty$, under the assumption that solutions exist when $k\to\infty$.

Necessary connection between the matrices H and Z, which satisfy imposed demands, can be established through the *discrete matrix Ljapunov equation* of the following type

$$A_L^T HA_L - H = -Z \tag{24}$$

where

$$A_L = A_1 + A_2L \tag{25}$$

The arbitrary, real, symmetric, positive definite matrix Z and the symmetric, positive definite matrix H, may be found as a unique solution of *discrete matrix Lyapunov equation* if and only if A_L is a *discrete stable matrix,* i.e. the matrix with all eigenvalues lying within the unit circle in the complex *z plane*. It should be noted that the matrix dimension H is $n_1\times n_1$, so that eq.(24) may be treated as a *reduced order discrete Lypunov matrix equation* with respect to the dimension of the initial descriptive system.

Now we are in position to present the following result.

Theorem 1. Let the rank condition (7) be satified. Then, the estimation $\mathbf{D}_e$ for the weak domain of attraction $\mathbf{D}$ of the null solution for the given discrete descriptive system, is given with eq.(19), on condition that the matrix L is any solution of eq.(15), and $A_L=(A_1+A_2L)$ is a discrete stable (Schur) matrix. Moreover, $\mathbf{D}_e$ is not singleton.

Proof. For the sake of brevity the proof is here omitted, and can be found in *Debeljković et al.* (1998) [16].

The second part of this paper is dedicated to the retrospective of results concerning contributions of different authors in the area of finite and practical stability. To this end, primarily, we present some useful definitions necessary for the latter understanding of complex conditions which are expressed for the better understanding of basic theorems providing sufficient conditions for the proposed and adopted concept of stability.

Stability definitions

Definition 1. System (1) is *practically stable* w.r.t. $\{\mathsf{K},\alpha,\beta,G\}$ if and only if there exists $\mathbf{y}_0 \in W_q$, which satisfies the following condition

$$\| \mathbf{y}_0 \|_G^2 < \alpha \tag{26}$$

implying that

$$\| \mathbf{y}(k) \|_G^2 < \beta, \ \ \forall k \in \mathsf{K} \tag{27}$$

Debeljković, Owens (1986) [18], *Owens, Debeljković* (1986) [30].

The previous definition is applicable in the frame of only *regular discrete descriptor systems analysis**).

In order to ensure the unique treatment of both *regular* and *irregular discrete descriptor systems,* the following definitions will be introduced in the sequel, being completely analogous to those introduced earlier for continuous linear singular systems, presented in papers of *Bajić* (1995) [3], *Debeljković et al.* (1995) [12,13].

Definition 2. Solutions of the system given in eqs.(2.a–2.b) are $\{\mathsf{K},\alpha,\beta,G\}$ *bounded* if and only if $\mathbf{x}_0 \in \varphi_I \cap S_G(\alpha)$ implying that $\mathbf{x}(k,\mathbf{x}_0) \in S_G(\beta)$ for $\forall k \in \mathsf{K}$.

In the previous definition the matrix G may be used in any structural form as, for example, it is natural to take it as: $G = E^T H E$, H being some real symmetric positively definite matrix such that $\| \mathbf{x}(k) \|_G = \sqrt{\mathbf{x}^T(k) G \mathbf{x}(k)}$ represents a norm on the subspace of *consistent initial conditions* W_q.

Having in mind the *special rank properties* of matrix E, it is useful to slightly reformulate the previous definitions in the following manner.

Definition 3. Solutions of the system given in eqs.(2a–2b) are $\{\mathsf{K},\alpha,\beta_1,\beta_2\}\}$ *bounded* if and only if $\mathbf{x}_0 \in \varphi_I \cap S_1(\alpha) \cap S_2(\alpha\beta_2/\beta_1)$ implying that $\mathbf{x}(k, \mathbf{x}_0) \in S_1(\beta_1) \cap S_2(\beta_2)$ for $\forall k \in \mathsf{K}$.

Definitions 1-3 may be treated as a special case of the so-called generic concept of practical stability, given in the paper of *Bajić* (1992b) [2].

Besides the concept of practical stability, it is of particular significance to discuss the *domain of practical stability*, having in mind that discrete descriptive systems, in general, may possess one or more solutions. It is obvious that if *all solutions* emanating from the set $\varphi_I \cap S_G(\alpha)$, $\{\mathsf{K},\alpha,\beta,G\}$ are *bounded*, then the system under consideration is also $\{\mathsf{K},\alpha,\beta,G\}$ *practically stable.* The previous considerations as well as the following discussions, are mostly taken from the paper *Bajić et al.* (1998) [5].

*) **Definition 1.a** System (1) is $\{\mathsf{K},\alpha,\beta,G\}$ *practically stable* if and only if $y_0 \in W_q \cap S_G(\alpha)$ $\ \mathbf{y}(k,\mathbf{y}_0) \in S_G(\beta)$ for $\forall k \in \mathsf{K}$.

If we recall the question related to the *domain of attraction,* it is preferable to use term *weak* since it is clear that the solutions of eqs.(1) or (2) need not be unique and every chosen initial condition $\mathbf{y}(k_0)$ or $\mathbf{x}(k_0)$ may not guarantee the required property in the sense of the adopted concept of stability. However, it is possible to guarantee that for each $\mathbf{y}(k_0)$ or $\mathbf{x}(k_0)$, taken from a corresponding domain, there exists at least one solution with a specific *practical stability* characterization. We will not prove that all solutions emanating from the concerned points $\mathbf{x}(k_0)$ possess the required property.

The potential (*weak*) domain $\{\mathsf{K},\alpha,\beta,G\}$ of *practical stability* for the same system is defined in an analogous manner as follows

$$\mathsf{P} = \left\{ \begin{matrix} \mathbf{x}_0 \in \varphi_I \cap S_G(\alpha) : \left(\exists \mathbf{x}(\cdot,\mathbf{x}_0)\right), \\ (\forall k \in \mathsf{K}), \quad \mathbf{x}(k,\mathbf{x}_0) \in S_G(\beta) \end{matrix} \right\} \tag{28}$$

We can define the potential (*weak*) domain $\{\mathsf{K},\alpha,\beta_1,\beta_2\}$ of *practical stability* for the same system

$$\mathsf{A} = \left\{ \begin{matrix} \mathbf{x}_0 \in \varphi_I \cap S_1(\alpha) \cap S_2(\alpha\beta_2/\beta_1) \\ \left(\exists \mathbf{x}(\cdot,\mathbf{x}_0)\right)(\forall k \in \mathsf{K})\mathbf{x}(k,\mathbf{x}_0) \in S_1(\beta_1) \cap S_2(\beta_2) \end{matrix} \right\} \tag{29}$$

Our task is to estimate the before-mentioned potential domains. We shall use the *Lyapunov direct method* to obtain the *estimations*: $\mathsf{P}_e \subseteq \mathsf{P}$ and $\mathsf{A}_e \subseteq \mathsf{A}$.

It should be pointed out that our development will not require the regularity condition of the matrix pair (*E-A*).

Some other aspects or the irregular discrete descriptor systems were considered in *Dziurla*, *Newcomb* (1987) [21] and *Dai* (1989) [11].

For all solutions of discrete descriptor systems given with eqs.(2a–2b) for which eq.(16) is valid the following conclusions are important:

1. These solutions of eqs.(2a–2b) have to belong to the set:

$$\mathbf{x}(k_0,\mathbf{x}_0) \in \aleph\left([L \ \ -I_{n_2}]\right) \tag{30}$$

2. If the *rank condition,* eq.(7), is satisfied, then there exist solutions $\mathbf{x}(k)$ of the system given by eqs.(2a–2b), which satisfy eq.(16) and their $\{\mathsf{K},\alpha,\beta,G\}$ *bondedness* is proved, then the *potential domain* $\{\mathsf{K},\alpha,\beta,G\}$ of *practical stability* of the system given with eqs.(2a–2b), may be estimated as

$$\mathsf{P}_e = \left\{ \mathbf{x}(k) \in \mathbf{R}^n : \ \mathbf{x}(k) \in S_G(\alpha) \cap \aleph([L \ \ -I_{n_2}]) \subseteq \mathsf{P} \right. \tag{31}$$

The last fact will be proved in one of the theorems which will be presented a little bit later.

For the system given in eqs.(2a–2b), the *Lyapunov – like function candidate* can be selected as

$$V\left(\mathbf{x}(k)\right) = \mathbf{x}_1^T(k) H \mathbf{x}_1(k) \tag{32}$$

where H is assumed to be a real positive definite matrix, i.e. $H = H^T > 0$

$$\Delta V\left(\mathbf{x}(k)\right) = V\left(\mathbf{x}(k+1)\right) - \rho V\left(\mathbf{x}(k)\right) \tag{33}$$

where $\rho \in \mathbf{R}$, calculated along the solutions of eqs.(2a–2b)

$$\begin{aligned} \Delta V\left(\mathbf{x}(k)\right) &= \kappa_{11} + \kappa_{12} + \kappa_{21} + \kappa_{22} \\ &\quad - \rho \mathbf{x}_1^T(k) H \mathbf{x}_1(k) \end{aligned} \tag{34}$$

where

$$\kappa_{ij} = \mathbf{x}_i^T(k) A_i^T H A_j x_j(k), \quad (i,j) = 1,2 \tag{35}$$

Combining eqs.(16) and (34), one can get

$$\begin{aligned}\Delta V(\mathbf{x}(k)) &= \mathbf{x}_1^T(k)\left((A_1 + A_2 L)^T H (A_1 + A_2 L\right)\mathbf{x}_1(k) \\ &\quad -\rho \mathbf{x}_1^T(k) H \mathbf{x}_1(k) \\ &= \mathbf{x}_1^T(k) Z \mathbf{x}_1(k)\end{aligned} \tag{36}$$

where

$$Z = (A_1 + A_2 L)^T H (A_1 + A_1 L) - \rho H \tag{37}$$

We should notice that Z is a real symmetric matrix.

Let $\overline{\lambda}(\cdot)$ and $\overline{\Lambda}(\cdot)$ denote the maximum and the minimum eigenvalue of the real symmetric matrix $(\cdot)$.

Stability theorems

Theorem 2. Let the *rank condition,* eq.(7), be satisfied. Let H be a real, symmetric, positive definite matrix. If L is any real matrix that satisfies eq.(15), then the system governed by eqs.(2a–2b) has solutions which are $\{\mathrm{K},\alpha,\beta_1,\beta_2\}$ *bounded*, with $\alpha \leq \beta_1$, if the following conditions are satisfied:

i) The matrix Z defined in eq.(37) is a negative semidefinite matrix

ii) $$\rho^k \Lambda(H)\cdot \alpha / \lambda(H) < \beta_1, \quad \forall k \in \mathrm{K} \tag{38}$$

iii) $$\| L \|^2 \leq \beta_2 / \beta_1 \tag{39}$$

Theorem 3. Let all conditions of *Theorem* **2** be satisfied. Then the estimation A_{e} of the potential domain $\mathsf{A}\{\mathrm{K},\alpha,\beta_1,\beta_2\}$ of practical stability for the system governed by eqs. (2a–2b), is determined with

$$\mathsf{A}_{\mathsf{e}} = \aleph([L \quad -I_{n_2}]) \cap S_1(\alpha) \cap S_2(\alpha\beta_2/\beta_1) \tag{40}$$

where the set A is given with eq.(29).

For the sake of brevity, the proofs of previous theorems are omitted here and can be found in the paper of *Debeljković et al.* (1998) [16].

Main results: *Robustness of practical stability*

For the needs of Lyapunov and non–Lyapunov stability robustness treatment, let discrete descriptor system be described by the perturbed differential equation:

$$E\mathbf{y}(k+1) = A\mathbf{y}(k) + A_p\,\mathbf{y}(k), \quad \mathbf{y}(k_0) = \mathbf{y}_0 \tag{41}$$

where A_p is a matrix representing perturbations in the system model.

As shown earlier, the basic system governed by eq.(41) may be transformed into its normal canonical form achieved by the usual linear nonsingular transformation, as follows

$$\mathbf{x}_1(k+1) = \left(A_1 + A_p^1\right)\mathbf{x}_1(k) + \left(A_2 + A_p^2\right)\mathbf{x}_2(k) \tag{42}$$

$$\mathbf{0} = A_3 \mathbf{x}_1(k) + A_4 \mathbf{x}_2(k) + A_p^{34}\mathbf{x}(k) \tag{43}$$

where $\mathbf{x}(k) = [\mathbf{x}_1^T(k) \quad \mathbf{x}_2^T(k)]^T \in \mathbf{R}^n$ need not represent the original variables of the system $\mathbf{y}(k) \in \mathbf{R}^n$ governed by eq.(41). State co-vectors are given with $\mathbf{x}_1(k) \in \mathbf{R}^{n_1}$ and $\mathbf{x}_2(k) \in \mathbf{R}^{n_2}$ with $n = n_1 + n_2$.

The given matrices have the following dimensions:

$$A_1, A_p^1 \in \mathbf{R}^{n_1\times n_1}, \quad A_2, A_p^2 \in \mathbf{R}^{n_1\times n_2}$$

$$A_3 \in \mathbf{R}^{n_2\times n_1}, \; A_4 \in \mathbf{R}^{n_2\times n_2}, \; A_p^{34} \in \mathbf{R}^{n\times n}$$

In order to simplify the formulation of the stability robustness results, we introduce the following assumption.

Assumption 1. Matrix A_p^{34} in eq.(43) is a null matrix.

The discussions presented in the sequel will be dedicated to the problem of robustness attractivity property of the phase space origin with respect to the solutions of the system governed by eqs.(42-43) *in the presence of unstructural perturbations*, *Debeljković et al.* (1998.a) [17].

To perform an analysis of robustness for the system, eqs. (42–43), we employ the Lyapunov function, defined by eq.(32).

Let the rank condition, eq.(7) be satisfied.

Then, by taking into account eqs.(16) and (22), the expression for the latter difference $\Delta V(\mathbf{x}(k))$ for the system given by eqs.(42-43), becomes

$$\begin{aligned}&\Delta V(\mathbf{x}(k)) = \\ &= \mathbf{x}_1^T(k)\left((F_1 + F_2 L)^T H (F_1 + F_2 L)\right)\mathbf{x}_1(k) \\ &- \mathbf{x}_1^T(k) H \mathbf{x}_1(k) = \mathbf{x}_1^T(k)\, Z_p\, \mathbf{x}_1(k)\end{aligned} \tag{44}$$

where

$$Z_p = (F_1 + F_2 L)^T H (F_1 + F_2 L) - H \tag{45}$$

$$F_i = A_i + A_p^i, \; (i=1,2) \tag{46}$$

with Z_p being a real and symmetric matrix.

Now we are in position to state the following result related to the unstructural perturbations present in the system and governed by eqs.(42-43).

Theorem 4. Let the *rank condition,* eq.(7), be satistied, as well as *Assumption* 1 and all conditions of *Theorem* 1.

Furthermore, let Z and H be two real, symmetric and positive definite matrices which satisfy the discrete Lyapunov matrix eq.(24), with the matrix A_L determined by eq.(25).

Then the estimation $\mathbf{D}_{\mathrm{e}}$ of the potential (weak) domain of attraction for the system governed by eqs.(42-43) is determined by eq.(19), if the following inequality is satisfied

$$\sigma_M(A_{PL}) < -\sigma_M(A_L) + \sqrt{\sigma_M^2(A_L) + \frac{\lambda(Z)}{\Lambda(H)}} \tag{47}$$

where

$$A_{PL} + A_p^1 + A_p^2 L \tag{48}$$

$\sigma_M(\cdot)$ being the maximum singular value of the matrix $(\cdot)$.

$\lambda(\cdot)$ and $\Lambda(\cdot)$ denote the minimum and maximum eigenvalues of any symmetric matrix under the consideration.

It is important to underline that the set $\mathbf{D}_e$ is not singleton.

Proof. The proof is based on the results presented in *Theorem* 1. The only difference is that we use expression for the latter difference $\Delta V(\mathbf{x}(k))$ given by eq.(44) instead of eq.(22) for the same system governed by eqs.(42-43). But the evident relationship between these equations established the following new significant connection, given with

$$\begin{aligned} Z_p &= (F_1+F_2L)^T H(F_1+F_2L)-H \\ &= (A_1+A_2L)^T H(A_1+A_2L) \\ &\quad +(A_p^1+A_p^2L)^T H(A_p^1+A_p^2L) \\ &\quad +(A_1+A_2L)^T H(A_p^1+A_p^2L) \\ &\quad +(A_p^1+A_p^2L)^T H(A_1+A_2L)-H \\ &= -Z+A_{PL}^T H A_{PL}+A_L^T H A_{PL}+A_{PL}^T H A_L \end{aligned} \tag{49}$$

so that

$$\begin{aligned} \mathbf{x}_1^T(k) Z_p \mathbf{x}_1(k) &= -\mathbf{x}_1^T(k) Z\mathbf{x}_1(k)+ \\ &+\mathbf{x}_1^T(k) A_{PL}^T H A_{PL}\mathbf{x}_1(k)+ \\ &+2\mathbf{x}_1^T(k) A_{PL}^T H A_L\mathbf{x}_1(k) \end{aligned} \tag{50}$$

but

$$\begin{aligned} &\mathbf{x}_1^T(k) A_{PL}^T H A_{PL}\,\mathbf{x}_1(k)+ \\ &\quad +2\,\mathbf{x}_1^T(k)\,A_{PL}^T H A_L\,\mathbf{x}_1(k) \le \\ &\quad \le \Lambda(H)\| A_{PL}\mathbf{x}_1(k)\|^2 \\ &\quad +2\Lambda(H)\| A_{PL}\,\mathbf{x}_1(k)\|\cdot\| A_L\,\mathbf{x}_1(k)\| \\ &\quad \le \Lambda(H)\,\sigma_M^2(A_{PL})\,\|\mathbf{x}_1(k)\|^2 \\ &+2\Lambda(H)\,\sigma_M(A_{PL})\|\mathbf{x}_1(k)\|\,\sigma_M(A_L)\|\mathbf{x}_1(k)\| \\ &\quad \le \Lambda(H)(\sigma_M^2(A_{PL}) \\ &+2\,\sigma_M(A_{PL})\,\sigma_M(A_L))\|\mathbf{x}_1(k)\|^2 \end{aligned} \tag{51}$$

and also

$$\lambda(Z)\|\mathbf{x}_1(k)\|^2 \le \mathbf{x}_1^T(k) Z\mathbf{x}_1(k) \tag{52}$$

so, combining eqs.(47-52), one can get that the matrix Z_p is negative definite under the following condition

$$\Lambda(H)(\sigma_M^2(A_{PL})+2\,\sigma_M(A_{PL})\,\sigma_M(A_L))<\lambda(Z) \tag{53}$$

The preceding equation is always fulfilled if the condition imposed by eq.(47) is satisfied, implying that the negative definiteness of matrix Z_p, and consequently the same property of the latter difference of function $\Delta V(\mathbf{x}(k))$.

The rest of the proof is identical to that carried out in the corresponding part of the proof in *Theorem*1, so it is omitted here.

In order to achieve a corresponding robustness stability consideration in the presence of *structural perturbations* for the system governed by eqs.(42–43), the following assumption should be introduced.

Assumption 2. Let

$$A_{PL}+A_p^1+A_p^2L=\left[a_{Pij} : i,j=1,2,\dots n_1\right] \tag{54}$$

where the matrix L is any solution of eq.(15).

Some constraints should be, also, imposed on the elements of matrix A_{PL} such as

$$|\,a_{Pij}\,| \le \pi_{ij} \tag{55}$$

where π_{ij} are known constants with particular values.

Theorem 5. Let the *rank condition,* eq.(7), be satisfied, as well as *Assumption* 1 and 2, and all conditions of *Theorem* 1. Furthermore, let Z and H be two real, symmetric and positive definite matrices which satisfy the discrete Lyapunov matrix equation (24), with the matrix A_L determined by eq.(25).

Let

$$\pi = \max_{1\le i,j\le n_1} \pi_{ij} \tag{56}$$

Then the estimation $\mathbf{D}_e$ of the potential (waek) domain of attraction for the system governed by eqs.(42-43) is determined by equation (19), if the following inequality is satisfied

$$\pi < \frac{1}{n_1}\left(-\sigma_M(A_L)+\sqrt{\sigma_M^2(A_L)+\frac{\lambda(Z)}{\Lambda(H)}}\right) \tag{57}$$

Moreover, set $\mathbf{D}_e$ is not singleton.

Proof. The proof is based on the paper of *Kolla et al.* (1989) [24], where the following relations were exactly proved:

$$\sigma_M(\Pi_1)=n_1\pi \text{ for the matrix } \Pi_1$$

with all identical elements being π and

$$\sigma_M(\Pi) \le \sigma_M(\Pi_1), \text{ since } \pi_{ij} \le \pi$$

The rest of the proof is identical to that presented in *Theorem* 4 and omitted here for the sake of brevity.

Now we are in position to present another result which represents a significant contribution to the analyzing stability robustness problem in the context of *practical stability* of the system governed by eqs.(42,43).

In order to achieve simplified and condensed results we introduce the following assumption:

Assumption 3. The following conditions are valid for the given matrices

$$\|A_p^1\| \le \varepsilon_1,\ \|A_p^2\| \le \varepsilon_2,\ \|L\| \le \varepsilon_3 \tag{58}$$

where ε_i =1,2,3, are real, positive numbers.

In order to carry out an appropriate analysis of robustness properties for this class of system, we adopt Lyapunov approach and the corresponding agregation function, given by eq.(32). Suppose that the *rank condition,* eq.(7), is satisfied.

Then, taking into consideration eq.(16), as well as eq.(36), the expression for the latter difference $\Delta V(\mathbf{x}(k))$, eqs.(42-43), becomes:

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= \mathbf{x}_1^T(k)\Big((F_1+F_2L)^T H(F_1+F_2L)\Big)\mathbf{x}_1(k)- \\ &-\rho\mathbf{x}_1^T(k) H\mathbf{x}_1(k)=\mathbf{x}_1^T(k) Z_{p\rho}\mathbf{x}_1(k) \end{aligned} \tag{59}$$

where

$$Z_{p\rho}=(F_1+F_2L)^T H(F_1+F_2L)-\rho H \tag{60}$$

$$F_i = A_i + A_p^i,\quad (i=1,2) \tag{61}$$

where $Z_{p\rho}$ is a real symmetric matrix.

The previous results enable us to promote the next significant contribution.

Theorem 6. Let the *rank condition,* eq.(7), be satisfied, as well as *Assumption* 1 and 3. Furthermore, let H be a real and symmetric matrix. If A_L is any matrix satisfying eq.(15), then the system governed by eqs.(42-43), has solutions which are $\{\mathrm{K},\alpha,\beta_1,\beta_2\}$ *practically stable* and the estimation A_e of the potential (weak) domain of $\{\mathrm{K},\alpha,\beta_1,\beta_2\}$ *practical stability* can be determined on the basis of eq.(40), if the following conditions are satisfied:

(i)
$$\begin{aligned}&\Lambda(Z)+\Lambda(H)(\varepsilon_1+\varepsilon_2\varepsilon_3)^2+\\&+2\Lambda\left(\Omega_H^T\Omega_H\right)(\varepsilon_1+\varepsilon_2\varepsilon_3)\leq 0\end{aligned} \tag{62}$$

(ii)
$$\rho^k\Lambda(H)\alpha/\lambda(H)<\beta_1,\quad \forall k\in\mathrm{K} \tag{63}$$

(iii)
$$\|L\|^2\leq\beta_1/\beta_2 \tag{64}$$

the matrix Z being determined by eq.(37), and the matrix Ω_H defined as

$$\Omega_H=H(A_1+A_2L) \tag{65}$$

Proof. The proof is based on the proof of *Theorem 2.*

The only difference is in the used expression for the latter difference $\Delta V(\mathbf{x}(k))$ along the trajectories of system.

On the basis of eqs.(59-61), one can get

$$\begin{aligned}Z_{p\rho}&=(F_1+F_2L)^TH(F_1+F_2L)-\rho H\\&=(A_1+A_2L)^TH(A_1+A_2L)+\\&+\left(A_p^1+A_p^2L\right)^TH\left(A_p^1+A_p^2L\right)+\\&+\left(A_p^1+A_p^2L\right)^TH(A_1+A_2L)-\rho H\end{aligned} \tag{66}$$

and using the condition (i) of this theorem, one can finally get

$$\begin{aligned}&\mathbf{x}_1^T(k)Z_{p\rho}\mathbf{x}_1(k)=\\&\mathbf{x}_1^T(k)\left((A_1+A_2L)^TH(A_1+A_2L)-\rho H\right)\mathbf{x}_1(k)\\&+\mathbf{x}_1^T(k)\left(A_p^1+A_p^2L\right)^TH\left(A_p^1+A_p^2L\right)\mathbf{x}_1(k)\\&+2\mathbf{x}_1^T(k)\left(A_p^1+A_p^2L\right)^TH(A_1+A_2L)\mathbf{x}_1(k)\\&=\mathbf{x}_1^T(k)Z\mathbf{x}_1(k)\\&\mathbf{x}_1^T(k)\left(A_p^1+A_p^2L\right)^TH\left(A_p^1+A_p^2L\right)\mathbf{x}_1(k)\\&2\mathbf{x}_1^T(k)\left(A_p^1+A_p^2L\right)^T\Omega_H\mathbf{x}_1(k)\leq\\&\leq\Lambda(Z)\|\mathbf{x}_1(k)\|^2\\&+\Lambda(H)\left\|A_p^1+A_p^2\right\|^2\cdot\|\mathbf{x}_1(k)\|^2\\&+2\Lambda\left(\Omega_H^T\Omega_H\right)\left\|A_p^1+A_p^2\right\|\|\mathbf{x}_1(k)\|^2\leq\\&\leq\Lambda(Z)\|\mathbf{x}_1(k)\|^2\\&+\Lambda(H)(\varepsilon_1+\varepsilon_2\varepsilon_3)^2\|\mathbf{x}_1(k)\|^2\\&+2\Lambda\left(\Omega_H^T\Omega_H\right)(\varepsilon_1+\varepsilon_2\varepsilon_3)^2\|\mathbf{x}_1(k)\|^2\\&\leq(\Lambda(Z)+\Lambda(H)(\varepsilon_1+\varepsilon_2\varepsilon_3)^2\\&+2\Lambda\left(\Omega_H^T\Omega_H\right)(\varepsilon_1+\varepsilon_2\varepsilon_3)^2)\|\mathbf{x}_1(k)\|^2\leq 0\end{aligned} \tag{67}$$

On the basis of the last inequality and other used relations, one can finally get

$$V(\mathbf{x}(k+1))\leq\rho V\mathbf{x}(k) \tag{68}$$

so the rest of the proof is identical to that carried out in the proof of *Theorem* 2.

The estimation A_e of the *domain of practical stability* is given by eq.(40), and consequently follows from the proof of *Theorem* 2 and indisputable fact that every initial condition with the property $\mathbf{x}_0\in\aleph([L-I_{n_2}])\subseteq\varphi_I$ generates at least one solution which is $\{\mathrm{K},\alpha,\beta_1,\beta_2\}$ *practically stable*.

This *Theorem* gives only the sufficient conditions which guarantee $\{\mathrm{K},\alpha,\beta_1,\beta_2\}$ *practical stability robustness* of the system under consideration and the *robustness of estimated* A_e potential (weak) domain $\{\mathrm{K},\alpha,\beta_1,\beta_2\}$ of *practical stability*. The maximum perturbations of model matrices presented in eqs.(42-43) are determined by *Assumption* 3 and constraints expressed directly by *Theorem* 6.

It is necessary to admit that the same expressions have been derived for the estimation of potential (*weak*) domains of practical stability in two cases: for the perturbed and nominal system. This is obviously the consequence of the adopted *Assumptions* 1 and 2.

Conclusion

On the basis of theoretical explanations, some important features of this particular class of systems have been exposed.

In the first part of the paper it has been shown that the use of Lyapunov's direct method allows simple sufficient algebraic conditions to be derived for testing the existence of solutions of linear discrete descriptive systems (LDDS) which converge toward the origin of the system's phase space. The determination of the potential domain of attraction of the origin is also analyzed for a class of time–invariant *regular* and *irregular* LDDS.

These results could be a basis for the future development of similar existence analysis for completely general nonlinear and time–dependent discrete descriptor systems. The results presented in the first part of the paper give an indication for a possible convenient approach in that sense. The results are adapted to cater for the robustness of attraction property of the phase–space origin for two different classes of perturbed LDDS.

The second part of the paper concerns the same class of systems, treating the problem of investigating simple sufficient algebraic conditions for the existence of particular solutions with specific practical stability constraints. The estimation of the potential (*weak*) domain of practical stability is obtained. The results could serve as a basis for the further development of similar existence analyses for other classes of LDDS. The results are adopted for the classical investigations of perturbed systems stability robustness.

Appendix *B concerns the problem of attraction property* of the phase-space origin for this class of systems, using the same methodology, but starting from the other state space system representations, including quite different expression for perturbed terms in the system models.

The main intention of these expositions was to underline that the results derived in connection with robustness are mostly dependent on initial data concerning the system under consideration and the structural form of adopted perturbed terms.

Appendix A - Notations

Singular values of the matrix D are denoted with $\sigma(D)$ and are defined in the following manner

$$\sigma(D)=\sqrt{\lambda(DD^T)} \tag{A1}$$

with denotations $\sigma_{\max}(D)$ and $\sigma_{\min}(D)$ for the maximum and minimum singular value, respectively.

$|D|$ will denote the matrix all entries of which represent the absolute values of its elements d_{ij}.

The symmetric matrix D_s is a symmetric part of the square matrix D, such that

$$D_s=\frac{D+D^T}{2} \tag{A2}$$

$D\geq 0$ denotes the positive semi-definite matrix, $D>0$ the positive definite and $D<0$ the negative definite matrix.

Notation $D_1 \leq D_2$ will be used for those matrices the elements of which satisfy the following relation: $d_{1ij} \leq d_{2ij}$ $\forall i,j$.

Appendix B - An alternative test for the investigation of phase space origin attractivity properties of the linear discrete descriptor systems

Consider the linear descriptive discrete system represented by its state space model in the following form

$$E\mathbf{y}(k+1)=A\mathbf{y}(k)+\mathbf{f_p}(\mathbf{y}),\ \mathbf{y}(k_0)=\mathbf{y}_0 \tag{B1}$$

usually with $k_0=0$.

Function $\mathbf{f_p}(\mathbf{y})$ represents the vector of general system perturbations.

Introducing a suitable nonsingular linear transformation

$$T\mathbf{x}(k)=\mathbf{y}(k), \quad \det T\neq 0 \tag{B2}$$

a broad class of linear descriptive discrete systems, eq.(B1), can be transformed in to the following form

$$\mathbf{x_1}(k+1)=A_1\mathbf{x}_1(k)+A_2\mathbf{x}_2(k)+\mathbf{f_{1p}}(T\mathbf{x}) \tag{B3a}$$

$$\mathbf{0}=A_3\ \mathbf{x}_1(k)+A_4\ \mathbf{x}_2(k)+\mathbf{f_{2p}}(T\mathbf{x}) \tag{B3b}$$

$$ET=\begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix},\quad AT=\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4\end{bmatrix} \tag{B4}$$

where $\mathbf{x}(t)=[\mathbf{x}_1^T(t)\ \mathbf{x}_2^T(t)]^T\in\mathbf{R}^n$ is the state decomposed vector, with $\mathbf{x}_1^T(t)\in\mathbf{R}^{n_1}$ and $\mathbf{x}_2^T(t)\in\mathbf{R}^{n_2}$ with $n=n_1+n_2$.

The matrices A_i, $i=1,2\ldots,4$, have appropriate dimensions.

Moreover, it is clear that

$$\det(ET)=\det E\det T=0 \tag{B5}$$

with $\det T\neq 0$.

Under the applied transformation, the perturbation vector can be expressed as:

$$\begin{aligned}\mathbf{f_p}(\mathbf{y})&=\mathbf{f_p}(T\mathbf{x})=\\ &=\mathbf{f}(\mathbf{x})=\begin{bmatrix}\mathbf{f}_1(\mathbf{x})\\ \mathbf{f}_2(\mathbf{x})\end{bmatrix}=\begin{bmatrix}\mathbf{f_{1p}}(T\mathbf{x})\\ \mathbf{f_{2p}}(T\mathbf{x})\end{bmatrix}\end{aligned} \tag{B6}$$

The vector $\mathbf{f}(\mathbf{x})$ being decomposed on two subvectors $\mathbf{f}_1(\mathbf{x})$ and $\mathbf{f}_2(\mathbf{x})$.

Stability robustness of discrete descriptor systems with unstructured perturbations

Let us consider the system governed by eqs.(B3a-B3b) under the following assumption.

Assumption B1. The perturbation vector can be adopted in the following form

$$\mathbf{f}(\mathbf{x})\left[\mathbf{f}_1^T(\mathbf{x})\ \ \mathbf{0}^T\right]^T \tag{B7}$$

Quasi-Lyapunov function is adopted as in the form given in eq.(20).

Having in mind that H is a symmetric matrix, implying that

$$\mathbf{x_1}^T A_1 H\mathbf{f}_1(\mathbf{x})=\mathbf{f}_1^T(\mathbf{x})HA_1\mathbf{x}_1 \tag{B8a}$$

and

$$\mathbf{x_2}^T A_2^T HA_1\mathbf{x}_1=\mathbf{x_1}^T A_1^T HA_2\mathbf{x}_2 \tag{B8b}$$

the forward difference is given with

$$\begin{aligned}&\Delta V(\mathbf{x})=\\ &=\mathbf{x}_1^T\left(A_L^T HA_L-H\right)\mathbf{x}_1+2\mathbf{x}_I^T A_L^T H\mathbf{f}_I(\mathbf{x})+\mathbf{f}_I^T(\mathbf{x})H\mathbf{f}_I(\mathbf{x})\end{aligned} \tag{B9}$$

Theorem B1. Let the *rank condition,* eq.(7), be satisfied and let L be any solution of the matrix equation (15), so that (16) is valid, too. The perturbation subvector may be adopted in the following manner

$$\mathbf{f_1}(\mathbf{x})=P\mathbf{x_1} \tag{B10}$$

with P being the perturbation matrix.

Let *Assumption* B1 be satisfied.

Then the system (B3) possesses a subset of solutions covergent to the origin of phase space if the following condition is satisfied:

$$\sigma_{\max}(P)<-\sigma_{\max}(A_L)+\left(\left(\sigma_{\max}(A_L)\right)^2+\frac{\sigma_{\min}(Z)}{\sigma_{\max}(H)}\right)^{1/2} \tag{B11}$$

$A_L=A_1+A_2L$ being a *discrete stable matrix*, with a real symmetric positive definite matrix $H^T=H>0$ being the solution of the *discrete Lyapunov matrix equation* given by eq.(25) for any symmetric matrix $Z=Z^T>0$.

Proof. Let all conditions of *Theorems* B1 be fulfilled.

Then one can easily write

$$\mathbf{x_1}(k+1)=(A_L+P)\mathbf{x_1}(k) \tag{B12a}$$

$$\mathbf{0}=(A_3+A_4L)\mathbf{x_1}(k) \tag{B12b}$$

Under the introduced assumption equation (B10), having in mind the validity of eq.(16), the following result can be written

$$\mathbf{f_1}(\mathbf{x})=\mathbf{f_1}\begin{bmatrix}\mathbf{x_1}\\ \mathbf{x_2}\end{bmatrix}=\mathbf{f_1}\begin{bmatrix}\mathbf{x_1}\\ L\mathbf{x_1}\end{bmatrix}=\mathbf{f_1}(\mathbf{x}) \tag{B13}$$

On the basis of previous results, one can write:

$$\Delta V(\mathbf{x})=\mathbf{x}_1^T(k)\left(-Z+2(P^T HA_L)_S+P^T HP\right)\mathbf{x}_1(k) \tag{B14}$$

In order to have the asymptotic stability of the system governed by eq. (B12), it is necessary that

$$-Z + 2(P^T H A_L)_S + P^T H P < 0 \tag{B15}$$

This is valid if

$$\sigma_{\max}(2(P^T H A_L)_S + P^T H P) < \sigma_{\min}(Z) \tag{B16}$$

Using the very well-known inequality, one can easily get

$$\sigma_{\max}(D_1 + D_2) \le \sigma_{\max}(D_1)\sigma_{\max}(D_2) \tag{B17}$$

so from (B16) directly follows

$$\sigma_{\max}(H)\left\{2\sigma_{\max}(P)\sigma_{\max}(A_L) + (\sigma_{\max}(P))^2\right\} < \\ < \sigma_{\min}(Z) \tag{B18}$$

Now we shall discuss both roots of eq.(B18).

For $P \neq 0$, $\sigma_{\max}(P) > 0$ and eq.(B18) gives eq.(B11). Therefore, when the condition given by eq.(B11) is fulfilled, $V(\mathbf{x})$ is a quasi-Lyapunov function for the system governed by eq.(B12). Since we have the *rank condition* to be satisfied and having in mind the *Assumption* B1 it follows that the perturbed system (B3) possesses a subset of solutions which converge to the origin of phase – space.

Remark B1. As well as in the case of time-invariant time continuous systems, *Patel and Toda,* (1980) [32], the constraint given with eq.(B11) has its maximum when $Z=I$, i.e. when $\sigma_{\min}(Z) = 1$.

Proof. For the given matrix A_L, eq.(B11) has its maximum when

$$\psi(Z) = \frac{\sigma_{\min}(Z)}{\sigma_{\max}(H)} \tag{B19}$$

has the maximum, where the matrices H and Z are connected by equation (24).

Since the relation, given by eq.(24) is linear upon H and Z, it follows that if the matrix H is the solution of that equation, for the particular matrix Z, then qH, is also the solution of the same equation for qZ, where q is any positive number.

Now for any matrix $Z>0$, let us select q, such that $q = 1/\sigma_{\min}(Z)$.

Then

$$\psi(qZ) = \frac{\sigma_{\min}(qZ)}{\sigma_{\max}(qH)} = \psi(Z) \tag{B20}$$

Indeed, ψ is unvariable for any constant value of q.

Since we have: $\sigma_{\min}(qZ) = 1$, it follows:

$$\psi(qZ)\frac{1}{\sigma_{\max}(qH)} \tag{B21}$$

Let matrix $\tilde{H} > 0$ be the solution of the following matrix equation

$$A_L^T \tilde{H} A_L - \tilde{H} = -I \tag{B22}$$

where

$$\psi(I) = \frac{\sigma_{\min}(I)}{\sigma_{\max}(\tilde{H})} = \frac{1}{\sigma_{\max}(\tilde{H})} \tag{B23}$$

Using the compact solution of Lyapunov matrix discrete equation, *Ogata* (1987) [29], one can get:

$$qH - \widetilde{H} = \sum_{k=0}^{\infty} (A_L^T)^k \cdot (qH - I) A_L^k \tag{B24}$$

Since $\sigma(qH) \ge 1$, it follows that $qH - I \ge 0$, so we find $qH - \tilde{H} \ge 0$ and $\sigma_{\max}(qH) \ge \sigma_{\max}(\widetilde{H})$.

Therefore, from eqs.(B20-B21) and (B23) directly follows

$$\psi(I) \ge \psi(H) \tag{B25}$$

for any matrix with the property $H = H^T > 0$.

So it has been proved that $Z=I$ gives the maximum for eq.(24).

Stability robustness of discrete descriptor systems with structured perturbations

Structural independent perturbations

Let the rank condition (7) be satisfied, under the condition that matrix L is any solution of eq.(15), so eq.(16) is also fulfilled.

The perturbation subvector, having in mind *Assumption* B1, may be adopted in a convenient form such as

$$\mathbf{f_1}(\mathbf{x}) = \mathrm{P}(k)\,\mathbf{x_1}(k) \tag{B26}$$

Constants π_{ij} and π are defined in the following manner, in a way that elements $p_{ij}(k)$ of the matrix $P(k)$ fulfiu

$$p_{ij}(k) \le |p_{ij}|_{\max} = \pi_{ij} \quad \pi = \max \pi_{ij} \tag{B27}$$

Theorem B2. System given with (B3) is stable if the following condition is satisfied

$$\pi < \frac{1}{n}\left(-\sigma_{\max}(A_L) + \left((\sigma_{\max}(A_L))^2 + \frac{\sigma_{\min}(Z)}{\sigma_{\max}(H)}\right)^{1/2}\right) \tag{B28}$$

It should be noted that $\sigma_{\max}(P_1) = n\pi$ for any matrix all elements of which are π, and it is obviuos that $\sigma_{\max}(P) \le \sigma_{\max}(P_1)$ since $\pi_{ij} \le \pi$, so the result follows directly from *Theorem* B1.

Theorem B3. System (B3) is stable if the following condition is satisfied

$$\pi < -\frac{\sigma_{\max}(U^T|HA_L|)_S}{\sigma_{\max}(U^T|H|U)} + \\ + \left(\left(\frac{\sigma_{\max}(U^T|HA_L|)_S}{(U^T|H|U)}\right)^2 + \frac{\sigma_{\min}(Z)}{\sigma_{\max}(U^T|H|U)}\right)^{1/2} \tag{B29}$$

the matrix U having all negative elements, so that $|P(k)| \le \le \pi U$.

Proof. The left side of eq. (B16) satisfies

$$\sigma_{\max}(P^T HP + 2(P^T HA_L)_S) \le \\ \le \sigma_{\max}(|P^T HP|) + 2\,\sigma_{\max}(|P^T HA_L|)_S \le \\ \le \pi^2\,\sigma_{\max}(|U^T HU|) + 2\pi\,\sigma_{\max}(|U^T HU|)_S \tag{B30}$$

So, according to the eq.(B16), the system governed by eq.(B3), i.e. eq.(B12), possesses the subset of solutions which converge to the origin of phase–space if

$$\begin{aligned}&\pi^2\ \sigma_{\max}\left(|U^T HU|\right)+\\&+2\pi\,\sigma_{\max}\left(|U^T HU|\right)_S<\sigma_{\max}(H)\end{aligned} \tag{B31}$$

wherefrom follows eq.(B29), since π>0.

If the matrix $P(k)$ is known or we can estimate the maximum values of all elements in eq.(B27), then the matrix U may be formed as: $U=[u_{ij}]$, $u_{ij}=\pi_{ij}/\pi$.

In this case it is obvious that $0\le u_{ij}\le 1$.

If the perturbation p_{ij} is not explicitly known, u_{ij} can be used simply as a positive real number.

If the perturbation p_{ij} of a_{Lij} elements of the matrix A_L is equal to zero, then directly follows: $u_{ij}=0$.

Structural dependent perturbations

In some classes of problems there are problems with a relatively small number of unknown parameters. In such cases, the uncertain time-dependent matrix P may be formed in the following way

$$P=k_i P_i \tag{B32}$$

where P_i are constant matrices and k_i are uncertain parameters which can vary independently.

Let us define $mn\times mn$ and $n\times n$ symmetric matrices

$$H_{pp}=\begin{bmatrix}(P_1^T HP_1) & (P_1^T HP_2)_S & \cdots & (P_1^T HP_m)_S\\ (P_1^T HP_2)_S & (P_2^T HP_2)_S & \cdots & (P_2^T HP_m)_S\\ \vdots & \vdots & \vdots & \vdots\\ (P_1^T HP_m)_S & (P_2^T HP_m)_S & \cdots & (P_m^T HP_m)\end{bmatrix} \tag{B33}$$

and

$$H_{api}=(A_L^T HP_i)_S \tag{B34}$$

Theorem B4. System (B3) with the structural perturbation term given by eq.(B32) possesses the subset of solutions which converge to the origin of phase–space, if the following condition is satisfied

$$\sum_{i=1}^{m}|k_i|^2\ \sigma_{\max}(H_{pp})+2\sum_{i=1}^{m}|k_i|\,\sigma_{\max}(H_{api})<\sigma_{\min}(Z) \tag{B35}$$

or

$$\begin{aligned}|k_{ij}|<&-\left(\frac{\sigma_{\max}(H_{api})}{m\sigma_{\max}(|H_{pp}|)}\right)+\\&+\left(\left(\frac{\sigma_{\max}\sum_{i=1}^{m}|H_{api}|}{m\sigma_{\max}(|H_{pp}|)}\right)^2+\frac{\sigma_{\min}(Z)}{m\sigma_{\max}(|H_{pp}|)}\right)^{1/2}\end{aligned} \tag{B36}$$

Proof. From the proof of *Theorem* B1 and using eq. (B32), the condition given with eq.(B16) is given in the following form

$$\begin{aligned}&\sigma_{\max}\left(\sum_{i=1}^{m}\sum_{j=1}^{m}k_i k_j(P_i^T HP_j)\right)+\\&+2\sigma_{\max}\left(\sum_{i=1}^{m}k_i(A_L^T HP_i)_S\right)<\sigma_{\max}(Z)\end{aligned} \tag{B37}$$

Let

$$K=[k_1 I\quad k_2 I\quad \cdots\quad k_m I]^T \tag{B38}$$

Then

$$\sum_{i=1}^{m}\sum_{j=1}^{m}k_i k_j(P_i^T HP_j)=K^T H_{pp}K \tag{B39}$$

and

$$\sigma_{\max}\left(\sum_{i=1}^{m}\sum_{j=1}^{m}k_i k_j(P_i^T HP_j)\right)\le\sigma_{\max}\left(H_{pp}\right)\sum_{i=1}^{m}|k_i|^2 \tag{40}$$

Also

$$\sigma_{\max}\left(\sum_{i=1}^{m}k_i(A_L^T HP_i)_S\right)\le\sum_{i=1}^{m}|k_i|\,\sigma_{\max}(H_{api}) \tag{B41}$$

Now, taking into acount eqs.(B39) and (B40), equation (B35) implies eq. (B37).

For the second part of the theorem one may write:

$$\begin{aligned}&\sigma_{\max}\left(\sum_{i=1}^{m}\sum_{j=1}^{m}k_i k_j(P_i^T HP_j)\right)\le\\&\le[\max|k_j|^2\cdot m]\,\sigma_{\max}\ (|H_{pp}|)\end{aligned} \tag{B42}$$

and

$$\begin{aligned}&\sigma_{\max}\left(\sum_{i=1}^{m}k_i H_{api}\right)\le\sigma_{\max}\left(\sum_{i=1}^{m}|k_i H_{api}|\right)\le\\&\le\max|k_j|\ \sigma_{\max}\left(\sum_{i=1}^{m}|H_{api}|\right)\end{aligned} \tag{B43}$$

Furthermore

$$\begin{aligned}&\max|k_j|^2\cdot m\cdot\sigma_{\max}\left(|H_{pp}|\right)+\\&+2\max|k_j|\sigma_{\max}\left(\sum_{i=1}^{m}|H_{ypi}|\right)<\sigma_{\min}(Z)\end{aligned} \tag{B44}$$

Equations (B42-B44) show that eq.(B35) implies eq. (B36), which concludes the proof.

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Analiza robusnosti stabilnosti linearnih stacionarnih deskriptivnih diskretnih sistema

Diskretni deskriptivni sistemi predstavljeni su u matematičkom smislu kombinacijom diferencnih i algebarskih jednačina, pri čemu ove druge predstavljaju ograničenje, koje opšte rešenje mora da zadovolji u svakom trenutku. Osnovna dinamička analiza ove klase sistema u vremenskom domenu podrazumeva ispitivanje stabilnosti, kako sa pozicija Ljapunova, tako i sa pozicija stabilnosti na konačnom vremenskom intervalu. Mimo toga, od posebne je važnosti i očuvanje ove važne osobine sistema i u prisustvu različitih perturbacija kako bi se i u krajnje nepredvidljivim uslovima obezbedilo kvalitetno ponašanje sistema. Ova složena problematika danas je predmet oblasti upravljanja, poznatija kao teorija robusnosti.

Ključne reči: **linearni diskretni sistemi, deskriptivni sistemi, stabilnost u smislu Ljapunova, stabilnost na konačnom vremenskom intervalu, robusnost stabilnosti.**

Analyse de la robustesse de stabilité chez les systèmes discrets, descriptifs, stationnaires et linéaires

Les systèmes discrets et descriptifs sont présentés, mathématiquement, par la combinaison des équations différences et des équations algébriques. Une contrainte des équations algébriques doit être satisfaite par une solution générale. La principale analyse dynamique de ces systèmes dans le domaine de temps comprend l'analyse de la stabilité par la méthode de Lyapunov et dans l'intervalle de temps limité. Il est très important de garder cette caractéristique du système en présence des perturbations différentes afin d'assurer le comportement satisfaisant du système. Cette problématique complexe est le sujet d'un domaine de contrôle connu comme la théorie de robustesse.

Mots-clés: **systèmes discrets et linéaires, systèmes descriptifs, stabilité selon Lyapunov, stabilité dans l'intervalle de temps limité, robustesse de stabilité.**