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Position error of a laser-illuminated object

Žarko Barbarić, PhD (Eng)[1)]

Position error of a laser-illuminated object (target) is analyzed. The relations for the error signal and the probability density function for positioning error signal are given. The probability of positioning and the position error are derived. The position error of a laser-illuminated object depends on the signal-to-noise ratio and the mean value of the error signal. The position error is inversely proportional to the square root of the signal-to-noise ratio. Also, the position error increases if the mean value of the error signal increases. The minimum position error is obtained for the mean value of the error signal equal to zero, and goes to the maximum when the mean value equals one. The minimum position error is the square root two times smaller than the maximum value, for a constant signal-to-noise ratio.

Key words: **quadrant photodiode, probability density function, probability of positioning, estimating of position accuracy.**

Introduction

THERE are a number of applications in which accurate positioning is needed, for example in industry and army [1]. The main characteristic in these applications is accuracy of positioning of an object. Laser positioning systems have special treatment, because their resolution is higher than the resolution in radar and other systems. A number of applications of laser positioning include tracking of illuminated target and measuring its angular position [1], estimating vibration effects of satellite [2], and measuring multidimensional displacement in space [3]. These and other applications imply the use of the quadrant photodiode (QPD) to measure lateral displacement in two perpendicular planes. A high-resolution multidimensional displacement monitoring system is developed with four quadrant photodiodes [3] that can be used for a monitoring with six degrees of freedom. The results have shown that the lateral resolution is better than 50 nm and the angular displacement is better than 0.25 micro-radians [3].

Theoretical analysis of position error is discussed in [1] and [4]. Both papers have pointed out that the main parameter, is the signal-to-noise ratio, which limits accuracy of positioning in laser systems with the quadrant photodiode. Also, the position error was found to change with the mean value of the error signal of positioning. The probability density function for the Gaussian distribution of noise current correlation among quadrants is presented in [5]. The probability density function for the Gaussian distribution and uncorrelated noise between quadrants current in each of them is derived in [6] and the same expression is obtained as the one presented in [7].

In this paper two approaches will be given to estimate the error of positioning of laser-illuminated object by the quadrant photodiode. The first one is deterministic, based on the calculation of the position error from the error signal of positioning. The second approach is statistic, where the position error is calculated on the basis of the probability density function and the probability that the error signal will be found between a given limit around the mean value of the error signal of positioning.

Error signal

The positioning of a laser-illuminated object is based on focusing the reflected laser beam from an object by the receiving optics. Fig.1 shows two main components – a thin lens, quadrant photodiode, and the basic optical geometry. The incident reflected laser energy arrives onto the thin lens and is focused on the quadrant photodiode surface. In Fig.1 incident radiation is incoming with the angle δ with respect to the normal on the lens surface and producing a spot with a center (C) on the quadrant photodiode (QPD). The cartesian coordinate system $x0y$ is set in the center of photodiode, and the center of the spot C has the coordinates x_0, y_0. Fig.1 shows that the point C is a function of the incident angle δ, for a constant distance (d) between the thin lens and the photodiode. From the geometry showed in Fig.1, the following is obtained: $\mathrm{tg}\delta_h = x_0/d$ and $\mathrm{tg}\delta_v = y_0/d$, where δ_h and δ_v are the angles in horizontal and vertical planes, respectively.

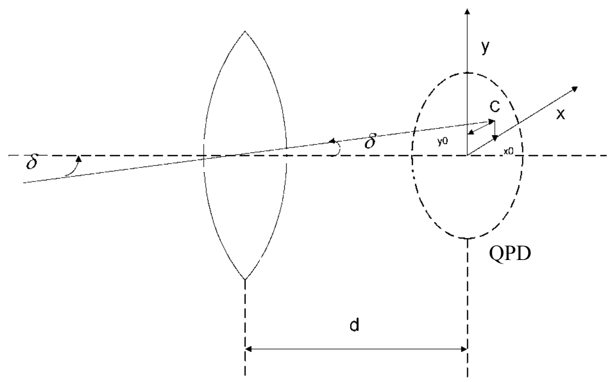

Figure 1. The geometry between the thin lens and the quadrant photodiode

[1)] Military Technical Institute (VTI) of the Yugoslav Army, Katanićeva 15, 11000 Beograd

A spot position on the quadrant photodiode with diameter 2a is shown in Fig.2. The diameter of the spot $2r$, is a function of the distance d, and the focal length of the thin lens.

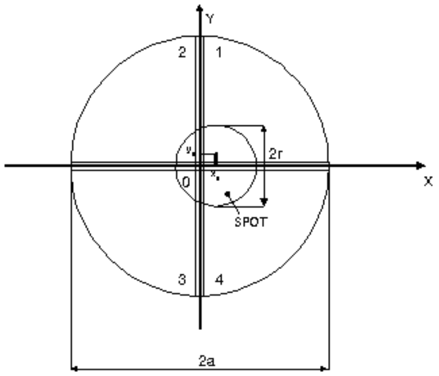

Figure 2. A spot position on the quadrant photodiode surface

The error signals, in a normalized form, for the horizontal and vertical plane, from Fig.2, are

$$\begin{aligned} \varepsilon_x &= \frac{(i_1+i_4)-(i_2+i_3)}{(i_1+i_4+i_2+i_3)} = \frac{i_x}{i_\Sigma} \\ \varepsilon_y &= \frac{(i_1+i_2)-(i_3+i_4)}{(i_1+i_2+i_3+i_4)} = \frac{i_y}{i_\Sigma} \end{aligned} \qquad (1)$$

where i_i is the current of the i-th quadrant (i =1,2,3,4).

The current of the i-th quadrant is $i_i=\Re q P_0$, where P_0 is the total received flux, q is a fraction ($1\geq q\geq 0$), and $\Re$ is the responsivity of the photodiode. The total current is the sum of four currents ($i_\Sigma=\Re P_0$).

There are a number of models for calculating the error signal as a function of displacement of the spot center with respect to the center of the photodiode. One of the models assumes a constant distribution of the irradiance on the sensitive photodiode surface and the limited spot size with square and circle shapes [4]. Other models assume the Gaussian or the sinc distribution of the irradiance on the sensitive photodiode surface.

From the first simple model, the normalized error signals are [4]

$$\begin{aligned} \varepsilon_x &= \frac{x_0}{r} = \frac{dtg\delta_h}{r}, |x_0| \leq r \\ \varepsilon_y &= \frac{y_0}{r} = \frac{dtg\delta_v}{r}, |y_0| \leq r \end{aligned} \qquad (2)$$

where r is the half size of the square spot.

Expressions (2) are the linear approximation of the error signal, for the displacement center of the spot and the center of the photodiode, smaller than the spot radius. From (2) the error signal is approximately directly proportional to the incident angle, because $\mathrm{tg}\delta \approx \delta$ for a small value of δ ($\delta\leq 10^0$). The ratio d/r is the constant of the position sensor ($d/r = K_D$).

The normalized forms of the error signal for the constant irradiance distribution on the circle spot, from Fig.2, are [4]

$$\begin{aligned} \varepsilon_x &= \frac{2}{\pi}\left(\frac{x_0}{r}\sqrt{1-\frac{x_0^2}{r^2}} + \arcsin(\frac{x_0}{r})\right), \quad |x_0| \leq r \\ \varepsilon_y &= \frac{2}{\pi}\left(\frac{x_0}{r}\sqrt{1-\frac{y_0^2}{r^2}} + \arcsin(\frac{y_0}{r})\right), \quad |y_0| \leq r \end{aligned} \qquad (3)$$

where the displacement of the center of the spot and the center of the photodiode, is smaller than the radius of spot.

The normalized forms of the error signal, for the Gaussian distribution of the irradiance with the center (x_0,y_0) and the square shape of the photodiode with size 2a, are [4]

$$\begin{aligned} \varepsilon_x &= \frac{2erf\left(\frac{x_0}{\sqrt{2}\sigma}\right) + erf\left(\frac{a-x_0}{\sqrt{2}\sigma}\right) - erf\left(\frac{a+x_0}{\sqrt{2}\sigma}\right)}{erf\left(\frac{a-x_0}{\sqrt{2}\sigma}\right) + erf\left(\frac{a+x_0}{\sqrt{2}\sigma}\right)} \\ \varepsilon_y &= \frac{2erf\left(\frac{y_0}{\sqrt{2}\sigma}\right) + erf\left(\frac{a-y_0}{\sqrt{2}\sigma}\right) - erf\left(\frac{a+y_0}{\sqrt{2}\sigma}\right)}{erf\left(\frac{a-y_0}{\sqrt{2}\sigma}\right) + erf\left(\frac{a+y_0}{\sqrt{2}\sigma}\right)} \end{aligned} \qquad (4)$$

where σ- is the standard deviation of the two-dimensional Gaussian distribution irradiance on the sensitive photodiode surface, and $erf(x)$ is the error function, where

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-u^2)du$$

The standard deviation σ in (4) and the radius of the spot r, in (3) are related. The part of flux in the circle of the radius r, with respect to the total incident flux is

$$\frac{1}{2\pi\sigma^2}\int_0^{2\pi} d\varphi \int_0^r \exp(-\frac{\rho^2}{2\sigma^2})\rho d\rho = 1-\exp(-\frac{r^2}{2\sigma^2}) \qquad (5)$$

For example, from eq.(5), 90% of the total flux is within the circle with the radius r=2.1456σ.

The normalized error signals from (2), (3) and (4) are given in Fig.3, for r=2.1456σ (r=2.25 mm, σ=1.048 mm).

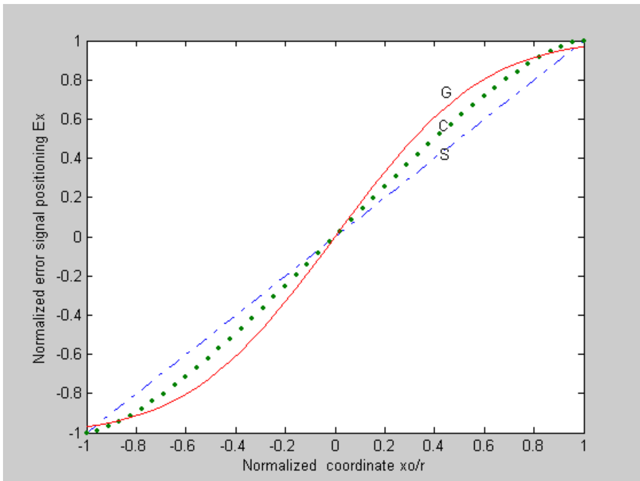

Figure 3. Normalized error signal ε_x as a function of displacement of the spot center and the photodiode center, for the constant (S-square, C-circle spot shape) and the Gaussian distribution (G) of irradiance on the photodiode surface

Fig.3 shows that the error signal changes from minus one up to one. The diagrams in Fig.3 show very good agreement of models for small displacements of the spot center and the photodiode center. The model assuming a constant irradiance and a circle shape of the spot (diagram C) can be used as a good approximation of the Gaussian irradiance distribution (diagram G). The simplest linear model can be used for small displacements, as shown in the diagram S in Fig.3.

Position error

Expressions for the error signal (2), (3) and (4) are derived with assumptions for the spot geometry and the irradiance distribution. The error signal in reality is a sum of signal and noise. The noise added to the signal has influence on changing a spot center around a point x_0,y_0. The position error can be understood as the changing of the spot center, which can not be detected. These changes of the spot center Δx_0 and Δy_0 can be estimated from the expressions for the error signal.

The change of the error signal from (1) and (2), for the x-axis is

$$\Delta\varepsilon_x = \frac{\Delta i_x}{i_\Sigma} = \frac{\Delta x_0}{r} \tag{6}$$

where i_Σ and r are constants.

The difference in the current Δi_x is assumed to be equal to the total noise current in the sum channel. The error signal change becomes the position error. The position error, from (6), for the x-axis is

$$\Delta\varepsilon_x = \frac{\Delta x_0}{r} = \frac{1}{\sqrt{SNR}} \tag{7}$$

where the SNR is the signal-to-noise ratio in the sum channel.

In a similar way, the first derivation of (1) and (3), the position error for the x-axis is

$$\Delta\varepsilon_x = \frac{1}{\sqrt{SNR}} = \frac{4}{\pi}\frac{\Delta x_0}{r}\sqrt{1-(x_0/r)^2} \tag{8}$$

Eqs.(7) and (8) show that the position error of a laser-illuminated object is inversely proportional to the square root of the signal-to-noise ratio in the sum channel. These equations show the relationship between the error signal and the change of the spot center. Eq.(8) also shows that the change of the spot center is a function of the spot center.

The error signals ε_x and ε_y in (1) are a combination of the current of incident irradiance and the photodiode noise. The error signal in (1), for one axis, can be written as

$$\varepsilon = \frac{S_D + N_1}{S_S + N_2} = \frac{u - v}{u + v} \tag{9}$$

where

S_D - is the signal pair-wise difference for a given axis
S_S - is the sum of signals of all (four) quadrants
N_1 - is the root mean square (rms) noise associated with the signal of difference
N_2 - is the rms noise of the sum signal
u,v - are variables, which represent signals taken pair-wise

Each quadrant is assumed to generate noise with the Gaussian distribution with zero mean value and the variance σ_n^2. Then u and v from (9) represent the signals from the pair quadrants of the photodiode

$$\begin{aligned} u &= \bar{u} + N_u \\ v &= \bar{v} + N_v \end{aligned} \tag{10}$$

where $\bar{u}$ and $\bar{v}$ are the mean values, and N_u and N_v are the fluctuations of u and v, respectively.

On the basis of a known theory, in [5] there is a probability density function derived for the error signal in a form

$$f(\varepsilon) = \frac{\sqrt{1-\rho^2}}{\pi(1+\varepsilon^2 - 2\rho\varepsilon)} \cdot \exp\left(-\frac{\bar{u}^2 + \bar{v}^2 - \rho(\bar{u}^2 - \bar{v}^2)}{2\sigma_p^2(1-\rho^2)}\right) \cdot \left[1 + \sqrt{\frac{\pi}{2}} \cdot B\left(erf(\frac{B}{\sqrt{2}})\right)\exp\frac{B^2}{2}\right] \tag{11}$$

where ρ is the correlation between N_1 end N_2 , $\sigma_p^2 = 2\sigma_n^2$, and B is defined as

$$B = \frac{\bar{u}(1+\varepsilon) + \bar{v}(1-\varepsilon) - \rho\left(\bar{u}(1+\varepsilon) + \bar{v}(\varepsilon - 1)\right)}{\sqrt{2}\sigma_p\sqrt{(1-\rho^2)(1+\varepsilon^2 - 2\rho\varepsilon)}}$$

Eq.(11) shows that the probability density function *(pdf)* of the error signal is approximately Gaussian. The analysis of *pdf* is given in [5] and [6]. In [5] it was shown that the value of ρ changes both the maximum and the width of *pdf*. The maximum width and the minimum amplitude of *pdf* were obtained for the correlation coefficient equal to zero. That means, that the worst case for the position error is when the noise between pair quadrants is uncorrelated (ρ=0).

A useful way to find the position error is to substitute the mean value pair (u,v) with the mean value of the error signal ($\bar{\varepsilon}$). The mean value of the error signal , $\bar{\varepsilon}$, from (9) and (10) is

$$\bar{\varepsilon} = \frac{\bar{u} - \bar{v}}{\bar{u} + \bar{v}} \tag{12}$$

The signal-to-noise ratio in the channel sum is

$$SNR = \frac{(\bar{u} + \bar{v})^2}{4\sigma_n^2} \tag{13}$$

After substituting (12) and (13) in (11) and rearranging the equation for B, for ρ=0, the following expression is obtained

$$B = \frac{\bar{u}(1+\varepsilon) + \bar{v}(1-\varepsilon)}{\sqrt{2}\sigma_p\sqrt{1+\varepsilon^2}} = \frac{1+\varepsilon\bar{\varepsilon}}{\sqrt{1+\varepsilon^2}}\sqrt{SNR} \tag{14}$$

Using (12) and (13) it can be written

$$\frac{\bar{u}^2 + \bar{v}^2}{4\sigma_n^2} = \frac{1}{2}SNR(1+\bar{\varepsilon}^2) \tag{15}$$

The probability density function from (11), for ρ =0, after incorporating the values from (14), (15) and the approximation for the complementary error function erfc(x)≈exp($-x^2$)/($x\pi^{1/2}$), becomes

$$f(\varepsilon) \approx \sqrt{\frac{SNR}{2\pi}}\frac{1+\varepsilon\bar{\varepsilon}}{(1+\varepsilon^2)^{3/2}}\exp\left(-\frac{SNR(\varepsilon - \bar{\varepsilon})^2}{2(1+\varepsilon^2)}\right) \tag{16}$$

The probability density function (16) is a good approximation for *pfd* (11). The error approximation is smaller than 0.8% for the worst case: $\bar{\varepsilon}$ =0, and for low SNR values (SNR=5) and decreases very fast with the increasing SNR.

In Fig.4 the probability density function (16) is shown graphically versus the error signal ε, for the mean value $\bar{\varepsilon}$ as a parameter, and the SNR=10.

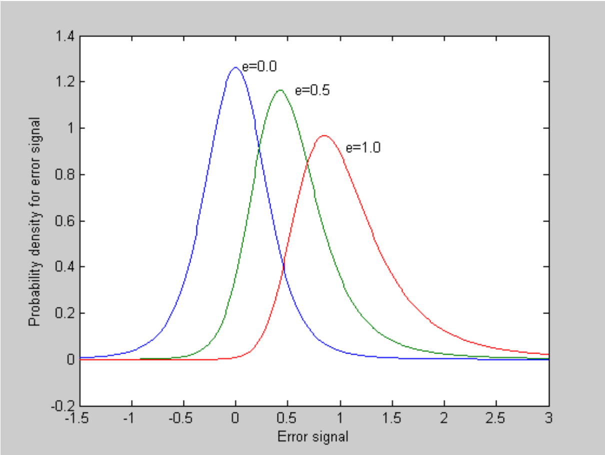

Figure 4. Diagrams of the *pdf* for SNR=10 and $\bar{\varepsilon}$ =0; 0.5 and 1

Fig.4 shows that the width of semi-Gaussian curves spreads and the level of the maximum value decreases with the increase of the mean value $\bar{\varepsilon}$.

From pdf of ε the position error can be derived. The accuracy of positioning presents a small range $\Delta\varepsilon$ around the mean value $\bar{\varepsilon}$ in which the probability of positioning is required. This range $\Delta\varepsilon$, is usually calculated for the probability of 50% (CEP-the Circular Error Probability). The probability of positioning is given by

$$P_p = \int_{\bar{\varepsilon}-\Delta\varepsilon}^{\bar{\varepsilon}+\Delta\varepsilon} f(\varepsilon)d\varepsilon \tag{17}$$

where $\Delta\varepsilon$ is the position error.

Integral (17) is solved in a closed form for the probability density function (16). After substituting $\frac{(\varepsilon-\bar{\varepsilon})^2}{1+\varepsilon^2}=x$, the integral from (17) in the unlimited form is obtained

$$\int f(\varepsilon)d\varepsilon = \int \sqrt{\frac{SNR}{2\pi}}\frac{1}{2\sqrt{x}}\exp\left(-\frac{SNR}{2}x\right)dx = \\ = \frac{1}{2}erf\left(\sqrt{\frac{SNR}{2}}\frac{\varepsilon-\bar{\varepsilon}}{\sqrt{1+\varepsilon^2}}\right) \tag{18}$$

From (17) and (18), the probability of positioning, defined as the error signal ε, lies in the range $\Delta\varepsilon$ around $\bar{\varepsilon}$, and it becomes

$$P_p = \frac{1}{2}erf\left(\frac{\sqrt{SNR}}{\sqrt{2}}\frac{\Delta\varepsilon}{\sqrt{1+(\bar{\varepsilon}+\Delta\varepsilon)^2}}\right)+ \\ +\frac{1}{2}erf\left(\frac{\sqrt{SNR}}{\sqrt{2}}\frac{\Delta\varepsilon}{\sqrt{1+(\bar{\varepsilon}-\Delta\varepsilon)^2}}\right) \tag{19}$$

The probability of positioning P_p, as the function of the mean value $\bar{\varepsilon}$ and the SNR, is shown in Fig.5.

Fig.5 shows that the probability of positioning increases with the SNR and decreases with the mean value $\bar{\varepsilon}$.

The probability of positioning from (19), for $\bar{\varepsilon}$ =0 becomes.

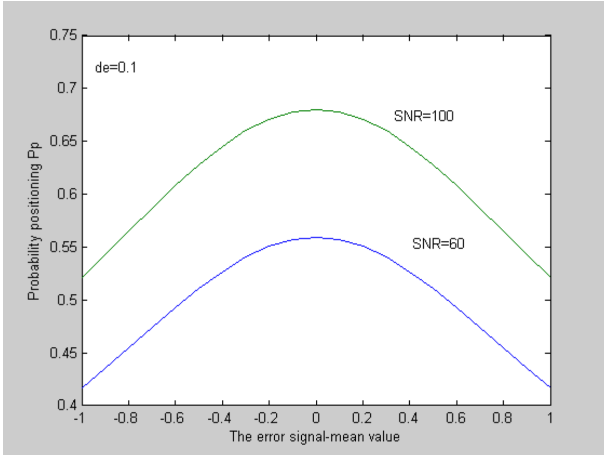

Figure 5. Probability of positioning *Pp* as a function of $\bar{\varepsilon}$, for $\Delta\varepsilon$=0.1 and the SNR=60 and 100

$$P_p = erf\left(\sqrt{\frac{SNR}{2}}\frac{\Delta\varepsilon}{\sqrt{1+(\Delta\varepsilon)^2}}\right) \tag{20}$$

The position error $\Delta\varepsilon$, from (20), is

$$\Delta\varepsilon = \frac{1}{\sqrt{\frac{SNR}{2C^2}-1}} \tag{21}$$

where $C=erf^{-1}(P_p)$.

Eq.(21) shows that the position error is defined for $SNR>2C^2$. This equation is a special case of a general approximation obtained from (19), in the following form

$$\Delta\varepsilon \approx \frac{\sqrt{2}C}{\sqrt{SNR}}\sqrt{1+\bar{\varepsilon}^2} \tag{22}$$

where an approximation error is smaller than 10^{-3}.

Eq.(22) shows that the maximum value of the position error is √2 times of the minimum value, for a given SNR. For example, the position error for P_p =0.5, from (22), lies in the range between the minimum value $\Delta\varepsilon=0.675/\sqrt{SNR}$, for $\bar{\varepsilon}$ =0, and the maximum value $\Delta\varepsilon=0{,}954/\sqrt{SNR}$, for $\bar{\varepsilon}$ =±1.

Fig.6 shows the position error as a function of the mean value of the error signal, and the SNR as a parameter.

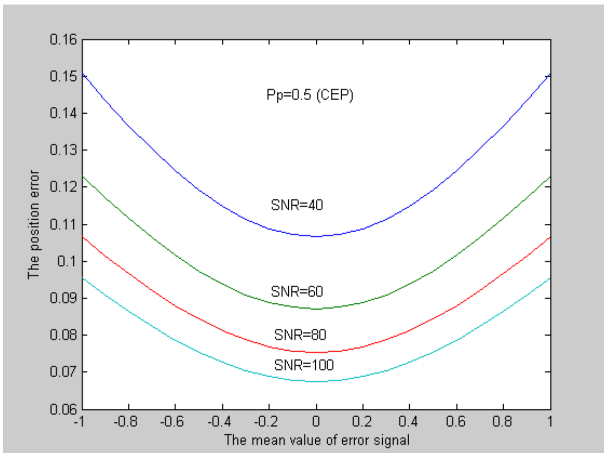

Figure 6. The position error as a function of the mean value, for the positioning probability of 50% and the signal-to-noise ratio as a parameter SNR= 40, 60, 80, and 100.

The position error changes with the mean value of the error signal as shown in Fig.6. The minimum position error value is always for the zero mean value, and the position error increases from the minimum up to the maximum value for the mean value equal to one. The maximum value is the square root of two times bigger than the minimum value.

Now there are two results for the position error. The first result is given in (7) and (8), which is derived directly from the error signal of positioning. The position error in (7) and (8) depends on the signal-to-noise ratio. The second result (22) is derived from the probability density function of the error signal. The position error (22) is a function of both the signal-to-noise ratio and the mean value of the error signal, for a given positioning probability.

The position error angle can be obtained if Δx from (8) is substituted with $\Delta\delta d$, and $\Delta\varepsilon$ from (22) with $\Delta\delta K_D$. The position error angle as a function of the normalized angle of positioning is shown in Fig.7.

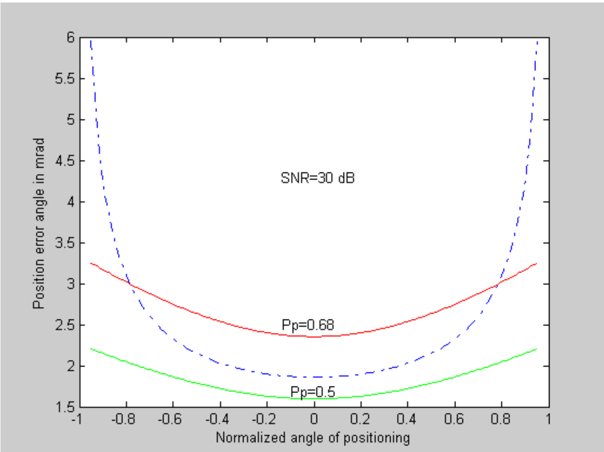

Figure 7. Position error angle as a function of the angle of positioning; deterministic (-) and statistics aproach for: P_p=0.5 and 0.68, SNR=1000, r=2.25 mm, and d=30 mm

The position error angle obtained from deterministic and statistic approaches is shown in Fig.7. The position error angle has the least value for the angle of positioning equal to zero. From Fig.7 it is clear that the position error angle, obtained from the deterministic model, increases faster than the position error angle obtained from the statistic model, at the ends of the positioning range. Also, from Fig.7 it is clear that the deterministic model (8) is a good approximation up to 80% of the positioning angle range.

Conclusion

The position error for a laser-illuminated target is estimated directly from the error signal and from the probability density function of the error signal. Both results show that the position error depends on the signal-to-noise ratio. But, the position error derived from the probability density function of the error signal shows that the position error changes to both signal-to-noise ratio and the mean value of the error signal, for a given positioning probability.

The position error of a laser-illuminated object is a function of both signal-to-noise ratio in the sum channel and the mean value of the error signal. The position error is inversely proportional to the square root of the signal-to-noise ratio. The maximum value of the position error is the square root of two times bigger than the minimum value, for a constant probability and a constant signal-to-noise ratio. The position error stands between the minimum value $0.675/\sqrt{SNR}$, for $\bar{\varepsilon}=0$, and the maximum value $0.954/\sqrt{SNR}$, for $\bar{\varepsilon}=\pm 1$, for the positioning probability of 50%.

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Određivanje greške pozicioniranja laserom ozračenog objekta

Analizirano je pozicioniranje laserom ozračenog objekta pomoću kvadrantne fotodiode. Određena je verovatnoća greške u pozicioniranju i pokazano je da greška u pozicioniranju zavisi od odnosa signal – šum, kao i od srednje vrednosti signala pozicioniranja. Greška pozicioniranja je obrnuto proporcionalna kvadratnom korenu iz odnosa signal – šum, a vrlo malo se menja s promenom srednje vrednosti signala pozicioniranja, aproksimativno za koren iz dva za opseg vrednosti signala pozicioniranja. U radu je dato poređenje aproksimativne greške izvedene iz signala pozicioniranja i greške određene iz verovatnoće pozicioniranja. Pokazano je da navedena aproksimacija može da se koristi u kvalitativnim analizama i da nije prihvatljiva u okolini granica jednoznačnog pozicioniranja objekta.

Ključne reči: **kvadrantna fotodioda, verovatnoća greške, estimacija greške.**

Détermination de l'erreur du positionnement d'un object illuminé par laser

L'erreur du positionnement d'um object illuminé par laser est analysée. On a demontré quelle dépend du rapport signal–bruit et de la valeur moyenne du signal de l'erreur de positionnement. L'erreur du positionnement est inversement proportionelle de la racine quadratique du rapport signal–bruit et elle augmente si la valeur du signal d'erreur augmente. L'erreur du positionnement minimale est obtenue pour la valeur moyenne du signal de positionnement qui est égale à zéro et l'erreur et maximale pour la valeur moyenne égale à un.

Mots-clés: **fotodiode de quadrant, probabilité de l'erreur, estimation de l'erreur.**