UDC: 681.511.2:517.932(047)=20
COSATI: 14-02, 14-07

Dynamic analysis of linear singular systems using orthogonal functions

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Singular systems are those the dynamics of which is governed by a mixture of algebraic and differential equations. In that sense the algebraic equations represent the constraints to the solution of the differential part. These systems, also known as descriptor, semi-state and generalized systems, arise naturally as a linear approximation of system models, or linear system models in many applications such as electric networks, aircraft dynamics, neural delay systems, chemical, thermal and diffusion processes, large-scale systems, interconnected systems, economics, optimization problems, feedback systems, robotics, biology, etc. For an elementary dynamic analysis of singular systems their solution in state space is necessary. In classical sense it means that there is a need for calculating general or pseudo inversions of system matrices. On the other hand this is too complicated in numerical sense. So this paper investigates another possibility of solving system equations using different aproximations based on strict applications of very well-known orthogonal functions. Some numerical examples have been worked out to show the applicability of the presented results.

Key words: **linear singular systems, dynamic analysis, orthogonal functions.**

Introduction

LINEAR singular systems are those the dynamics of which is governed by a mixture of algebraic and differential equations. In that sense the algebraic equations represent the constraints to the solution of the differential part.

The existence (solvability), uniqueness and smoothness of solutions of linear singular systems, as well as their possible canonical forms, are the questions that must be carefully treated. They differ significantly from those established for normal systems. In that sense, our primary task is, before discussing any questions concerning stability problems for this class of systems, to indicate and demonstrate these problems clearly.

A particular problem is always connected with a need to find a state response of linear singular systems. In that sense two approaches have been usually accepted, both based on a strict application of general or so-called pseudo inversions of system matrices, which leads in the former case to the implementation of *Drazin* and in the latter case of *Moore-Penrose* inversions, see *Debeljković et al* [27,28]. The given examples show all general complexity of the proposed procedures.

In some cases, when *linear descriptive systems* are investigated, a state space response can be achieved by using a fundamental matrix which enables finding system solutions using the Laurent expansion only.

The problem presented, in the sequel, extends some known analytic techniques for the solutions of state-space equations of normal systems to the case of linear singular systems. This problem appears to introduce a serious computational task. To alleviate these numerical efforts, an approximate solution of linear singular systems by using orthogonal functions is proposed. This approximate solution is a direct extension of known results for normal systems by using orthogonal functions of type Walsh, Block-pulse, Laguerre, etc.

Since the basic paper of *Chen, Hsiao* appeared in 1975 [16], the system dynamic analysis by using orthogonal functions has become very popular.

This approach shows that the differential-algebraic system equations may be converted in to pure algebraic equations that can be solved in the terms of the orthogonal basic functions. The further numerical treatment of this problem is a very simple one.

The implementation of this approach has been used in several feedback singular control problems, *Rao* [55], *Paraskevopoulos* [52], *Campbell* [9], *Marszalek* [43], *Paraskevopoulos* [54], *Rao, Tzaffestast* [56], in optimal control *Chen, Shih* [21], in signal processing, telecommunications and pattern recognition *Ahmed, Rao* [1], linear system analysis and design *Chen, Hsiao* [17,18,19] and by some authors such as *Gantmacher* [30], *Luenberger* [38], *Campbell* [9], *Christodoulou et al.* [22], for some other purposes.

Although the solutions obtained by the use of orthogonal functions are approximative, necessary accuracy may be achieved with a sufficient number of basic functions. The

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advantages that can be achieved by this method are more than evident.

The expression *orthogonal functions* denotes piece-wise functions such as *Hamadar, Haar, Laguerre, Walsh and Block-pulse* or orthogonal polynomials such *Chebyshev, Legendre, Hermite* and some others.

The application of these functions in the system dynamic analysis and control is presented in the papers of *Rao* [55] and *Paraskevopoulos* [54].

Let there be prescribed a linear time invariant singular system of the form

$$E\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

where the state vector $\mathbf{x} \in \Re^n$, the control vector $\mathbf{u} \in \Re^m$, with the *singular matrix* $E \in \Re^{n\times n}$ and with the constant matrices $A \in \Re^{n\times n}$ and $B \in \Re^{n\times n}$.

In order that eq.(1) has a unique solution there must be $\det(sE + A) \neq 0$. Furthermore, when $\mathbf{x}_0$ is a consistent initial condition, then the solution eq.(1) is unique and contains no impulses, *Debeljkovic et. al* [25,26]. If $\mathbf{x}_0$ does not belong to the subspace of the consistent initial conditions, the solution of eq.(1) involves impulses, which is undesirable.

Clearly, working with the solution eq.(1) expressed using *Drazin inverse*, *Debeljkovic et. al* [24] requires a great amount of computational work. To overcome this difficulty there will be presented a method which reduces the problem of solving eq.(1) to that of solving an algebraic system of equations.

This method is based on the idea of approximating the solution $\mathbf{x}(t)$ by a truncated orthogonal series as follows

$$\mathbf{x}(t)=F\phi_r(t) \qquad F\in \Re^{n\times r} \tag{2}$$

$$\phi_r^T(t) = [\phi_0(t), \phi_1(t), ..., \phi_{r-1}(t)] \tag{3}$$

where $\phi_0(t)$, $\phi_1(t)$, …, $\phi_{r-1}(t)$ is a set of *r* basic functions, orthogonal at a certain interval [α,β], which is usually addopted as [0,1), *r being* an integer.

F is the $n\times r$ *constant matrix* to be determined.

The basic idea of eqs.(2) and (3) is to replace the unknown vector $\mathbf{x}(t)$ by the matrix F. This approach has already been applied to normal systems many times before.

Let $\mathbf{u}(t)$ be approximated by the following truncated orthogonal series

$$\mathbf{u}(t)=G\phi(t) \qquad G \in \Re^{n\times r} \tag{4}$$

where *G* is the $m\times r$ *known* matrix.

The known initial condition $\mathbf{x}_0 = \mathbf{x}(0)$, may be written as

$$\mathbf{x}_0 = Q\phi(t) \qquad Q \in \Re^{n\times r} \tag{5}$$

The choice of orthogonal series to be used depends on a problem to be solved. It is well known − *Chen, Hsiao* [16], *Chen, Shih* [21] − that a great number of basic orthogonal functions such as *Walsh, Block-pulse, Laguerre, Chebyshev, Fourier* and *Hermite* function have the following integral property

$$\int_0^t \phi_r(s)ds \approx P_r\phi_r(t) \qquad P_r \in C^{r\times r} \tag{6}$$

where P_r is the nonsingular constant matrix, often called *operational matrix*. It depends on both selected orthogonal functions and the number *r*.

Integrating eq.(1) gives

$$E\mathbf{x} - E\mathbf{x}_0 = \int_0^t A\boldsymbol{x}(s)ds - \int_0^t B\boldsymbol{u}(s)ds \tag{7}$$

Using eqs.(2), (3) and (4), one can get

$$EF\phi_r(t) - EQ\phi_r(t) = \int_0^t AF\phi_r(s)ds + \int_0^t BG\phi_r(s)ds \tag{8}$$

Substituting eq.(6) in eq.(8), yields

$$EF\phi_r(t) - EQ\phi_r(t) = AF P_r\ \phi_r(t) + BG P_r\ \phi_r(t) \tag{9}$$

and finally

$$AFP - EF = -EQ - BGP \tag{10}$$

In this way differential eq.(1) in $\mathbf{x}(t)$ is transformed into an algebraic eq.(10), which should be solved upon *F*, and when the solution is obtained it should be returned to eq.(2) which enables obtaining an approximative solution for $\mathbf{x}(t)$.

In should be noted that eq.(10) represents a so-called generalized Lyapunov algebraic equation, showing when it has solutions and what this means in terms of the admissible (consistent) initial subspace of system (1). Let us point out now that if *E=I* then eq.(10) is a *discrete-time Lyapunov equation* and if *A=I*, then it *is continuous-time Lyapunov equation*, the properties of both of which are well understood, *Chen, Hsiao* [16].

Determination of the system response using the Walsh functions and the Kronecker product

Suppose now that the initial condition, given as eq.(5), can be rewritten

$$\mathbf{x}(0) = [\,\mathbf{x}_0\ \mathbf{0}\ldots\mathbf{0}\,]\,\phi_r(t) = Q\,\phi_r(t) \tag{11}$$

The Walsh functions, due to their simple form, are especially attractive from the numerical point of view. In that case $r = 2^p$ should be chosen to form a complete set of orthogonal functions.

Then the Walsh operational matrix P_r has the form, *Chen, Shih* [21]:

$$P_r = \begin{bmatrix} P_{r/2} & -(1/2)I_{r/2} \\ (1/2r)I_{r/2} & 0_{r/2} \end{bmatrix} \tag{12}$$

where $I_r\,(0_r)$ is the $r\times r$ unit (null) matrix, respectively.

The recurssion procedure starts with the matrix $P_1 = \frac{1}{2}$.

Then eq.(10) can be written as

$$AFP - EF = D \tag{13}$$

where

$$D = -EQ - BGP \qquad D \in \Re^{n\times r} \tag{14}$$

If *X* is the ($p\times q$) matrix, then $\mathbf{v}(X)$ is the ($pq\times 1$) matrix formed by listing the *q* columns of the matrix *X* in order.

Then eq.(10) can be written as

$$M\mathbf{v}(F) = \mathbf{v}(D) \tag{15}$$

with

$$\mathbf{f} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{r-1} \end{bmatrix} \quad \mathbf{d} = \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{r-1} \end{bmatrix} \tag{16}$$

where f_i i d_i, i = 0, 1, 2,…, r-1 are the i-th columns of the matrix F and the matrix D, respectively.

The ($rn\times rn$) matrix M is given by

$$\begin{aligned} M &= A \otimes P^T - E \otimes I^T \\ &= A \otimes P^T - E \otimes I \end{aligned} \tag{17}$$

where ⊗ denotes the Kronecker product defined as follows

$$A \otimes P^T = \begin{bmatrix} p_{11}A & p_{21}A & \cdots & p_{r1}A \\ p_{12}A & p_{22}A & \cdots & p_{r2}A \\ \vdots & \vdots & \cdots & \vdots \\ p_{1r}A & p_{2r}A & \cdots & p_{rr}A \end{bmatrix} \tag{18}$$

Similary for $E\otimes I$.

Now it can be written

$$M = \begin{bmatrix} p_{11}A - E & p_{21}A & \cdots & p_{r1}A \\ p_{12}A & p_{22}A - E & \cdots & p_{r2}A \\ \vdots & \vdots & \cdots & \vdots \\ p_{1r}A & p_{2r}A & \cdots & p_{rr}A - E \end{bmatrix} \tag{19}$$

An alternative way for writting eqs.(13), (14) and (18) is given in *Barnet,* [2].

The solution of eq.(15) can be easily found so

$$\mathbf{v}(F) = M^{-1}\mathbf{v}(D) \tag{20}$$

The main difficulty of eq.(17) is that, because of the presence of the Kronecker products, a matrix with the dimensions ($nr\times nr$) has to be inverted.

To overcome this difficulty *Chen, Hsiao* [16], for the case of the Walsh functions, *Marszalek* [42], for the case of the Block-pulse functions and *Paraskevopulos* [52], for the case of the Chebyshev polynomials presented an algorithm which considerably eleminates the effort of solving eq.(17).

The second and more serious difficulty with eq.(17) is that it may be ill-conditioned or even singular. That it could be ill-conditioned follows from the fact that it can be singular for some nonzero values of the matrices A i E.

These cases appear to be difficult to be detected by establishing some types of criteria or conditions on the matrix M, owing to the complexity of the structure and the dependence of the structure of the matrix M on the number of expansion terms r.

Indeed, consider, for example, the matrix M for the (r+1) expansion terms

$$M_{r+1} = A \otimes P_{r+1}^T - E \otimes I =$$

$$= \begin{bmatrix} p_{11}A - E & p_{21}A & \cdots & p_{r1}A & p_{r+1,1}A \\ p_{12}A & p_{22}A - E & \cdots & p_{r2}A & p_{r+1,2}A \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ p_{1r}A & p_{2r}A & \cdots & p_{rr}A & p_{r+1,r}A \\ p_{1,r+1}A & p_{2,r+1}A & \cdots & p_{r,r+1}A & p_{r+1,r+1}A - E \end{bmatrix} \tag{21}$$

$$M_{r+1} = \begin{bmatrix} M_r & p_{r+1,q}A \\ P_{q,r+1}A & p_{r+1,r+1}A - E \end{bmatrix} \tag{22}$$
$$q = 1,2,...,r$$

The above expression shows that, if r is increased by one, then all the columns and rows of the matrix M undergo a change. This means that, if the matrix M_r is not invertible, it is easy to say whether the matrix M_{r+1} is invertible or not.

Using the attractive shifted Chebyshev polynomials, the matrix M may has this form

$$M_2 = \begin{bmatrix} \frac{3}{4} & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & -\frac{1}{4} \\ 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which is clearly not singular.

The example which illustrates the difference in possible approximation errors among different classes of orthogonal functions is given in *Paraskevopoulos* [53].

We will consider eq.(1) with $\det E = 0$ and to make it *tractable* we assume eq.(1) to be regular, eq.

$$\Delta(s) \equiv |sE - A| \neq 0$$

The regularity is equivalent to the existence and the uniqueness of the solution $\mathbf{x}(t)$, given $\mathbf{x}(0)$ and $\mathbf{u}(t)$. The roots of $\Delta(s)$ are called the *finite relative eigenvalues* of the matrix pair (E, A). These are simply the *finite zeros* of the pencil ($sE - A$).

The *infinite* zeros of the matrix pencil ($sE-A$) are the *infinite relative eigenvalues* of the pencil (E,A).

The *relative spectrum* of the matrix pencil (E,A) is the union of finite and infinite zeros.

The finite is denoted by $\sigma(E,A)$ spectrum of the matrix pencil (E, A).

The spectrum of the single matrix J is denoted by $\sigma(J)$.

The explanation of the mentioned difficulties (a problem when eq.(10) has solutions and what this means in the subspace of the consistent initial conditions for system (1)), is given in *Lewis* [33], *Wong* [59].

Theorem 1. Let the matrix pencil (E,A) be regular. Suposse λ_i is a finite relative eigenvalue of the matrix pencil (E,A) and μ_j is an eigenvalue of matrix P. Then the generalized Lyapunov eq.(10) has a unique solution for F, for all matrices B, G and Q, if and only if: $\lambda_i \mu_j \neq 1$ for all i and j.

Proof. Since eq.(1) is regular, there is no loss of generality in assuming that it is in the Weierstrass form

$$\dot{\boldsymbol{x}}_1 = J\boldsymbol{x}_1 + B_1\boldsymbol{u} \tag{23}$$

$$N\dot{\boldsymbol{x}}_2 = \boldsymbol{x}_2 + B_2\boldsymbol{u} \tag{24}$$

where J is in the Jordan form and N is the nilpotent matrix consisting of the Jordan blocks with the eigenvalue zero.

Let $\boldsymbol{x}_1 \in \Re^{n_1}$ and $\boldsymbol{x}_2 \in \Re^{n_2}$

Based on this, eq.(10) takes the form

$$JF_1P - F_1 = -Q_1 - B_1GP \tag{25}$$

$$F_2P - NF_2 = -NQ_2 - B_2GP \tag{26}$$

where matrices G and F in the new basis have been portioned to conform to the *slow* and *fast subsystems* (23) and (24), respectively.

Eq.(25) is now recognized as a *discrete Lyapunov equation* the properties of which are well-known. T has a unique solution F_1, for all matrices B_1, G i Q_1, if and only if $\lambda_i \in \sigma(J)$ and $\mu_j \in \sigma(P)$, then $\lambda_i\ \mu_j \neq 1$. However, $\sigma(J)$ coincides with $\sigma(E, A)$.

The *continous Lyapunov equation* has a unique solution F_2, for all matrices B_2, G i Q_2, if and only if $\lambda_i \in \sigma(N)$ and $\mu_j \in \sigma(P)$, then $\lambda_i - \mu_j \neq 1$. However, this is guaranteed since N is nilpotent and P is nonsingular.

The condition of the *Theorem* is equivalent to the nonsingularity of any matrix representation of the linear operator $f(F) = AFP - EF$, including the traditional Kronecker product representation. Since the matrix P depends on the basis set $\phi(t)$ selected and on the number of functions r in the set, it is clear that all choices of r may not be allowed for the given matrices E and A.

The next result gives an explicit expression for the solution F to eq.(10).

Theorem 2. Suppose that the matrix pencil (E,A) is regular and $\sigma(E,A) \cap \sigma(P^{-1})$ is an empty set.

Let $\Delta(s)=det(sE-A)= s^{n_1} + \alpha_1 s^{n_1-1} + ... + \alpha_{n_1}$ and k is the index of the matrix pencil (E,A), i.e. N in eq.(24) satisfies the expression $N^{k-1} \neq 0,\ N^k = 0)$.

Then the solution to eqs.(25) and (26) is given by

$$F_1 = \left[H_1\ JH_1\ ...\ J^{n_1-1}H_1\right]\times \begin{bmatrix} \alpha_{n_1-1} & \alpha_{n_1-1}I \cdots I \\ \alpha_{n_1-2} & \cdots & 0 \\ \vdots & & 0\ 0 \\ I & 0 & 0\ 0 \end{bmatrix} \begin{bmatrix} K \\ K_1P^{-1} \\ \vdots \\ K_1P^{-n_1+1} \end{bmatrix} \Delta^{-1}\left(P^{-1}\right) \tag{27}$$

$$F_2 = -\left[H_2\ NH_2\ ...\ N^{k-1}H_2\right] \begin{bmatrix} K_2 \\ K_2P^{-1} \\ \vdots \\ K_2P^{-k+1} \end{bmatrix} \tag{28}$$

where

$$H_1 \equiv [I \quad B_1],\ K_1 \equiv \begin{bmatrix} Q_1 \\ G \end{bmatrix} \tag{29}$$

$$H_2 \equiv [N \quad B_2],\ K_2 \equiv \begin{bmatrix} Q_2 \\ GP \end{bmatrix} \tag{30}$$

Proof. Write eq.(25) as $JF_1 - F_1P^{-1} = -H_1K_1$ and eq.(26) as $F_2P - NF_2 = -H_2K_2$.

Now the result follows by a trivial modification of the derivation in *Chen* [15].

Determination of the system response using the Block-pulse functions

It is well-known that the Walsh functions are closely related to the Block-pulse functions. The Walsh operational matrix of the integraton P may be given in the following way *Chen at al.* [16,18,19].

$$P = \frac{1}{r}WKW \tag{31}$$

where $1/r$ is the test period, $W \in \Re^{r\times r}$ is the well-known Walsh matrix consisting of +1 and −1 in a dyadic order and K is the Block-pulse matrix of the integration

$$K = \begin{bmatrix} 1\backslash 2 & 1 & 1 & \cdots & 1 \\ 0 & 1\backslash 2 & 1 & \cdots & 1 \\ 0 & 0 & 1\backslash 2 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1\backslash 2 \end{bmatrix} \tag{32}$$

Using the matrix K instead of the matrix P in the analysis of eq.(1), one can get

$$A\bar{F}K - E\bar{F} = D \tag{33}$$

$$\bar{D} = -E\bar{Q} - B\bar{G}K \tag{34}$$

where the matrices $\bar{F}$, $\bar{G}$ i $\bar{K}$ are the Block-pulse representations of state, initial condition, input $\mathbf{x}(t)$, $\mathbf{x}(0)$ and $\mathbf{u}(t)$, respectively.

It was shown in *Marszalek* [42] that the calculation of the piecewise constant solution of eq.(1) with the matrix K is equivalent to applying the trapezoidal rule of the integration.

Thus, using the basic results given in *Marszalek* [42], for $\mathbf{X}(s) = \boldsymbol{L}[\mathbf{x}(t)]$, one can get

$$\begin{aligned} &\mathbf{X}^s(z) \overset{\Delta}{=} \sum_{i=0}^{\infty} \mathbf{x}_i z^{-i} = \\ &= \mathbf{X}\left(\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}\right)\frac{2}{T}(1+z^{-1})^{-1},\quad x_i \in \Re \end{aligned} \tag{35}$$

where z^{-1} is the delay, $T = 1/r$.

Applying eq.(35) to eq.(1), written in the s domain, one can get, *Marszalek* [42]

$$\begin{aligned} &Es\mathbf{X}(s) - E\mathbf{X}(0) = A\mathbf{X}(s) + B\mathbf{U}(s) \\ &E\frac{2}{T}\frac{z-1}{z+1}\frac{t}{2}\left(1+z^{-1}\right)\mathbf{X}^s(z) - E\mathbf{x}(0) = \\ &= \frac{T}{2}B\left(1+z^{-1}\right)\mathbf{X}^s(z) + \frac{T}{2}B\left(1+z^{-1}\right)\mathbf{U}^s(z) \\ &\mathbf{U}^s(z) \square\ u_i z^{-1}, u_i \in \Re \end{aligned} \tag{36}$$

Then from eq.(36) one can have

$$\begin{aligned} &E\sum_{i=0}^{\infty} \mathbf{x}_i z^{-i}(z-1) - E\mathbf{x}(0) = \\ &= \frac{T}{2}A\sum_{i=0}^{\infty} \mathbf{x}_i z^{-i}(z+1) + \frac{T}{2}B\sum_{i=0}^{\infty} \mathbf{x}_i z^{-i}(z+1) \end{aligned} \tag{37}$$

Equating the coeficients of the like powers of z^{-i}, one can get

$$\text{For } z^{-1}: E\mathbf{x}_0 - E\mathbf{x}(0) = \frac{T}{2}A\mathbf{x}_0 - \frac{T}{2}B\mathbf{u}_0 \tag{38}$$

$$\begin{aligned} &\text{For } z^0, z^{-1}, ...: \\ &E\left(\mathbf{x}_i - \mathbf{x}_{i-1}\right) = \frac{T}{2}A\mathbf{x}_0 + \frac{T}{2}B\left(\mathbf{u}_i + \mathbf{u}_{i-1}\right) \end{aligned} \tag{39}$$

Next, for $\det(E-\frac{T}{2}A)\neq 0$, one can have

$$\mathbf{x}_0=(E-\frac{T}{2}A)^{-1}\ (E\mathbf{x}(0)+\frac{T}{2}B\mathbf{u}_0) \tag{40}$$

$$\mathbf{x}_i=\left(E-\frac{T}{2}A\right)^{-1}\cdot \cdot\left(\left(E+\frac{T}{2}A\right)\mathbf{x}_{i-1}+\frac{T}{2}B\left(\mathbf{u}_i+\mathbf{u}_{i-1}\right)\right),\ i=1,2,... \tag{41}$$

$\mathbf{x}_i$ (i=0,1,…) represents the Block-pulse solution of eq.(1), but it is easy now to express the piecewise constant solution in terms of the Walsh functions.

It can be done with

$$F=(\frac{1}{2})\bar{F}W \tag{42}$$

where $\bar{F}=[\mathbf{x}_0,\mathbf{x}_1,...,\mathbf{x}_r]$.

Determination of the system response using the single-term Walsh functions

Time invariant systems

The proposed method is a very simple one and can be easily implemented on digital computers. It is also highly stable because it is based on the trapezoidal rule.

Rao et al. (1980) and *Rao* [55] introduced the single-term Walsh series (STWS) to remove the inconveniences in the Walsh functions (WF) and the Block-pulse functions (BPF).

Palanisamy [48], *Palanisamy, Balachandran* [49,50] and *Palanisamy, Rao* (1983) introduced the STWS approach to the analysis and optimal control of linear and non-linear systems. The STWS method, also finds its application in the analysis of time varying and non-linear networks and soothing circuits, *Palanisamy* [48].

The STWS method provides the Block-pulse and discrete solutions of problems for any lenght of time in an easy manner. This is not possible with the WF and BPF techniques.

Consider the linear singular system given by eq.(1).

With the STWS approach, the given function is expanded as a single-term Walsh series in the normalized time interval $\tau\in[0,1)$, which corresponds to the interval $t\in[0,\frac{1}{r})$.

by defninig $t=\frac{\tau}{r}$, r being an integer.

In the normalized interval, eq.(1) becomes

$$E\dot{\mathbf{x}}(\tau)=\frac{A}{r}\mathbf{x}(\tau)+\frac{A}{r}\mathbf{u}(\tau) \tag{43}$$

Now expanding $\dot{\mathbf{x}}(\tau)$, $\mathbf{x}(\tau)$ and $\mathbf{u}(\tau)$, in the STWF as

$$\dot{\mathbf{x}}(\tau)=C_i\Psi_0(\tau),\ \ \mathbf{x}(\tau)=B_i\Psi_0(\tau)$$
$$\mathbf{u}(\tau)=H_i\Psi_0(\tau) \tag{44}$$

the following recursive relationship is obtained with E=1/2

$$C_i=(E-A/2r)^{-1}G_i$$
$$B_i=(1/2)C_i+\mathbf{x}(i-1) \tag{45}$$
$$\mathbf{x}(i)=C_i+\mathbf{x}(i-1)$$

where $G_i=\frac{A}{r}\mathbf{x}(i-1)+\frac{B}{r}H_i$, $i=1,2,\ldots$

The $\mathbf{x}(i)$ gives the discrete values of the state and B_i gives the BPF values of the state for any lenght of time.

This is the main advantage of the presented method. Even though the matrix E is singular, the difference (E–A/2r) turns out to be non-singular. The value of r can be selected to be large to increase the accuracy of the results and each unit interval consists of r Block-pulses.

Time varying systems

Let us now consider a linear time-varying singular system

$$E(t)\dot{\mathbf{x}}(t)=A(t)\mathbf{x}(t)+B(t)\mathbf{u}(t),\ \ \mathbf{x}(0)=\mathbf{x}_0 \tag{46}$$

with the ($n\times n$) matrix $E(t)$ being singular for any t, so it can not be written in a classical form, $A(t)$ is the ($n\times n$) time dependent matrix, B is also the ($n\times m$) time dependent matrix, $\mathbf{x}(t)$ is the ($n\times 1$) state vector and $\mathbf{u}(t)$ is the ($m\times 1$) input (control) vector.

With the STWS approach, the given function is expanded as a single-term Walsh series in the normalized time interval $\tau\in[0,1)$, which corresonds to the interval $t\in[0,\frac{1}{r})$

by defining $t=\frac{\tau}{r}$, r being an integer.

Equation (46), at the normalized interval, becomes

$$E(\tau)\dot{\mathbf{x}}(\tau)=\frac{A(\tau)}{r}\mathbf{x}(\tau)+\frac{B(\tau)}{r}\mathbf{u}(\tau) \tag{47}$$

Now by expanding $E(\tau)$, $A(\tau)$, $B(\tau)$, $\dot{\mathbf{x}}(\tau)$, $\mathbf{x}(\tau)$ and $\mathbf{u}(\tau)$ in the STWS, as

$$\dot{\mathbf{x}}(\tau)=C_i\Psi_0(\tau),\ \ \mathbf{x}(\tau)=B_i\Psi_0(\tau) \tag{48a}$$

$$\mathbf{u}(\tau)=H_i\Psi_0(\tau),\ \ E(\tau)=M_i\Psi_0(\tau) \tag{48b}$$

$$A(\tau)=S_i\Psi_0(\tau),\ \ B(\tau)=Y_i\Psi_0(\tau) \tag{48c}$$

the following recursive relationship is obtained with E=1/2

$$C_i=[M_i-S_i/2r]^{-1}G_i \tag{49a}$$

$$B_i=(1/2)C_i+\mathbf{x}(i-1) \tag{49b}$$

$$\mathbf{x}(i)=C_i+\mathbf{x}(i-1) \tag{49c}$$

where

$$G_i=[S_i\mathbf{x}(i-1)+Y_iH_i]/r,\quad i=1,2,... \tag{50}$$

The $\mathbf{x}(i)$ gives the discrete values of the state and B_i gives the BPF values of the state for any lenght of time. This is the main advantage of this method.

In eq.(49), the matrix $[M_i-S_i/2r]$ has to be inverted at each step and all other operations are matrix additions and/or multiplications. The value of r can be selected to be large enough to increase the accuracy of the results and each unit interval consists of r Block-pulses.

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Appendix A - Notation

A	– matrix
a	– elements of matrix A
B	– matrix, control or input matrix
b	– elements of matrix B, positive number
C	– matrix, output matrix
c	– elements of matrix C, positive number
D	– matrix
d	– elements of matrix D, positive number
E	– singular matrix
e	– elements of matrix E, positive number
F	– matrix
f	– elements of matrix F, function
G	– matrix
g	– elements of matrix G, positive number
H	– matrix
h	– elements of matrix H
I	– unit matrix
i	– current number
J	– matrix, matrix in the Jordan form
j	– index of matrix, scalar, current number
K	– matrix
M	– matrix
m	– positive number
N	– nilpotent matrix
n	– system order
P	– matrix
p	– elements of matrix P, positive number
$p(\cdot)$	– polunomial
Q	– matrix
q	– rank of matrix Q, generaliyed system order, positive number
R	– set of real numbers
$R(s)$	– polynomial
r	– positive number
s	– complex variable
S	– matrix
t	– time
U	– matrix
$\mathbf{u}(t)$	– input vector, control vector
$\mathbf{v}$	– vector
V	– linear transformation matrix
W	– matrix
W_k	– subspace of consistent initial conditions
$\mathbf{w}$	– vector
$\mathbf{x}(t)$	– state vector
$\mathbf{x}_i$	– output vector
Z	– matrix
α	– real, positive scalar
β	– real, positive scalar
δ	– small positive number
$\delta(t)$	– impulse function
ε	– small positive number
$\phi(t)$	– orthogonal function
ν	– multiplicity
λ	– complex number, scalar, eigenvalue
μ	– constant, eigenvalue
π	– constant
$\sigma([\cdot])$	– singular value of matrix $[\cdot]$
$\sigma\{[\cdot]\}$	– eigenvalue matrix spectar $[\cdot]$
τ	– time constant, dimensionless time, time
φ	– set of orthogonal functions
$\varphi(t)$	– time dependent function
ϕ	– orthogonal function
$\Delta(s)$	– characteristic polynomial
Ψ_0	– single-term orthogonal function.

Particular notations

$\aleph[\cdot]$	– null space of matrix (kernel)
□	– range of matrix
(E,A)	– matrix pencil degree degree of polynomial
det$[\cdot]$	– matrix determinant
$[\cdot]$	– diagonal matrix
ind$[\cdot]$	– index of matrix
rank$[\cdot]$	– rank of matrix
tr$[\cdot]$	– trace of matrix
$\otimes$	– Kronecker product
$\oplus$	– direct sum
Σ	– sum
□	– given by definition
$\equiv$	– identical
$\Rightarrow$	– follows
$\in$	– belongs
$\forall$	– for every
$\subseteq$	– subset
$\subset$	– real subset
$\cap$	– intersection
$\cup$	– union

Appendix B – Numerical examples: On using the Walsh functions

Example B1. For r = 2 one can have

$$P_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{4} & 0 \end{bmatrix} \tag{B1}$$

For $r = 4$ one can have

$$P_4 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{8} & 0 \\ \frac{1}{4} & 0 & 0 & -\frac{1}{8} \\ \frac{1}{8} & 0 & 0 & 0 \\ 0 & \frac{1}{8} & 0 & 0 \end{bmatrix} \tag{B2}$$

Example B2. Consider a linear singular system, in the form of eq.(1), where

$$E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{B3}$$

with the initial condition $\mathbf{x}_0 = [1 \ \ 0]^T$.

Let the input function u(t) equal 1. To find the solution $\mathbf{x}(t)$ using the algorithm proposed, one can use eq.(15) direcetly. In this example we are going to use the Walsh functions. For numerical computations we adopt $r = 2$ which gives the matrix P in the following form

$$P=\begin{bmatrix}\frac{1}{2} & -\frac{1}{4}\\ \frac{1}{4} & 0\end{bmatrix} \tag{B4}$$

Also $\mathbf{u}(t)=\mathbf{h}^T\phi(t)$, where $\mathbf{h}^T=[1\ \ 0]$. Then the matrix M and the vector $\mathbf{d}$, in eq.(15), become

$$M=A\otimes P^T-E\otimes I^T= \\ =\begin{bmatrix}0 & -1 & 0 & 0\\ -\frac{1}{2} & 0 & -\frac{1}{4} & 0\\ 0 & 0 & 0 & -1\\ \frac{1}{4} & 0 & 0 & 0\end{bmatrix} \tag{B5}$$

and

$$\mathbf{d}=\begin{bmatrix}0\\ -\frac{1}{2}\\ 0\\ \frac{1}{4}\end{bmatrix} \tag{B6}$$

Solving eq.(15), one can obtain

$$\mathbf{f}^T=[1000] \tag{B6}$$

So, finally

$$x_1=[1\ 0]\phi(t)=\phi_0(t) \tag{B7a}$$

$$x_2=[0\ 0]\phi(t)=0 \tag{B7b}$$

It should be noted that if r is greater, the results will still be the $x_1=1$ and $x_2=0$.

To check, from the state equation $\dot{x}_2=0$ and $-x_1+u=0$, are obtained so it is obvious that $x_2=(x_2)_0$ and $x_1=u$. But since the initial condition $(x_2)_0=0$, it follows that $x_2=0$, as well. Besides that, using the complex domain, it can be shown

$$\mathbf{X}(s)=(Es-A)^{-1}\mathbf{b}U(s)=\begin{bmatrix}1/s\\ 0\end{bmatrix} \tag{B8}$$

i.e. $X_1(s)=1/s$ and $X_2(s)=0$.

So it follows: $x_1=1$ and $x_2=0$, whic had to be shown.

Example B3. Let us make a slight change of the previous example

$$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad E=\begin{bmatrix}1/4 & 0\\ 0 & 0\end{bmatrix} \tag{B9}$$

Using the method of the previous example and with r=2 we can use again the Walsh functions, such that

$$P=\begin{bmatrix}\frac{1}{2} & -\frac{1}{4}\\ \frac{1}{4} & 0\end{bmatrix} \tag{B10}$$

From eqw.(19) we can obtain the matrix M

$$M=\begin{bmatrix}\frac{1}{2}A-E & -\frac{1}{4}A\\ \frac{1}{4}A & -E\end{bmatrix}=\begin{bmatrix}\frac{1}{4} & 0 & -\frac{1}{4} & 0\\ 0 & \frac{1}{2} & 0 & \frac{1}{4}\\ \frac{1}{4} & 0 & -\frac{1}{4} & 0\\ 0 & \frac{1}{4} & 0 & 0\end{bmatrix} \tag{B11}$$

which is obviously nonsingular.

It can be shown that eq.(19) may be singular even if the system is nonsingular.

Example B4. Let us take

$$A=\begin{bmatrix}4 & 0\\ 0 & 0\end{bmatrix}\quad E=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{B12}$$

and the matrix P and r as in the previous example.

Then

$$M=\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix} \tag{B13}$$

The fact that the matrix M may be singular or even ill-conditioned always makes trouble. If for a moment we suppose that the matrix M is nonsingular, then the approach based on the orthogonal functions gives an estimatie of the solution for any initial condition $\mathbf{x}_0$.

But we must always have in mind that eq.(1) has only *smooth solutions* whenever $\mathbf{x}_0$ belongs to the subspace of the consistent initial conditions W_k, e.i. $\mathbf{x}_0\in W$, for the given input function $\mathbf{u}(t)$, *Campbell* [7]. In that way we are going to have the solutions regardless they exist or not for a given initial condition. Particularly, if the input function $\mathbf{u}(t)$ is piecewise we should not have any distributional response, which usually happens in the presence of discontinuity. Of course if the input function $\mathbf{u}(t)$ is quite enough *smooth* and $\mathbf{x}_0$ belongs to the subspace of the consistent initial conditions an approach which ignores the presence of impulses in the system solution has obvious advantages.

Example B5. Let us examine the *Example* B.3, with

$$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\quad E=\begin{bmatrix}1/4 & 0\\ 0 & 0\end{bmatrix} \tag{B14}$$

By the use of the Walsh functions, the matrix M, for r=2 is singular, which can be seen from *Example* B.3, but for r=4 it is

$$M_4=\begin{bmatrix}\frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{8} & 0 & 0 & 0\\ 0 & \frac{1}{2} & 0 & \frac{1}{4} & 0 & \frac{1}{8} & 0 & 0\\ -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 0 & 0 & \frac{1}{8} & 0\\ 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & \frac{1}{8}\\ -\frac{1}{8} & 0 & 0 & 0 & -\frac{1}{4} & 0 & 0 & 0\\ 0 & -\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{1}{8} & 0 & 0 & 0 & -\frac{1}{4} & 0\\ 0 & 0 & 0 & -\frac{1}{8} & 0 & 0 & & 0\end{bmatrix}$$

which is obviously nonsingular.

Appendix C - Numerical examples : On using the Block-pulse functions

Example C1. Let us consider eq.(1) with

$$E=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\quad A=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\quad \mathbf{b}=\begin{bmatrix}1/2\\ 1\end{bmatrix} \tag{C1}$$

$$u(t)=1(t)\quad P=\begin{bmatrix}\frac{1}{2} & -\frac{1}{4}\\ \frac{1}{4} & 0\end{bmatrix}\quad \mathbf{x}(0)=\mathbf{0} \tag{C2}$$

The matrices D and M are in the form (r=2, $\mathbf{u}_0=\mathbf{u}_1=1$, Q=0)

$$D=\begin{bmatrix}-\frac{1}{4} & \frac{1}{8}\\ -\frac{1}{8} & \frac{1}{4}\end{bmatrix} \tag{C3}$$

$$M=\begin{bmatrix}-\frac{1}{2} & 0 & -\frac{1}{4} & 0\\ 0 & \frac{1}{2} & 0 & \frac{1}{4}\\ \frac{1}{4} & 0 & -1 & 0\\ 0 & -\frac{1}{4} & 0 & 0\end{bmatrix} \tag{C4}$$

So

$$\mathbf{v}(F)=M^{-1}\mathbf{v}(D)=\begin{bmatrix}\frac{9}{50} & -1 & -\frac{2}{25} & 0\end{bmatrix}^T \tag{C5}$$

$$F=\begin{bmatrix}9/50 & -2/25\\ -1 & 0\end{bmatrix} \tag{C6}$$

On the other hand, from eq.(42), for T = 1/r = 1/2, one can obtain

$$\mathbf{x}_0=\begin{bmatrix}1/10\\ -1\end{bmatrix}\quad \mathbf{x}_1=\begin{bmatrix}13/50\\ -1\end{bmatrix} \tag{C7}$$

and from (43)

$$F=\frac{1}{2}\begin{bmatrix}1/10 & 13/50\\ -1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}= \\ =\begin{bmatrix}9/50 & -2/25\\ -1 & 0\end{bmatrix} \tag{C8}$$

These results show that the procedure of solving eq.(1) with the Walsh operational matrices may be achieved by using very simple recursive algorithms. Besides, if follows from eq.(42) that the recursive schema gives a piecewise solution if $\det(E-\frac{T}{2}A)\neq 0$.

Appendix D - Numerical examples : On using the single-term Walsh functions

Example D1. Let us consider a linear, time invariant singular system, given by eq.(1), where

$$E=\begin{bmatrix}1 & 0 & -2\\ -1 & 0 & 2\\ 2 & 3 & 2\end{bmatrix}\quad B=\begin{bmatrix}1 & 2 & 1\\ 2 & -1 & -3\\ 0 & 1 & 1\end{bmatrix} \tag{D1}$$

$$A=\begin{bmatrix}0 & -1 & -2\\ 27 & 22 & 17\\ -18 & -14 & -10\end{bmatrix}\quad \mathbf{u}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix} \tag{D2}$$

$$\mathbf{x}_0=\begin{bmatrix}0.4123\\ 0.0769\\ -1.2500\end{bmatrix} \tag{D3}$$

The exact solution of eq.(1) is given by *Campbell et al.* [26]

$$\begin{aligned} x_1(t)&=\frac{7}{52}e^{\frac{2}{3}t}-t+\frac{5}{18}\\ x_2(t)&=\frac{14}{13}e^{\frac{2}{3}t}+2t-1\\ x_3(t)&=-\frac{7}{4}e^{\frac{2}{3}t}-t+0.5\end{aligned} \tag{D4}$$

By using eq.(45) and eq.(D.1), the discrete time solution $\mathbf{x}^*(t)$ and the exact solution $\mathbf{x}(t)$ are calculated for m = 100.
The results are shown in Tables 1 - 3.

Table D1. Solution of eq.(45) and eq.(D4) for $x_1(t)$

Solution No.	Time	$x_1(t)$ (exact solution)	$x_1{}^*(t)$ (STWS, m=100)
1	0	0.4123	0.4123
2	0.5	-0.0343	-0.0348
3	1.0	-0.4600	-0.4610
4	1.5	-0.8563	-0.8577
5	2.0	-1.2115	-1.2137
6	2.5	-1.5095	-1.5122
7	3.0	-1.7275	-1.7308

Table D2. Solution of eq.(45) and eq.(D4) for $x_2(t)$

Solution. No.	Time	$x_2(t)$ (exact solution)	$x_2{}^*(t)$ (STWS, m=100)
1	0	0.0769	0.0769
2	0.5	1.5029	1.5034
3	1.0	3.0975	3.0985
4	1.5	4.9273	4.9288
5	2.0	7.0854	7.0876
6	2.5	9.7017	9.7045
7	3.0	12.9574	12.9607

Table D3. Solution of eq..(45) and eq.(D4) for $x_3(t)$

Solution No.	Time	$x_3(t)$ exact solution	$x_3{}^*(t)$ (STWS, m=100)
1	0	-1.2500	-1.2500
2	0.5	-2.4423	-2.4425
3	1.0	-3.9085	-3.9090
4	1.5	-5.7569	-5.7577
5	2.0	-8.1389	-8.1400
6	2.5	-11.2653	-11.2667
7	3.0	-15.4308	-15.4324

Example D2. Let us consider a linear, *time varying* singular system, given by eq.(46), where

$$E(t)=\begin{bmatrix}0 & 0\\ 1 & t\end{bmatrix}\quad A(t)=\begin{bmatrix}-1 & 1-t\\ 0 & -2\end{bmatrix} \tag{D5}$$

$$B(t) = \begin{bmatrix} e^t & 1 \\ t^2 & 2 \end{bmatrix} \quad \mathbf{x}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{D6}$$

The exact solution of eq.(46) is given by

$$\begin{aligned} x_1 &= (1+t)e^t - t^3 \\ x_2 &= t^2 - e^t \end{aligned} \tag{D7}$$

Using eq.(49) and eq.(D7), the discrete solution $\mathbf{x}^*(t)$ and the exact solution are calculated for m=16 and m=100.

The results are shown in Tables D4 anf D5.

The approximated solutions agree quite well with the exact solutions.

Table D4. Solution of eq.(49) and eq.(D7) for $x_1(t)$

Solution No.	Time	$x_1(t)$ (exact solution)	$x_1^*(t)$ (STWS, m=16)	$x_1^*(T)$ (STWS, m=100)
1	0.00	1.0000	1.0000	1.0000
2	0.25	1.5894	1.5894	1.5877
3	0.50	2.3480	2.3477	2.3480
4	0.75	3.2828	3.2815	3.2878
5	1.00	4.4365	4.4335	4.4365
6	1.25	5.9001	5.8943	5.8917
7	1.50	7.8292	7.8192	7.8290
8	1.75	10.4658	10.4495	10.4770
9	2.00	14.1671	14.1420	14.1665

Table D5. Solution of eq.(49) and eq.(D7) for $x_2(t)$

Solution No.	Time	$x_2(t)$ (exact solution)	$x_2^*(t)$ (STWS, m=16)	$x_2^*(T)$ (STWS, m=100)
1	0.00	-1.0000	-1.0000	-1.0000
2	0.25	-1.2215	-1.2211	-1.2149
3	0.50	-1.3987	-1.3977	-1.3987
4	0.75	-1.5545	-1.5527	-1.6511
5	1.00	-1.7182	-1.7155	-1.7182
6	1.25	-1.9278	-1.9238	-1.9211
7	1.50	-2.2316	-2.2260	-2.2316
8	1.75	-2.6921	-2.6844	-2.6985
9	2.00	-3.3890	-3.3787	-3.3887

Received: 28.02.2002

Primena ortogonalnih funkcija u dinamičkoj analizi linearnih singularnih sistema

Singularni sistemi predstavljeni su u matematičkom smislu kombinacijom diferencijalnih i algebarskih jednačina, pri čemu ove druge predstavljaju ograničenje koje opšte rešenje mora da zadovolji u svakom trenutku. Primera singularnih sistema ima skoro u svim granama nauke i tehnike. Javljaju se često u elektromagnetnim kolima, dinamici robota i letelica, optimizacionim problemima i u graničnom slučaju singularno-perturbovanih sistema. Sa stanovišta elementarne dinamičke analize, uvek je potrebno poznavati njihovo kretanje u prostoru stanja. U klasičnom smislu to podrazumeva izračunavanje generalisanih inverzija sistemskih matrica što predstavlja veoma složenu numeričku proceduru. U ovom radu dat je jedan drugi prilaz, koji koristeći dobro poznate ortogonalne funkcije pruža dobru mogućnost da se, korišćenjem aproksimativnog prilaza baziranog na pomenutim funkcijama, odredi traženo kretanje singularnog sistema. Teorijska izlaganja propraćena su sa nekoliko pažljivo odabranih primera.

Ključne reči: **linearni sistemi, singularni sistemi, dinamička analiza sistema, ortogonalne funkcije.**

Application des fonctions orthogonales dans l'analyse dynamique des systèmes singulaires et linéaires

Les systèmes singulaires sont présentés, mathématiquement, comme la combinaison des équations différentielles et algébriques. Les équations algébriques sont la contrainte pour la solution des équations différentielles. Tels systèmes sont souvent appliqué dans les réseaux électro-magnétiques, robotique, dynamique d' aéronefs, problèmes d'optimisation et le cas limite des systèmes singulaires et perturbés. Pour l'analyse dynamique élémentaire, il est nécessaire de savoir leur solution dans l'espace d'état, c'est-à-dire de calculer les inversions généralisées des matrices de système − un procédé numériquement très compliqué. Une autre solution est ici proposée, une possibilité de resoudre les équations de système en utilisant les approximations différentes basées sur les applications des fonctions orthogonales três connues. Le discours théorique est suivi par quelques exemples soigneusement choisis.

Mots-clés: **systèmes linéaires, systèmes singulaires, analyse dynamique du système, fonctions orthogonales.**