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Dynamic analysis of generalized nonautonomous state space systems

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Generalized state space systems are those the dynamics of which is governed by a mixture of algebraic and differential equations. Some mathematical models have been shown to document this fact. The complex nature of generalized state space singular systems causes many difficulties in the analytical and numerical treatment of such systems, particularly when there is a need for their control. In that sense the question of their stability deserves great attention. A brief survey of the results concerning the stability of a particular class of these systems, operating in free as well as in forced regimes, in the sense of Lyapunov, is presented as a basis for their high quality dynamic investigation.

Key words: **generalized state space systems, asymptotic stability, Lyapunov equation.**

Introduction

GENERALIZED state space systems are those the dynamics of which is governed by a mixture of algebric and differential equations. In that sense the algebraic equations present the constraints to the solution of the differential part.

These systems are also known as descriptor and semi--state systems and arise naturally as a linear approximation of system models, or linear system models in many applications such as electrical networks, *aircraft dynamics*, neutral delay systems, chemical, thermal and diffusion processes, large-scale systems, interconnected systems, economics, optimization problems, feedback systems, robotics, biology, etc.

Possibilities of the dynamic analysis of generalized state space systems: asymptotic system stability

Let us consider linear generalized state space systems (GLSS) represented by

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t)\ , \quad \mathbf{x}(t_o) = \mathbf{x}_0 \qquad \mathbf{y}(t) = C\mathbf{x}(t) \tag{1}$$

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)\ , \quad \mathbf{x}(t_o) = \mathbf{x}_0 \qquad \mathbf{y}(t) = C\mathbf{x}(t), \tag{2}$$

with the matrix E possibly singular, where $\boldsymbol{x}(t) \in \mathbf{R}^{n}$ is a generalized state-space vector and $\boldsymbol{u}(t) \in \mathbf{R}^{m}$ is a control variable.

The matrices A, B and C are of appropriate dimensions and are defined over the field of real numbers.

The system given by eq.(1) operates in a free regime and the system given by eq.(2) operates in a forced regime, i.e. some external force is applied on it. It should be stressed that, in the general case, the initial conditions need not be the same for an autonomus system and a system operating in the forced regime. System models of this form have some important advantages in comparison with the models in the *normal form, e.g. when E=I* and an appropriate discussion can be found in *Bajić* [1] *and Debeljković et al.* [13,14,15].

The complex nature of generalized state space systems causes many difficultes *in analytical and numerical treatment* that do not occur when systems in the normal form are considered. In this sense the questions of existence, solvability, uniqueness, and smothness are present and must be solved in a satisfactory manner. A short and concise, acceptable and understandable explanation of all these questions may be found in the papers of *Lazarević et al.* [17].

The survey of updated results for generalized state space systems and a broad bibliography can be found in *Bajić* [1], *Campbell* [4,5], *Lewis* [19,20], *Debeljković et al.* [13,14,15] and in two special issues of the journal *Circuits, Systems and Signal Procesing* [7,8].

Asymptotic system stability

Stability plays a central role in the theory of systems and control engineering. There are different kinds of stability problems that arise in the study of dynamic systems, such as Lyapunov stability, finite time stabilty, practical stability, technical stabilty and BIBO stability. The first part of this section is concerned with the asymptotic stability of the equlibrium points of *linear autonomous generalized state space systems.* As we treat the linear systems this is equivalent to the study of the stabilty of the systems. The Lyapunov direct method is well exposed in a

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number of very well known references. Here we present some different and interesting approaches to this problem, including the contributions of the authors of this paper.

Linear autonomous generalized state space systems

Stability definitions

Definition 1. Eq.(1) is exponentially stable if one can find two positive constants α, β such that every solution of eq.(1), satisfy: $\|\mathbf{x}(t)\| \le \alpha e^{-\beta t}\|\mathbf{x}_o\|$, *Pandolfi* [23].

Definition 2. The system given by eq.(1) will be termed *asymptotically stable* if, for all consistent initial conditions $\mathbf{x}_o$, $\mathbf{x}(t) \to \mathbf{0}$ as $t \to \infty$, *Owens, Debeljković* [22].

Definition 3. We call the system given by eq.(1) *asymptotically stable* if all roots of det *(sE - A)* , *i.e.* all finite eigenvalues of this matrix pencil, are in the open left - half complex plane, and the system under consideration is *impulsive free* if there is no $\mathbf{x}_0$ such that $\mathbf{x}$(t) exibits discontinuous behavior in the free regime, *Lewis* [19].

Definition 4. The system given by eq.(1) is called *asymptotically stable* if all finite eigenvalues λ_i, i=1,…,n_1, of the matrix pencil (λE −A) have negative parts, *Muller* [21].

Definition 5. The equilibrium $\boldsymbol{x} = \mathbf{0}$ of the system given by eq.(1) is said to be *stable* if for every ε >0, and for any $t_0 \in J$, there exists an $\delta = \delta(\varepsilon, t_0) > 0$, such that $\|\mathbf{x}(t, t_0, \mathbf{x}_0)\| < \varepsilon$ hold for all $t \ge t_0$ whenever $\mathbf{x}_0 \in W_k$ and $\|\mathbf{x}_0\| < \delta$, where J denotes a time interval such that $J = [t_0, +\infty)$, $t_0 \ge 0$, *Chen, Liu* [6].

Definition 6. The equilibrium $\boldsymbol{x} = \mathbf{0}$ of the system given by eq.(1) is said to be *unstable* if there exists an ε > 0, and $t_0 \in J$, for any δ > 0, such that there exists a $t^* \ge t_0$, for which $\|\mathbf{x}(t^*, t_0, \mathbf{x}_0)\| \ge \varepsilon$ holds, although $\mathbf{x}_0 \in W_k$ and $\|\mathbf{x}_0\| < \delta$, *Chen, Liu* [6].

Definition 7. The equilibrium $\boldsymbol{x} = \mathbf{0}$ of the system given by eq.(1) is said to be *attractive* if for every $t_0 \in J$, there exists an $\eta = \eta(t_0) > 0$, such that $\lim_{t\to\infty} \mathbf{x}(t, t_0, \mathbf{x}_0) = 0$, whenever $\mathbf{x}_0 \in W_k$ and $\|\mathbf{x}_0\| < \eta$, *Chen, Liu* [6].

Definition 8. The equilibrium $\boldsymbol{x} = \mathbf{0}$ of the singular system given by eq.(1) is said to be *asymptotically stable* if it is stable and attractive, *Chen, Liu* [6].

Lemma 1. The equilibrium $\boldsymbol{x} = \mathbf{0}$ of the linear singular system given by eq.(1) is *asymptotically stable* if and only if it is *impulsive-free*, and $\sigma(E,A) \subset \mathbb{C}^-$ *Chen, Liu* [6].

Lemma 2. The equilibrium $\boldsymbol{x} = \mathbf{0}$ of the system given by eq.(1) is *asymptotically stable* if and only if it is *impulsive-free, and* $\lim_{t\to\infty} \mathbf{x}(t) = 0$, *Chen, Liu* [6].

Stability theorems

Theorem 1. Eq.(1), with *A=I, I* being the identity matrix, is *exponentially stable* if and only if the eigenvalues of *E* have non positive real parts.

Proof. The state response of a singular system under consideration is given by

$$\mathbf{x}(t) = e^{-\hat{E}^D \hat{A}(t-t_0)} \hat{E}\hat{E}^D \mathbf{q},\ \mathbf{q} \in \mathbf{C}^n \tag{3}$$

with the restriction on the vector of consistent initial conditions, given by the following equation

$$\mathbf{x}_0 = \hat{E}\hat{E}^D \mathbf{x}_0 \tag{4}$$

If *E* is written in a diagonal form, then

$$e^{-\hat{E}^D \hat{A}(t-t_0)} \hat{E}\hat{E}^D = \begin{bmatrix} e^{Q_0^{-1}t} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{q} \tag{5}$$

which decays exponentially when $\lambda \in \sigma(0)$ implying that $\mathrm{Re}(\lambda) < 0$, where $\sigma(\lambda_i)$ denotes the eigenvalue spectar of the appropriate matrix. We use the upper index "D" to indicate the Drazin inverz. Because the eigenvalues of Q_0 are those eigenvalues of *E* which are not zero, it has completed the proof.

Theorem 2. Let I_Ω be the matrix which represents the operator on $\mathbf{R}^n$ which is the identity on Ω and the zero operator on Λ. Eq.(1), with *A=I*, is stable if an $n \times n$ matrix *P* exists, which is the solution of the matrix equation

$$E^T P + PE = -I_\Omega \tag{6}$$

with the following properties

$$\begin{gathered} P = P^T \\ P\boldsymbol{q} = 0,\quad \boldsymbol{q} \in \Lambda \\ \boldsymbol{q}^T P \boldsymbol{q} > 0,\quad \boldsymbol{q} \ne 0,\quad \boldsymbol{q} \in \Omega \end{gathered} \tag{7}$$

where

$$Q = W_k = \aleph(I - EE^D),\quad \Lambda = \aleph(EE^D) \tag{8}$$

where W_k is the subspace of the consistent intial conditions. ℵ denotes the kernel or null space of the matrix ( ).

Proof. If eq.(6) has a solution *P* as above, *E* cannot have eigenvalues with positive real parts. Hence, eq.(1) is stable. Conversely, assume that eq.(1) is stable. Let *P* be defined by

$$\mathbf{q}^T P \mathbf{q} = \int_0^{+\infty} \left\| \exp(Et) E^D \mathbf{q} \right\|^2 dt \tag{9}$$

The integral is zero if $\mathbf{q} \in \Lambda$ and is a finite number if $\mathbf{q} \in \Omega$. It is clear that the matrix P is the solution of eq.(6) with the properties, a), b), c), *Pandolfi* [23].

Theorem 3. The system given by eq.(1) is *asymptotically stable* if and only if:

a) *A* is invertible and

b) a positive-definite, self-adjoint operator *P* on $\mathbf{R}^n$ exists, such that

$$A^T PE + E^T PA = -Q \tag{10}$$

where *Q* is self-adjoint and positive in the sense that

$$\mathbf{x}^T(t) Q \mathbf{x}(t) > 0 \text{ for all } \boldsymbol{x} \in \boldsymbol{W_k} . \backslash \{\mathbf{0}\} \tag{11}$$

Owens, Debeljković [22].

Proof. To prove sufficiency, note that $W_k \cap \aleph(E) = \{0\}$ indicates that

$$V(\boldsymbol{x}) = \mathbf{x}^T(t) E^T P E \mathbf{x}(t) \tag{12}$$

is a positive-definite quadratic form on W_{k^*}. All smooth solutions $\mathbf{x}(t)$ evolve in W_{k^*} so $V(\mathbf{x})$ can be used as a *"Lyapunov function"*.

Clearly, using the equation of motion eq.(1), one can have

$$\begin{aligned} V &= \dot{\mathbf{x}}^T(t)E^T PE\mathbf{x}(t) + \mathbf{x}^T(t)E^T PE\dot{\mathbf{x}}(t) = \\ &= (E\mathbf{x}(t))^T PE\mathbf{x}(t) + x^T(t)E^T PE\dot{\mathbf{x}}^T = \\ &= (A\mathbf{x}(t))^T E^T PE\mathbf{x}(t) + \mathbf{x}^T(t)E^T PA\mathbf{x}(t) = \\ &= \mathbf{x}^T(t)A^T PE\mathbf{x}(t) + \mathbf{x}^T(t)E^T PA\mathbf{x}(t) = \\ &= -\mathbf{x}^T(t)Q\mathbf{x}(t) \le -\lambda V \end{aligned} \tag{13}$$

where

$$\lambda = \min\left\{\mathbf{x}^T(t)Q\mathbf{x}(t) : V(\boldsymbol{x}) = 1, \boldsymbol{x} \in \boldsymbol{W}_{k^*}\right\} \tag{14}$$

is strictly positive by eq.(11).

Clearly

$$0 \le V(\boldsymbol{x}(t)) \le V(\boldsymbol{x}_0)e^{\lambda t} \to 0(t \to \infty) \tag{15}$$

so that $E\mathbf{x}(t)$ and $\mathbf{x}(t)$ tend to zero as $t\to\infty$ as required, *Debeljković et al.* [14].

Theorem 4. The system given by eq.(1) is *asymptotically stable* if and only if:

a) A is invertible and

b) positive-definite, self-adjoint operator P exists, such that

$$\mathbf{x}^T(t)\left(A^T PE + E^T PA\right)\mathbf{x}(t) = -\mathbf{x}^T(t)I\mathbf{x}(t) \tag{16}$$

Owens, Debeljković [22].

Theorem 5. Let *(E,A)* be regular and *(E,A,C)* be observable. Then *(E,A)* is *impulsive free* and *asymptotically stable* if and only if a positive definite solution P to

$$A^T PE + E^T PA + E^T C^T CE = 0 \tag{17}$$

exists and if P_1 and P_2 are two such solutions, then $E^T P_1 E = E^T P_2 E$, *Lewis* [19].

Theorem 6. If there are symmetric matrices P, Q satisfying

$$A^T PE + E^T PA = -Q \tag{18}$$

and if

$$\mathbf{x}^T E^T PE\,\mathbf{x} > 0 \quad \forall \mathbf{x} = S_1\mathbf{y}_1 \neq \mathbf{0} \tag{19}$$

$$\mathbf{x}^T Q\mathbf{x} \ge 0 \quad \forall \mathbf{x} = S_1\mathbf{y}_1 \tag{20}$$

then the system described by eq.(1) is *asymptotically stable* if

$$rank\begin{bmatrix} sE - A \\ S_1^T \end{bmatrix} = n \quad \forall s \in \mathbf{C} \tag{21}$$

and marginally stable if the condition given by eq.(21) does not hold, *Muller* [21].

Proof. Assume P, Q according to eqs.(19) and (20), then by transformation

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \; S = \begin{bmatrix} S_1 & S_2 \end{bmatrix}, \tag{22}$$

$$RES = \begin{bmatrix} I_1 & 0 \\ 0 & N_\nu \end{bmatrix}, \; RAS = \begin{bmatrix} A_1 & 0 \\ 0 & I_2 \end{bmatrix} \tag{23}$$

where the identity matrices I_1 and I_2 are of dimensions n_1 and n_2 with $n_1 + n_2 = n$ and the n_2 x n_2 matrix N_ν is the nilpotent of the index ν, one has

$$A_1^T P_1 + P_1 A_1 = -S_1^T Q S_1 = -Q_1 \tag{24}$$

with

$$P_1 = P_1^T > 0, \quad Q_1 = Q_1^T \ge 0 \tag{25}$$

Therefore the system given by eq.(1) is stable in the sense of Lyapunov and is *asymptotically stable* if and only if

$$rank\begin{bmatrix} sI_1 - A_1 \\ Q_1 \end{bmatrix} = n_1 \quad \forall s \in \mathbf{C} \tag{26}$$

So, it is necessary to show that the condition

$$rank\begin{bmatrix} sE - A \\ S_1^T Q \end{bmatrix} = n_1 \quad \forall s \in \mathbf{C} \tag{27}$$

is equivalent to the expression given by eq.(26).

By the transformation of eqs.(22) and (23) one has

$$\begin{aligned} rank\begin{bmatrix} sE - A \\ S_1^T Q \end{bmatrix} &= rank\begin{bmatrix} sI_1 - A_1 & 0 \\ 0 & sN_\nu - I_2 \\ Q_1 & Q_{12} \end{bmatrix} = \\ &= n_2 + rank\begin{bmatrix} sI_1 - A_1 \\ Q_1 \end{bmatrix} \end{aligned} \tag{28}$$

showing the equivalence of eq.(26) and eq.(27).

Theorem 7. The equilibrium $\boldsymbol{x}=\mathbf{0}$ of the system given by eq.(1) is *asymptotically stable,* if an $n\times n$ symmetric positive definite matrix P exists, such that along the solutions of the system given by eq.(1), the derivative of the function $V(E\mathbf{x})=(E\mathbf{x})^T P(E\mathbf{x})$, is a negative definite for the variates of $E\mathbf{x}$, *Chen, Liu* [6].

Proof. First, the regularity of (E,A) means that the $n\times n$ nonsingular matrices U and V exist, such that

$$\text{UEV} = \begin{pmatrix} I_1 & 0 \\ 0 & N \end{pmatrix}, \; \text{UAV} = \begin{pmatrix} A_1 & 0 \\ 0 & I_2 \end{pmatrix} \tag{29}$$

and eq.(1) is equivalent to

$$\begin{aligned} \dot{z}_1 &= A_1 z_1 + \mathbf{0} \\ N\dot{z}_2 &= \mathbf{0} + z_2 \end{aligned} \tag{30}$$

here $Q\begin{pmatrix} z_1 & z_2 \end{pmatrix}^T = \boldsymbol{x}$, A_1 is an $n_1 \times n_1$ nonsingular matrix and N is an $n_2 \times n_2$ nilpotent matrix, $n_1 + n_2 = n$.

Next, the fact that $V(E\mathbf{x})$ is a negative definite quadratic form for the variates of $E\mathbf{x}$ means that an $n\times n$ symetric matrix W exists with $E^T WE$ is a positive semi definite with the rank of $E^T WE$ being equal to r, such that

$$V(E\mathbf{x}) = -(E\mathbf{x})^T W(E\mathbf{x}) \tag{31}$$

or

$$A^T PE + E^T PA = -E^T WE \tag{32}$$

Letting

$$P = U^T\begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix}U \tag{33}$$

$$W = U^T\begin{pmatrix} W_{11} & W_{12} \\ W_{12}^T & W_{22} \end{pmatrix}U \tag{34}$$

one has

$$\begin{aligned} P_{11}A_1 + A_1^T P_{12} &= -W_{11} \\ P_{22}N + N^T P_{22} &= -N^T W_{22} N \\ P_{12} + A_1^T P_{12} N &= -W_{12} N \end{aligned} \quad (35)$$

where P_{11}, P_{22} and W_{11} are all positive definite matrices.

In the sequel we prove that $N = 0$. Suppose that the form of the nilpotent matrix N is

$$N = \begin{pmatrix} J_1 & & & \\ & \ddots & & \\ & & J_i & \\ & & & 0 \end{pmatrix} \quad (36)$$

where J_i is a Jordan block matrix in which the diagonal elements are all zero (i=1,…,s), then all elements of the first row of both $N^T P_{22}$ and $N^T W_{22} N$ are zero. It follows from the second formula of eq.(35) that all elements of the first row $P_{22} N$ are zero. If N=0 is not true, without loss of generality, this suposes that $J_1 \neq 0$, then it can be deduced that the element of the first row and the first column of the matrix P_{22} is zero. This is not true since P_{22} is a positive definite. Thus it must be that N=0, in other words, and the linear singular system described by eq.(1) is impulse-free. The positive definity of the matrix W_{11} and the first formula of eq.(35) imply that A_1 is an *asymptotically stable matrix*. It follows from eq.(30) and N=0 that $\lim_{t\to+\infty} \mathbf{x}=\mathbf{0}$ holds from $\mathbf{x}=Q\begin{pmatrix} z_1 & z_2 \end{pmatrix}^T$ and the conclusion of *Theorem 7* follows directly from *Lemma 1*.

Theorem 8. If an (n×n) symmetric, positive definite matrix P exists, such that along with the solutions of the system, given by eq.(1), the derivative of the function $V(E\mathbf{x})=(E\mathbf{x})^T P(E\mathbf{x})$ i.e. $\dot{V}(E\mathbf{x})$ is a positive definite for all variates of $E\mathbf{x}$, then the equilibrium $\mathbf{x}=\mathbf{0}$ of the system given by eq.(1) is *unstable, Chen, Liu* [6].

Theorem 9. If an n×n symmetric, positive definite matrix P exists, such that along with the solutions of the system given by eq.(1), the derivative of the function $V(E\mathbf{x}) = (E\mathbf{x})^T P(E\mathbf{x})$ i.e. $\dot{V}(E\mathbf{x})$ is a negative semidefinite for all variates of $E\mathbf{x}$, then the equilibrium $\mathbf{x} = \mathbf{0}$ of the system given by eq.(1), is *stable, Chen, Liu* [6].

Theorem 10. Let (E,A) be regular and (E,A,C) be impulse observable and finite dynamics detectable. Then (E,A) is stable and impulse-free if and only if a solution (P,H) to the generalized *Lyapunov equations (GLE)* exists.

$$A^T P + H^T A + C^T C = 0 \quad (37)$$

$$H^T E = E^T P \geq 0 \quad (38)$$

Proof. We assume that E, A, C are given by a Weierstrass form

$$E = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \quad (39)$$

where r is the number of finite dynamic modes, and N is a nilpotent Jordan form.

Sufficiency. Partitioning

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \quad (40)$$

one obtains

$$\begin{aligned} & H_{11}^T A_1 + A_1^T P_{11} + C_1^T C_1 = 0 \\ & H_{11}^T = P_{11} \geq 0 \end{aligned} \quad (41)$$

$$\begin{aligned} & H_{12}^T A_1 + P_{21} + C_2^T C_1 = 0 \\ & H_{12}^T = N^T P_{21} \end{aligned} \quad (42)$$

$$\begin{aligned} & H_{21}^T + A_1^T P_{12} + C_1^T C_2 = 0 \\ & H_{21}^T N = P_{12} \end{aligned} \quad (43)$$

$$\begin{aligned} & H_{22}^T + P_{22} + C_2^T C_2 = 0 \\ & H_{22}^T N = N^T P_{22} \end{aligned} \quad (44)$$

Note that (E, A, C) is impulse observable if and only if

$$\Re(N^T) + \Re(C_2^T) + \aleph(N^T) = \mathbf{R}^{n-r} \quad (45)$$

Let

$$\alpha := \min\{k \mid (N^T)^k = 0, k > 0\} \quad (46)$$

Then

$$\begin{aligned} \Re(N^T)^{\alpha-1} &= \Re(N^T)^{\alpha-1} C_2^T + \Re(N^T)^{\alpha} \\ &= \Re(N^T)^{\alpha-1} C_2^T \end{aligned} \quad (47)$$

Pre-multiplying eq.(45) by $(N^T)^{\alpha-1}$ and post-multiplying by $(N)^{\alpha-1}$ yields

$$\begin{aligned} & (N^T)^{\alpha-1} H_{22}^T (N)^{\alpha-1} + (N^T)^{\alpha-1} P_{22} (N)^{\alpha-1} = \\ & = -(N^T)^{\alpha-1} C_2^T C_2 (N)^{\alpha-1}. \end{aligned} \quad (48)$$

It follows again from eq.(45) that both terms in the left-hand side of eq.(48) are zero, so that $(N^T)^{\alpha-1} C_2^T$=0. Hence, from eq.(48), one obtains $\Re\ (N^T)^{\alpha-1}$=0, contradicting the minimality of α. This implies that N=0, so that (E,A) is impulse-free. Also, since (A_1, C_1) is detectable, one can see from eq.(41) that A_1 is stable. Hence (E,A) is stable, *Takaba et al.* [24].

Necessity. Suppose that (E,A) is stable and impulse-free. Then eqs.(41), (42), (43) and (44) are with N=0. From the hypotheses, there exists a solution $P_{11} \geq 0$ to eq.(40). Moreover, $P_{12} = H_{12}$=0, $P_{21} = H_{21} = -C_2^T C_1$, and P_{22}, H_{22} are arbitrary satisfying eq.(45). Thus it has been shown that a solution (P,H) exists to eqs.(37) and (38) with

$$E^T P = \begin{bmatrix} P_{11} & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad (49)$$

Takaba et al. [24].

Some assumptions and preliminaries are needed for further exposition.

Suppose that the matrices E and A commute, that is: EA=AE. Then a real number λ exists such that $\lambda E - I = A.$, otherwise, from the property of regularity, one may multiply eq.(1) by $(\lambda E - A)^{-1}$ so one can obtain a system that satisfies the above assumption and keep the stability the same as the original system.

It is well known that there always exists linear nonsingular transformation, with the invertible matrix T, such that

$$\begin{bmatrix} TET^{-1} & TAT^{-1} \end{bmatrix} = \{diag\begin{bmatrix} E_1 & E_2 \end{bmatrix} \quad diag\begin{bmatrix} A_1 & A_2 \end{bmatrix}\} \tag{50}$$

where E_1 is of full rank and E_2 is a nilpotent matrix, satisfying

$$E_2^h \neq 0, \ E_2^{h+1} = 0, \quad h \geq 0 \tag{51}$$

In addition, it is evident

$$A_1 = \lambda E_1 - I_1, \quad A_2 = \lambda E_2 - I_2 \tag{52}$$

The system given by eq.(1) is equivalent to

$$E_1 \dot{\mathbf{x}}_1(t) = A_1 \mathbf{x}_1(t) + B_1 \mathbf{u}(t) \tag{53a}$$

$$E_2 \dot{\mathbf{x}}_2(t) = A_2 \mathbf{x}_2(t) + B_2 \mathbf{u}(t) \tag{53b}$$

where $\mathbf{x}^T = \begin{bmatrix} \mathbf{x}_1^T & \mathbf{x}_2^T \end{bmatrix}$.

Lemma 3. The system given by eq.(1) is *asymptotically stable* if and only if the *"slow" sub - system*, eq.(53a) is asymptotically stable, *Zhang et al.* [27].

Lemma 4. $\mathbf{x}_1 \neq \mathbf{0}$ is equivalent to $E^{h+1}\mathbf{x} \neq \mathbf{0}$, *Zhang et al.* [27].

Define the Lyapunov function as

$$V(E^{h+1}\mathbf{x}) = \mathbf{x}^T (E^{h+1})^T P E^{h+1} \mathbf{x} \tag{54}$$

where

$P > 0, \quad P \in \mathbf{R}^{nxn}$ satisfying: $V(E^{h+1}\mathbf{x}) > 0$ if $E^{h+1}\mathbf{x} \neq \mathbf{0}$, when $V(0) = 0$.

From eq.(1) and eq.(53), bearing in mind that *EA=AE,* one can obtain

$$(E^h)^T A^T P E^{h+1} + (E^{h+1})^T P A E^h = = -(E^{h+1})^T W E^{h+1} \tag{55}$$

where $W > 0, \quad W \in \mathbf{R}^{nxn}$.

Eq.(55) is said to be the Lyapunov equation for the system given by eq.(1).

Denote with

$$\deg\det(sE - A) = rankE_1 = r \tag{56}$$

Theorem 11. The system, given by eq.(1), is *asymptotically stable* if and only if for any matrix *W>0*, eq.(55) has a solution $P \geq 0$ with a positive external exponent *r*, *Zhang et al.* [27].

Proof.

Necessity. Eq.(53) with $\mathbf{u}$(t)=$\mathbf{0}$ is substituted into eq.(55), gives

$$\begin{aligned}&\begin{bmatrix} (E_1^h)^T & 0 \\ 0 & (E_2^h)^T \end{bmatrix}\begin{bmatrix} A_1^T & 0 \\ 0 & A_2^T \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_4 \end{bmatrix}\begin{bmatrix} E_1^{h+1} & 0 \\ 0 & 0 \end{bmatrix} \\ &+\begin{bmatrix} (E_1^{h+1} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} E_1^h & 0 \\ 0 & E_2^h \end{bmatrix} \\ &= -\begin{bmatrix} (E_1^{h+1})^T & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_1 & W_2 \\ W_2^T & W_3 \end{bmatrix}\begin{bmatrix} E_1^{h+1} & 0 \\ 0 & 0 \end{bmatrix}\end{aligned} \tag{57}$$

Notice that E_1 is of full rank, so the equivalent form can be obtained

$$A_1^T P_1 E_1 + E_1^T P_1 A_1 = -E_1^T W_1 E_1 \tag{58}$$

$$P_2 A_2 E_2^h = 0 \tag{59}$$

where

$$T^T P T^{-1} = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}, \quad \mathrm{T}^{\mathrm{T}} W T^{-1} = \begin{bmatrix} W_1 & W_2 \\ W_2^T & W_3 \end{bmatrix} \tag{60}$$

If the matrix pair *(E,A) is asymptotically stable,* this implies that (E_1, A_1) is *asymptotically stable as well.*

Let $W > 0$, then $W_1 > 0$. Then eq.(58) has a solution $P_1 > 0$ with an internal exponent *r*. Let $P_2 = 0$ then $P_3 = 0$, and the necessity is proved.

Sufficiency. Since any $W > 0$ implies $W_1 > 0$, eq.(55) has a solution if and only if eq.(58) and eq.(59) have solutions respectively, and $P_1 > 0$. Therefore (E_1, A_1) is asymptotically stable. Then the sufficiency follows immediately from *Lemma 3.*

One can choose $P_3 > 0$ since P_3 is not restricted and one can have the following result immediately.

Theorem 12. The system given by eq.(1) is *asymptotically stable* if and only if for any given $W > 0$ the Lyapunov eq.(55) has the solution $P > 0$, *Zhang et al* [27].

The conclusion is the same as in the case of the very well known Lyapunov stability theory if *E* is of full rank. If the matrix *E* is singular, then there is more than one solution.

It should be noted that the results of the preceeding theorems are very similar in some way and are derived only for *regular linear generalized state space systems*.

In order to investigate the stability of irregular generalized state space systems, the folowing results can be used, *Bajić at al.* [1]. For this case, the linear singular system is used in a suitable canonical form, i.e.

$$\dot{\mathbf{x}}_1(t) = A_1 \mathbf{x}_1(t) + A_2 \mathbf{x}_2(t) \tag{61}$$

$$\mathbf{0} = A_3 \mathbf{x}_1(t) + A_4 \mathbf{x}_2(t) \tag{62}$$

Herewith, we examine the problem of the existence of solutions which converge toward the origin of the systems phase-space for *regular and irregular singular linear systems.* By a suitable nonsingular transformation, the original system is transformed to a convenient form. This form of system equations enables development and easy application of *Lyapunov's diect method (LDM)* for the intended existence analysis for a subclass of solutions. In the case when the existence of such solutions is established, an underestimation of the weak domain of the attraction of the origin is obtained on the basis of *symmetric positive definite solutions of a reduced order matrix Lyapunov equation.* The estimated weak domain of attraction consists of the phase space points, which generate at least one solution convergent to the origin.

First, let the set of the consistent initial values of eqs.(61) and (62) be denoted by W_{k^*}. Also, consider the manifold $\boldsymbol{m} \subseteq \mathbf{R}^{nxn}$. determined by eq.(62) as $\boldsymbol{m} = \aleph\left(\begin{bmatrix} A_3 & A_4 \end{bmatrix}\right)$. For the system governed by eqs.(61) and (62) the set W_{k^*} of the consistent initial values is equal to the manifold $\boldsymbol{m}$, that is $W_{k^*} = \boldsymbol{m}$.

It is easy to see that the convergence of the solutions of the system given by eq.(1) and system, given by eqs.(61) and (62) toward the origin is an equivalent problem, since the proposed transformation is nonsingular.

Thus, for the null solution of eqs.(61) and (62) the weak domain of attraction is going to be estimated. The weak domain of attraction of the null solution $\mathbf{x}(t) \equiv \mathbf{0}$ of the system given by eqs.(61) and (62) is defined by

$$\boldsymbol{D} \triangleq \left\{ \mathbf{x}_0 \in \Re^n : \mathbf{x}_0 \in \boldsymbol{m}, \exists \mathbf{x}(t,\mathbf{x}_0), \lim_{t\to\infty} \| \mathbf{x}(t,\mathbf{x}_0) \| \to \mathbf{0} \right\} \quad (63)$$

The term *weak* is used because the solutions of eqs. (61) and (62) need not to be unique, and thus for every $\mathbf{x}_0 \in \boldsymbol{D}$ there may also exist solutions which do not converge toward the origin. In our case $\boldsymbol{D}=\boldsymbol{m}=W_{k^*}$, and the weak domain of attraction may be thought of as the weak global domain of attraction Note that this concept of global domain of attraction used in the paper, differs considerably with respect to the global attraction concept known for state variable systems, *Bajić et al.* [2], *Debeljković et al.* [13].

Our task is to estimate the set $\boldsymbol{D}$. We will use LDM to obtain the underestimate $\boldsymbol{D}_e$ of the set $\boldsymbol{D}$ (i.e. $\boldsymbol{D}_e \subseteq \boldsymbol{D}$). Our development will not require the regularity condition of the matrix pencil ($sE-A$).

For the systems in the form of eqs.(61) and (62) the Lyapunov-like function can be selected as

$$V(\mathbf{x}(t)) = \mathbf{x}_1^T(t) P \mathbf{x}_1(t), \quad P = P^T \quad (64)$$

where P will be assumed to be a positive definite and real matrix.

The total time derivative of V along the solutions of eqs.(61) and (62) is then

$$\begin{aligned} V(\mathbf{x}(t)) &= \mathbf{x}_1^T(t)(A_1^T P + PA_1)\mathbf{x}_1(t) + \\ &+ \mathbf{x}_1^T(t) PA_2 \mathbf{x}_2(t) + \mathbf{x}_2^T(t) A_2^T P \mathbf{x}_1(t) \end{aligned} \quad (65)$$

A brief consideration of the attraction problem shows that if eq.(65) is a negative definite, then for every $\mathbf{x}_0 \in W_{k^*}$ we have $\| \mathbf{x}_1(t) \| \to 0$ as $t\to\infty$. Then $\| \mathbf{x}_2(t) \| \to 0$ as $t\to\infty$, for all those solutions for which the following connection between $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ holds

$$\mathbf{x}_2(t) = L\mathbf{x}_1(t) \quad \forall t \in \mathbf{R} \quad (66)$$

The main question is if the relation given in eq.(66) can be established in a way so as not to contradict the constraints. Since it is not possible for irregular singular linear systems, then we have to reformulate our task to establish the relation given in eq.(66) so that it does not pose to many additional novel constraints to eq.(62).

In order to efficiently use this fact for the analysis of the attraction problem, we introduce the following consideration that also proposes a construction of the matrix L.

Let eq.(66) hold. Assume that the rank condition

$$\text{rank}\begin{bmatrix} A_3 & A_4 \end{bmatrix} = \text{rank}\, A_4 = r \le n_2 \quad (67)$$

is satisfied. Then a matrix L exist, being any solution of the matrix equation

$$0 = A_3 + A_4 L \quad (68)$$

where *0* is a null matrix of the dimensions the same as A_3.

On the basis of eq.(66), eq.(68) and eq.(62), it becomes evident that whenever a solution $\mathbf{x}(t)$ fufills eq.(66), then it has also has to fulfill eq.(62). One can investigate in more details the implications of the introduced equations. When they hold, then all solutions of the system eqs.(61) and (62), which satisfy eq.(66), are subjected to algebraic constraints

$$F\mathbf{x}(t) = \begin{bmatrix} A_3 & A_4 \\ L & -I \end{bmatrix} \mathbf{x}(t) = 0 \quad (69)$$

Assuming that $\dot{V}(\mathbf{x}(t))$ determined by eq.(65) is a negative definite, the following conclusions are important:

1. The solution of eqs.(61) and (62) has to belong to set $\aleph(\begin{bmatrix} A_3 & A_4 \end{bmatrix}) \cap \aleph([L \quad -I])$;
2. If the rank $F=n$ then the judgement on the domain of attraction of the null solution is not possible on the basis of the adopted approach, or more precisely, in this case the estimate of the weak domain $\boldsymbol{D}$ of attraction is a singleton: $\{\mathbf{x}(t) \in \aleph(\begin{bmatrix} A_3 & A_4 \end{bmatrix}) : \mathbf{x}(t) \equiv 0\}$;
3. If the rank $F<n$, then the estimates of the weak domain of attraction needs to be a singleton and it is estimated as

$$\boldsymbol{D}_e = \{\mathbf{x}(t) \in \Re^n : \mathbf{x}(t) \in \aleph(\begin{bmatrix} A_3 & A_4 \end{bmatrix}) \cap \aleph([L \quad -I])\} \subseteq \boldsymbol{D} \quad (70)$$

Now eq.(65) and eq.(66) are employed to obtain

$$\begin{aligned} \dot{V}(\mathbf{x}(t)) &= \mathbf{x}_1^T(t)((A_1 + A_2 L)^T P + \\ &+ P(A_1 + A_2 L))\mathbf{x}_1(t) \end{aligned} \quad (71)$$

which is a negative definite with respect to $\mathbf{x}_1(t)$ if and only if

$$\Omega^T P + P\Omega = -Q, \Omega = A_1 + A_2 L \quad (72)$$

where Q is a real symmetric positive definite matrix. We are now in the position to state the following result.

Theorem 13. Let the rank condition eq.(67) hold and let the rank $F < n$, where the matrix F is defined in eq.(69). Then, the underestimate $\boldsymbol{D}_e$ of the weak domain $\boldsymbol{D}$ of the attraction of the null solution of system given by eqs.(61) and (62), is determinated by eq.(70), providing $(A_1 + A_2 L)$ is a ***Hurwitz matrix***. If $\boldsymbol{D}_e$ is not a singleton, then there are solutions of eq.(61) and (62) different from the null solution, $\mathbf{x}(t) \equiv \mathbf{0}$, which converge towards the origin as time $t\to+\infty$.

Proof. If the rank condition is satisfied, then for all the solutions of eqs.(61) and (62) that satisfy eq.(66), one can have $\mathbf{x}(t) \in \aleph([L \quad -I])$ and simultaneously, these solutions $\mathbf{x}(t) \in \boldsymbol{m} \equiv \aleph(\begin{bmatrix} A_3 & A_4 \end{bmatrix})$. Hence, according to eq.(69), $\mathbf{x}(t) \in \aleph(\begin{bmatrix} A_3 & A_4 \end{bmatrix}) \cap \aleph([L \quad -I])$. However, eq.(65) and eq.(66) imply eq.(71). Since $(A_1 + A_2 L)$ is a *Hurwitz matrix*, then according to the well known results of the Lyapunov matrix equation, a unique symmetric positive definite matrix P satisfying eq.(72) exists. Hence, V defined by eq.(64) is a positive definite function with respect to $\mathbf{x}_1(t)$, and its total time derivative along the solutions of eqs.(61-62) constrained by eq.(66) is a negative definite, so $\lim \| \mathbf{x}_1(t) \| \to 0$ as $t\to+\infty$, as long as $\mathbf{x}_0 \in \aleph(\begin{bmatrix} A_3 & A_4 \end{bmatrix}) \cap \aleph([L \quad -I])$. But eq.(66) implies also $\lim \| \mathbf{x}_2(t) \| = \lim \| \mathbf{x}_1(t) \| \to 0$ as $t\to+\infty$. So, with the rank $F<n$, more than one value of $\mathbf{x}(t)$ satisfies eq.(69).

Hence, as $\aleph([A_3 \quad A_4]) \cap \aleph([L \quad -I])$ is not a singleton, there are solutions different from the null solutions which converge towards, the origin as time $t \to +\infty$. This proves the theorem, *Bajić et al.* [1].

Linear nonautonomous generalized state space systems

In the sequel, the *generalized Lyapunov equations* (GLE) given by *Bender* [3] are further studied for continuous-time generalized state space systems. Under a rank condition, the stability of continuous-time generalized state space systems is related to the uniqueness of the solutions of the Lyapunov equations, provided that the systems are controlable. Furthemore, under certain conditions, the *controllability Grammians* obtained from the *Lyapunov equations* are guaranteed to be a positive definite. All the results are valid for both impulsive and non-impulsive generalized state space systems. Many definitions of cotrollability of the infinite-frequency modes of generalized state space systems have been presented in the literature. However, for time-invariant systems with a *regular pencil* ($sE-A$), all these definitions reduce down to two definitions of controllability at infinity. *These are analogous to the difference between controllability and reachability.*

The parameters of *the Laurent expansion* of the generalized resolvent matrix $(sE-A)^{-1}$ are a very useful tool for analyzing generalized state space systems. This is because they separate the subspace spanned by solutions in the eigenspace associated with finite eigenvalues of the pencil ($sE-A$) from the subcpace spanned by solutions associated with infinite eigenvalues. The infinite-eigenspace solutions can be termed as a "impulsive" solutions in a continuous-time system.

The Laurent parameters can thus be used to split the system, given by eq.(2) into causal (*non-impulsive*) and non-causal (*impulsive*) subsystems.

The *Laurent parameters*, also known as fundamental matrices, have played an important part in the analysis of singular systems. Based on these parameters, *Lewis* [18] defined the controllability matrices for the analysis of the controllability of descriptor systems. *Bender* [3] introduced the *reachability Grammians* and associated them with Lyapunov-like equations without the nonimpulsive or causality restriction.

Suppose that ($sE-A$) is a regular pencil. The system given by eq.(2) is denoted by (E,A,B,C). It is known that the *Laurent parameters* $\{\phi_k, -\mu \le k < \infty\}$ specify the unique series expansion of the resolvent matrix about $s=\infty$

$$(sE-A)^{-1} = s^{-1}\sum_{k=-\mu}^{\infty} \phi_k s^{-k}, \mu \ge 0 \tag{73}$$

valid in some set $0<|s|\le\delta$, $\delta>0$. The positive integer μ is the nilpotent index. Two square invertible matrices U and V exist such that (E,A,B,C) is transformed to the Weierstrass canonical form

$$\bar{E} = U^{-1}EV^{-1}, \quad \bar{A} = U^{-1}AV^{-1} \tag{74a}$$

$$\bar{B} = U^{-1}B, \quad \bar{C} = CV^{-1} \tag{74b}$$

with

$$s\bar{E}-\bar{A} = \begin{bmatrix} sI-J & 0 \\ 0 & sN-I \end{bmatrix}, \; \bar{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \; \bar{C} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}^T \tag{75}$$

where J and N are in the Jordan canonical form and N is nilpotent.

Also, the corresponding Laurent parameters in the Weierstrass form are

$$\bar{\phi}_k = V\phi_k U = \begin{cases} \begin{bmatrix} J^k & 0 \\ 0 & 0 \end{bmatrix}, k \ge 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & -N^{-k-1} \end{bmatrix}, k < 0 \end{cases} \tag{76}$$

Remark 1. If E is nonsingular, the singular system given by eq.(2) can be premultiplied by E^{-1} to derive an equivalent state-space system. In this case the following simplifications occur

$$\phi_0 = I, U = E, V = I, J = E^{-1}A, B, B_1 = E^{-1}B, C_1 = C \tag{77}$$

and N, B_2 and C_2 do not exist (i.e., N is a zero-dimensional matrix).

In this case the eigenvalues of the pencil ($sE-A$) are the eigenvalues of $E^{-1}A$ and are obviously finite. If $E=I$, eq.(2) is already in the Weierstrass canoical form and one can have

$$U = I, J = A, \; B_1 = B \tag{78}$$

We now summarize some useful propreties of the *Laurent parameters*

$$E\phi_k - A\phi_{k-1} = \phi_k E - \phi_{k-1}A = \delta_{0k} I \tag{79}$$

$$\phi_0 E \phi_0 = \phi_0 \tag{80}$$

$$\phi_{-1} A \phi_{-1} = -\phi_{-1} \tag{81}$$

$$\phi_k = \begin{cases} (\phi_0 A)^k \phi_0, & k \ge 0 \\ (-\phi_{-1}E)^{-k-1}\phi_{-1}, & k < 0 \end{cases} \tag{82}$$

$$E\phi_k A = A\phi_k E, \quad \forall k \tag{83}$$

$$\phi_k E \phi_j = \phi_j E \phi_k = \phi_k A \phi_j = \phi_j A \phi_k \tag{84}$$

if $k<0$, $j\ge0$

$$\left.\begin{matrix} (-\phi_{-1}E)^{\mu} = (-E\phi_{-1})^{\mu} = 0 \\ (-\phi_{-1}E)^{\mu-1} \ne 0, \; (-E\phi_{-1})^{\mu-1} \ne 0 \end{matrix}\right\} \tag{85}$$

$\phi_0 E$ and $E\phi_0$ are the projections on H_F along H_I (86a)

$-\phi_{-1}A$ and $-A\phi_{-1}$ are the projections on H_I along H_F (86b)

where H_F and H_I are the spaces spanned by the eigenvectors ν_i satisfying $\lambda_i E\nu_i = A\nu_i$ corresponding to the finite and infinite eigenvalues λ_i, respectively. That is, H_F is the subspace spanned by causal solutions and H_I is the subspace spanned by noncausal or *"infinite frequency"* or *"impulsive"* solutions. Note that if E is nonsingular, $H_F = \mathbf{R}^n$, $H_I=0$, $\phi_0=I$, $\phi_0 E=E=E\phi_0$, and $\phi_{-1}=\phi_{-1}A = A\phi_{-1}=0$.

The solution of a singular system can be expressed directly in terms of the Laurent parameters.

$$\mathbf{x} = \phi_0 E\mathbf{x} - \phi_{-1}A\mathbf{x}(t)$$
$$\left(e^{\phi_0 At}\mathbf{x}_0 + \int_0^t e^{\phi_0 A(t-\tau)}\phi_0 B\mathbf{u}(\tau)d\tau \right) - \left((-\phi_{-1}E)^m \mathbf{x}^{(m)}(t) + \sum_{k=0}^{m-1}(-\phi_{-1}E)^k \phi_{-1}B\mathbf{u}^{(k)}(t) \right) \tag{87}$$

$$\mathbf{y}(t) = C(\phi_0 E - \phi_{-1}A)\mathbf{x}(t) \tag{88}$$

where $i \geq 0$ and $m \geq 0$. As indicated by the property of eq.(87), the Laurent parameters can be used to separate the causal solution subspace from the noncausal solution subspace.

Definition 9. If the integral exists, *the causal continuous-time singular system reachability Grammian is*

$$G_c^{cr} = \int_0^\infty \phi_0 e^{A\phi_0 t} BB^T e^{\phi_0^T A^T t}\phi_0^T dt \tag{89}$$

Bender [3].

The noncausal continuous-time singular system *reachability Grammian is*

$$G_{nc}^{cr} = -\sum_{k=-\mu}^{-1} \phi_k BB^T \phi_k^T \tag{90}$$

The continuous-time singular system *reachability Grammian is*

$$G^{er} = G_c^{er} + G_{nc}^{cr} \tag{91}$$

If the integral does not exist, only G_{nc}^{cr} is defined, *Bender* [3].

The columns of $\phi_0 E G_c^{cr} E^T \phi_0^T = G_c^{cr}$ span the causal reachable subspace, and the columns of G_{nc}^{cr} span the noncausal reachable subspace, which is the subspace "reachable at ∞". By the same argument the columns of G^{cr} span the reachable subspace for the entire system.

Theorem 14.

i) If G_c^{cr} exists, it satisfies

$$\phi_0 (E G_c^{cr} A^T + A G_c^{cr} E^T)\phi_0^T = -\phi_0 B B^T \phi_0^T \tag{92}$$

ii) G_{nc}^{cr} always exists and satisfies

$$\phi_{-1}(E G_{nc}^{cr} E^T - A G_{nc}^{cr} A^T)\phi_{-1}^T = \phi_{-1} B B^T \phi_{-1}^T \tag{93}$$

iii) Suppose the range of R^c (see Apendix *B*) contains the range of ϕ_0 E (i.e.,the pair (J, B_1) is reachable). Then if all finite eigenvalues of the pencil (*sE–A*) have the real part less than zero, eq.(92) has a symmetric solution G_c^{cr} which satisfies $\mathbf{x}^T G_c^{cr}$ x > 0 for all x such that

$$\mathbf{x} = E^T \phi_0^T \mathbf{w} \neq \mathbf{0} \tag{94}$$

Furthermore, $\phi_0 E G_c^{cr} E^T \phi_0^T$ is unique.

Conversely, if eq.(92) has a symmetric solution satisfying eq.(94), then $\phi_0 E G_c^{cr} E^T \phi_0^T$ is unique and all finite eigenvalues of the pencil (*sE–A*) have the real part less than zero.

iv) If the range of R_{nc} contains the range of $\phi_{-1}A$ (i.e., if the pair (N, B_2) is reachable), then eq.(93) has a symmetric solution G_{nc}^{cr} satisfying $\mathbf{x}^T G_{nc}^{cr} \mathbf{x} < 0$, for all $\mathbf{x}$ such that

$$\mathbf{x} = A^T \phi_{-1}^T \mathbf{w} \neq \mathbf{0} \tag{95}$$

Furthermore, $\phi_{-1} A G_{nc}^{cr} A^T \phi_{-1}^T$ is unique.

For the sake of brevity the proof is ommited and can be found in *Bender* [3].

Definition 10. A singular system is *asymptotically stable* if and only if its slow subsystem (I, J, B_1, C_1) is asymptotically stable. The slow subsystem is *controllable*, or equivalently, the descriptor system is *R-controllable*, if and only if

$$\text{rank}\left[B_1, JB_1, ..., J^{n_1 - 1}B_1\right] = n_1 \tag{96}$$

where n_1 =degree(det(*sE – A*)) is the dimension of the slow subsystem.

The fast subsystem is *controllable* if and only if

$$\text{rank}[B_2, NB_2, \ldots, N^{\mu-1} B_2] = n - n_1 \tag{97}$$

Dai [10].

The *controllability* of a singular system implies both its slow and fast subsystems are *controllable*.

Definition 11. For the continuous-time descriptor system (*E*,*A*,*B*,*C*), the *slow controllability Grammian is*

$$G_s^c = \int_0^\infty \phi_0 e^{A\phi_0 t} BB^T e^{\phi_0^T A^T t}\phi_0^T dt \tag{98}$$

provided that the integral exists. The *fast controllability Grammian is*

$$G_f^c = \sum_{k=-\mu}^{-1} \phi_k BB^T \phi_k^T \tag{99}$$

The *controllability Grammian is*

$$G^c = G_s^c + G_f^c \tag{100}$$

Zhang et al. [28].

It can be seen that there is no significant difference between *Definition 11* and *Definition 9*.

In the Weierstrass canonical form, given by eq.(75), the corresponding Grammians of G_s^c and G_f^c are denoted by $\bar{G}_s^c$ and $\bar{G}_f^c$ respectively. From eq.(75) and eq.(76), it can be easily shown that

$$\bar{G}_s^c = V G_s^c V^T, \quad \bar{G}_f^c = V G_f^c V^T. \tag{101}$$

Proposition 1.

$$\text{i) } \phi_0 E G_s^c E^T \phi_0^T = G_s^c \tag{102}$$

$$\text{ii) } \phi_{-1} A G_f^c A^T \phi_{-1}^T = G_f^c \tag{103}$$

Proof.

i) From eqs.(79-84), one can have

$$\phi_0 E G_s^c E^T \phi_0^T = \int_0^\infty \phi_0 e^{A\phi_0 t} BB^T e^{\phi_0^T A^T}\phi_0^T dt = \int_0^\infty \phi_0 e^{A\phi_0 t} BB^T e^{\phi_0^T A^T}\phi_0^T dt = G_s^c \tag{104}$$

ii) From eqs.(79-84), one can also have

$$\begin{aligned}\phi_{-1}AG_f^cA^T\phi_{-1}^T &= \sum_{k=-\mu}^{-1}\phi_{-1}A\phi_kBB^T\phi_k^TA^T\phi_{-1}^T = \\ &= \sum_{k=-\mu}^{-1}\phi_kBB^T\phi_k^T = G_f^c\end{aligned} \tag{105}$$

In relation to the *Grammians* defined for (*E*,*A*,*B*,*C*), the corresponding *Lyapunov equations will be stated*.

Theorem 15.

i) G_s^c satisfies

$$G_c^sA^T\phi_0^T + \phi_0AG_s^c = -\phi_0BB^T\phi_0^T \tag{106}$$

ii) G_f^c uniquely satisfies

$$G_c^s - \phi_{-1}EG_f^cE^T\phi_{-1}^T = \phi_{-1}BB^T\phi_{-1}^T \tag{107}$$

iii) If the system given by eq.(2) is *asymptotically stable*, then the slow subsystem *is controllable if and only if* eq.(106) has the unique solution $G_s^c \geq 0$ which satisfies

$$\text{ran}\left(G_s^c\right) = \text{degree}\left(\det\left(sE - A\right)\right) \tag{108}$$

iii) The fast subsystem is *controllable if and only if*

$$\text{rank}\left(G_f^c\right) = n - \text{degree}\left(\det\left(sE - A\right)\right) \tag{109}$$

iv) If the system given by eq.(2) is *asymptotically stable*, then the system given by eq.(2) *is controllable if and only if*

$$G^c = G_s^c + G_f^c > 0 \tag{110}$$

Proof.

i) and ii) can be easily established from *Bender* [3] with eq.(102).

iii) When eq.(2) is in the Weierstrass canonical form, given by eq.(75), such that

$$\bar{G}_s^c = \begin{bmatrix} G_{11}^c & G_{12} \\ G_{12}^T & G_{22} \end{bmatrix} \tag{111}$$

then eq.(106) reduces to:

$$\begin{aligned}&\begin{bmatrix} G_{11}^c & G_{12} \\ G_{12}^T & G_{22} \end{bmatrix}\begin{bmatrix} J^T & 0 \\ 0 & 0 \end{bmatrix} + \\ &+\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}\times\begin{bmatrix} G_{11}^c & G_{12} \\ G_{12}^T & G_{22} \end{bmatrix} = \begin{bmatrix} -B_1B_1^T & 0 \\ 0 & 0 \end{bmatrix}\end{aligned} \tag{112}$$

That is

$$G_{11}^cJ^T + JG_{11}^c = -B_1B_1^T \tag{113}$$

$$JG_{12} = 0 \tag{114}$$

Since eq.(2) is *asymptotically stable*, then $G_{12} = 0$ and it is obvious that G_{11}^c>0 is the unique solution of eqs.(113) and (114) if and only if the slow subsystem is controlable. Condition given by eq.(109) ensures that G_{22} =0, and hence

$$\bar{G}_s^c = \begin{bmatrix} G_{11}^c & 0 \\ 0 & 0 \end{bmatrix} \tag{115}$$

is the unique solution of eq.(112).

v) When eq.(2) is in Weierstrass canonical form, given by eq.(75), such that

$$\bar{G}_f^c = \begin{bmatrix} G_{11} & G_{21} \\ G_{21}^T & G_{22}^c \end{bmatrix} \tag{116}$$

then eq.(108) reduces to

$$\begin{bmatrix} G_{11} & G_{21} \\ G_{21}^T & G_{22}^c \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & NG_{22}^cN^T \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & B_2B_2^T \end{bmatrix} \tag{117}$$

Hence $G_{11} = G_{21}$ =0. Notice that *N* is nilpotent and G_{22}^c ≥ 0 is the unique solution of

$$G_{22}^c - NG_{22}^cN^T = B_2B_2^T \tag{118}$$

The uniqueness of

$$\bar{G}_f^c = \begin{bmatrix} 0 & 0 \\ 0 & G_{22}^c \end{bmatrix} \tag{119}$$

then follows. Furthemore, G_{22}^c>0 if and only if the fast subsystem is controllable, and now $\bar{G}_f^c$ satisfies eq.(109).

v) From eq.(115) and eq.(119) it follows

$$\bar{G}^c = \bar{G}_s^c + \bar{G}_f^c = \begin{bmatrix} G_{11}^c & 0 \\ 0 & G_{22}^c \end{bmatrix} \tag{120}$$

If the system given by eq.(2) *is controllable*, both the slow and fast subsystem *are controllable*. Hence if system given by eq.(2) *is stable*, then eq.(2) *is controllable* if and only if $\bar{G}^c$>0.

Remark 2. If *E* is nonsingular, then ϕ_0=*I* and ϕ_{-1}=0. In this case, the *controllability Grammian* G^c becomes

$$G^c = \int_0^\infty e^{At}BB^Te^{A^Tt}dt \tag{121}$$

It can be seen that G^c satisfies

$$G^c\ A^T + A\ G^c = -B\ B^T \tag{122}$$

Therefore, normal systems and generalized state space systems have unified Grammian form and Lyapunov equations, *Zhang et al* [28].

Conclussion

To assure *asymptotical stability for linear generalized state space systems* it is not enough only to have the eigenvalues of the matrix pair *(E,A)* in the left half complex plane, but also to provide an impulse-free motion of the system under consideration. Some different approaches have been shown in order to construct Lyapunov stability theory for a particular class of linear generalized state space systems operating in free and forced regimes.

APPENDIX A - Usual notations

With $\aleph(F)$ and $\Re(F)$ we will denote the kernel (null space) and range on any operator *F*, respectively, i.e.

$$\aleph(F) = \{\mathbf{x}: F\mathbf{x} = \mathbf{0},\ \forall\ \mathbf{x}\in\mathbf{R}^n\} \tag{A1}$$

$$\Re(F) = \{\mathbf{y}\in\Re^m,\ \mathbf{y}=F\mathbf{x},\ \mathbf{x}\in\mathbf{R}^n\} \tag{A2}$$

with

$$\dim\aleph(F)+\dim\Re(F)=n \tag{A3}$$

APPENDIX B - Reachability Grammians

We begin this section by defining the *reachable subspace* in terms of the Laurent parameters. We follow the deveopment of *Lewis* [18]. We shall define the reachable subspace in terms of the following *reachability matrices*

$$R_c = \left(\phi_0 B..\phi_{n-1} B\right) \tag{B1}$$

$$R_{nc} = \left(\phi_{-\mu} B..\phi_{-1} B\right) \tag{B2}$$

and

$$R = \left(R_{nc} \quad R_c\right) \tag{B3}$$

The subscript *c* implies that the columns of R_c span the reachable part of the causal solution subspace, and the subscript *nc* implies that the columns of R_{nc} span the reachable part of the noncausal solution subspace.

Definition B1. For a continuous-time singular system, the *causal reachable subspace* is the space spanned by the columns of R_c, *the noncausal reachable subspace* is the space spanned by the columns of R_{nc}, and the reachable subspace is the space spanned by the columns of *R*, *Lewis* [18].

Remark B1:

1. If the reachable subspace defined here for the continuous-time system, given by eq.(2) is equal to $\mathbf{R}^n$, the singular system is "controllable" in the sense of *Cobb* [9]. That means there is a $(\mu-1)$ - times continuously differentiable input $\mathbf{u}(t)$ which will steer the descriptor vector $\mathbf{x}(t)$ from any initial condition in the range of $\phi_0 E$ to any arbitrary location in the descriptor space $\mathbf{R}^n$ in finite time. This is an extension of (and if *E=I* is equivalent to) the usual definition of reachability for state-space systems.
2. If and only if the causal subsystem is reachable, i.e., if the pair (J, B_1) is reachable, do the columns of R_c span the range of $\phi_0 E$. That is, the columns of R_c span the causal solution subspace.
3. If and only if the noncausal subsystem is reachable, i.e., if the pair (N, B_2) is reachable, do the columns of R_{nc} span the range of $\phi_{-1} A$. That is, the columns of R_{nc} span the noncausal solution subspace.

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Dinamička analiza generalisanih neautonomnih sistema u prostoru stanja

Generalisani sistemi u prostoru stanja, u matematičkom smislu, predstavljeni su kombinacijom sistema diferencijalnih i sistema algebarskih jednačina. Priroda ovih sistema je takva da prouzrokuje mnoge poteškoće u njihovoj dinamičkoj analizi, posebno u prilikama kada je potrebno i njima upravljati. U tom smislu, pitanje njihove stabilnosti ima poseban značaj. U ovom radu dat je pregled osnovnih rezultata koji se bave stabilnošću ove klase sistema u smislu Ljapunova, kako za slobodni tako i za prinudni radni režim, a kao podloga za njihovu visokokvalitetnu dinamičku analizu.

Ključne reči: **generalisani sistemi u prostoru stanja, asimptotska stabilnost, jednačina Ljapunova.**

Analyse dynamique des systèmes de l'espace d'état généralisés et non-autonomes

La dynamique des systèmes généralisés de l'espace d'état est déterminée par un ensemble des équations algébriques et différentielles, ce qui était soutenu par quelques modèles mathématiques. La nature complexe des systèmes singulaires et généralisés de l'espace d'état provoque beaucoup de difficultés pendant les traitements analytiques ou numériques de tels systèmes, surtout quand leur contrôle est en question. Par conséquent, leur stabilité est d'une grande importance. Les résultats concernant la stabilité d'une classe particulière de tels systèmes fonctionnant en régime libre ou forcé, au sens de Lyapounov, sont présentés comme la base pour les recherches dynamiques plus approfondies.

Mots-clés: **systèmes généralisés de l'espace d'état, stabilité asymptotique, équation de Lyapounov.**