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Single-frequency nonlinear vibrations of antisymmetric angle-ply laminated plates

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In this paper single-frequency vibrations of antisymmetric angle-ply laminated rectangular plates freely supported along the edges are analysed. By using Krylov-Bogolyubov-Mitropol'skij's method, the asymptotic approximation of the two-parameter family of solutions in the first approximation is given. A numerical example includes the analysis of single-frequency plate vibrations in stationary and non-stationary conditions under the activity of time-dependence outer impulses. Amplitude-frequency and phase-frequency characteristics of the plate in stationary and non-stationary conditions for different laminate characteristics are presented graphically.

Key words: **nonlinear vibrations, laminated composite, asymptotic method, amplitude, phase, single-frequency.**

Introduction

LAMINATED composite vibrations have been the object of consideration during the past five decades. The equations of laminated plate vibrations are basically identical to those for a single-layer orthotropic plate. Jones [1] gives the fundamentals of the tension-deformation state of laminated plates and the differential equations of the plate linear vibrations. By invoking the Galerkin method, Ghaza-rian and Locke [2] determine equations of laminated plate vibrations which are simple for analysis. Gorman and Ding [3] determine the value of eigenfrequencies for different angles of lamination, number of lamina, boundary conditions and different ratios of the Young's modulus. Khdeir and Reddy [4] consider the free vibrations of laminated composite plates for different boundary conditions, comparing the Kirchhoff theory with the applied one. Tylikovski [5] considers the stability of the nonlinear symmetrical laminated cross-ply plates. The equation of the cross-ply laminated plate vibration is derived by introducing the Airy function.

A widely applicable asymptotic method of [6] for solving nonlinear vibra-tions continuum problems is applied in the papers of Pavlović [10] and K.Hedrih and others (1974), (1978), (1986). Pavlović (1984) has published a study about the analysis of resonant regime two-frequency vibrations of shallow shells; K.Hedrih and the others (1986) analyse four-frequency vibrations of thin shells with an initial irregularity.

In this paper the single-frequency vibrations of the laminated plate under time dependent external force effect are considered. Also, the influence of mechanical and other characteristics on the amplitude and phase of the asymptotic solution is given in the first approximation.

Problem formulation

The components of the deformation tensor and the components of the curvature of the plate middle surface are defined as follows

$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u}{\partial x}+\dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2\\ \dfrac{\partial v}{\partial y}+\dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2\\ \dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}+\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\end{Bmatrix},\quad \{\kappa\}=\begin{Bmatrix}\kappa_x\\ \kappa_y\\ \kappa_{xy}\end{Bmatrix}=\begin{Bmatrix}-\dfrac{\partial^2 w}{\partial x^2}\\ -\dfrac{\partial^2 w}{\partial y^2}\\ -2\dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y}\end{Bmatrix} \tag{1}$$

where u(x,y,t), v(x,y,t) are the in-plane displacements whether w(x,y,t) is a displacement normal to the middle surface of the plate or not.

The model is applicable to a plate that satisfies the following conditions:

a) Thickness of the plate is significantly small,
b) Plate thickness is either uniform or varies slowly so that the three-dimensional stress effects are ignored,
c) Normals of the material to the original reference surface remain straight, as well as the normal to the deformed reference surface,
d) Applied transverse loads are distributed over the plate surface areas, and
e) Support conditions are such that no significant extension of the midsurface can occur.

The membrane force, moments of bending and torsion moment in the cross section along the axes can be presented as

$$\{N\}=\begin{Bmatrix}N_x\\ N_y\\ N_{xy}\end{Bmatrix},\quad \{M\}=\begin{Bmatrix}M_x\\ M_y\\ M_{xy}\end{Bmatrix} \tag{2}$$

The relation between the forces and the moments in the middle surface of the plate is expressed by the equation

$$\begin{Bmatrix}\{N\}\\ \{M\}\end{Bmatrix}=[C]\begin{Bmatrix}\{\varepsilon\}\\ \{\kappa\}\end{Bmatrix}=\begin{bmatrix}[A] & [B]\\ [B] & [D]\end{bmatrix}\begin{Bmatrix}\{\varepsilon\}\\ \{\kappa\}\end{Bmatrix} \tag{3}$$

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The matrix of stiffness [C] for antisymmetric angle-ply laminates has the form

$$[C]=\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\ A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\ 0 & 0 & B_{16} & \mathbf{D}_{11} & \mathbf{D}_{12} & 0 \\ 0 & 0 & B_{26} & \mathbf{D}_{12} & \mathbf{D}_{22} & 0 \\ B_{16} & B_{26} & 0 & 0 & 0 & \mathbf{D}_{66} \end{bmatrix} \quad (4)$$

and the matrices of extensional stiffness [A], coupling stiffness [B] and bending stiffness [D] are defined as

$$[A]=\begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \quad [B]=\begin{bmatrix} 0 & 0 & B_{16} \\ 0 & 0 & B_{26} \\ B_{16} & B_{26} & 0 \end{bmatrix}$$
$$[D]=\begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \quad (5)$$

The elements of the matrix of stiffness are defined as

$$\left(A_{ij}, B_{ij}, D_{ij}\right)=\int_{-h/2}^{h/2}\left(1, z, z^2\right)\bar{\bar{Q}}_{ij}\, dz$$

where h is the thickness of the plate and $\bar{\bar{Q}}_{ij}$ is the reduced in-plane stiffness of an individual lamina.

$$\bar{\bar{Q}}_{11}=Q_{11}\cos^4\theta+2\left(Q_{12}+2Q_{66}\right)\sin^2\theta\cos^2\theta+Q_{22}\sin^4\theta$$
$$\bar{\bar{Q}}_{12}=\left(Q_{11}+Q_{22}-4Q_{66}\right)\sin^2\theta\cos^2\theta+ +Q_{12}\left(\sin^4\theta+\cos^4\theta\right)$$

$$\bar{\bar{Q}}_{22}=Q_{11}\sin^4\theta+2\left(Q_{12}+2Q_{66}\right)\sin^2\theta\cos^2\theta+Q_{22}\cos^4\theta$$
$$\bar{\bar{Q}}_{16}=\left(Q_{11}-Q_{12}-2Q_{66}\right)\sin\theta\cos^3\theta+ +\left(Q_{12}-Q_{22}+2Q_{66}\right)\sin^3\theta\cos\theta$$

$$\bar{\bar{Q}}_{26}=\left(Q_{11}-Q_{12}-2Q_{66}\right)\sin^3\theta\cos\theta+ +\left(Q_{12}-Q_{22}+2Q_{66}\right)\sin\theta\cos^3\theta$$

$$\bar{\bar{Q}}_{66}=\left(Q_{11}+Q_{22}-2Q_{12}-2Q_{66}\right)\sin^2\theta\cos^2\theta+ +Q_{66}\left(\sin^4\theta+\cos^4\theta\right)$$

The reduced in-plane stiffnesses of an individual lamina are expressed in terms of the lamina principal material properties [1]

$$Q_{11}=\frac{E_1}{1-\nu_{12}\nu_{21}}, \quad Q_{12}=\frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}=\frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}$$
$$Q_{22}=\frac{E_2}{1-\nu_{12}\nu_{21}}, \quad Q_{66}=G_{12}$$

where θ is the angle of lamina and E_1, E_2, G_{12} and ν_{12} are the major Young's modulus, minor Young's modulus, the shear modulus and major Poisson ratio, respectively.

Differential equations of the plate vibration are obtained provided that the forces and the moments in the coordinate direction are balanced dynamically

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0, \quad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0$$
$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+N_x\frac{\partial^2 w}{\partial x^2}+2N_{xy}\frac{\partial^2 w}{\partial x\partial y}+ \quad (6)$$
$$+N_y\frac{\partial^2 w}{\partial y^2}+q(x,y,t)=\rho h\frac{\partial^2 w}{\partial t^2}+2\beta\rho h\frac{\partial w}{\partial t}$$

where ρ is the density of the plate material, β is the damping coefficient and q(x,y,t) is the external disturbing force.

From eq.(3) the components of the moment of bending as well as the moment of torsion can be expressed in terms of transverse displacement of the middle surface of the plate

$$M_x=B_{16}\gamma_{xy}-D_{11}\frac{\partial^2 w}{\partial x^2}-D_{12}\frac{\partial^2 w}{\partial y^2},$$
$$M_y=B_{26}\gamma_{xy}-D_{12}\frac{\partial^2 w}{\partial x^2}-D_{22}\frac{\partial^2 w}{\partial y^2}, \quad (7)$$
$$M_{xy}=B_{16}\varepsilon_x+B_{26}\varepsilon_y-2D_{66}\frac{\partial^2 w}{\partial x\partial y}$$

The function of tension (, ,)

From eqs.(8), (11) and (12), the components of the deformation tensor can be expressed in terms of the function of tension

$$\varepsilon_x = A_{11}^* \frac{\partial^2 \psi}{\partial y^2} + A_{12}^* \frac{\partial^2 \psi}{\partial x^2} + B_{16}^* M_{xy}$$

$$\varepsilon_y = A_{12}^* \frac{\partial^2 \psi}{\partial y^2} + A_{22}^* \frac{\partial^2 \psi}{\partial x^2} + B_{26}^* M_{xy} \qquad (13)$$

$$\gamma_{xy} = -A_{66}^* \frac{\partial^2 \psi}{\partial x \partial y} + B_{16}^* M_x + B_{26}^* M_y$$

Substituting eqs.(7) and (8) into the third equation of the system eq.(6) and including eq.(13), after its differentiation, into the left side of eq.(10) gives

$$\rho h \frac{\partial^2 w}{\partial t^2} + 2\beta\rho h \frac{\partial w}{\partial t} + L_{AU}(w) + $$
$$+ e_1 \frac{\partial^4 \psi}{\partial x^3 \partial y} + e_2 \frac{\partial^4 \psi}{\partial x \partial y^3} - L(w,\psi) = q(x,y,t) \qquad (14)$$

$$\Theta_{AU}\psi = -\frac{1}{2} L(w,w) - k_1 \frac{\partial^4 w}{\partial x^3 \partial y} - k_2 \frac{\partial^4 w}{\partial x \partial y^3} \qquad (15)$$

with the following denotations

$$L_{AU}(w) = g_{11} \frac{\partial^4 w}{\partial x^4} + g_{12} \frac{\partial^4 w}{\partial x^2 \partial y^2} + g_{22} \frac{\partial^4 w}{\partial y^4} \qquad (16)$$

$$L(w,\psi) = \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \psi}{\partial x^2} - 2\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \psi}{\partial x \partial y} \qquad (17)$$

$$b_6 = \frac{1}{1 - B_{16}B_{16}^* - B_{26}B_{26}^*}$$

$$g_{11} = b_6 B_{16}\left(B_{16}^* D_{11} + B_{26}^* D_{12}\right) + D_{11}$$

$$g_{12} = b_6 B_{16}\left(B_{16}^* D_{12} + B_{26}^* D_{22}\right) +$$
$$+ b_6 B_{26}\left(B_{16}^* D_{11} + B_{26}^* D_{12}\right) + 2\left[D_{12} + 2b_6 D_{66}\right]$$

$$g_{22} = b_6 B_{26}\left(B_{16}^* D_{12} + B_{26}^* D_{22}\right) + D_{22}$$

$$e_1 = b_6\left(B_{16}A_{66}^* - 2\left(B_{16}A_{12}^* + B_{26}A_{22}^*\right)\right)$$

$$e_2 = b_6\left(B_{26}A_{66}^* - 2\left(B_{16}A_{11}^* + B_{26}A_{12}^*\right)\right)$$

$$\Theta_{AU}\psi = h_{11}\frac{\partial^4 \psi}{\partial x^4} + h_{12}\frac{\partial^4 \psi}{\partial x^2 \partial y^2} + h_{22}\frac{\partial^4 \psi}{\partial y^4}$$

$$h_{11} = b_6 B_{26}^*\left(B_{16}A_{12}^* + B_{26}A_{22}^*\right) + A_{22}^*$$

$$h_{12} = b_6 B_{16}^*\left(B_{16}A_{12}^* + B_{26}A_{22}^*\right) +$$
$$+ b_6 B_{26}^*\left(B_{16}A_{11}^* + B_{26}A_{12}^*\right) + \left(b_6 A_{66}^* + 2A_{12}^*\right)$$

$$h_{22} = b_6 B_{16}^*\left(B_{16}A_{11}^* + B_{26}A_{12}^*\right) + A_{11}^*$$

$$k_1 = b_6\left(\left(B_{16}^* D_{11} + B_{26}^* D_{12}\right) - 2B_{26}^* D_{66}\right)$$

$$k_2 = b_6\left(\left(B_{16}^* D_{12} + B_{26}^* D_{22}\right) - 2B_{16}^* D_{66}\right)$$

Eqs.(14) and (15) are the differential equations of the plate vibration. By solving eqs.(14) and (15), bearing in mind the boundary and initial conditions, one can determine the transverse displacements of the middle surface w(x,y,t) of the laminated plate, as well as the function of tension ψ(x,y,t). Also according to eqs.(7), (8) and (13), all the necessary tensors of the deformation components, force components and moments are determined. Eqs.(14) and (15) are identical to those given in the reference [2].

Single-frequency vibrations of antisymmetric laminated angle-ply plates

Let us consider the plate vibrations described by the system of differential eqs.(14) and (15). We presume that the disturbing force q(x,y,t) is acting on the system. The force is 2π -periodical in θ_1(t) with the constant amplitude P_1^* in the form

$$q(x,y,t) = \varepsilon P_1^* \sin\theta_1 \cdot w_{11}(x,y) \qquad (18)$$

where $\frac{d\theta_1}{dt} = \nu_1(t)$ is the instantaneous frequency and ε is a small positive parameter. For the laminated plate, freely supported along the edges, the boundary conditions are

$$\left.\begin{matrix} x=0 \\ x=a \end{matrix}\right\} w=0;\ M_x = B_{16}\gamma_{xy} - D_{11}\frac{\partial^2 w}{\partial x} - D_{12}\frac{\partial^2 w}{\partial y} = 0,$$
$$\left.\begin{matrix} y=0 \\ y=b \end{matrix}\right\} w=0;\ M_y = B_{26}\gamma_{xy} - D_{12}\frac{\partial\ w}{\partial x} - D_{22}\frac{\partial^2 w}{\partial y} = 0$$

Let initial conditions be

$$w(x,y,t)\Big|_{t=0} = \sum_i \sum_j p\ \ w_{i\,j}(x,y),$$
$$\left.\frac{\partial w(x,y,t)}{\partial t}\right|_{t=0} = \sum\sum_{11} q\ \ w_i\ (x_j, y) \qquad (20)$$

where $w\ \ (x,y) = \sin(\frac{i\pi}{a}x)\sin(\frac{j\pi}{b}y)$ are any normal functions and p_{ij} and q_{ij} are real numbers. According to the boundary and initial conditions (eqs.(19) and (20)) we presume that, in the single frequency regime of the plate vibration transverse displacement w(x,y,t), the solution of the system of eqs.(14) and (15) would be in the form

$$w(x,y,t) = f(t)w\ \ (x,y) \quad f(t)\sin(\frac{\pi x}{a})\sin(\frac{\pi y}{b}) \qquad (21)$$

where $f(t)$ is an unknown function of time, which will be determined from the equation of vibration.

Taking eq.(16) into consideration, the function L(w,w) is evaluated in the form

$$\Psi(x,y,t) = \frac{\lambda}{32 \cdot h} f\ \ (t)\cos(\frac{2\pi x}{a}) +$$
$$+ \frac{1}{32 \cdot \lambda\ \ \cdot h} f\ \ (t)\cos(\frac{2\pi y}{b}) + \qquad (22)$$
$$+ \frac{\lambda\left(k\ \ + \lambda\ k\ \ \right)}{h\ \ + \lambda\ h\ \ + \lambda\ h} f(t)\,\mathrm{c\ \ s}(\frac{\pi x}{a})\cos(\frac{\pi y}{b})$$

where λ=a/b is the ratio of the plate sides.

(19)

Multiply eq.(14) by $w_{11}(x,y)dxdy$, after substituting the disturbing force eq. (18) and expressions eqs.(21) and (22) in their right and left side respectively, to integrate by the plate surface ($x \in (0,a)$, $y \in (0,b)$). If we introduce the substitution

$$\xi_1(t) = \frac{f_1(t)}{h} \tag{23}$$

after the integration, we will obtain a differential equation in the unknown function $\xi_1(t)$

$$\ddot{\xi}_1 + \omega_1^2 \xi_1 = -2\beta \dot{\xi}_1 + \alpha_1 \xi_1^3 + \varepsilon P_1 \sin\theta_1 \tag{24}$$

where

$$\omega_1^2 = \frac{1}{\rho h}\frac{\pi^4}{a^4}\left(g_{11} + \lambda^2 g_{12} + \lambda^4 g_{22} - \lambda^2 \frac{\left(k_1 + \lambda^2 k_2\right)\left(e_1 + \lambda^2 e_2\right)}{h_{11} + \lambda^2 h_{12} + \lambda^4 h_{22}}\right) \tag{25}$$

$$\alpha_1 = -\frac{1}{16}\frac{h}{\rho}\frac{\pi^4}{a^2 b^2}\left(\frac{\lambda^2}{h_{11}} + \frac{1}{\lambda^2 h_{22}}\right) \tag{26}$$

$$P_1 = \frac{P_1^*}{\rho h^2} \tag{27}$$

Eq.(24) represents the differential equation of the compulsive vibration of the plate in the single-frequency regime with the frequency given by eq.(25).

For the composition of the asymptotic approximations of the solution of the perturbed vibration eq.(24), which corresponds to the single-frequency vibrations, it is indispensable that $\nu_1(t) \approx \omega_{sr}$, where ω_{sr} is is the circular frequency "unperturbed" vibration. Suppose the following conditions are fulfilled (Mitropol'skij and Mossenkov [7], Hedrih [9]):

a) in the unperturbed system, undamped harmonic oscillation with the frequency ω_{sr}, depending only on two arbitrary constants, are possible to occur,
b) a unique solution of eq.(24), corresponding to the equilibrium state in the unperturbed system, is the trivial solution w(x,y,t)=0,
c) in the unperturbed system, there are no internal resonant states, i.e. $\omega_{sr} \neq (p/q)\ \omega_{mn}$ where m,n=2, 3, 4, ... and p and q are mutually prime numbers.

With these assumptions, the asymptotic solution of the equation of perturbed vibrations is [6]

$$\begin{aligned}\xi_1 &= a_1\cos(\theta_1 + \varphi_1) + \varepsilon u^{(1)}(\tau,\theta_1,a_1,\varphi_1) + \\ &+ \varepsilon^2 u_j^{(2)}(\tau,\theta_1,a_1,\varphi_1) + \dots\end{aligned} \tag{28}$$

where $\tau=\varepsilon t$ is the "slowly-changed time" and, $u_j^{(1)}\left(\tau,\theta_1,a_1,\varphi_1\right)$, $u_j^{(2)}\left(\tau,\theta_1,a_1,\varphi_1\right)$, ... are the periodical functions with the following arguments: θ_1 and φ_1 with the period 2π; the amplitude and phase of solution (eq.(28)) can be found from differential equations

$$\begin{aligned}\frac{da_1}{dt} &= \varepsilon A_1(\tau,a_1,\varphi_1) + \varepsilon^2 A_2(\tau,a_1,\varphi_1) + \dots.. \\ \frac{d\varphi_1}{dt} &= \omega_1 - \nu_1 + \varepsilon B_1(\tau,a_1,\varphi_1) + \varepsilon^2 B_2(\tau,a_1,\varphi_1) + \dots...\end{aligned} \tag{29}$$

where A_1, B_1, A_2, B_2, ... are the unknown functions in the "slowly-changed time" and amplitude and phase. These functions can be determined from the supposed solution of (eq.(28)) in eq.(24) and equalizing the coefficient by the same harmonics. Staying at the first approximation, the solution of eq.(24) will have the form

$$\xi_1 = a_1\cos(\nu_1 t + \varphi_1) \tag{30}$$

where the differential equations in the first approximation will be

$$\begin{aligned}\frac{da_1}{dt} &= -\beta a_1 - \frac{P_1}{\omega_1 + \nu_1}\cos\varphi_1, \\ \frac{d\varphi_1}{dt} &= \omega_1 - \nu_1 - \frac{3}{8}\frac{\alpha_1}{\omega_1}a_1^2 + \frac{P_1}{a_1(\omega_1 + \nu_1)}\sin\varphi_1\end{aligned} \tag{31}$$

Numerical analysis of compulsive vibrations of the laminated plate in stationary conditions

If expressions (31), which are the first approximation differential equations of the solution (30), equal zero, i.e.

$$\begin{aligned}-\beta a_1 - \frac{P_1}{\omega_1 + \nu_1}\cos\varphi_1 &= 0, \\ \omega_1 - \nu_1 - \frac{3}{8}\frac{\alpha_1}{\omega_1}a_1^2 + \frac{P_1}{a_1(\omega_1 + \nu_1)}\sin\varphi_1 &= 0,\end{aligned} \tag{32}$$

the equations, which define the relationship of the amplitude and phase of the asymptotic solution (30), will be obtained.

Solving these equations in the amplitude $a_1=f_1(\nu)$ and the phase $\varphi_1= f_2(\nu)$ we obtain the amplitude-frequency and the phase-frequency characteristics of the laminated plates vibration for the stationary conditions. For all numerical examples the following characteristics of the laminated plate are taken: the dimensions of the plate a=2 m, b=1 m; the thickness of the plate h=1 cm; the specific density of the plate material ρ=2600 kg/m^3; the damping coefficient β=6 s^{-1} and the amplitude of disturbance P_1^*=1300 N/m^2. The changes of the amplitude-frequency and phase-frequency characteristics due to the changes of some laminate characteristics are given in the following examples.

The amplitude-frequency characteristics of four-layered antisymmetric laminate with lamina orientation 30°/-30°/30°/-30° and thickness 0.2h/0.3h/0.3h/0.2h while changing the relation of longitudinal and transverse modulus of elasticity (E_1/E_2=5;10;40) are shown in Fig.1a. With the relation E_1/E_2 increasing, the amplitude-frequency curves are getting displaced to higher amplitudes and lower frequencies. The frequency region in which, for the same frequencies of the external force, there is a possibility of three stationary states (two are stable and one unstable) is getting displaced to lower frequencies. This is a frequency region between the resonant jumps. The examination of the phase frequency characteristics shown in Fig.1b gives the same conclusions.

In the next example the laminate analysis for the ratio E_1/E_2=10, and for the same orientation of individual lamina 30°/-30°/30°/-30° is performed. During the analysis, the thickness of laminae is changed. The amplitude-frequency and phase-frequency characteristics for different thicknesses are shown in Fig.2. It is obvious that with decreasing of the external lamina thickness and increasing of internal lamina thickness, the amplitudes have lower values and the frequency curves shift to the higher frequencies of the external disturbing force.

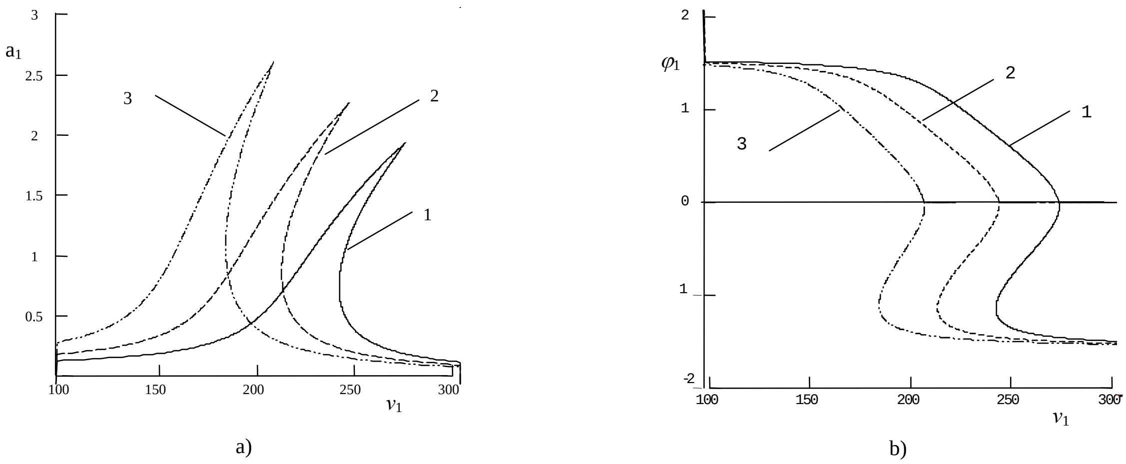


Figure 1. Amplitude-frequency (a) and phase-frequency (b) characteristics for a different ratio E_1/E_2 (1-E_1/E_2=5, 2- E_1/E_2=10, 3-E_1/E_2=4[redacted]

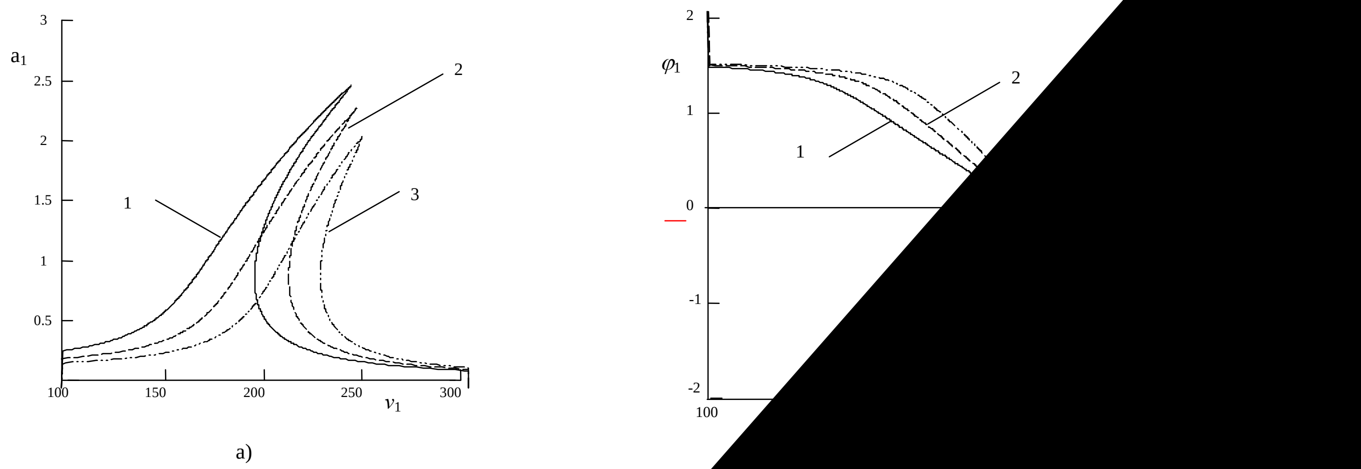
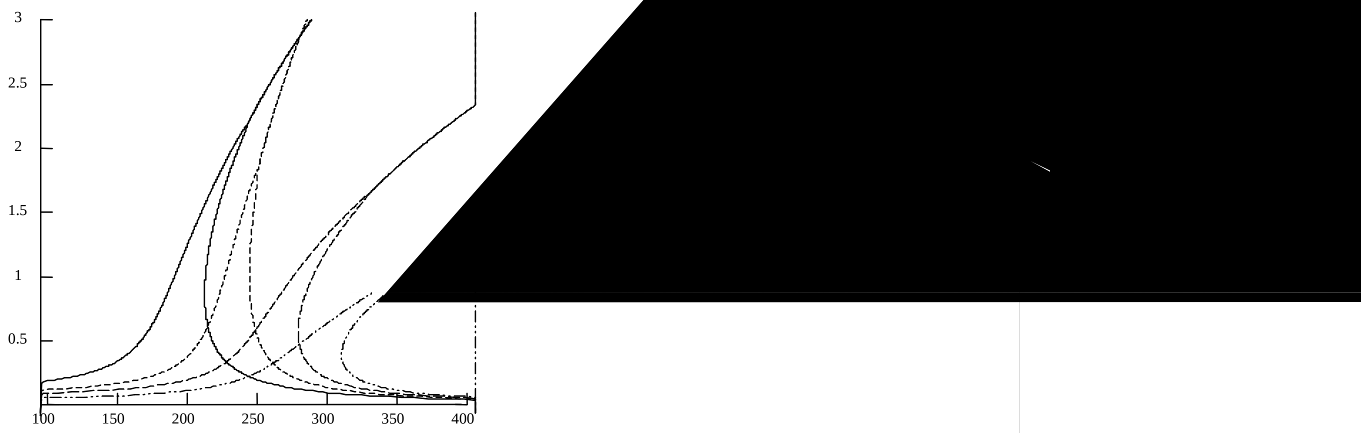


Figure 2. Amplitude-frequency (a) and phase-frequency (b) character[redacted] (1-0.4h/0.1h/0.1h/0.4h, 2-0.3h/0.2h/0.2h/0.3h, 3-0.2h/0.3h/0.3h/0.[redacted]



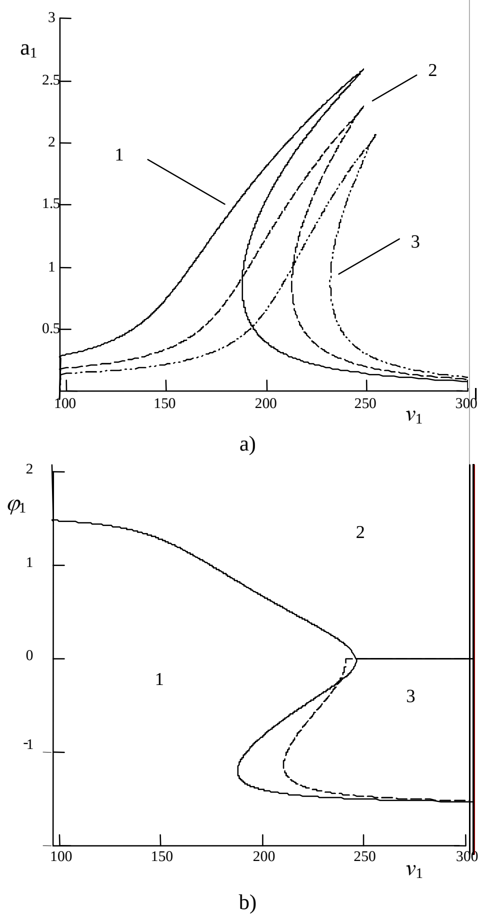


Figure 4. Amplitude-frequency (a) and phase-frequency (b) characteristics for different angles of laminae (1-φ=30°, 2-φ=45°, 3-φ=60°, 4-φ=75°)

Numerical analysis of compulsive vibrations of the laminated plate in nonstationary conditions

Eqs. (31) are the first approximation differential equations of the asymptotical solution of differential equation (24). Numerical solving of these equations by means of the Runge-Kutta method (the fourth order), gives amplitude frequency characteristics of a single frequency regime of the laminated plate vibration in nonstationary conditions. The dependence of these curves on certain laminate characteristics change is given in the following examples.

The amplitude-frequency characteristics of the four-layered laminate (30°/-30°/30°/-30°, 0.2h/0.3h/0.3h/0.2h) for different ratios of the longitudinal and transverse modulus of elasticity are shown in Fig.5. The amplitude-frequency characteristics at linear increasing of the external force frequency are shown in Fig.5a. Passing through the resonant state is realized by decreasing the external force frequency as in Fig.5b. On the basis of both diagrams, it can be concluded that by the increase of the ratio E_1/E_2, the maximums of the amplitudes are growing too, and are being displaced towards lower frequencies.

The amplitude-frequency characteristics for different thicknesses of longitudinal and transverse lamina orientation with E_1/E_2=10 are shown in Fig.6. It is obvious that with the increase of the internal lamina thickness, the maximums of the amplitudes are decreasing.

The influence of the number of laminae on amplitude-frequency characteristics in the nonstationary state is shown in Fig.7. At the ratio E_1/E_2=10, it is taken that the total thickness of the longitudinal laminae is the same as the thickness of the transverse laminae (by h/2). For two-layered laminate it is (30°/-30°, 0.5h/0.5h), for four-layered

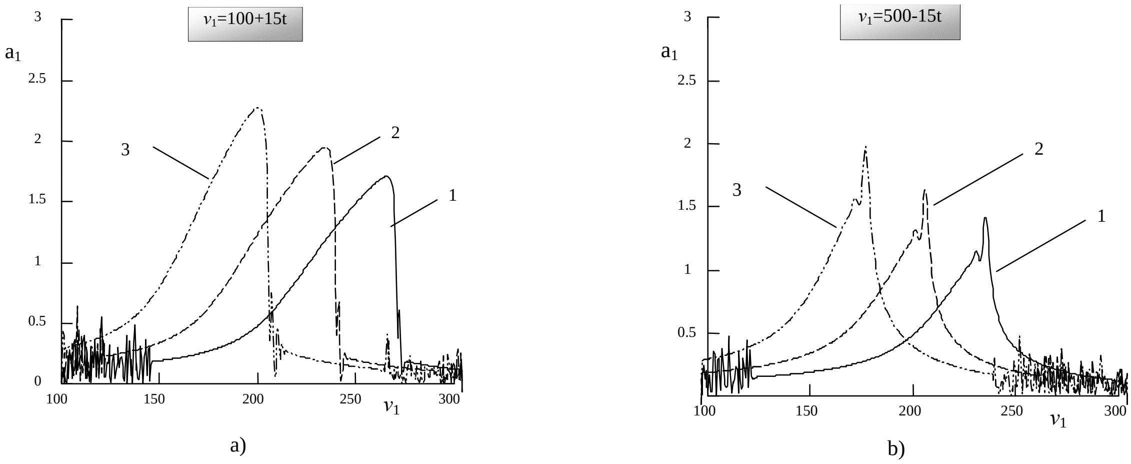


Figure5. Amplitude-frequency characteristics for a different ratio E_1/E_2 (1-E_1/E_2=5, 2- E_1/E_2=10, 3-E_1/E_2=40)

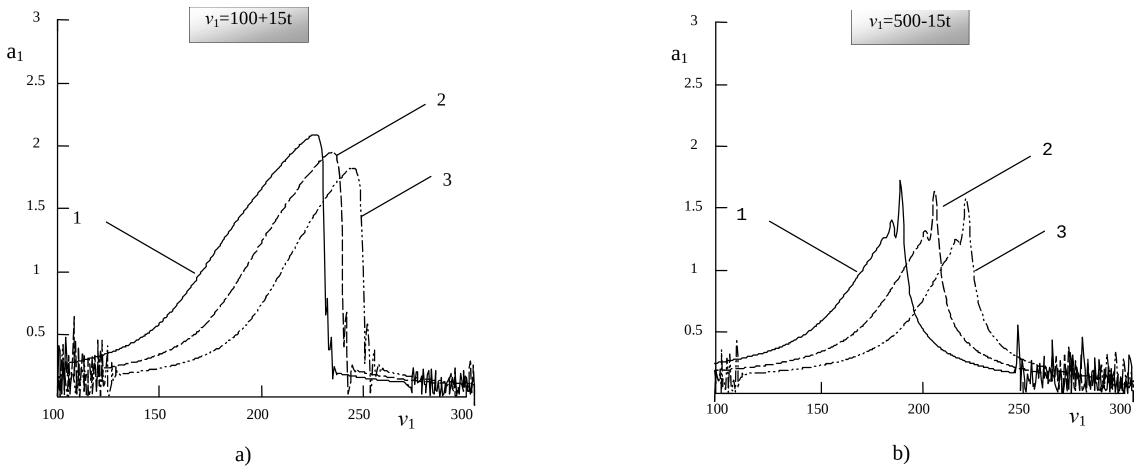
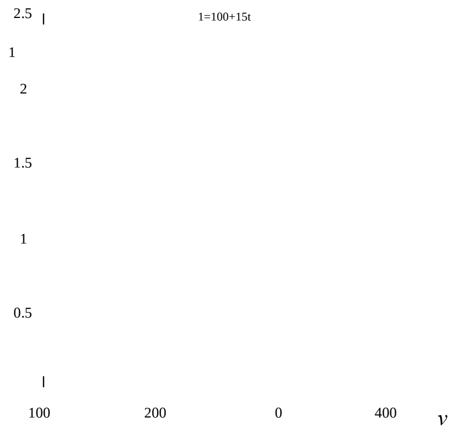
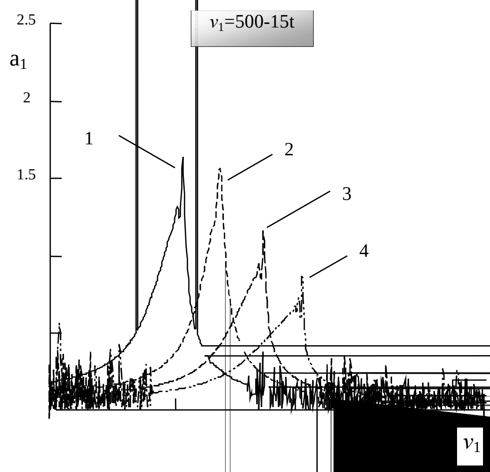
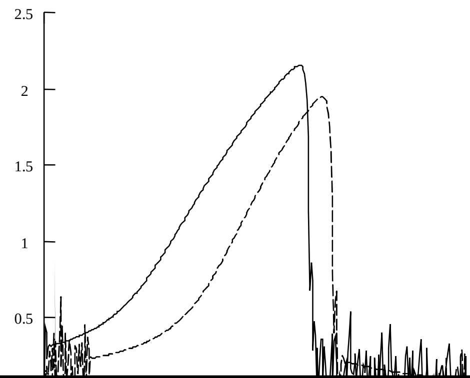


Figure6. Amplitude-frequency characteristics for different thicknesses of laminae (1-0.4h/0.1h/0.1h/0.4h, 2-0.3h/0.2h/0.2h/0.3h, 3-0.2h/0.3h/0.3h/0.2h)


1=100+15t
2.5
1
2
1.5
1
0.5
100
200
0
400
v

a)


v_1=500-15t
2.5
a_1
2
1.5
1
2
3
4
v_1


2.5
2
1.5
1
0.5

[redacted]

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Jednofrekventne nelinearne oscilacije antisimetričnih ugaonih lamelastih ploča

Analizirane su jednofrekventne oscilacije antisimetrične ugaone lamelaste ploče pravougaonog oblika, slobodno oslonjene na svojim krajevima. Date su asimptotske aproksimacije dvoparametarske familije rešenja u prvoj aproksimaciji, korišćenjem metode Krilova-Bogoljubova-Mitropoljskog. Numerički primer obuhvata analizu oscilovanja ploče u stacionarnim i nestacionarnim uslovima pod dejstvom vremenski zavisne spoljašnje pobude. Grafički su prikazane amplitudno-frekventne i fazno–frekventne karakteristike oscilovanja ploče u stacionarnim i nestacionarnim uslovima za različite karakteristike lamelata.

Ključne reči: **Nelinearne oscilacije, lamelasti kompoziti, asimptotska metoda, amplituda, faza, frekvencija, jednofrekventnost.**

Oscillations non-linéaires à fréquence unique chez les plaques laminées, angulaires et antisymètriques

Le papier analyse les oscillations à fréquence unique chez les plaques laminées, angulaires et antisymétriques en forme rectangulaire, appuées librement sur deux bouts. Les approximations asymptotiques de la groupe de solutions à deux paramètres en première approximation sont données en utilisant la méthode de Krylov-Bogolybov-Mitropol'skij. L' exemple numérique comprend l'analyse des oscillations de la plaque dans les conditions stationnaires et non-stationnaires sous l'effet de l'excitation extérieure dépendante de temps. Les caractéristiques en fréquence d'amplitude et en fréquence de phase des oscillations de la plaque dans les conditions données sont présentées par la voie graphique pour les caractéristiques différentes des lamelles.

Mots-clés: **oscillations non-linéaires, composites laminés, méthode asymptotique, amplitude, phase, fréquence, fréquence unique.**