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Forces of the natural and programmed motion

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On the basis of perceived attributes of motion of a body or a system of bodies the force compelling the body to realize this motion has been searched for and found. Following the same logic and mathematical model in problem solving, the forces realizing the motion according to the prescribed program are searched for. The two-body problem is solved by determining the general formula for the force of reciprocal attraction. From the general formula for reciprocal attraction of two bodies, under Kepler's laws conditions, Newton's law of gravitation is obtained.

Key words: **forces, control, laws of motion, program, inverse problem, two-body problem.**

Introduction

DIFFERENT motions are visible and noticeable in the nature and technical practice. These motions are frequently a subject of natural and mathematical as well technical sciences. In some phenomena the causes of the motion are found, while in others only certain attributes of the motion are measurable: a trace or a trajectory of the motion, velocities, accelerations, distances, mutual distances, … , constraints forcing a body to move in a certain way. The classical and celestial mechanics resolve the above-mentioned problem with a sufficiently high level of precision. According to numerous opinions these theories describe all motions with a precision equal to the mathematical excellence. However, there are dissonant opinions and statements concerning even the foundations of these theories. No matter which principle, one starts from Newton's axioms or Lagrange's variational principle, two fundamental problems arise: 1) to study and determine the motion if the forces are known and 2) the inverse problem, i.e. to determine the forces if the needed attributes of the motion are known. This second task, although mathematically simpler than the first one, is considered impossible by some scientists. Theorems concerning the control and optimal control ([1], pp. 148-169) demonstrate that, even in the general case, it is possible to determine the control forces for the motion according to the prescribed program. There are not opposing remarks to these statements probably due to the complexity of the corresponding differential, varational, integral and algebraical finite equations of motion. On the contrary to this, an example of the determination of the force of the mutual attraction of two bodies arouses the suspicion here and there because it leads to a more general expression in comparison with the formula for the universal force of gravitation where this force is reciprocally proportional to the second degree of the displacement. In order to clarify this problem before a generalisation we shall solve some simple well-known examples.

The forces of the well-known motions

Example 1. With respect to an orthonormal coordinate system which is at rest or moves uniformly, let us consider a body (as a material point) of the mass m moving in the plane $z=0$ along the parabolic path

$$y=-\frac{g}{2c^2}x^2+h,\ \dot{x}=\dot{x}_0:=x\left(t_0\right) \tag{1}$$

Let us determine the magnitude F of the force $\mathbf{F}(X,Y,Z)$ that compels the body to move along this trajectory.

Solution: According to the second Newton's axiom or law, the differential equations of motion read

$$\left.\begin{aligned} m\ddot{x}&=X\\ m\ddot{y}&=Y\\ m\ddot{z}&=Z\end{aligned}\right\} \tag{2}$$

Besides these three equations with three unknown components of the force (X, Y, Z) and the three unknown functions of time ($x(t)$, $y(t)$, $z(t)$) representing the coordinates of the center of inertia of the body, there are three more equations

$$\left.\begin{aligned} &z=0\\ &\dot{x}=\dot{x}_0=:c\\ &y=-\frac{g}{2c^2}x^2+h\end{aligned}\right\} \tag{3}$$

By differentiating these equations up to the second derivatives from equations (2), we obtain

$$\left.\begin{aligned} &\ddot{z}=0\\ &\ddot{x}=0\\ &\ddot{y}=-\frac{g}{c^2}\dot{x}^2-\frac{g}{c^2}x\ddot{x}=-\frac{g}{c^2}\dot{x}^2=-g\end{aligned}\right\} \tag{4}$$

Substituting the second derivatives from equations (2) in equations (3) we obtain the wanted force

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$$\left.\begin{aligned} X &= 0 \\ Y &= -mg \\ Z &= 0 \end{aligned}\right\} \Rightarrow F = -mg \tag{5}$$

The same force would be also obtained by the same procedure for the material point moving along the parabola of the form (Fig.1)

$$y = -\frac{g}{2\dot{x}_0^2}x^2 + \frac{\dot{y}_0}{\dot{x}_0}x, \quad \dot{x} = \dot{x}_0 \tag{6}$$

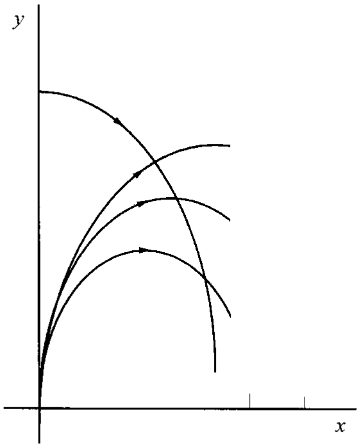

Figure 1.

Example 2. What force $\mathbf{F}_1(X_1,Y_1)$ should be applied to a heavy body of the mass m_1 moving along the path

$$y_1 = \frac{g}{2\dot{x}_{10}^{\;2}}x_1^2 \tag{7}$$

in order to meet the body of the mass m from the previous example, where $\dot{y}_{10} = 0$ is the vertical initial velocity and the horizontal initial velocity of the body m_1 is as in the previous example $\dot{x}_1 = \dot{x}_{10}$.

Solution: The differential equations in this case read

$$\left.\begin{aligned} m_1\ddot{x}_1 &= X_1 \\ m_1\ddot{y}_1 &= -m_1 g + Y_1 \end{aligned}\right\} \tag{8}$$

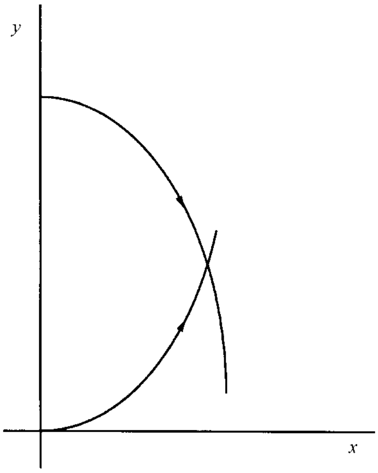

Figure 2.

From the condition $\dot{x}_1 = \dot{x}_{10}$, i.e. $\ddot{x}_1 = 0$, from the condition $\ddot{y}_1 = \frac{g}{\dot{x}_{10}^{\;2}}\dot{x}_1^2 = g$ and equations (8) we obtain

$$\left.\begin{aligned} X_1 &= 0 \\ Y_1 &= 2m_1 g \end{aligned}\right\} \tag{9}$$

The meeting of two bodies is possible at the intersection of two curves, depending on the kinematical and initial geometrical conditions: $\dot{x} = \dot{x}_0$, $\dot{x}_1 = \dot{x}_{10}$ and $x_{10} = y_{10} = \dot{y}_{10} = 0$

On the artificial satellites motion

The first example is simular to Galileo's experiment which establishes the constancy of the acceleration of bodies dropping onto the Earth.

On the other side, in the study concerning the artificial satellites motion around the Earth, the author of paper [1] (on p.68) starts from Newton's force of gravitation

$$F = -\frac{f\ M\ m}{r^2} \tag{10}$$

where $f = 4\pi^2 a^3 / MT^2$ is the gravitational constant, M is the mass of the Earth, m is the satellitess oe me m

$$\left.\begin{aligned} M\ \ddot{x}_1 &= X_1 \\ M\ \ddot{y}_1 &= Y_1 \end{aligned}\right\}, \quad \left.\begin{aligned} m\ddot{x}_2 &= X_2 \\ m\ddot{y}_2 &= Y_2 \end{aligned}\right\}; \tag{15}$$
$$X_1 = 0\ , \quad \ddot{x}_1 = 0\ ; \quad Y_1 = 0\ , \quad \ddot{y}_1 = 0$$

and

$$r = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2} = \text{const.} \tag{16}$$

is the distance between the Earth's center of inertia ant the corresponding center of the satellite on the circular trajectory.

By two times differentiating equations (16) with respect to time we obtain

$$\begin{aligned} \left(x_2 - x_1\right)\ddot{x}_2 + \left(y_2 - y_1\right)\ddot{y}_2 &= -v_{or}^2 \\ v_{or}^2 = \left(\dot{x}_2 - \dot{x}_1\right)^2 + \left(\dot{y}_2 - \dot{y}_1\right)^2 & \end{aligned} \tag{17}$$

Substituting the second derivatives (the components of the acceleration vector) from the equations of motion (15)

$$\left.\begin{aligned} \ddot{x}_2 &= \frac{X_2}{m} \\ \ddot{y}_2 &= \frac{Y_2}{m} \end{aligned}\right\} \tag{18}$$

into equation (16), after the allowed changes of variables

$$\left.\begin{aligned} x_2 - x_1 &= r\cos\varphi \\ y_2 - y_1 &= r\sin\varphi \end{aligned}\right\} \quad \left.\begin{aligned} X_2 &= F\cos\varphi \\ Y_2 &= F\sin\varphi \end{aligned}\right\} \tag{19}$$

we obtain

$$\frac{rF}{m} = -v_{or}^2 \tag{20}$$

i.e. the force we have looked for

$$F = -m\frac{v_{or}^2}{r} \tag{21}$$

This formula for the centripetal force coincides with the fourth Newton's theorem and all its consequences ([3], pp.78-79) because, without changing the essence of this formula, we can write for F

$$\begin{aligned} F &= -m\frac{v_{or}^2}{r} = -mr\dot{\varphi}^2 = -mr\frac{4\pi^2}{T^2} \\ &= -\frac{4\pi^2 r^3}{T^2}\frac{m}{r^2} = -\frac{4\pi^2 r^3}{MT^2}\frac{Mm}{r^2} \end{aligned} \tag{22}$$

or, taking into account (11),

$$F = -f\frac{Mm}{r^2} \tag{23}$$

where

$$f = \frac{4\pi^2 r^3}{MT^2} \tag{24}$$

In addition, the supposition on the uniform Earth's motion only approximatively corresponds to the reality and it remains to us to test formulae (10) and (11) with as few hypothesis as possible.

The more general formula for reciprocal action of two bodies

If we accept the third Kepler's law in the form $r^3/T^2 = \text{const.}$ it is obvious ([1], p.71) that

$$f = \frac{4\pi^2 a^3}{MT^2} \tag{25}$$

is a constant quantity, where M is the Earth's mass, T is the time of rotation of the satellite around the Earth and a is the mean distance between the Earth's and satellite's centers of inertia. This form of the formula for constancy of gravitation can be found in text-books (see. for example [4], p.18), where M is the Sun's mass, r =a is the average distance of the planets from the Sun and T is the time interval of the planets rotation around the Sun.

However, for the Keplerian motions, it is already clear from his first law that the distances between the bodies change during time, i.e. $\rho = \rho(t)$. Without loss of generality, for the sake of brevity of writing, let us consider plane motion. The differential equations of motion, according to the second and third Newton's axiom, read

$$\left.\begin{aligned} M\ \ddot{x}_1 &= X_1 \\ M\ \ddot{y}_1 &= Y_1 \end{aligned}\right\} \quad \left.\begin{aligned} m\ddot{x}_2 &= X_2 \\ m\ddot{y}_2 &= Y_2 \end{aligned}\right\} \tag{26}$$
$$X_1 = -\,X_2 \quad Y_1 = -\,Y_2$$

The unique restriction, which we impose to the motion of two bodies, is that the distance between their centers of inertia changes during the time

$$\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 = \rho^2(t) \tag{27}$$

On the basis of the quoted differential equations of motion and algebraic equation (27) it is possible to determine the components X and Y of the force in a few ways. Let us do it as in the first example. By differentiating equation (27) up to the second derivatives with respect to time, we obtain

$$\left(x_2 - x_1\right)\left(\ddot{x}_2 - \ddot{x}_1\right) + \left(y_2 - y_1\right)\left(\ddot{y}_2 - \ddot{y}_1\right) = \dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2 \tag{28}$$

where

$$v_{or}^2 = \left(\dot{x}_2 - \dot{x}_1\right)^2 + \left(\dot{y}_2 - \dot{y}_1\right)^2 \tag{29}$$

Substituting the second derivatives $\ddot{x}_1, \ddot{y}_1$ and $\ddot{x}_2, \ddot{y}_2$ from equations (26) into equations (28) it follows

$$\begin{aligned} &\left(x_2 - x_1\right)\left(\frac{X_2}{m} - \frac{X_1}{M}\right) + \\ &+\left(y_2 - y_1\right)\left(\frac{Y_2}{m} - \frac{Y_1}{M}\right) = \dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2 \end{aligned} \tag{30}$$

Introducing the allowed substitutions

$$\left.\begin{aligned} x_2 - x_1 &= \rho\cos\varphi \\ y_2 - y_1 &= \rho\sin\varphi \end{aligned}\right\} \quad \left.\begin{aligned} X_2 &= -X_1 = F_2\cos\varphi \\ Y_2 &= -Y_1 = F_2\sin\varphi \end{aligned}\right\} \tag{31}$$

we obtain

$$F = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{M + m}\frac{Mm}{\rho} = \chi\frac{Mm}{\rho} \tag{32}$$

or

$$F = -\frac{Mm}{M+m}\left(\ddot{\rho} - \rho\dot{\varphi}^2\right) = -\frac{Mm}{M+m} w_\rho \tag{33}$$

where

$$\chi = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{M+m} \ , \quad w_\rho = \ddot{\rho} - \rho\ddot{\varphi}^2 \tag{34}$$

This formula for the force is more general than the one in (10). Let us demonstrate that by two or three well-known but very important examples. The simplest example can be the motion of two bodies with the masses m_1 and m_2 along the x-axis according to the model in Fig.3 and with the reciprocal distance changing according to the following law

$$\begin{aligned} &\rho = x_2 - x_1 = \rho_0 + \varepsilon \sin \omega t \\ &\rho_0,\ \varepsilon,\ \omega = \text{const.} \quad \dot{\varphi} = 0 \end{aligned} \tag{35}$$

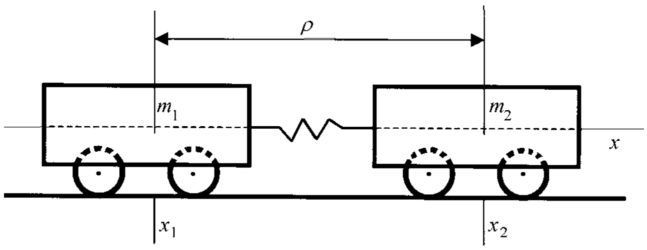

Figure 3.

By the substitution of

$$\ddot{\rho} = \ddot{x}_2 - \ddot{x}_1 = -\varepsilon\omega^2 \sin \omega t \tag{36}$$

in formula (33), we obtain

$$F = -\frac{m_1 m_2}{m_1 + m_2}\omega^2\left(\rho - \rho_0\right) \tag{37}$$

as it has been expected.

Similarly, for the curvilinear motion

$$\rho = \rho_0 + \varepsilon \cos \Omega t \ , \quad \dot{\varphi} = \Omega = \frac{2\pi}{T} \tag{38}$$

from formula (33) it follows that the force reads

$$F = -\frac{m_1 m_2}{m_1 + m_2}\Omega^2\left(2\rho - \rho_0\right) \tag{39}$$

For the motion according to the first and second Kepler's laws from formula (33) expression (10) for the force will be obtained but with a more general coefficient f^*. Indeed, in accordance with the first Kepler's law for the planet motion along eccentric ellipses

$$\rho = \frac{p}{1 + e\cos\vartheta} \tag{40}$$

and the second law concerning the constancy of the sectorial velocity

$$\rho^2\dot{\vartheta} = \text{const.} = C = \frac{2\pi ab}{T} \tag{41}$$

the second derivative of the distance ρ with respect to time reads

$$\ddot{\rho} = C\frac{e\cos\vartheta}{p}\dot{\vartheta} = \frac{C^2}{\rho^3} - \frac{C^2}{p\rho^2} \tag{42}$$

Substituting $\ddot{\rho}$ and $\dot{\vartheta}$ in formula (33) we obtain "Newton's force of gravitation"

$$F = -\frac{4\pi^2 a^3}{\left(m_1 + m_2\right)T^2}\frac{m_1 m_2}{\rho^2} = -f^*\frac{m_1 m_2}{\rho^2} \tag{43}$$

by which the Sun attracts the planets. Therefore, this formula, known as "Newton's universal law of gravitation" is accurate as much as Kepler's laws; it should not be forgotten, though, that Kepler has formulated his laws only for the motion of the principal planets around the Sun, while Newton has formulated a particular group of theorems for the principal planets motion, the other group for the motion of the Moon around the Earth, the third one for the motion of satellites around the Jupiter …

The numerical calculations show that the quantity χ has the order of magnitude 10^{-22} versus Cavendish's constant (G:= f^*=6.67×10^{-11}; [5]) with the order of 10^{-11}; the following table convincingly shows this

Table 1. Table of the comparative reduced values of Cavendish's constant f^* and the coefficient of the proportionality χ

Planets	χ (10^{-22} kg^{-1}m^{2}s^{-2})	f^* (10^{-11} kg^{-1}m^{3}s^{-2})
Mercury	11.530009763	6.675926528
Venus	6.162068773	6.667358417
Earth	4.461522471	6.6744376
Mars	2.927237003	6.6771173131
Jupiter	0.856673545	6.6674902
Saturn	0.467060718	6.6649564
Uranium	0.233141020	6.6909141
Neptune	0.148224707	6.6650721
average values	3.348253233	6.672425

Comparing the quantities χ and f^* for the same points of the trajectories, we obtain, as it follows from the table, a new formula for the "constant of gravitation"

$$f^* = \frac{av_{or}^2}{m_1 + m_2} = a\chi \tag{44}$$

where a is the large semiaxis of the ellyptic trajectory, i.e. this is the mean distance between the centers of inertia of the Sun (the Earth) and the planets (the satellite).

Conclusion

On the basis of the comparison of the cited papers it follows that there is no unique attitude concerning the force of reciprocal attraction of two bodies.

It has been proved that two bodies, moving with the mutual distance ρ, act one to an other with the force

$$F = \chi\frac{m_1 m_2}{\rho} \tag{45}$$

where

$$\chi = \frac{\dot{\rho}^2 + \rho\ddot{\rho} - v_{or}^2}{m_1 + m_2} \tag{46}$$

The formula for Newton's force of gravitation (43) follows from formula (33) for the Keplerian motions.

In the case that the mutual distance between the centers of inertia of two bodies changes according to the law

$$\rho = \rho_0 + \varepsilon \cos \Omega t \ , \quad \dot{\varphi} = \Omega = \frac{2\pi}{T} \tag{47}$$

we obtain that the corresponding force is directly proportional to the distance.

For the free vertical fall of a heavy body with the mass m we have, according to Galileo, $y = -\frac{1}{2} g t^2$ and therefore from formula (32) it follows that the force acting to the body is equal

$$F = -\mathrm{M}\ g \approx -mg \tag{48}$$

where $\mathrm{M} \equiv \dfrac{Mm}{M+m}$ is a so-called reduced mass. In the case of the fall of the body having mass of one kilogram it follows

$$\mathrm{M} \equiv \frac{m}{1 + \dfrac{1}{5.9742 \times 10^{24}}} = \frac{m}{1 + 0.167 \times 10^{-24}} \approx m \tag{49}$$

For new conditions, formula (32) gives (generates) new functions of forces.

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Sile prirodnog i programiranog kretanja

Na osnovu uočenih atributa kretanja nekog tela ili sistema tela, traži se i određuje sila koja primorava telo da ostvaruje posmatrano kretanje. Po istoj logici i matematičkom načinu rešavanja problema, traže se sile pod čijim dejstvom se može realizovati kretanje po zadatom programu. Rešen je problem dva tela određivanjem opšte formule za silu međusobnog privlačenja. Za uslove Keplerovih zakona iz opšte formule uzajamnog dejstva dva tela dobija se, kao posledica, Njutnov zakon gravitacije.

Ključne reči: **sile, upravljanje, zakoni kretanja, program, inverzni problem, problem dva tela**

Forces du mouvement naturel et programmé

A la base des paramètres apperçus du mouvement d’un corps ou d’un système de corps, on cherche et trouve la force qui contraint le mouvement au corps. En suivant la même logique et le modèle mathématique pour la solution des problèmes, on cherche les forces dont l’action provoque un mouvement selon le programme present. Le problème de deux corps était résolu par déterminant la formule générale pour la force d’attraction mutuelle. Sous les conditions des lois de Kepler, la loi de gravitation de Newton est obtenue comme la conséquence de la formule générale pour la force d’attraction mutuelle.

Mots-clés: **forces, commande, lois de mouvement, programme, probleme inverse, problème de deux corps.**