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Error and noise analyses and their influence on the air target tracking and coordinates estimation

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Normal coordinates evaluation during tracking of aerial targets has been presented. Errors and the space model of state of matrix for the measured values are also presented. A new method for error estimation has been suggested for aerial, targets motion courses and directions in AA defense equipment.

Key words: **target tracking, air targets, coordinate estimations, errors of tracking, state of space.**

Introduction

IN order to estimate target position in the space modern air defense systems usually use different sensors as radar or optoelectronic observing and sighting devices. These systems pointed and directed aerial target automatic taking spherical coordinates for distances and angles, to the target, by direct measuring in the real time.

Radar equipment for tracking and targets estimation, has excellent acquisition range and tracking reliability but it is very sensitive to jamming. In order to avoid detection, jamming and destruction, optoelectronic tracking systems of the passive type are used more frequently. They usually have laser range finders, CCD and IR cameras and the possibility to estimate bore sight line of camera and the two positioning servo platforms.

Target tracking and estimation in the air space is the process that consists of operations such as directing a sighting device, positioning line and coordinate measurements, and finally, evaluating target kinematic parameters.

Pointing of the sighting device is realized by the automatically-servo device, around two perpendicular axes. Measurement of the target position in the air space is realized by measuring the range and the sphere angles of the range sight axis.

Kinematic performances of the target motion are the vectors of the target position and velocity as well as the second derivation of the target position. Determination of kinematic values is also coupled with mathematical space of statement models adapted for dynamical systems estimation.

The values that determine the target motion are simply the values of the target state. The basic assumption for the correct estimation of the target state values is a good precision of coordinate measurements, and the correct statistical evaluation of the errors properties. This paper presents a method of coordinates estimation for the target state values, based on the measured data by an appropriate mathematical model.

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Coordinate systems, its connections and transformations

The position of the target in the air or on the Earth surface can be determined by geographical or central coordinates in relation to the Earth fixed point (Fig.1). These coordinate systems are the Earth coordinates, and are usually used for studying the relative motion of flying vehicles [1].

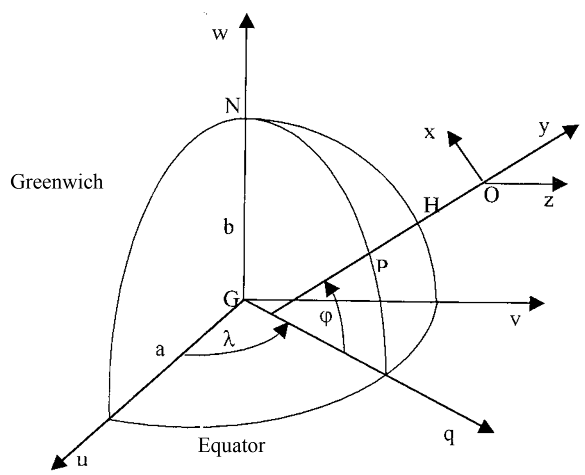

Figure 1. Coordinate systems on the Earth ellipsoid

The spherical Earth shape cannot be used for approximations that are correct enough which is not the case with ellipsoidal shape. More than a hundred mathematical expressions of ellipsoidal models of the Earth have been used in the world until today. Bessel, Hayford, Krasowsky created some of them, but the most frequently used one is the Bessel ellipsoid with the following characteristics:

- big axis **a = 6 377 397.155 m**
- small axis **b = 6 356 078.963 m**

The position of any point O in the space of the Earth (Fig.1), is determined by the ***geographical coordinates*** **(*H*,φ,λ)**:

H (m) – Sea altitude
φ (°) – geographical width (Position to the GREENWICH line)
λ (°) – geographical length (Position to the equatorial line)

The geocentric coordinates G (u,v,w) (Fig.1) are used for the transient operations of geographical coordinates transforming them to the geodetic coordinates, and also for the long range trajectories of projectile and other flying vehicles.

The origin of this system is the Center of the Earth ellipsoid *G*. The axis *Gu* lies in the equatorial plane in Greenwich meridian initial direction. The axis *Gv*, lies in the equatorial plane but perpendicular to the plane *Guv* forming the right-orientated coordinate space.

The equation of the Earth ellipsoid in the geocentral coordinates system is

$$\frac{u^2}{a^2}+\frac{v^2}{a^2}+\frac{w^2}{b^2}=1 \tag{1}$$

The geodetic coordinates O (x,y,z) (Fig.1), are used for the determination of sighting devices position for the weapons and also for the flying vehicles trajectories. The initial point of this coordinate system is the point *O* with the geographical coordinates $O\,(H_o,\varphi_o,\lambda_o)$. The axis *Ox* is orientated to the North (N), and the axis *OZ* to the East (E). The plane *OXZ* is the parallel plane with the tangential line of the ellipsoid plane at the point *P* of the intersection of the perpendicular line to the ellipsoid at the point *O*. The axis *Oy* overlaps with this perpendicular line (and plane *OXY* overlaps with the meridian plane of the *O* point).

If the geographical coordinates (H,φ,λ) of an arbitrary point *M* are known, their perpendicular coordinates $M(u,v,w)$ in the geocentric coordinate system are determined by

$$\begin{aligned} u &= q_p\cos\lambda + H\cos\varphi\cos\lambda \\ v &= q_p\sin\lambda + H\cos\varphi\sin\lambda \\ w &= \frac{b}{a}\sqrt{a^2-q_p^2}+H\sin\varphi \end{aligned} \tag{2}$$

The parameter q_p is determined by

$$q_p=\frac{a}{\sqrt{1+(b/a)^2 tg^2\varphi}} \tag{3}$$

If the perpendicular coordinates of the point $M(u,v,w)$ in the geocentric system *Guvw* are known, the same point $M(H,\varphi,\lambda)$, has the geographical coordinates

$$\begin{aligned} \lambda &= \mathrm{arctg}\left(\frac{v}{u}\right) \\ \varphi &= \mathrm{arctg}\left(\frac{a}{b}\frac{\sqrt{a^2-q_p^2}}{q_p}\right) \\ H &= \frac{\sqrt{u^2+v^2}-q_p}{\cos\varphi} \end{aligned} \tag{4}$$

The parameter q_p is possible to be evaluated by the method of successive approximation as a solution

$$a_4q_p^4+a_3q_p^3+a_2q_p^2+a_1q_p+a_0=0 \tag{5}$$

The coefficients of the polynomial depend on the known parameters and are determined by

$$\begin{aligned} a_0 &= -a^2q^2 \\ a_1 &= -2a^2q\left(\frac{b^2}{a^2}-1\right) \\ a_2 &= \frac{b^2}{a^2}w^2+q^2-a^2\left(\frac{b^2}{a^2}-1\right)^2 \\ a_3 &= 2q\left(\frac{b^2}{a^2}-1\right) \\ a_4 &= \left(\frac{b^2}{a^2}-1\right)^2 \\ q &= \sqrt{u^2+v^2} \end{aligned} \tag{6}$$

The transformation of the coordinates of the point M from the geocentric system *Guvw* into the geodetic coordinate system *Oxyz*, and vice versa, can be performed by one translation and two rotations of the geocentric coordinate system.

The origin of the geodetic system *Oxyz* is determined by the geographical coordinates $O(H_o,\varphi_o,\lambda_o)$. Its coordinates in the geocentric coordinate system $O(u_o,v_o,w_o)$ are given by expression (2).

The relations between the point *M* coordinates in geocentric and geodetic systems are expressed in the space-state form as

$$\begin{bmatrix} x\\ y\\ z\end{bmatrix}=A_o\begin{bmatrix} u-u_0\\ v-v_0\\ w-w_0\end{bmatrix} \tag{7}$$

The rotation matrix of the system (A_o) has the form

$$A_o=\begin{bmatrix} -\cos\lambda_0\sin\varphi_0 & -\sin\lambda_0\sin\varphi_0 & \cos\varphi_0\\ \cos\lambda_0\cos\varphi_0 & \sin\lambda_0\cos\varphi_0 & \sin\varphi_0\\ -\sin\lambda_0 & \cos\lambda_0 & 0\end{bmatrix} \tag{8}$$

The arbitrary point *M* with coordinates in the geocentric system *Guvw*, when its coordinates in the geodetic system *Oxyz* are known, is obtained from matrix eq.(7) in the form

$$\begin{bmatrix} u\\ v\\ w\end{bmatrix}=\begin{bmatrix} u_0\\ v_0\\ w_0\end{bmatrix}+A_0^T\begin{bmatrix} x\\ y\\ z\end{bmatrix} \tag{9}$$

In this equation A_o^T is the transported matrix of the orthogonal matrix A_o.

It is usually necessary to evaluate the connection between the coordinates of an arbitrary point in two geodetic coordinate systems *Rxyz* and $T\xi\eta\zeta$ with the origins determined by the geographical coordinates $R(H_r,\varphi_r,\lambda_r)$ and $T(H_t,\varphi_t,\lambda_t)$.

The normal (perpendicular) coordinates in the geocentric system of the points *R* and *T* is $R(u_r,v_r,w_r)$ and $T(u_t,v_t,w_t)$, determined by eq.(2).

The coordinates of the point *T* in the geodetic plane *R*(*xyz*) are determined by eq.(7) in the form

$$\begin{bmatrix} x_t\\ y_t\\ z_t\end{bmatrix}=A_r\begin{bmatrix} u_t-u_r\\ v_t-v_r\\ w_t-w_r\end{bmatrix} \tag{10}$$

where $A_t=[a_{ij}(\varphi_t,\lambda_t)]$ is the matrix equation of relation (8).

The coordinates of the point *R* in the geodetic system $T\xi\eta\zeta$, according to the same relation, has the expression

$$\begin{bmatrix} \xi_r \\ \eta_r \\ \zeta_r \end{bmatrix} = A_t \begin{bmatrix} u_r - u_t \\ v_r - v_t \\ w_r - w_t \end{bmatrix} \quad (11)$$

From this equation we can also obtain the matrix expression of A_t in the form determined by (8).

The relation between the coordinates at the point T in the system $Rxyz$ and the coordinates of the point R in the system $T\xi\eta\zeta$ has the form

$$\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = -A_r A_t^T \begin{bmatrix} \xi_r \\ \eta_r \\ \zeta_r \end{bmatrix} = -B_{rt} \begin{bmatrix} \xi_r \\ \eta_r \\ \zeta_r \end{bmatrix} \quad (12)$$

where

$$B_{rt} = A_r A_t^T \quad (13)$$

is the rotation matrix of the system $T\xi\eta\zeta$ into the $Rxyz$ coordinates, and its elements have the following scalar form

$$\begin{aligned}
b_{11} &= \cos(\lambda_r - \lambda_t)\sin\varphi_r \sin\varphi_t + \cos\varphi_r \cos\varphi_t \\
b_{12} &= -\cos(\lambda_r - \lambda_t)\sin\varphi_r \cos\varphi_t + \cos\varphi_r \sin\varphi_t \\
b_{13} &= -\sin(\lambda_r - \lambda_t)\sin\varphi_r \\
b_{21} &= -\cos(\lambda_r - \lambda_t)\cos\varphi_r \sin\varphi_t + \sin\varphi_r \cos\varphi_t \\
b_{22} &= \cos(\lambda_r - \lambda_t)\cos\varphi_r \cos\varphi_t + \sin\varphi_r \sin\varphi_t \\
b_{23} &= \sin(\lambda_r - \lambda_t)\cos\varphi_r \\
b_{31} &= \sin(\lambda_r - \lambda_t)\sin\varphi_t \\
b_{32} &= -\sin(\lambda_r - \lambda_t)\cos\varphi_t \\
b_{33} &= \cos(\lambda_r - \lambda_t)
\end{aligned} \quad (14)$$

If the coordinates of the point $M(\xi,\eta,\zeta)$, in the geodetic coordinate system $T\xi\eta\zeta$, are known, the same point M, in the geocentric coordinate system, according to eq.(9), is

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} u_t \\ v_t \\ w_t \end{bmatrix} + A_t^T \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \quad (15)$$

The coordinates of the point $M(u,v,w)$ in the geodetic coordinate system $Rxyz$, according to the eq.(7), are

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A_r \begin{bmatrix} u \\ v \\ w \end{bmatrix} - A_r \begin{bmatrix} u_r \\ v_r \\ w_r \end{bmatrix} \quad (16)$$

Replacing relation (15) into eq.(16) the coordinates in the matrix expressions are

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A_r \begin{bmatrix} u_t - u_r \\ v_t - v_r \\ w_t - w_r \end{bmatrix} + A_r A_t^T \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \quad (17)$$

Considering relations (10)-(13) we obtain the needed correlations between the coordinates of the point M in the geodetic coordinate system $Rxyz$ and $T\xi\eta\zeta$ in the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} + Brt \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = Brt \begin{bmatrix} \xi - \xi_r \\ \eta - \eta_r \\ \zeta - \zeta_r \end{bmatrix} \quad (18)$$

When the origins of both geodetic coordinate system are at small distances, (R is close to T), i.e. $\varphi_r \cong \varphi_t$ and $\lambda_r \cong \lambda_t$, the rotation matrix B_{rt}, in eq.(18), becomes the unit diagonal matrix form (b_{ii}=1 and b_{ij}=0), which means reducing the transformations of coordinates by the relations of translatory-displaced systems.

The rotation matrix B_{rt}, for the transformation from one geodetic coordinate system to the other is the function of the geographical angles (coordinates) and their coordinate points. The fixed points of any sighting device of the weapon is useful to be put in the position measured by geographical coordinates.

Space target estimation and attitude measurement

For the positioning of a flying vehicle as a target, the usualu equipment is an automatically tracking system. This system measures the coordinates of inclined distance (d), azimuth angle (α), and elevation angle (β), according to its own position in the self coordinate system (Fig.2).

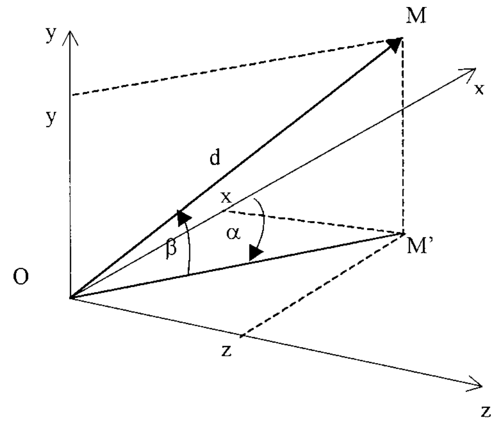

Figure 2. Geodetic coordinate system and the tracking point position

According to the measured coordinates of the fixed point $M(d,\alpha,\beta)$ - the tracking point, in the geodetic coordinates (system $Oxyz$), are:

$$\begin{aligned} x &= d\cos\beta\cos\alpha \\ y &= d\sin\beta \\ z &= d\cos\beta\sin\alpha \end{aligned} \quad (19)$$

Sighting equipment for tracking flying targets is rotateable over the perpendicular axes. The sensors on the axes measure azimuth angles (α_s) and elevation (β_s) of the boresight optical lines of sensors (CCD, TV or IR camera). Laser rangefinders and radar sensors estimate the target distance (d).

Signals obtained from the sighting radar equipment are proportional to the target angles real time range.

Optical signals from TV or IR cameras are usually digitalized in the matrix pixels (rasters) form, thus enabling the estimation of the target center (ξ,η) according to the sensor bore sight axis C (see Fig.3).

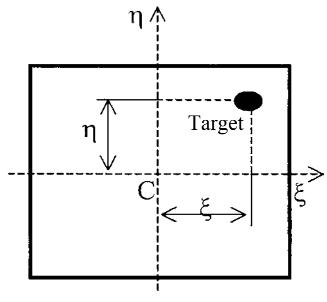

Figure 3. Target on the sensor display

The azimuth and elevation angles of bore sight optical axis *OC* and the target line *OH* in the optical field of the display are shown in Fig.4. The optical axis *OC* is normal to the plane of $C\xi\eta$ which is at the distance of $OC=d$ from the camera used for measuring the lateral inclination ξ and η (mm).

The coefficient of the correlation usually takes the value $k=1/d$. The best correlation of the target line angles is obtained from the triangles ΔOAF ΔOCD, ΔOHF and ΔOAF (Fig.4)

$$\begin{aligned} tg\Delta\alpha &= \frac{\xi}{OA} = \frac{k\xi}{\cos\beta s - k\eta\sin\beta s} \\ \alpha &= \alpha s + \Delta\alpha \\ tg\Delta\beta &= \frac{\eta}{d} = k\eta \\ tg\beta &= \frac{HF}{OF} = \frac{AD}{OA}\cos\Delta\alpha = tg(\beta s + \Delta\beta)\cos\Delta\alpha \end{aligned} \tag{20}$$

Small angles of optical devices give the following approximate expressions

$$tg\Delta\alpha = \Delta\alpha,\ \cos\Delta\alpha = 1,\ tg\Delta\beta = \Delta\beta \tag{21}$$

The local angles of target lines give the following equations

$$\begin{aligned} \alpha &= \alpha_s + \Delta\alpha \\ \beta &= \beta_s + \Delta\beta \end{aligned} \tag{22}$$

The angles of displacement of the target line to the sighting line given by the optical zero bore sight line, are determined by

$$\begin{aligned} \Delta\alpha &= \frac{k\xi}{\cos\beta_s - k\eta\sin\beta_s} \\ \Delta\beta &= k\eta \end{aligned} \tag{23}$$

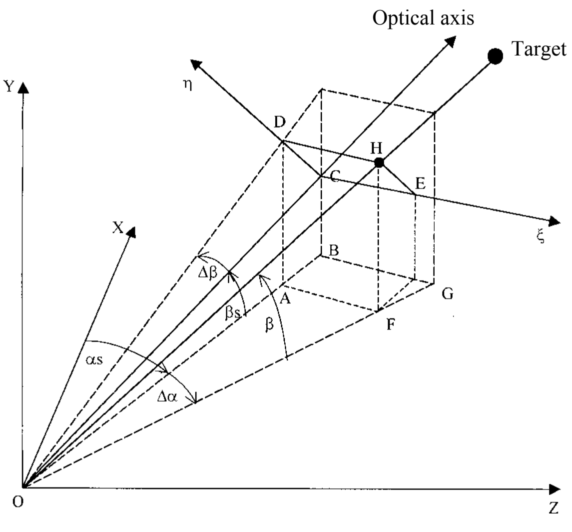

Figure 4. Observing field of the optical sighting device.

Model and types of errors in measuring target tracking distance (range) and angles

Error analysis has an important role for target tracking because most of the tracking methods need information on errors. Errors of measurement are random values, known as noise values that underlie the process and its exit parameters. The exit signal of the process is superposed on the noise as the additional value during measurement, and its expression is [2,3]

$$r_m = r + v \tag{24}$$

This equation is known as the “observing model”, and determines a method of data collection. The value r_m is the measured exit signal and r is the possible acceptable signal and v is the noise signal during measurement. For the polar spherical coordinates of the target, measured by sensor–platform equipment, equation (24) applied as space of state is

$$\begin{bmatrix} \alpha_m(t) \\ \beta_m(t) \\ d_m(t) \end{bmatrix} = \begin{bmatrix} \alpha(t) \\ \beta(t) \\ d(t) \end{bmatrix} + \begin{bmatrix} v_\alpha(t) \\ v_\beta(t) \\ v_d(t) \end{bmatrix} \tag{25}$$

Using equation (22), for the errors of target tracking angles we get

$$\begin{aligned} v_\alpha &= v_{\alpha s} + v_{\Delta\alpha} \\ v_\beta &= v_{\beta s} + v_{\Delta\beta} \end{aligned} \tag{26}$$

The most important assumption for the error estimation is that the system errors are excluded, i.e the rectification of the platform and sensors has average errors of zero state, equal to zero. This form is

$$E[v_d]=0,\ E[v_{\alpha S}]=0,\ E[v_{\beta S}]=0,\ E[v_{\Delta\alpha}]=0,\ E[v_{\Delta\beta}]=0 \tag{27}$$

That means that the mathematical expectations of all measured values are equal to their correct values

$$E\left[r_m\right] = r \tag{28}$$

Random errors (noise) in the measurement equipment are the sum of a large quantity of noise that underlie the Gausian distribution.

The angle and range (distance) estimation processes are independent, which gives an assumption of noncorelation of their noise. The constants of periods of sensors are a few hundred times smaller than the time constants for target tracking. That means that, in any moment of the tracking period, the errors are independent from two-step times intervals). For these assumptions, the deviations of errors is

$$\begin{aligned} \sigma_\alpha^2 &= \sigma_{\alpha s}^2 + \sigma_{\Delta\alpha}^2 \\ \sigma_\beta^2 &= \sigma_{\beta s}^2 + \sigma_{\Delta\beta}^2 \end{aligned} \tag{29}$$

Since the white noise of the Gaussian random zero expiration models is the function of errors and time, the following applies:

$$\begin{aligned} &E\left[v_\alpha(t_1)v_\beta(t_2)\right] = 0, E\left[v_\alpha(t_1)v_d(t_2)\right] = 0 \\ &E\left[v_\beta(t_1)v_d(t_2)\right] = 0, E\left[v_\alpha(t_1)v_\alpha(t_2)\right] = 0 \\ &E\left[v_\beta(t_1)v_\beta(t_2)\right] = 0, E\left[v_d(t_1)v_d(t_2)\right] = 0 \\ &t_1 \neq t_2, t_1, t_2 \in R \end{aligned} \tag{30}$$

All noises of measurement are Gaussian, i.e. they have a normal distribution of probability with the zero systematic error, $v \in N(0,V_v)$. Here V_v is a variance matrix of space for the noise v. Their diagonal elements are dispersions of the space vector V, and other elements point out expressed correlations. This is expressed in the following form

$$V_v = \begin{bmatrix} \sigma_\alpha^2 & 0 & 0 \\ 0 & \sigma_\beta^2 & 0 \\ 0 & 0 & \sigma_d^2 \end{bmatrix} \tag{31}$$

The errors of angles and their displacements $\Delta\alpha$ and $\Delta\beta$ can be obtained in the linear form by the approximation of the Taylor order. This expression has a general form as

$$\begin{aligned}\Delta(\Delta\alpha) &= \frac{\partial\Delta\alpha}{\partial\xi}\Delta\xi + \frac{\partial\Delta\alpha}{\partial\eta}\Delta\eta + \frac{\partial\Delta\alpha}{\partial\beta s}\Delta\beta s \\ \Delta(\Delta\beta) &= \frac{\partial\Delta\beta}{\partial\xi}\Delta\xi + \frac{\partial\Delta\beta}{\partial\eta}\Delta\eta + \frac{\partial\Delta\beta}{\partial\beta s}\Delta\beta s\end{aligned} \qquad (32)$$

The angle displacement dispersions are

$$\begin{aligned}\sigma_{\Delta\alpha}^2 &= \left(\frac{\partial\Delta\alpha}{\partial\xi}\right)^2\sigma_\xi^2 + \left(\frac{\partial\Delta\alpha}{\partial\eta}\right)^2\sigma_\eta^2 + \left(\frac{\partial\Delta\alpha}{\partial\beta s}\right)^2\sigma_{\beta s}^2 \\ \sigma_{\Delta\beta}^2 &= \left(\frac{\partial\Delta\beta}{\partial\xi}\right)^2\sigma_\xi^2 + \left(\frac{\partial\Delta\beta}{\partial\eta}\right)^2\sigma_\eta^2 + \left(\frac{\partial\Delta\beta}{\partial\beta s}\right)^2\sigma_{\beta s}^2\end{aligned} \qquad (33)$$

Equation (33) gives the functional correlation of the average squared errors of angle displacements, from the elevation angle of the instrument optical axis

$$\sigma_{\Delta\alpha} = f(\beta_S),\ \ \sigma_{\Delta\beta} = f(\beta_S) = k\sigma_\eta \qquad (34)$$

These values are [4,5]

- for the sighting radar
$\sigma_{\alpha s} = \sigma_{\beta s} = 0.25$ mrad $= 50''$ i $\sigma_d = 5$ m

(35)

- for the optoelectronic device
$\sigma_{\alpha s} = \sigma_{\beta s} = 0.025$ mrad $= 5''$ i $\sigma_d = 2$ m

If the normal resolution of the system is 50 lines/mm, then the average squared error of the optical system evaluation is

$$\sigma_\xi = \sigma_\eta = \frac{1}{50} = 0.02 \text{ mm} \qquad (36)$$

For the proportion parameter k the reciprocal value of the lens focal distance is f =1500 mm, or f =3000 mm. Relations (34) are determined by the values $\beta \in (0, 80^o)$, and shown in Fig.5 and Fig.6.

The eror of the elevation angle $\Delta\beta$ determination depends on the estimation technology after measurement. Besides the estimation error, the elevation angle β_s of the theodolite optical axis is also important for the angle error determination Δ_α . This additional correlation gives infinite increasing of the error $\sigma_{\Delta\alpha}$, with the bore sight angle closing to 90 degrees. But, for the elevations 60-70^0 this influence can be taken as unimportant, and errors are not in the correlation.

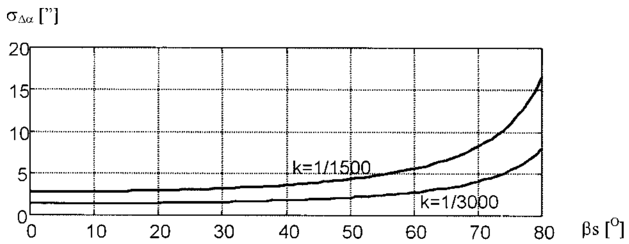

Figure 5. Eror dispersion of the azimuth angle

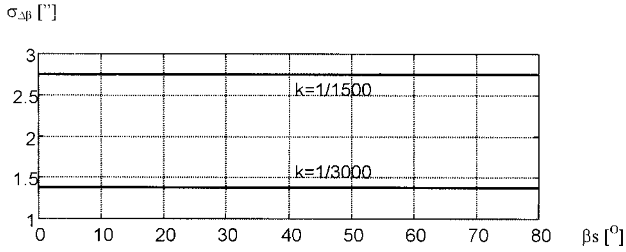

Figure 6. Error dispersion of the elevation angle.

Perpendicular coordinates of target tracking

For tracking and flight target estimation, the geodetic coordinate system $T\xi\eta\zeta$, is fixed for the measurement equipment on the ground. This coordinate system is a local system. Weapons usually have a referent coordinate system *Rxyz* which is also geodetic. The initial coordinate system is *RXYZ* [1,4,5] in Fig.7.

Fixed points of sighting devices and weapon are determined in the geographical coordinates $R(Hr,\varphi r,\lambda r)$ and $T(Ht,\varphi t,\lambda t)$, that enables avoiding the errors of the Earth sphere as well as using the derived transformations of coordinate systems.

Target estimation and tracking measurements on the target trajectory are performed in the local coordinate system $T\xi\eta\zeta$ of the high resolution device and are given in the form of (d,α,β). These values have to be calculated in the referent coordinate system *Rxyz*. For the coordinates of the target *M* in the referent coordinate system it is necessary to evaluate the weapon position *R* and the sighting position *T* in the geocentric coordinates system *Guvw* based on eq.(2). The coordinates of $T\ (x_t, y_t, z_t)$, i.e. the position of the instrument are given by relation (10).

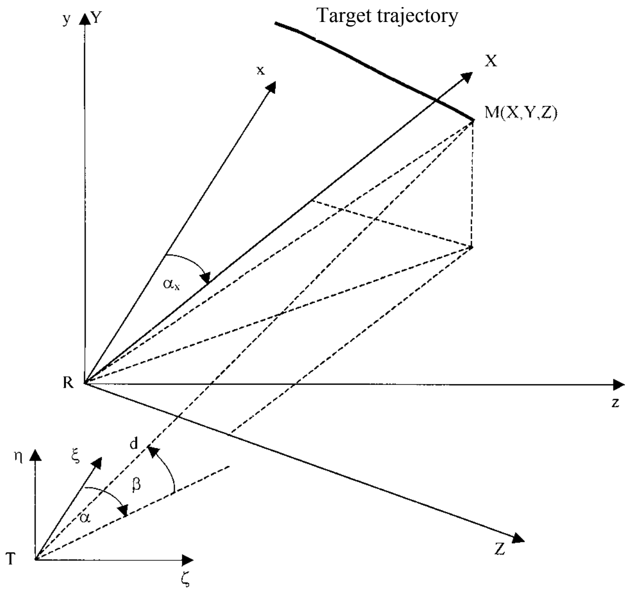

Figure 7.

The measured point coordinates in the referent geodetic coordinate system are given by eq.(18) and transformed for this case

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} + Brt\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \qquad (37)$$

The coordinates x_t,y_t,z_t are the position of the instrument, *Brt* is the matrix of rotation of the local coordinate system into the referent geodetic coordinate system, and, ξ,η,ζ are the measured point coordinates in the *T* coordinate system.

These coordinates have the following expressions

$$\begin{aligned}\xi &= d\cos\beta\cos\alpha\\ \eta &= d\sin\beta\\ \zeta &= d\cos\beta\sin\alpha\end{aligned} \tag{38}$$

The position of the start (initial) coordinate system *RXYZ*, in relation to the referent system *Rxyz* is determined by the azimuth of the *X*-axis, α_x. The relation of the coordinates of *M* (*X,Y,Z*) in the initial coordinate system and the same target point *M* (*x,y,z*) in the geodetic referent system *Rxyz* is given by

$$\begin{bmatrix}X\\Y\\Z\end{bmatrix} = C\begin{bmatrix}x\\y\\z\end{bmatrix} \tag{39}$$

The value *C* is the space of the state in the matrix form that determines the rotation of the geodetic coordinates *Rxyz* into the *RXYX*. This matrix is given by the following expression

$$C = \begin{bmatrix}\cos\alpha_x & 0 & \sin\alpha_x\\ 0 & 1 & 0\\ -\sin\alpha_x & 0 & \cos\alpha_x\end{bmatrix} \tag{40}$$

In the case that the measurements of distances are missing, the coordinates of the space target point *M* can be determined by the intersection of two or three sighting points measured angles of elevation and azimuth.

The directions of sighting from two or more sighting devices into the same space target point are not generally intersected.

The basic hypothesis of the intersection method of two or more sighting lines is that the target point is placed in the middle of the minimum distance between two pointed sight lines. (Fig.8, between the points A and B).

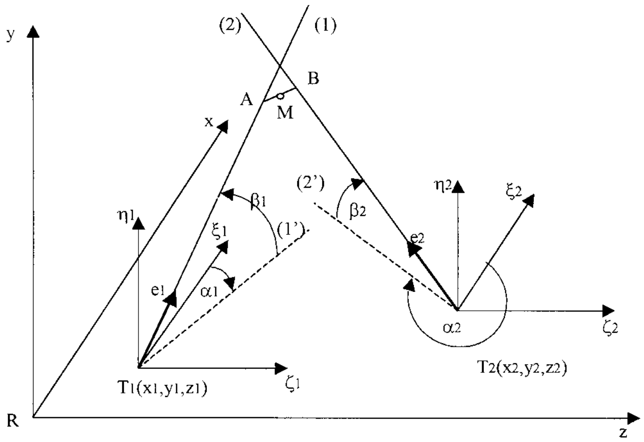

Figure 8. Intersection method for target distance evaluation

The components of the unit vector of the direction of sighting $e_i = [e_{\xi i}\ e_{\eta i}\ e_{\zeta i}]^T$ in the line space of state form are

$$\begin{aligned}e_{\xi i} &= \cos\beta_i\cos\alpha_i\\ e_{\eta i} &= \sin\beta_i\\ e_{\zeta i} &= \cos\beta_i\sin\alpha_i\end{aligned} \tag{41}$$

where α_i and β_i are the local angles of tracking (sighting) directions of the target for the number of devices *i*=1,2.

To evaluate the equations of tracking directions in the referent geodetic coordinate system *Rxyz*, unit vectors have to be taken as the component of scalars, placed on the parallel lines with the referent coordinate system.

In this mathematical method, the process of evaluation has to be used by the rotation matrix B_{rt} with no translation correction.

The components of the unit vector of tracking directions are

$$\begin{bmatrix}e_{xi}\\e_{yi}\\e_{zi}\end{bmatrix} = B_{rt}\begin{bmatrix}e_{\xi i}\\e_{\eta i}\\e_{\zeta i}\end{bmatrix} \tag{42}$$

The equations of tracking directions are

$$\begin{aligned}\frac{x-x_1}{e_{x1}} &= \frac{y-y_1}{e_{y1}} = \frac{z-z_1}{e_{z1}} = k_1\\ \frac{x-x_2}{e_{x2}} &= \frac{y-y_2}{e_{y2}} = \frac{z-z_2}{e_{z2}} = k_2\end{aligned} \tag{43}$$

where x_i,y_i,z_i are the coordinates of the tracking instrument *Ti* and e_{xi}, e_{yi}, e_{zi} are the components of the unit vector $\boldsymbol{e}_i$ for tracking direction in the referent system *Rxyz* (*i*=1,2).

The coordinates of *A* and *B* points taken in eq.(43) give the equations

$$\begin{aligned}x_A &= x_1 + k_1 e_{x1}\\ y_A &= y_1 + k_1 e_{y1}\\ z_A &= z_1 + k_1 e_{z1}\end{aligned} \tag{44}$$

The parameters k_1 and k_2 are determined from the condition that the distance *AB*=*d* (represented in Fig.8)

$$d^2 = (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 \tag{45}$$

The value d is minimal, if the following condition is fulilled

$$\frac{\partial(d^2)}{\partial k_1} = 0, \frac{\partial(d^2)}{\partial k_2} = 0 \tag{46}$$

The solution is given by the linear form

$$\begin{aligned}k_1 - a_1 k_2 + b_1 &= 0\\ k_2 - a_1 k_1 + b_2 &= 0\end{aligned} \tag{47}$$

The cooefficients in eq.(47) are

$$\begin{aligned}a_1 &= e_{x1}e_{x2} + e_{y1}e_{y2} + e_{z1}e_{z2}\\ b_1 &= e_{x1}(x_1 - x_2) + e_{y1}(y_1 - y_2) + e_{z1}(z_1 - z_2)\\ b_2 &= e_{x2}(x_1 - x_2) + e_{y2}(y_1 - y_2) + e_{z2}(z_1 - z_2)\end{aligned} \tag{48}$$

The solution of system (47) is

$$\begin{aligned}k_1 &= \frac{a_1 b_2 - b_1}{1 - a_1^2}\\ k_2 &= \frac{-a_1 b_1 + b_2}{1 - a_1^2}\end{aligned} \tag{49}$$

Equations (44) and (49) give the solution of the coordinates of the points *A* and *B* which have a minimum distance. The middle of this distance is a point of target tracked by two tracking (sighting) lines. Its coordinates, evaluated by this intersection model, are

$$x = \frac{1}{2}(x_A + x_B) = \frac{1}{2}(x_1 + x_2 + k_1 e_{x1} + k_2 e_{x2})$$
$$y = \frac{1}{2}(y_A + y_B) = \frac{1}{2}(y_1 + y_2 + k_1 e_{y1} + k_2 e_{y2}) \qquad (50)$$
$$z = \frac{1}{2}(z_A + z_B) = \frac{1}{2}(z_1 + z_2 + k_1 e_{z1} + k_2 e_{z2})$$

The perpendicular coordinates of *M* (*X,Y,Z*) in the initial coordinate system are *RXYZ* and they are expressed by relation (39).

Determination of errors in measuring the target rectangular coordinates

The target trajectory and its coordinates can be evaluated by the statistical „dispersion", model. The measurement of the target trajectory coordinates *M*(*X,Y,Z*) is done by transforming the measurement of polar coordinates (α, β, d). The errors of measuring perpendicular coordinates are, therefore, mutually correlated.

The errors measured in the local coordinate system fixed for the sensor equipment are given in the matrix form (31). The perpendicular coordinates of the target trajectory in the local system are given by eq.(38). The coordinates ξ, η, ζ obtained from the Taylor order, are given in the form of linear errors.

For the matrix equation of perpendicular coordinates and its errors, the covariance matrix equation is

$$V_\xi = F\, V_\alpha F^T \qquad (51)$$

The element of this big matrix product are

$$V_\xi = \begin{bmatrix} \sigma_\xi^2 & \sigma_{\xi\eta}^2 & \sigma_{\xi\zeta}^2 \\ \sigma_{\xi\eta}^2 & \sigma_\eta^2 & \sigma_{\eta\zeta}^2 \\ \sigma_{\xi\zeta}^2 & \sigma_{\eta\zeta}^2 & \sigma_\zeta^2 \end{bmatrix} \qquad (52)$$

and

$$F = \frac{\partial(\xi,\eta,\zeta)}{\partial(\alpha,\beta,d)} = \begin{bmatrix} \frac{\partial\xi}{\partial\alpha} & \frac{\partial\xi}{\partial\beta} & \frac{\partial\xi}{\partial d} \\ \frac{\partial\eta}{\partial\alpha} & \frac{\partial\eta}{\partial\beta} & \frac{\partial\eta}{\partial d} \\ \frac{\partial\zeta}{\partial\alpha} & \frac{\partial\zeta}{\partial\beta} & \frac{\partial\zeta}{\partial d} \end{bmatrix} \qquad (53)$$

The scalar solution of the space of state (51) is

$$\begin{aligned}
\sigma_\xi^2 &= \zeta^2\sigma_\alpha^2 + \frac{\xi^2\eta^2}{\xi^2+\zeta^2}\sigma_\beta^2 + \frac{\xi^2}{d^2}\sigma_d^2 \\
\sigma_\eta^2 &= (\xi^2+\zeta^2)\sigma_\beta^2 + \frac{\eta^2}{d^2}\sigma_d^2 \\
\sigma_\zeta^2 &= \xi^2\sigma_\alpha^2 + \frac{\eta^2\zeta^2}{\xi^2+\zeta^2}\sigma_\beta^2 + \frac{\zeta^2}{d^2}\sigma_d^2 \\
\sigma_{\xi\eta}^2 &= -\xi\eta\sigma_\beta^2 + \frac{\xi\eta}{d^2}\sigma_d^2 \\
\sigma_{\xi\zeta}^2 &= -\xi\zeta\sigma_\alpha^2 + \xi\zeta\frac{\eta^2}{\xi^2+\zeta^2}\sigma_\beta^2 + \frac{\xi\zeta}{d^2}\sigma_d^2 \\
\sigma_{\xi\eta_L}^2 &= -\eta\zeta\sigma_\beta^2 + \frac{\eta\zeta}{d^2}\sigma_d^2
\end{aligned} \qquad (54)$$

The matrix of errors for coordinate measurements in the initial coordinate system *RXYZ* is obtained by introducing the rotation matrix of the local geodetic coordinate system into the referent system of B_{rt} and the *C* matrix

$$V_X = C\, B_{rt} V_\xi B_{rt}^T C^T \qquad (55)$$

The relation between B_{rt} and *C* is given by eqs.(14) and (40). For the method of "intersection" by two tracking devices, the errors are given by the matrix in the form

$$V\alpha_i = \begin{bmatrix} \sigma_{\alpha i}^2 & 0 \\ 0 & \sigma_{\beta i}^2 \end{bmatrix} \qquad (56)$$

where i=1,2.

The matrices of covariance of unit vector components for the line of tracking in the local geodetic system $T_i\xi_i\eta_i\zeta_i$, after the linearization, given by eq.(41), are determined by

$$V_{e\xi i} = \Phi_i V_{\alpha i} \Phi_i^T \qquad (57)$$

where

$$\Phi_i = \frac{\partial(e_{\xi i}, e_{\eta i}, e_{\zeta i})}{\partial(\alpha_i, \beta_i)} = \begin{bmatrix} \frac{\partial e_{\xi i}}{\partial\alpha_i} & \frac{\partial e_{\xi i}}{\partial\beta_i} \\ \frac{\partial e_{\eta i}}{\partial\alpha_i} & \frac{\partial e_{\eta i}}{\partial\beta_i} \\ \frac{\partial e_{\zeta i}}{\partial\alpha_i} & \frac{\partial e_{\zeta i}}{\partial\beta_i} \end{bmatrix} \qquad (58)$$

The matrix equation (57) and its scalar solution are given by the following expressions

$$\begin{aligned}
\sigma_{e\xi i}^2 &= (\cos\beta_i \sin\alpha_i)^2 \sigma_{\alpha i}^2 + (\sin\beta_i \cos\alpha_i)^2 \sigma_{\beta i}^2 \\
\sigma_{e\eta i}^2 &= (\cos\beta_i)^2 \sigma_{\beta i}^2 \\
\sigma_{e\zeta i}^2 &= (\cos\beta_i \cos\alpha_i)^2 \sigma_{\alpha i}^2 + (\sin\beta_i \sin\alpha_i)^2 \sigma_{\beta i}^2 \\
\sigma_{e\xi i\eta i}^2 &= -\sin\beta_i \cos\beta_i \cos\alpha_i \sigma_{\beta i}^2 \\
\sigma_{e\xi i\zeta i}^2 &= -\cos^2\beta_i \sin\alpha_i \cos\alpha_i \sigma_{\alpha i}^2 + \sin^2\beta_i \sin\alpha_i \cos\alpha_i \sigma_{\beta i}^2 \\
\sigma_{e\eta i\zeta i}^2 &= -\cos\beta_i \sin\beta_i \sin\alpha_i \sigma_{\beta i}^2
\end{aligned} \qquad (59)$$

By eq.(42), the unit vector matrix $\boldsymbol{e_i}$ given by the referent geodetic system is

$$V_{exi} = B_{rti} V_{e\xi i} B_{rti}^T \qquad (60)$$

where B_{rti} is also a matrix form, given by eq.(14) for any tracking position.

The method of intersection with two tracking devices is formed from the matrix elements V_{ex1} and V_{ex2} in the following way

$$V_e = \begin{bmatrix} V_{ex1} & 0 \\ 0 & V_{ex2} \end{bmatrix} \qquad (61)$$

The matrix of covariance for the errors of the intersection method in the referent system *Rxyz*, after linearizing eq.(50) into the Taylor order, is obtained by the formula

$$V_x = F\, V_e F^T \qquad (62)$$

where *F* is

$$F = \frac{\partial(x,y,z)}{\partial(e_{x1}, e_{y1}, e_{z1}, e_{x2}, e_{y2}, e_{z2})} \qquad (63)$$

It is also a matrix form of particular derivations (3x6 matrix) determined, in the same manner as the function Φ, in eq.(58).

Finally, the errors of coordinate measurement in the initial coordinate system *RXYZ* are obtained, taking into account the rotation of coordinates (39), in the form

$$V_X = C\, V_x C^T \tag{64}$$

where C is the rotation matrix of the referent geodetic coordinate system for the azimuth of firing direction given by eq.(40).

Measurement system errors evaluation

The error estimation of a measurement system is always necessary if the tracking process is considered and if a new flow chart of tracking is formed. For any particular measurement, using matrix equations, at any of the measured points on the trajectory, it is possible to evaluate the average error of coordinates determination and other flight parameters in the real time. For any dynamical value of the target, it is possible to evaluate the average squared error in any moment of time.

The equations for variance matrices, provide error estimation for a predicted trajectory or a target motion plane. The average squared errors for coordinates estimation are the functions of the coordinates of a point in the space under measurement. These values of error dispersions are

$$\sigma_{X,Y,Z} = f(X,Y,Z) \tag{65}$$

That means, it is possible to evaluate squared errors in the given plane. If the flight of the target is in the horizontal plane, of the given interval, then it is possible, by variating X and Z coordinates, to get the average values of the errors of the horizontal target flight.

On the basis of the calculated values of the average squared errors in the given plane, it is possible to draw line diagrams as the curves of constant values of the squared average errors.

These diagrams represent the estimation of the measured errors for the given configuration of the measured equipment. Diagrams are used for:

- precision prediction analysis for the choice of measurement equipment
- flow chart of the equipment line
- flight trajectories choice in the experimental testing process, or during the primary analysis for antiaircraft defense.

Simulation of errors of target tracking estimation in the horizontal plane

For a target that flies in the horizontal plane at the altitude Y=1000 m and the latitude $Z\in$[-5000 m, 5000 m] of the range $X\in$[0,20000 m], the errors are estimated and given in the diagrams. The following values of flight parameters are taken.

- geographical coordinates of the launching pad (reference coordinate system center). *RXYZ.*

$$R{:}\ Hr{=}100\text{ m},\ \varphi_r = 44^o\ 45'\ 46.789'',\ \lambda_r = 20^o\ 30'\ 40.506'' \tag{65a}$$

- geographical coordinates of measuring sensor equipment

$$T1{:}Ht{=}100\text{m},\ \varphi_t = 44^o\ 42'\ 41.403'',\ \lambda_t = 20^o\ 27'\ 28.291'' \tag{65b}$$

- azimuth of the shooting direction $\alpha_x = 0^o$.

The squared average errors, of the target coordinate estimation in the referent coordinate system are determined for the following given conditions:

a) Trajectory of the target is measured by the radar [5]. The average squared errors of the replacements of the angles and distance are known. These values are

$$\sigma_\alpha = \sigma_\beta = 50''\ \text{ i }\ \sigma_d = 5\text{ m} \tag{65c}$$

b) Trajectory of the target is measured by a sighting device of optical type with the laser range finder [4]. The average squared errors of the azimuth, elevation and range (distance) of the target position in the space are

$$\begin{aligned} \sigma_\alpha &= \sqrt{\sigma^2_{\alpha s} + \sigma^2_{\Delta\alpha}} = \sqrt{5^2+2^2} = \sqrt{29}\ ('') \\ \sigma_\beta &= \sqrt{\sigma^2_{\beta s} + \sigma^2_{\Delta\beta}} = \sqrt{5^2+2^2} = \sqrt{29}\ ('') \\ \sigma_d &= 2\text{ m} \end{aligned} \tag{65e}$$

The values of $\sigma_{\Delta\alpha}$ and $\sigma_{\Delta\beta}$ are only two angle seconds, in accordance with the diagrams in Fig.5 and Fig.6.

c) Trajectory of the projectile is measured by two optical sighting devices where one has the same postion as the case b), while the other has the position

$$T2{:}Ht{=}100\text{m},\ \varphi_t = 44^o\ 43'\ 42.414'',\ \lambda_t = 20^o\ 33'\ 34.356'' \tag{65d}$$

The fixed points of the instruments (optical devices) (*T*1 and *T*2) for target tracking are shown in Fig.9, in the reference coordinate system *RXYZ.*

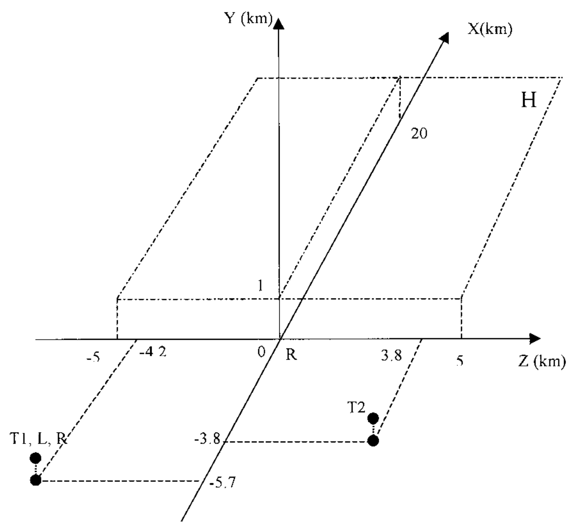

Figure 9. Horizontal plane disposition of the trackers pad *T*1 and *T*2

The calculation of the average square errors for the coordinate target estimation has been realized by computer simulation. The calculation results are given in diagrams 10, 11, and 12.

The simulation test shows the following results:

- Maximum precision (minimum errors) can be obtained by two optical sensors by the intersection method, for the ranges not higher than 15 km.
- For the ranges lighter between 15 km and 20 km maximum precision (minimal errors) is obtained by the method applying one optical device and the laser range finder.
- In radar measurements, errors of angles and distances are greater, then measured by optical devices.

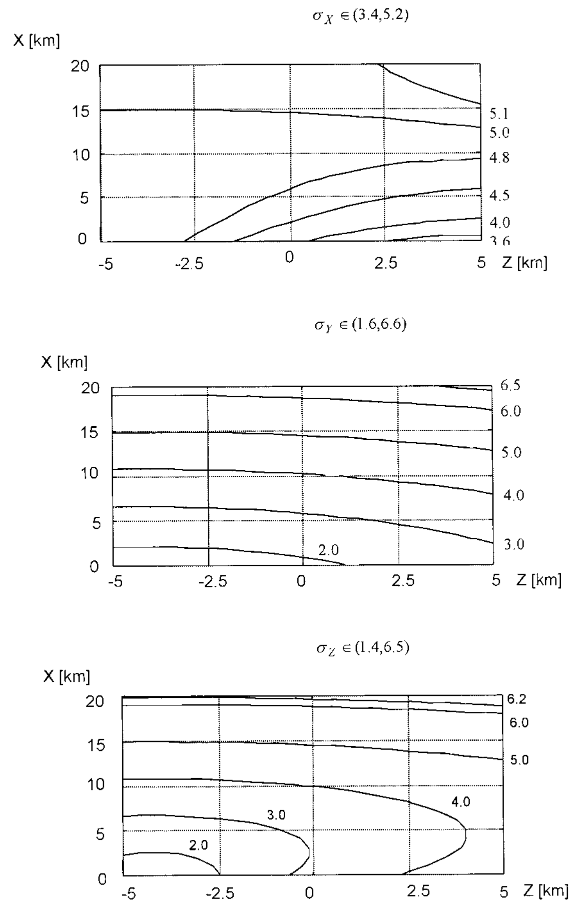

Figure 10. Average square errors of the coordinates estimation in the horizontal plane during measurement by the radar (case a).

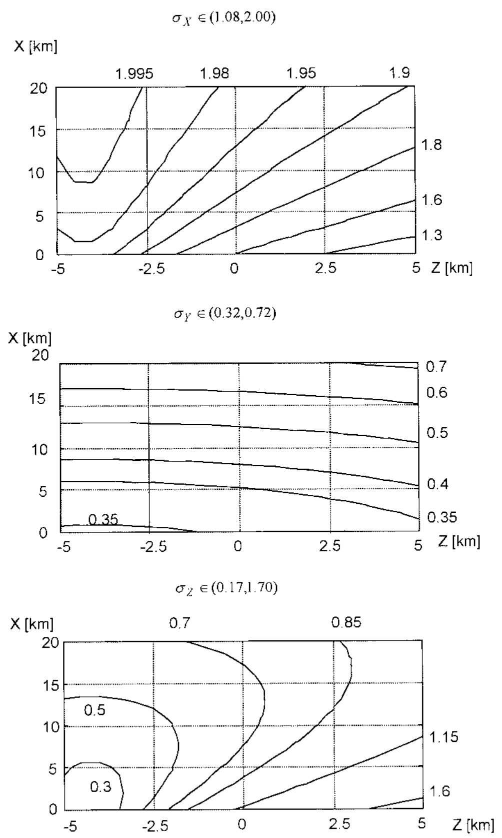

Figure 11. Average square errors of the coordinates estimation in the horizontal plane during measurement by the optical device and the laser range finder (case b).

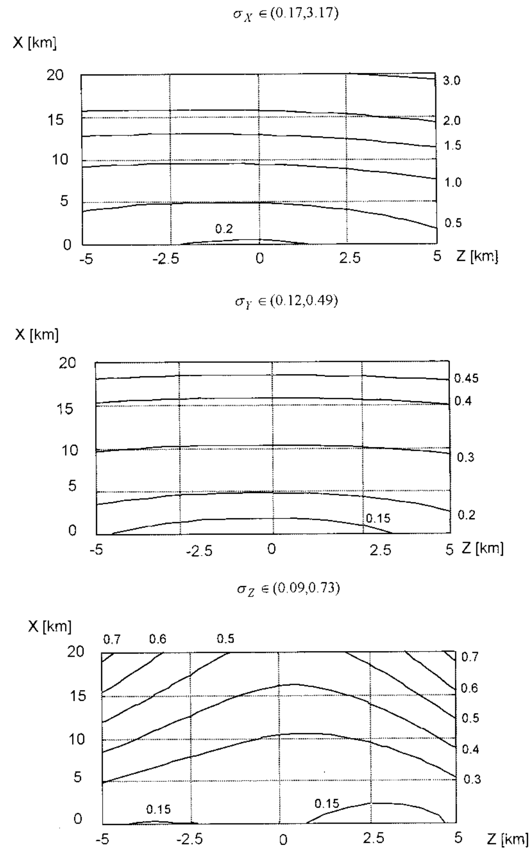

Figure 12. Average square errors of the coordinates estimation in the horizontal plane during measurement by two optical devices (case c).

Conclusion

Quality of target tracking and target position estimation, is one of the most important parameters influencing the performance of any FCS antiaircraft system.

For any configuration of the tracking equipment with the target correlation, by using the matrix of covariance, it is possible to estimate errors in any point of the tracking space in the real time.

Mathematical models of geographical and geodetic coordinates, rearanging and calculation of rectangular coordinates, and transformations in any position of the tracking (and sighting) device, and weapon on the pad, provide direct relations with any iterative process.

The model shows the area of elevation angles where errors and coordinates measurements have practically independent relations between them.

This means that angles of azimuth measurement, and angles of elevation measurement, and their errors are independent, in some areas of values.

For the hypothesis of target trajectory and its motion in the antiaircraft defense, this model provides an advance estimation of the errors in any space point of potential target motion. This provides choosing of an optimal method of tracking, tactics of defense, and equipment of an sensors composition, in the real FCS system design. That means practically the cheapest tracking equipment using prediction evaluation of the erros of tracking.

The main contribution of the tracking and coordinate estimation of the whole area is in the mathematical direct errors modeling, over the direct model of coordinate transformations, that enables composition of the errors no matter which type of tracking equipment, FCS system uses.

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Analiza tačnosti određivanja položaja cilja u toku praćenja sistemima sa Zemlje

Predstavljen je postupak određivanja pravougaonih koordinata u procesu praćenja vazdušnih ciljeva. Analizirane su greške merenja položaja cilja i date su matematičke relacije za izračunavanje matrica kovarijansi grešaka merenja. Predložen je postupak izračunavanja grešaka merenja položaja cilja u prostoru obuhvaćenom merenjima ili za unapred određene pravce kretanja vazdušnih ciljeva, radi analize uticaja tačnosti praćenja cilja na efikasnost sistema protivvazdušne odbrane.

Ključne reči: **praćenje cilja, merenje koordinata, greške merenja, kovarijansna matrica.**

Analyse de la précision de la poursuite d’une cible par l’équipement terrestre

L’évaluation des coordonnées rectangulaires pendant la poursuite des cibles aériennes est présentée. Les erreurs de mesure de la position d’une cible sont analysées et les relations mathématiques pour calculer leurs matrices de covariances sont données. On a proposé une nouvelle méthode pour l’estimation d’erreurs de mesure pendant la poursuite de la cible dans une zone ou une direction préalablement déterminée afin d’analyser l’effet de la précision de la poursuite sur l’efficacité de la défense antiaérienne.

Mots-clés: **poursuite de la cible, mesure des coordonnées, erreurs de mesure, matrice de covariance.**