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Finite time stability of linear discrete systems: retrospective of results

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A detailed chronological review of the results concerning finite time stability of linear discrete systems has been given along with numerous definitions. Through selectively chosen theorems the most recently published and up to date results have been exposed, specifying the conditions of stability and instability of linear discrete systems in free and forced regime. Most of the results are given in the form of sufficient conditions for this stability concept.

Key words: **discrete systems, finite time stability, practical stability, Bellman–Gronwall lemma.**

Introduction

Practical matters require that we concentrate not only on the system stability (e.g. in the sense of Lyapunov), but also on the boundaries of system trajectories. A system could be stable but completely useless if it possesses undesirable transient performances. Thus, it may be useful to consider the stability of such systems with respect to certain subsets of statespace, which are *a priori* defined in the given problem. Furthermore, it is of particular significance to analyze the behavior of dynamical systems exclusively over a finite time interval.

These boundness properties of system responses, i. e. the solution of system models, are very important from the engineering point of view. Realizing this fact, numerous definitions of the so-called technical and practical stability were introduced. Roughly speaking, these definitions are essentially based on the predefined boundaries for the perturbation of initial conditions and the allowable perturbation of the system response. In the engineering applications of control systems, this fact becomes very important and sometimes crucial, for the purpose of characterizing in advance, in quantitative manner, possible deviations of the system response. Thus, the analysis of these particular boundness properties of solutions is an important step, which precedes the design of control signals, when finite time or practical stability control is concerned.

Chronological review of recently published results

A specific concept of discrete time systems, practical stability operating on finite time interval, was investigated by *Hurt (1967)* with a particular emphasis on the possibilities of error arising in the numerical treatment of results.

A finite time stability concept was, for the first time, extended to discrete time systems by *Michel and Wu* (*1969*). Practical stabilty or "*set stability*", throughout estimation system trajectory behavior on finite time interval was given by *Heinen* (*1970, 1971*). He was the first who gave necessary and sufficient conditions for this concept of stability, using the Lyapunov approach based on the "*discrete Lyapunov functions*" application.

Even more detailed analysis of these results considering different aspects of discrete time systems practical stabilty as well as the questions of their realization and controllability, was given by *Weiss* (*1972*).

The same problems were treated by *Weiss and Lam* (*1973*), who extended them to the class of nonlinear complex discrete systems.

Efficient sufficient conditions of finite time stability of linear discrete time systems expressed through norms and/or matrices were derived by *Weiss and Lee* (*1971*).

Lam and Weiss (*1974*) were the first who applied the so-called concept of "*final stability*" on discrete time systems whose motions are scrolled within the time varying sets in the state space.

Some simple definitions connected to sets representing difference equations or at the same time discrete time systems, were given by *Shanholt* (*1974*). Only the sufficient conditions are given by the established theorems. These results are based on the Lyapunov stability and can be used, in a way, for a finite time stability concept, for which reason they are mentioned here.

Grippo and Lampariello (*1976*) have generalized all foregoing results and given the necessary and sufficient conditions of different concepts of finite time stability inspired by definitions of practical stability and instability, earlier introduced by *Heinen* (*1970*).

The same authors applied the before-mentioned results in the analysis of "*large-scale systems*", *Grippo, Lampariello* (*1978*).

Practical stabilty with settling time was for the first time introduced by *Debeljković* (*1979.a*) in connection with the analysis of different classes of linear discrete time systems, general enough to include time invariant and time varying systems, systems operated in free or forced operating regimes, as well as the systems whose dynamical behavior is expressed through the so-called "*functional system matrix*". In the mentioned paper, the sufficient conditions of practi-

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cal instability and a discrete version of a very well known Bellman-Gronwall lemma have also been derived.

Other papers, *Debeljković* (*1979.b, 1980.a, 1980.b, 1983*) deal with the same problems and mostly represent the basic results of the doctoral dissertation, *Debeljković (1979.a).*

For the particular class of discrete time systems with the functional system matrix, sufficient conditions have been derived in *Debeljković* (*1993*).

Finally, in his paper, using the concept of practical stability *Bajić* (*1983*) carries out the quantitative analysis of the transient response of a particular class of discrete time homogenous nonstationary bilinear systems with stationary linear parts. These results enable to determine limits of systems responses.

In the context of practical stability for linear discrete time singular systems, various results were first obtained in *Debeljković and Owens* (*1986*) and *Owens and Debeljković* (*1986*).

Basic results recapitulation

The concept of practical and finite time stability was for the first time extended to the general class of discrete time systems by *Michel and Wu (1969).*

Bearing in mind the importance of this paper and its influence on the further development and further application on particular classes of discrete time systems this paper will be presented in the sequel.

Systems to be considered are governed by the vector difference equation

$$\mathbf{x}(k+1) = \psi\left(k, \mathbf{x}(k)\right) \tag{1}$$

$$\mathbf{x}(k+1) = \psi\left(k, \mathbf{x}(k), \mathbf{f}(k)\right) \tag{2}$$

$$\mathbf{x}(k+1) = \psi\left(k, \mathbf{x}(k)\right) + \mathbf{f}\left(k, \mathbf{x}(k)\right) \tag{3}$$

and for the particular class of these systems

$$\mathbf{x}(k+1) = A(k)\mathbf{x}(k) \tag{4}$$

$$\mathbf{x}(k+1) = A\mathbf{x}(k) \tag{5}$$

$$\mathbf{x}(k+1) = A\left(k, \mathbf{x}(k)\right)\mathbf{x}(k) \tag{6}$$

$$\mathbf{x}(k+1) = A\left(\mathbf{x}(k)\right)\mathbf{x}(k) \tag{7}$$

$$\mathbf{x}(k+1) = A\mathbf{x}(k) + \mathbf{f}(k) \tag{8}$$

where $\mathbf{x}(k) \in \Re^n$ is the state vector and the vector function satisfies: $\mathbf{f}: \ \mathrm{K}_N x \quad \Re^n \ \rightarrow \ \Re$.

It is also assumed that **f** satisfies the adequate smoothnees requirements so that the solution of eqs.(1),(2) and (3) exists and is unique and continuous with respect to k and initial data and $\mathbf{x}(k)$ is the n-dimensional state vector defined in discrete time instances: k_0, $k_0 + 1$, $k_0 + 2$, … .

$\mathbf{f}(k) = \mathbf{f}(k, \ \mathbf{x}(k))$ is r-dimensional external force in eqs.(1),(2) and (3) such that: $r \leq n$.

It is obviuos that the vector functions satisfy $\psi(\cdot)$ and $\mathbf{f}(\cdot)$ satisfies:

$$\psi: \ \mathrm{K} \times \mathrm{R}^n \rightarrow \mathrm{R}, \quad \mathbf{f}: \ \mathrm{K} \times \mathrm{R}^r \rightarrow \mathrm{R} \tag{9}$$

Since both functions are bounded for all the bounded values of their arguments they need not have to satisfy the following conditions

$$\psi(k, \mathbf{0}) \equiv \mathbf{0}, \qquad \psi(k, \mathbf{0}, \mathbf{0}) \equiv \mathbf{0} \tag{10}$$

which means that the investigation of stability properties of the systems can be done with respect to the sets of points rather than one point.

Let R^n denote state space of systems given by eqs.(1), (2), (3), (4) and (5) and $\|(\cdot)\|$ Euclidean norm.

The solutions of eqs.(1), (2), (3), (4) and (5) are denoted by

$$\mathbf{x}(k, k_0, \mathbf{x}_0) \equiv \mathbf{x}(k) \tag{11}$$

and in a time instance k_0 with

$$\mathbf{x}(k_0) = \mathbf{x}(0) = \mathbf{x}(k_0, k_0, \mathbf{x}_0) \tag{12}$$

The discrete time interval is denoted, by K_N as a set of nonnegative integers:

$$\begin{aligned} \mathrm{K}_N &= \left\{ \ k_0, (k_0+1), (k_0+2) \ \ \ldots \ \ k_0 + k_N \right\} \\ &= \left\{ \ k: \ k_0 \leq k \leq k_0 + k_N \right\} \end{aligned} \tag{13}$$

Quantity k_N can be a positive integer or the symbol $+\infty$, so that finite time stability and practical stability can be treated simultaneously.

$k_s, \ \ k_s \in \left\{0, \ k_N\right\}$ is pre-specified settling time.

K_N^s denotes the discrete time interval as follows

$$\mathrm{K}_N^s = \left\{ \ k: \ \ k_0 + k_s < k < k_0 + k_N \right\} \tag{14}$$

The set difference is denoted by $\mathrm{K}_N \backslash \ \ \mathrm{K}_N^s$

$$\mathrm{K}_N \backslash \mathrm{K}_N^s = \left\{ \ k: \ \ k_0 \leq k \leq k_0 + k_N \right\} \tag{15}$$

Let $V: \ \mathrm{K}_N \ \mathrm{x} \ \mathrm{R}^n \rightarrow \mathrm{R}$, so that $V(k, \mathbf{x})$ is bounded for $" k \in K_N$ and for which $\|\mathbf{x}\|$ is also bounded.

Define the total difference of *V*/*k*, **x** (*k*)/ along the trajectory of the sy

)

In (16) " · " denotes the dot product of two vectors and

$$\Delta \mathbf{x}(k) = \mathbf{x}(k+1) - \mathbf{x}(k) \tag{18}$$

is the finite difference.

Definition 1. Function $V(k,\mathbf{x}(k))$ is said to posses the property Γ if the vector $\nabla V(k,\mathbf{x}(k))$ is unique regardless of the particular path taken when going from one specific point to another in state space R^n, *Michel, Wu* (*1969*).

Next, let

$$\Delta V\left(k,\mathbf{x}(k)\right) = \Delta V_{\mathbf{f}}\left(k,\mathbf{x}(k)\right) + \nabla V\left(k,\mathbf{x}(k)\right)\cdot \mathbf{f}\left(k,\mathbf{x}(k)\right) \tag{19}$$

where

$$\Delta V_{\mathbf{f}}\left(k,\mathbf{x}(k)\right) \equiv \Delta V\left(k,\mathbf{x}(k)\right)_{\mathbf{f}\equiv\mathbf{0}} \tag{20}$$

with the function f(·,·) in the linear combination presented in eq.(3).

Besides that, we use the following notation

$$\begin{aligned}
V_M^{\alpha}(k) &= \max_{\|\mathbf{x}(k)\|<\alpha} V\left(k,\mathbf{x}(k)\right)\\
V_m^{\alpha}(k) &= \min_{\|\mathbf{x}(k)\|<\alpha} V\left(k,\mathbf{x}(k)\right)\\
V_M^{\beta\alpha}(k) &= \max_{\alpha\le\|\mathbf{x}(k)\|<\beta} V\left(k,\mathbf{x}(k)\right)\\
V_m^{\beta\alpha}(k) &= \min_{\alpha\le\|\mathbf{x}(k)\|<\beta} V\left(k,\mathbf{x}(k)\right)\\
V_M^{(\alpha+\rho),\alpha}(k) &= \max_{\alpha\le\|\mathbf{x}(k)\|<(\alpha+\rho)} V\left(k,\mathbf{x}(k)\right)\\
V_m^{(\alpha+\rho),\alpha}(k) &= \min_{\alpha\le\|\mathbf{x}(k)\|<(\alpha+\rho)} V\left(k,\mathbf{x}(k)\right)\\
V_M^{\alpha,(\alpha-\rho)}(k) &= \max_{(\alpha-\rho)\le\|\mathbf{x}(k)\|<\alpha} V\left(k,\mathbf{x}(k)\right)\\
V_m^{\alpha,(\alpha-\rho)}(k) &= \min_{(\alpha-\rho)\le\|\mathbf{x}(k)\|<\alpha} V\left(k,\mathbf{x}(k)\right)
\end{aligned} \tag{21}$$

$$\begin{aligned}
V_{\bar{M}(\cdot)}(k) &= \max_{\mathbf{x}\in\bar{S}_{(\cdot)}} V\left(k,\mathbf{x}(k)\right)\\
V_{\bar{m}(\cdot)}(k) &= \min_{\mathbf{x}\in\bar{S}_{(\cdot)}} V\left(k,\mathbf{x}(k)\right)\\
V_{M(\cdot)}(k) &= \max_{\mathbf{x}\in S_{(\cdot)}} V\left(k,\mathbf{x}(k)\right)\\
V_{m(\cdot)}(k) &= \min_{\mathbf{x}\in S_{(\cdot)}} V\left(k,\mathbf{x}(k)\right)\\
V_{M(\cdot)}^{\partial}(k) &= \max_{\mathbf{x}\in\partial S_{(\cdot)}} V\left(k,\mathbf{x}(k)\right)\\
V_{m(\cdot)}^{\partial}(k) &= \min_{\mathbf{x}\in\partial S_{(\cdot)}} V\left(k,\mathbf{x}(k)\right)\\
V_{\underline{M}(\cdot)}(k) &= \max_{\mathbf{x}\in S_{(\cdot)}^c} V\left(k,\mathbf{x}(k)\right)\\
V_{\underline{m}(\cdot)}(k) &= \min_{\mathbf{x}\in S_{(\cdot)}^c} V\left(k,\mathbf{x}(k)\right)
\end{aligned} \tag{22}$$

For the time-invariant sets it is assumed: $S_{(\cdot)}$ are bounded, open sets.

The closure and boundary of $S_{(\cdot)}$ are denoted by $\bar{S}_{(\cdot)}$ and $\partial S_{(\cdot)}$, respectively, so: $\partial S_{(\cdot)} = \bar{S}_{(\cdot)} \setminus S_{(\cdot)}$. $S_{(\cdot)}^c$ denotes the complement of $S_{(\cdot)}$.

Let S_β be a given set of all allowable states of the system for $\forall k \in \mathrm{K}_N \setminus \mathrm{K}_N^s$ and S_γ is a set of all allowable states of the system for $\forall k \in \mathrm{K}_N^s$ $S_\gamma \subset S_\beta$.

The set S_α, $S_\alpha \subset S_\beta$ denotes the set of all alowable initial states and S_ε the corresponding set of disturbances.

The sets S_α, S_β, S_γ are connected and a priori known.

$\lambda(\cdot)$ denotes the eigenvalues of the matrix $(\cdot)$.

Λ is the maximum eigenvalue.

We, also, define the open set S_δ such that $S_\delta \subseteq S_\gamma$ with the following property

$$V_{M\delta}(k) \le V_{\underline{m}\delta}(k) = V_{\bar{m}\rho}(k), \qquad \forall k \in \mathrm{K}_N^s \tag{23}$$

where

$$S_\rho = S_\beta \setminus S_\delta \tag{24}$$

We introduce the set S^* such that

$$S^* = \left\{ k,\ \mathbf{x}(k):\ \ x(k) \in S_\beta \setminus S_\delta \cap S_\alpha, \quad k \in \mathrm{K}_{N-1} \right\} \tag{25}$$

Instead of general sets, let the sets be defined as

$$S_{\kappa\xi} = \left\{ \mathbf{x}(k) \in \mathrm{R}^n :\ \|\mathbf{x}\| < \xi \right\} \tag{26}$$

$$\bar{S}_{\kappa\xi} = \left\{ \mathbf{x}(k) \in \mathrm{R}^n :\ \|\mathbf{x}\| \le \xi \right\} \tag{27}$$

$$\partial S_{\kappa\xi} = \bar{S}_{\kappa\xi} \setminus S_{\kappa\xi} = \left\{ \mathbf{x}(k) \in \mathrm{R}^n :\ \|\mathbf{x}\| = \xi \right\} \tag{28}$$

The consequences are as follows

$$\begin{aligned}
V_{M\xi}(k) &= \max_{\|\mathbf{x}\|<\xi} \ V(k,\mathbf{x})\\
V_{\bar{M}\xi}(k) &= \max_{\|\mathbf{x}\|\le\xi} \ V(k,\mathbf{x})\\
V_{M\xi}^{\partial}(k) &= \max_{\|\mathbf{x}\|=\xi} \ V(k,\mathbf{x})\\
V_{\underline{M}\xi}(k) &= \max_{\|\mathbf{x}\|>\xi} \ V(k,\mathbf{x})
\end{aligned} \tag{29}$$

$$\begin{aligned}
V_{m\xi}(k) &= \min_{\|\mathbf{x}\|<\xi} \ V(k,\mathbf{x})\\
V_{\bar{m}\xi}(k) &= \min_{\|\mathbf{x}\|\le\xi} \ V(k,\mathbf{x})\\
V_{m\xi}^{\partial}(k) &= \min_{\|\mathbf{x}\|=\xi} \ V(k,\mathbf{x})\\
V_{\underline{m}\xi}(k) &= \min_{\|\mathbf{x}\|>\xi} \ V(k,\mathbf{x})
\end{aligned} \tag{30}$$

Stability definitions

Definition 2. System (1) is stable *w.r.t.* $\{\alpha, \beta, k_0, k_N, \|(\cdot)\|\}$, $\alpha \le \beta$, if for any trajectory $\mathbf{x}(k)$ the initial condition $\|\mathbf{x}(k_0)\| < \alpha$, implies that $\|\mathbf{x}(k)\| < \beta$ for $\forall k \in \mathrm{K}_N$.

Definition 3. System (1) is quasi-*contractively stable w.r.t.* $\{\alpha, \beta, \gamma, k_0, k_N, \|(\cdot)\|\}$, $\alpha \le \beta < \gamma$, if for any trajectory $\mathbf{x}(k)$ with the initial condition $\|\mathbf{x}(k_0)\| < \alpha$, the following conditions are satisfied

i) stable *w.r.t* to $\{\alpha, \gamma, k_0, k_N, \|(\cdot)\|\}$

and

ii) there exists a moment $k_p \in [k_0, (k_0+k_N)[$, such that $\|\mathbf{x}(k)\| < \beta$ for $\forall k \in\ ]k_p, (k_0+k_N)]$.

Definition 4. System (1) is *contractively stable w.r.t.* $\{\alpha, \beta, \gamma, k_0, k_N, \|(\cdot)\|\}$, $\alpha < \beta \le \gamma$, if for any trajectory $\mathbf{x}(k)$ with the initial condition $\|\mathbf{x}(k_0)\| < \alpha$, the following conditions are satisfied

i) stable *w.r.t* to $\{\alpha, \gamma, k_0, k_N, \|(\cdot)\|\}$

and

ii) there exists a moment $k_p \in [k_0, (k_0+k_N)[$, such that $\|\mathbf{x}(k)\| < \beta$ for $\forall k \in\ ]k_p, (k_0+k_N)]$.

Definition 5. System (2) is *stable w.r.t* $\{\alpha, \beta, \varepsilon, k_0, k_N, \|(\cdot)\|\}$, $\alpha \le \beta$, if for any trajectory $\mathbf{x}(k)$ with the initial condition $\|\mathbf{x}(k_0)\| < \alpha$ and the condition $\|\mathbf{f}(k)\| \le \varepsilon$, for $\forall k \in \mathrm{K}$, $\mathbf{x}(k) \in \{S_\beta \setminus S_\alpha\}$ follows that $\|\mathbf{x}(k)\| < \beta$ for $\forall k \in \mathrm{K}_N$.

Definition 6. System (2) is quasi-*contractively stable w.r.t.* $\{\alpha, \beta, \gamma, \varepsilon, k_0, k_N, \|(\cdot)\|\}$, $\alpha \le \beta < \gamma$, if for any trajectory $\mathbf{x}(k)$ with the initial condition $\|\mathbf{x}(k_0)\| < \alpha$ and the condition $\|\mathbf{f}(k)\| \le \varepsilon$, for $\forall k \in \mathrm{K}$, $\mathbf{x}(k) \in \{S_\gamma \setminus S_\alpha\}$, the following conditions are satisfied

i) stable *w.r.t* to $\{\alpha, \gamma, \varepsilon, k_0, k_N, \|(\cdot)\|\}$

and

ii) there exists a moment $k_p \in [k_0, (k_0+k_N)[$, such that $\|\mathbf{x}(k)\| < \beta$ for $\forall k \in]k_p, (k_0+k_N)]$.

Definition 7. System (2), is quasi-*contractively stable w.r.t.* $\{\alpha, \beta, \gamma, \varepsilon, k_0, k_N, \|(\cdot)\|\}$, $\beta < \alpha \le \gamma$, if for any trajectory $\mathbf{x}(k)$ with the initial condition $\|\mathbf{x}(k_0)\| < \alpha$, and the condition $\|\mathbf{f}(k)\| \le \varepsilon$, for $\forall k \in \mathrm{K}$, $\mathbf{x}(k) \in \{S_\gamma \setminus S_\beta\}$, the following conditions are satisfied

i) stable *w.r.t* to $\{\alpha, \gamma, \varepsilon, k_0, k_N, \|(\cdot)\|\}$

and

ii) there exists a moment $k_p \in [k_0, (k_0+k_N)[$, such that $\|\mathbf{x}(k)\| < \beta$ for $\forall k \in]k_p, (k_0+k_N)]$.

The first three definitions concern the systems working in the free operating regime and the rest the systems working in the forced operating regime.

In the sequel we present some theorems which give sufficient conditions for the systems to be considered on the finite time interval, *Michel, Wu (1969)*.

Stability theorems

Theorem 1. System (1) is *stable w.r.t* $\{\alpha, \beta, k_0, k_N, \|(\cdot)\|\}$, $\alpha < \beta$, if there exists a function $V(k, \mathbf{x}(k))$: $\mathrm{K}_N \times \mathrm{R}^n \to \mathrm{R}$, which is bounded for $\forall k \in \mathrm{K}_N$ and for all bounded values $\mathbf{x}(k)$ and if there exists a function $\phi(k)$ which is bounded on K_N, such that the following conditions are satisfied

$$\|\Delta\mathbf{x}(k)\| < (\beta - \alpha)/q = \rho, \quad q \ge 2, \quad \forall k \in \mathrm{K}_N \tag{31}$$

$$\text{i)}\ \Delta V\left(k, \mathbf{x}(k)\right) < \phi(k), \quad \forall k \in \mathrm{K}_N, \quad \forall \mathbf{x} \in \{S_\beta \setminus S_\alpha\} \tag{32}$$

$$\text{ii)}\ \sum_{j=k_1}^{k_2-1} \phi(j) \le V_m^{\beta,(\beta-\rho)}(k_2) - V_M^{(\alpha+\rho),\alpha}(k_1), \ \forall k_1, k_2 \in \mathrm{K}_N, \ k_1 < k_2 \tag{33}$$

Proof. The proof is based on contradiction.

Let $\mathbf{x}(k)$ denote a given trajectory of the system given in eq.(1), at time t the initial point is a state which satisfies the given initial condition $\|\mathbf{x}(k_0)\| < \alpha$. Assuming that there exists a discrete instant (moment) $k_r \in \mathrm{K}_N$ the first such that $\|\mathbf{x}(k_r)\| \ge \beta$. Then, there exists a discrete moment k_2, $k_2 = (k_r - 1)$, $k_2 \in \mathrm{K}_N$, such that $\|\mathbf{x}(k_2)\| = \|\mathbf{x}(k_r - 1)\| < \beta$. Then, there exists a discrete moment $k_p \in \mathrm{K}_N$ such that $\mathbf{x}(k_p)$ is the last point for which one can write $\|\mathbf{x}(k_p)\| < \alpha$. Moreover, there exists a discrete moment k_1, $k_1 = (k_p + 1)$, $k_1 \in \mathrm{K}_N$, such that $\|\mathbf{x}(k_1)\| = \|\mathbf{x}(k_p + 1)\| \ge \alpha$. From the hypothesis i), as the first condition of *Theorem* 1, one can derive

$$k_0 \le k_p < k_1 < k_2 < k_r \tag{34}$$

Which leads to a logical conclussion that

$$\alpha \le \|\mathbf{x}(k_1)\| < (\alpha + \rho) \tag{35}$$

$$(\beta - \rho) < \|\mathbf{x}(k_2)\| < \beta \tag{36}$$

If we say that

$$V\left(k, \mathbf{x}(k)\right) = V\left(k_1, \mathbf{x}(k_1)\right) + \sum_{j=k_1}^{k_2-1} \Delta V\left(j, \mathbf{x}(j)\right), \quad k_0 < k_1 < k \le k_2 \tag{37}$$

and bearing in mind the definition of $V_M^{(\alpha+\rho),\alpha}$, we can write

$$V\left(k_2, \mathbf{x}(k_2)\right) \le V_M^{(\alpha+\rho),\alpha}(k_1) + \sum_{j=k_1}^{k_2-1} \Delta V\left(j, \mathbf{x}(j)\right) \tag{38}$$

From the condition ii)

$$V\left(k_2, \mathbf{x}(k_2)\right) < V_M^{(\alpha+\rho),\alpha}(k_1) + \sum_{j=k_1}^{k_2-1} \phi(j) \tag{39}$$

and finally, taking the condition iii)

$$V\left(k_2, \mathbf{x}(k_2)\right) < V_M^{(\alpha+\rho),\alpha}(k_1) + V_m^{\beta,(\beta-\rho)}(k_2) - V_M^{(\alpha+\rho),\alpha}(k_1) \tag{40}$$

or

$$V\left(k_2, \mathbf{x}(k_2)\right) < V_m^{\beta,(\beta-\rho)}(k_2) \tag{41}$$

but we must remember that eq.(36) is still valid. Hence, eq.(41) shows the contradiction with respect to the definition of $V_m^{\beta,(\beta-\rho)}(k_2)$, so the basic assumption concerning the initial moment k_r is wrong. This means that there is no discrete moment k_r which belongs to the discrete time interval K_N such that $\|\mathbf{x}(k_r)\| > \beta$.

Since this argument is independent of the chosen $\mathbf{x}(k_0)$ and bearing in mind that the proof has been taken for an arbitrary chosen trajectory $\mathbf{x}(k)$, which emanates from set S_α, it is obvious that the *Theorem* is proved. ■

Theorem 2. System (1), is quasi-*contractively stable w.r.t.* $\{\alpha, \beta, \gamma, k_0, k_N, \|(\cdot)\|\}$, $\alpha < \beta < \gamma$, if there exists a function $V(k, \mathbf{x}(k))$: $\mathrm{K}_N \times \mathrm{R}^n \to \mathrm{R}$, which is bounded for $\forall k \in \mathrm{K}_N$ and for all bounded values $\mathbf{x}(k)$ and if there exist functions $\phi_1(k)$ and $\phi_2(k)$ which are bounded on K_N, so that the following conditions are satisfied:

$$\text{i)}\ \|\Delta\mathbf{x}(k)\| < (\gamma - \alpha)/q = \rho, \quad q \ge 2, \quad \forall k \in \mathrm{K}_N, \quad (\beta - \alpha) > \rho \tag{42}$$

$$\text{ii)}\ \Delta V\left(k, \mathbf{x}(k)\right) < \phi_1(k), \quad \forall k \in \mathrm{K}_N, \quad \forall \mathbf{x} \in \{S_\gamma \setminus S_\alpha\} \tag{43}$$

$$\text{iii)}\ \Delta V\left(k, \mathbf{x}(k)\right) < \phi_2(k), \quad \forall k \in \mathrm{K}_N, \quad \forall \mathbf{x} \in \{S_\gamma \setminus \overline{S}_{(\beta-\rho)}\} \tag{44}$$

$$\text{i)}\ \sum_{j=k_1}^{k_2-1} \phi_1(j) \le V_m^{\gamma,(\gamma-\rho)}(k_2) - V_M^{(\alpha+\rho),\alpha}(k_1), \quad \forall k_1, k_2 \in \mathrm{K}_N, \quad k_1 < k_2 \tag{45}$$

$$\text{ii)}\ \sum_{j=k_3}^{k_0+k_N-1} \phi_2(j) < V_m^{\gamma,\beta}(k_0 + k_N) - V_M^{\beta,(\beta-\rho)}(k_3), \quad k_3 \in \mathrm{K}_N, \quad k_3 < (k_0 + k_N) \tag{46}$$

Theorem 3. System (1) is quasi-*contractively stab*[redacted]
w.r.t. $\{\alpha, \beta, \gamma, k_0, k_N, \|(\cdot)\|\}$, $\beta < \alpha < \gamma$, if there exists[redacted]
function $V(k, \mathbf{x}(k))$: $K_N \times R^n \rightarrow R$, which is bounded [redacted]
$\forall k \in K_N$ and for all bounded values $\mathbf{x}(k)$ and if there ex[redacted]
functions $\phi_1(k)$ and $\phi_2(k)$ which are bounded on K_N, su[redacted]
that the following conditions are satisfied

i) $\|\Delta\mathbf{x}(k)\| < (\gamma - \alpha)/q = \rho$
$q \geq 2, \quad \forall k \in K_N, \quad \beta > \rho$ (4[redacted]

ii) $\Delta V\left(k, \mathbf{x}(k)\right) < \phi_1(k)$
$\forall k \in K_N, \quad \mathbf{x} \in \{S_\gamma \setminus S\}$ (4[redacted]

iii) $\Delta V\left(k, \mathbf{x}(k)\right) < \phi_2(k)$
$\forall k \in K_N, \quad \forall \mathbf{x} \in \{S_\gamma \setminus \overline{S}_{(\beta-\rho)}\}$

The proofs of *Theorems 2-6* are basically in concagrement with those of the *Theorem 1,* and for the sake of brevity are omitted here. The reader should consult the reference *Michel, Wu* (*1969*), with a remark that the following relation should be used in proofs

$$V\left(k,\mathbf{x}(k)\right)=V\left(k_1,\mathbf{x}(k_1)\right)+\sum_{j=k_1}^{k-1}\Delta V_{\mathbf{f}}\left(k,\mathbf{x}(k)\right)+ \\ +\sum_{j=k_1}^{k-1}\nabla V_{\mathbf{f}}\left(k,\mathbf{x}(k)\right)\cdot\mathbf{f}\left(k,\mathbf{x}(k)\right),\ \ k>k_1 \tag{72}$$

where $\Delta V_{\mathbf{f}}\left(k,\mathbf{x}(k)\right)$ is defined in eq.(20).

Author's original contributions, which expand the concept of practical stability with settling time on general and particular classes of discrete time systems are presented in the sequel.

Basic results–recapitulation

Some of the earlier results, given in *Debeljković* (*1979.a, 1993*) are presented.

General theorems of practical stabilty with settling time of discrete systems

Let the systems (1-3) be defined on the discrete time interval K_N.

Definition 8. System (1), is *practically stable with the settling time* k_s, u *w.r.t.* $\{k_0, \mathrm{K}_N, S_\alpha, S_\beta, S_\gamma\}$, $S_\alpha\subset S_\beta$, $S_\gamma\subset S_\beta$, if and only if

$$\mathbf{x}(k_0)\equiv\mathbf{x}_0\in S_\alpha \tag{73}$$

implying that

$$\text{i)}\ \mathbf{x}(k,k_0,\mathbf{x}_0)\in S_\beta,\quad \forall k\in\mathrm{K}_N \tag{74}$$

$$\text{ii)}\ \mathbf{x}(k,k_0,\mathbf{x}_0)\in S_\gamma,\quad \forall k\in\mathrm{K}_N^s \tag{75}$$

Definition 9. System (3), is *practically stable with the settling time* k_s, u *w.r.t.* $\{k_0, \mathrm{K}_N, S_\alpha, S_\beta, S_\gamma\}$, $S_\alpha\subset S_\beta$, $S_\gamma\subset S_\beta$, if and only if

$$x_0\in S_\alpha\quad\wedge\quad \mathbf{f}(k,\mathbf{x}(k))\in S_\varepsilon,\forall k\in\mathrm{K}_N \tag{76}$$

implying that

$$\text{i)}\ \mathbf{x}(k,k_0,\mathbf{x}_0)\in S_\beta,\quad \forall k\in\mathrm{K}_N \tag{77}$$

$$\text{ii)}\ \mathbf{x}(k,k_0,\mathbf{x}_0)\in S_\gamma,\quad \forall k\in\mathrm{K}_N^s \tag{78}$$

Theorem 7. System (1) is *practically stable with the settling time* k_s, *w.r.t.* $\{k_0, \mathrm{K}_N, S_\alpha, S_\beta, S_\gamma\}$, $\alpha<\beta<\gamma$, if there exists a function $V(k, \mathrm{x}(k))$: $\mathrm{K}_N\times \mathrm{R}^n\rightarrow \mathrm{R}$, which is bounded for $\forall k\in\mathrm{K}_N$ and for all bounded values $\|\mathrm{x}\|$ and if there exists a function $\phi_1(k)$: $\mathrm{R}^n\rightarrow\mathrm{R}$, which is bounded for $\forall k\in\mathrm{K}_{N-1}$ and if the following conditions are fullfilled

$$\text{i)}\ \Delta V\left(k,\mathbf{x}(k)\right)<\phi(k),\qquad \forall(k,\mathbf{x})\in S^* \tag{79}$$

$$\text{ii)}\ \sum_{j=k_o}^{j=k-1}\phi(j)\le V_{\underline{m}\beta}(k)-V_{\bar{M}\alpha}(k_0), \\ \forall k\in\mathrm{K}_N\setminus\mathrm{K}_N^s \tag{80}$$

$$\text{iii)}\ \sum_{j=k_o}^{j=k-1}\phi(j)<V_{\underline{m}\delta}(k)-V_{\bar{M}\alpha}(k_0),\forall k\in\mathrm{K}_N^s \tag{81}$$

$$\text{iv)}\ V_{M\delta}\le V_{\underline{m}\delta}(k)=V_{\bar{m}\xi}(k),\forall k\in\mathrm{K}_N^s \tag{82}$$

Debeljković (*1979.a*).

Proof. Let us show that eqs.(79) and (80) guarantee the condition i) of *Definition* 8.

The proof is based on contradiction.

Let us take any system trajectory which emanates for any point M which belongs to the set S_α. If we assume that for some moment k_1, $k_1\in\mathrm{K}_N\setminus\mathrm{K}_N^s$, the first one on the above mentioned set, the condition $\mathrm{x}(k_1, k_0, \mathrm{x}_0)\notin S_\beta$. is fullfilled.

Based on the previous text and beforehand introduced notation, it can be written

$$V\left(k_1,\mathbf{x}(k_1)\right)=V(k_0,\mathbf{x}_0)+ \\ +\sum_{j=k_0}^{j=k_1-1}\Delta V\left(j,\mathbf{x}(j)\right),\qquad k_1\in\mathrm{K}_N\setminus\mathrm{K}_N^s \tag{83}$$

Using the definition for $V_{\bar{M}(\cdot)}(k)$ and eq.(79), one can get

$$V\left(k_1,\mathbf{x}(k_1)\right)<V_{\bar{M}\alpha}(k_0)+ \\ +\sum_{j=k_0}^{j=k_1-1}\phi(j),\qquad k_1\in\mathrm{K}_N\setminus\mathrm{K}_N^s \tag{84}$$

Using eq.(80) we finally get

$$V\left(k_1,\mathbf{x}(k_1)\right)<V_{\underline{m}\beta}(k_1),\qquad k_1\in\mathrm{K}_N\setminus\mathrm{K}_N^s \tag{85}$$

This inequality can be achieved only and only if $\mathbf{x}(k_1,k_0,\mathbf{x}_0)\notin S_\beta^c$. But this result is in contradiction with the basic assumption. Therefore, since the moment k_1 is chosen arbitrarily from the discrete time interval $\mathrm{K}_N\setminus\mathrm{K}_N^s$, it follows

$$\mathbf{x}(k,k_0,\mathbf{x}_0)\in S_\beta, \\ \forall k\in\mathrm{K}_N\setminus\mathrm{K}_N^s,\quad \forall\mathbf{x}_0\in S_\alpha \tag{86}$$

If we assume again that system (1) is not *practically stable with the settling time* k_s, then at least there is one certainly one discrete moment k_2, $\exists\ !\ k_2\in\mathrm{K}_N^s$, for which

$$\mathbf{x}(k_2,k_0,\mathbf{x}_0)\notin S_\delta,\quad \forall k_2\in\mathrm{K}_N^s \tag{87}$$

Then

$$V\left(k_2,\mathbf{x}(k_2)\right)=V(k_0,\mathbf{x}_0)+ \\ +\sum_{j=k_0}^{j=k_2-1}\Delta V\left(j,\mathbf{x}(j)\right),\qquad k_2\in\mathrm{K}_N^s \tag{88}$$

Using the definition for $V_{\bar{M}(\cdot)}(k)$ and eq.(79), one can get

$$V\left(k_2,\mathbf{x}(k_2)\right)<V_{\bar{M}\alpha}(k_0)+\sum_{j=k_0}^{j=k_2-1}\phi(j),\qquad k_2\in\mathrm{K}_N^s \tag{89}$$

Using eq.(81) we finally get

$$V\left(k_2,\mathbf{x}(k_2)\right)<V_{\underline{m}\delta}(k_2)=V_{\bar{m}\xi}(k_2),\qquad k_2\in\mathrm{K}_N^s \tag{90}$$

Eq.(90) can be satisfied only and if only $\mathrm{x}(k_2, k_0, \mathrm{x}_0)\notin S_\delta$. But this result is in contradiction with the starting assumption. This fact demands that, for the discrete moment $k_2\in\mathrm{K}_N^s$ the following condition is satisfied

$$\mathbf{x}(k,k_0,\mathbf{x}_0)\in S_\delta,\qquad \forall k\in\mathrm{K}_N^s \tag{91}$$

Since

$$S_\delta \subseteq S_\gamma, \tag{92}$$

then

$$\mathbf{x}(k,k_0,\mathbf{x}_0) \in S_\gamma, \quad \forall k \in \mathrm{K}_N^s \tag{93}$$

which concludes the proof. ■

Theorem 8. A system is *practically stable with the settling time k_s, w.r.t.* $\{k_0, \mathrm{K}_N, S_\alpha, S_\beta, S_\gamma, S_\varepsilon\}$, $\alpha < \beta > \gamma$, if there exists a function $V(k, \mathrm{x}(k))$: $\mathrm{K}_N \times \mathrm{R}^n \to \mathrm{R}$, with the property Γ, which is bounded for $\forall k \in \mathrm{K}_N$ and for all bounded values $\|\mathrm{x}\|$ if there exist functions $\phi_1(k)$: $\mathrm{R}^n \to \mathrm{R}$ i $\phi_2(k)$: $\mathrm{R}^n \to \mathrm{R}$, which are bounded for all values $k \in \mathrm{K}_{N-1}$ and if the following conditions are satisfied

$$\text{i)}\ \begin{aligned}&\Delta V(k,\mathbf{x}(k)) < \phi_1(k),\\ &\forall (k,\mathbf{x}) \in S^*, \mathbf{f}(k,\mathbf{x}) \in S_\varepsilon\end{aligned} \tag{94}$$

$$\text{ii)}\ \begin{aligned}&\nabla V(k,\mathbf{x}(k))\cdot \mathbf{f}(k,\mathbf{x}) \le \phi_2(k),\\ &\forall (k,\mathbf{x}) \in S^*,\ \mathbf{f}(k,\mathbf{x}) \in S_\varepsilon\end{aligned} \tag{95}$$

$$\text{iii)}\ \begin{aligned}&\sum_{j=k_o}^{j=k-1} (\phi_1(j)+\phi_2(j)) \le V_{\underline{m}\beta}(k) - V_{\overline{M}\alpha}(k_0),\\ &\forall k \in \mathrm{K}_N \setminus \mathrm{K}_N^s\end{aligned} \tag{96}$$

$$\text{iv)}\ \begin{aligned}&\sum_{j=k_o}^{j=k-1} (\phi_1(j)+\phi_2(j)) \le V_{\underline{m}\delta}(k) - V_{\overline{M}\alpha}(k_0),\\ &\forall k \in \mathrm{K}_N^s\end{aligned} \tag{97}$$

$$\text{v)}\ V_{M\delta} \le V_{\underline{m}\delta}(k) = V_{\overline{m}\xi}(k), \ \forall k \in \mathrm{K}_N^s \tag{98}$$

Debeljković (1979.a).

Proof. The proof is also based on contradiction.

For a system trajectory, such that $\mathrm{x}_0 \in S_\alpha$ and assuming that there exists a discrete moment k_1, $k_1 \in \mathrm{K}_N \setminus \mathrm{K}_N^s$ the first one for which the following condition is fullfilled $\mathrm{x}(k_1, k_0, \mathrm{x}_0) \notin S_\beta$.

Then, having in mind the relations expressed above, we can write

$$\begin{aligned}V(k_1,\mathbf{x}(k_1)) &= V(k_0,\mathbf{x}_0) + \sum_{j=k_0}^{j=k_1-1} \Delta V_{\mathbf{f}}(j,\mathbf{x}(j))\\ &+ \sum_{j=k_0}^{j=k_1-1} \nabla V(j,\mathbf{x}(j))\cdot \mathbf{f}(j,\mathbf{x}(j)), k_1 \in \mathrm{K}_N \setminus \mathrm{K}_N^s\end{aligned} \tag{99}$$

Using the definition $V_{\overline{M}(\cdot)}(k)$, it follows that

$$\begin{aligned}V(k_1,\mathbf{x}(k_1)) &\le V_{\overline{M}\alpha}(k) + \sum_{j=k_0}^{j=k_1-1} \Delta V_{\mathbf{f}}(j,\mathbf{x}(j))\\ &+ \sum_{j=k_0}^{j=k_1-1} \nabla V(j,\mathbf{x}(j))\cdot \mathbf{f}(j,\mathbf{x}(j)), k_1 \in \mathrm{K}_N \setminus \mathrm{K}_N^s\end{aligned} \tag{100}$$

Using the first two conditions of the *Theorem*

$$\begin{aligned}&V(k_1,\mathbf{x}(k_1)) \le V_{\overline{M}\alpha}(k) + \sum_{j=k_0}^{j=k_1-1} (\phi_1(j)+\phi_2 j)\\ &k_1 \in \mathrm{K}_N \setminus \mathrm{K}_N^s\end{aligned} \tag{101}$$

Having in mind eq.(96)

$$V(k_1,\mathbf{x}(k_1)) \le V_{\underline{m}\beta}(k), \quad k_1 \in \mathrm{K}_N \setminus \mathrm{K}_N^s \tag{102}$$

This equation can be satisfied only and only if $\mathbf{x}(k_1,k_0,\mathbf{x}_0) \notin S_\beta^c$. However, this result is in contradiction with the starting assumption, so it follows that there is no discrete moment $k_1, \neg\exists k_1 \in \mathrm{K}_N \setminus \mathrm{K}_N^s$ for which $\mathbf{x}(k_1,k_0,\mathbf{x}_0) \in S_\beta^c$. Therefore, the only possibility is

$$\begin{aligned}&\mathbf{x}(k,k_0,\mathbf{x}_0) \in S_\beta,\\ &\forall k \in \mathrm{K}_N \setminus \mathrm{K}_N^s,\ \forall \mathbf{x}_0 \in S_\alpha\end{aligned} \tag{103}$$

which guarantees that the system given by eq.(3) is practically stable.

If we suppose that the system (3) is not practically stable with the settling time k_s, it is bound to exist at least one discrete moment k_2, $\exists\ !\ k_2 \in \mathrm{K}_N^s$, for which $\mathbf{x}(k_2,k_0,\mathbf{x}_0) \notin S_\delta$, $k_2 \in \mathrm{K}_N^s$.

Then

$$\begin{aligned}V(k_2,\mathbf{x}(k_2)) &= V(k_0,\mathbf{x}_0) + \sum_{j=k_0}^{j=k_2-1} \Delta V_{\mathbf{f}}(j,\mathbf{x}(j))\\ &+ \sum_{j=k_0}^{j=k_2-1} \nabla V(j,\mathbf{x}(j))\cdot \mathbf{f}(j,\mathbf{x}(j)),\ k_2 \in \mathrm{K}_N^s\end{aligned} \tag{104}$$

Using the definition $V_{\overline{M}(\cdot)}(k)$ and eqs.(94) and (95), eq.(97), finally becomes

$$V(k_2,\mathbf{x}(k_2)) < V_{\underline{m}\delta}(k_2) = V_{\overline{m}\xi}(k_2), \quad k_2 \in \mathrm{K}_N^s \tag{105}$$

which is a contradiction in regard to the starting assumption and should give $\mathbf{x}(k_2,k_0,\mathbf{x}_0) \in S_\delta^c$. This implies that, since the discrete moment k_2 is arbitrary chosen from the discrete time interval K_N^s, then

$$\mathbf{x}(k,k_0,\mathbf{x}_0) \in S_\delta,\ \ \forall k \in \mathrm{K}_N^s \tag{106}$$

Since

$$S_\delta \subseteq S_\gamma, \tag{107}$$

it follows

$$\mathbf{x}(k,k_0,\mathbf{x}_0) \in S_\gamma, \quad \forall k \in \mathrm{K}_N^s \tag{108}$$

which concludes the proof. ■

Should we include $S_\varepsilon = \varnothing$, in the former *Theorem, it is reduced to Theorem* 7.

General theorems application to the particular classes of discrete time systems

We shall concentrate on the linear discrete systems expressed by eqs.(4), (5), (6), (7) and (8).

The application of previous results, expressed in the view of *General Theorems*, needs a basic definitions' slight preformulation, as given in the sequel.

Definition 10. System (1) is *practically stable with the settling time k_s, w.r.t.* $\{k_0, k_N, \alpha, \beta, \gamma, \|(\cdot)\|^2\}$, if and only if

$$\|\mathbf{x}(k_0)\|^2 = \|\mathbf{x}_0\|^2 < \alpha \tag{109}$$

which implies

$$\text{i)}\ \|\mathbf{x}(k)\|^2 < \beta, \quad \forall k \in \mathrm{K}_N \tag{110}$$

$$\text{ii)}\, \| \mathbf{x}(k) \|^2 < \gamma, \quad \forall k \in \mathrm{K}_N^s \tag{111}$$

at $\gamma < \alpha < \beta$.

Definition 11. The system given by eq.(3) is *practically stable with the settling time* k_s, *w.r.t.* $\{k_0, k_N, \alpha, \beta, \gamma, \varepsilon, \|(\cdot)\|^2\}$, if and only if

$$\| \mathbf{x}_0 \|^2 < \alpha \quad \wedge \quad \| \mathbf{f}(k, \mathbf{x}(k)) \| \le \varepsilon, \quad \forall k \in \mathrm{K}_N \tag{112}$$

which implies

$$\text{i)}\, \| \mathbf{x}(k) \|^2 < \beta, \quad \forall k \in \mathrm{K}_N \tag{113}$$

$$\text{ii)}\, \| \mathbf{x}(k) \|^2 < \gamma, \quad \forall k \in \mathrm{K}_N^s \tag{114}$$

at $\gamma < \alpha < \beta$.

Definition 12. The system given by eq.(1), is *practically unstable w.r.t.* $\{k_0, k_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if for

$$\| \mathbf{x}_0 \|^2 < \alpha \tag{115}$$

there exists a discrete moment $k = k^* \in \mathrm{K}_N$, so that

$$\| \mathbf{x}(k^*) \|^2 \ge \beta \tag{116}$$

Definition 13. The system given by eq.(3) is *practically unstable w.r.t.* $\{k_0, k_N, \alpha, \beta, \varepsilon, \|(\cdot)\|^2\}$, $\alpha < \beta$, if for

$$\| \mathbf{x}_0 \|^2 < \alpha \quad \wedge \quad \| \mathbf{f}(k, \mathbf{x}(k)) \| \le \varepsilon, \quad \forall k \in \mathrm{K}_N \tag{117}$$

there exists a discrete moment $k = k^* \in \mathrm{K}_N$, so that

$$\| \mathbf{x}(k^*) \|^2 \ge \beta \tag{118}$$

Theorems which give sufficient conditions of practical stability will be presented, *Debeljković* (*1979.a*).

Theorem 9. The system given by eq.(4) is *practically stable with the settling time* k_s, *w.r.t.* $\{k_0, k_N, \alpha, \beta, \gamma, \|(\cdot)\|^2\}$, $\gamma < \alpha < \beta$, if the following conditions are satisfied

$$\text{i)} \prod_{j=k_0}^{j=k_0+k-1} \Lambda(j) \le \beta / \alpha, \quad \forall k \in \mathrm{K}_N \setminus \mathrm{K}_N^s, \tag{119}$$

$$\text{ii)} \prod_{j=k_0}^{j=k_0+k-1} \Lambda(j) \le \gamma / \alpha, \quad \forall k \in \mathrm{K}_N^s \tag{120}$$

where $\Lambda(j)$ denotes the maximum eigenvalue of the matrix $A^T(k)A(k)$.

Proof. Let $V(\mathbf{x}(k)) = \ln \ \mathbf{x}^T(k)\mathbf{x}(k)$.

Then

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= \ln \ \mathbf{x}^T(k+1)\mathbf{x}(k+1) \\ -\ln \ \mathbf{x}^T(k)\mathbf{x}(k) &= \ln \ \frac{\mathbf{x}^T(k)A^T(k)A(k)\mathbf{x}(k)}{\mathbf{x}^T(k)\mathbf{x}(k)} \\ &\le \Lambda\left(A^T(k)A(k)\right) \end{aligned} \tag{121}$$

The summation $\sum_{k_0}^{k_0+k-1}$ of the previous equation for $\forall k \in \mathrm{K}_N \setminus \mathrm{K}_N^s$, gives

$$\begin{aligned} \ln \mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k) &\le \sum_{j=k_0}^{j=k_0+k-1} \ln \Lambda(j) \ + \\ + \ln \mathbf{x}^T(k_0)\mathbf{x}(k_0), &\quad \forall k \in \mathrm{K}_N \setminus \mathrm{K}_N^s \end{aligned} \tag{122}$$

Bearing in mind that $\|x_0\|^2 < \alpha$ and the first condition of *Theorem* 9, then

$$\ln \mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k) < \ln \beta, \quad \forall k \in \mathrm{K}_N \setminus \mathrm{K}_N^s \tag{123}$$

which confirms the practical stability of the system given by eq.(4) on the discrete time interval $\mathrm{K}_N \setminus \mathrm{K}_N^s$.

To prove the last condition the summation $\sum_{k_0}^{k_0+k-1}$ of eq.(121), on the discrete time interval K_N^s is made, confirming that eq.(115) is still valid.

It follows that

$$\begin{aligned} &\ln \mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k) \le \\ &\sum_{j=k_0}^{j=k_0+k-1} \Lambda(j) \ + \ln\alpha, \quad \forall k \in \mathrm{K}_N^s \end{aligned} \tag{124}$$

Using the second condition of the *Theorem*

$$\ln \mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k) < \ln\gamma, \quad \forall k \in \mathrm{K}_N^s \tag{125}$$

that was meant to be proved. ■

Theorem 10. The system given by eq.(5) is *practically stable with the settling time* k_s, *w.r.t.* $\{k_0, k_N, \alpha, \beta, \gamma, \|(\cdot)\|^2\}$, $\gamma < \alpha < \beta$, if the following conditions are satisfied

$$\Lambda^k \le \beta / \alpha, \quad \forall k \in \mathrm{K}_N \setminus \mathrm{K}_N^s \tag{126}$$

$$\Lambda^k \le \gamma / \alpha, \quad \forall k \in \mathrm{K}_N^s \tag{127}$$

where: $\Lambda = \Lambda(A^T A) = \lambda_{\max}(A^T A)$.

Proof. The proof follows directly from the proof of the previous *Theorem*, if $\Lambda\left(A^T(k)A(k)\right) = \Lambda(A^T A) = const.$ ■

Theorem 11. The system given by eq.(7) is *practically stable with the settling time* k_s, *w.r.t* $\{k_0, k_N, \alpha, \beta, \gamma, \|(\cdot)\|\}$, $\gamma < \alpha < \beta$, if there exists a real, scalar function $\phi(k)$, which is bounded for $\forall k \in \mathrm{K}_{N-1}$ and if the following conditions are satisfied

$$\| A(k, \mathbf{x}) \| < \phi(k), \quad \forall k \in \mathrm{K}_{N-1}, \quad \forall \mathbf{x} \in \{S_{\kappa\beta} \setminus S_{\kappa\gamma}\} \tag{128}$$

$$\prod_{j=k_0}^{j=k_0+k-1} \phi(j) \le \sqrt{\beta / \alpha}, \quad \forall k \in \mathrm{K}_N \setminus \mathrm{K}_N^s \tag{129}$$

$$\prod_{j=k_0}^{j=k_0+k-1} \Lambda(j) \le \sqrt{\beta / \gamma}, \quad \forall k \in \mathrm{K}_N^s \tag{130}$$

Proof. Let: $V(\mathbf{x}(k)) = \| \mathbf{x}(k) \|$.

Then

$$\begin{aligned} \Delta V(\mathbf{x}(k)) &= \ln \| \mathbf{x}(k+1) \| - \ln \| \mathbf{x}(k) \| \\ &= \ln \frac{\| \mathbf{x}(k+1) \|}{\| \mathbf{x}(k) \|} \le \ln \left\| A(k, \mathbf{x}(k)) \right\| \\ &\le \ln \phi(k), \ \forall k \in K_{N-1}, \ \forall \mathbf{x} \in \{S_{\kappa\beta} \setminus S_{\kappa\gamma}\} \end{aligned} \tag{131}$$

The method of the rest of the proof is the same as the previous ones, so it is omitted here for the sake of brevity, *Debeljković* (*1979.a*). ■

Theorem 12. The system given by eq.(4) is *practically stable with the settling time* k_s, *w.r.t.* $\{k_0, k_N, \alpha, \beta, \gamma, \|(\cdot)\|\}$, $\gamma < \alpha < \beta$, if the following conditions are satisfied

$$\| \Phi(k, k_0) \| \le \sqrt{\beta / \alpha}, \quad \forall k \in \mathrm{K}_N \tag{132}$$

$$\|\Phi(k,k_0)\| \le \sqrt{\gamma/\alpha}, \quad \forall k \in \mathrm{K}_N \setminus \mathrm{K}_N^s \tag{133}$$

where $\Phi(k, k_0)$ denotes the fundamental matrix of the system given by eq.(4), *Debeljković* (*1979.a*).

Proof. The solution of the system given by eq.(4) vector difference equation in the discrete moment k, is givent with

$$\mathbf{x}(k,k_0,\mathbf{x}_0) = \Phi(k,k_0)\mathbf{x}_0, \quad \forall k \in \mathrm{K}_N \tag{134}$$

Then

$$\|\mathbf{x}(k,k_0,\mathbf{x}_0)\| = \|\mathbf{x}(k)\| \le \|\Phi(k,k_0)\| \cdot \|\mathbf{x}_0\|, \quad \forall k \in \mathrm{K}_N \tag{135}$$

Bearing in mind that $\|x_0\|^2 < \alpha$ and taking into account the first condition of the *Theorem*

$$\begin{aligned}\|\mathbf{x}(k)\| &\le \|\Phi(k,k_0)\| \cdot \sqrt{\alpha} \\ &\le (\sqrt{\beta/\alpha}) \cdot \sqrt{\alpha} < \sqrt{\beta}, \\ &\forall k \in \mathrm{K}_N\end{aligned} \tag{136}$$

which proves the practical stability of the system considered on the discrete time interval K_N.

In the same way, it can be shown that the following condition is also satisfied

$$\|\mathbf{x}(k)\| < \sqrt{\gamma}, \quad \forall k \in \mathrm{K}_N^s \tag{137}$$

which concludes the proof. ■

If A in eq.(4) is a constant matrix, then condition (132) is changed to

$$\|A^k\| \le \sqrt{\beta/\alpha}, \quad \forall k \in \mathrm{K}_N \tag{138}$$

Weiss and Lee (*1971*), making the investigation of stability much easier.

If A is a square matrix of dimensions $n \times n$, then

$$\|A^k\| \le (\|A\|)^k \tag{139}$$

using the inequality sign when *A is a normal matrix.* This result may be used if necessary.

Lemma 1. Let the system under analysis be described by eq.(4).

If

$$\mathbf{q}(k,h) = \Phi(k,h)\mathbf{v}(h) \tag{140}$$

then

$$\|\mathbf{q}(k,h)\|^2 \le \|\mathbf{v}(h)\|^2 \cdot \prod_{j=h}^{j=k-1} \Lambda(j) \tag{141}$$

where $\Lambda(j) = \max \lambda(A^T(j)A(j))$.

Proof. If the first difference of eq.(140) with respect to k is found, we get

$$\mathbf{q}(k+1,h) = \Phi(k+1,h)\mathbf{v}(h) \tag{142}$$

Since

$$\Phi(k+1,h) = A(k) \cdot \Phi(k,h) \tag{143}$$

then, taking into account eq.(140)

$$\mathbf{q}(k+1,h) = A(k) \cdot \mathbf{q}(k,h) \tag{144}$$

Resolving the expression $\Delta \ln \mathbf{q}^T(k,h)\mathbf{q}(k,h)$, we get

$$\begin{aligned}\Delta \ln \mathbf{q}^T(k,h)\mathbf{q}(k,h) &= \ln \mathbf{q}^T(k+1,h) \;\; \mathbf{q}(k+1,h) \\ &\quad - \ln \mathbf{q}^T(k,h) \;\; \mathbf{q}(k,h) \\ &= \ln \frac{\mathbf{q}^T(k,h)A^T(k)A(k)\mathbf{q}(k,h)}{\mathbf{q}^T(k,h)\mathbf{q}(k,h)} \\ &\le \ln \Lambda\left(A^T(k)A(k)\right)\end{aligned} \tag{145}$$

Making a summation $\sum_{j=h}^{j=k-1}$ of the previous equation, taking into account that

$$\mathbf{q}^T(h,h)\mathbf{q}(h,h) = \mathbf{v}^T(h)\mathbf{v}(h) \tag{146}$$

because

$$\Phi(h,h) = I, \qquad \mathbf{q}(h,h) = \mathbf{v}(h) \tag{147}$$

it follows that

$$\sum_{j=h}^{j=k-1} \Delta \ln \mathbf{q}^T(j,h)\mathbf{q}(j,h) \le \sum_{j=h}^{j=k-1} \ln \Lambda(j) \tag{148}$$

So

$$\begin{aligned}&\ln \mathbf{q}^T(k,h)\mathbf{q}(k,h) - \ln \mathbf{q}^T(h,h)\mathbf{q}(h,h) \le \\ &\qquad \le \sum_{j=h}^{j=k-1} \ln \Lambda(j)\end{aligned} \tag{149}$$

or

$$\ln \frac{\mathbf{q}^T(k,h)\mathbf{q}(k,h)}{\mathbf{q}^T(h)\mathbf{q}(h)} \le \ln \prod_{j=k}^{j=k-1} \Lambda(j) \tag{150}$$

which finally gives

$$\|\mathbf{q}(k,h)\|^2 \le \|\mathbf{v}(h)\|^2 \cdot \prod_{j=h}^{j=k-1} \Lambda(j) \tag{151}$$

that was meant to be proved. ■

If the matrix $A(k) = A$, then $\Lambda(j) = \Lambda = const.$, so eq.(151) is transformed to

$$\|\mathbf{q}(k,h)\| \le \|\mathbf{v}(h)\| \, \Lambda^{0.5(k-h)} \tag{152}$$

and can be used when the time invariant system, given by eq.(5) is considered, *Debeljković* (*1997.a*).

The derived results represent a *discrete version of the very well known Bellman-Gronwall's* lemma, *Angelo* (*1974*).

Theorem 13. The system given by eq.(8) is *practically stable with the settling time* k_s, *w.r.t.* $\{k_0, k_N, \alpha, \beta, \gamma, \varepsilon, \|(\cdot)\|\}$, $\gamma < \alpha < \beta$, if the following conditions are satisfied

$$\text{i)}\ \Lambda^{0.5k} + k \cdot \varepsilon^* \Lambda^{0.5(k-1)} \le \sqrt{\beta/\alpha}, \quad \forall k \in \mathrm{K}_N \setminus \mathrm{K}_N^s \tag{153}$$

$$\text{ii)}\ \Lambda^{0.5k} + k \cdot \varepsilon^* \Lambda^{0.5(k-1)} \le \sqrt{\gamma/\alpha}, \quad \forall k \in \mathrm{K}_N^s \tag{154}$$

where $\varepsilon^* = \varepsilon/\sqrt{\alpha}$ and $\Lambda = \lambda_{\max}(A^T A)$.

Proof. The solution of eq.(8) is given with

$$\mathbf{x}(k) = \Phi(k,0)\mathbf{x}_0 + \sum_{j=0}^{j=k-1} \Phi(k-1,j)\mathbf{f}(j), \quad \forall k \in \mathrm{K}_N \tag{155}$$

or

$$\|\mathbf{x}(k)\| \leq \|\Phi(k,0)\|\cdot\|\mathbf{x}_0\| + \\ + \sum_{j=0}^{j=k-1}\|\Phi(k-1,j)\|\cdot\|\mathbf{f}(j)\|, \forall k\in \mathrm{K}_N \tag{156}$$

Using the basic result of *Lemma* 1.

$$\|\mathbf{x}(k)\| \leq \|\mathbf{x}_0\|\Lambda^{0.5k} + \sum_{j=0}^{j=k-1}\|\mathbf{f}(j)\|\Lambda^{0.5(k-1)}, \\ \forall k\in \mathrm{K}_N \setminus \mathrm{K}_N^s \tag{157}$$

Since

$$\|\mathbf{x}_0\| < \sqrt{\alpha} \quad \wedge \quad \|\mathbf{f}(k)\| \leq \varepsilon, \quad \forall k \in \mathrm{K}_N \tag{158}$$

then

$$\|\mathbf{x}(k)\| \leq \Lambda^{0.5k}\sqrt{\alpha} + k\cdot\varepsilon \quad \Lambda^{0.5(k-1)}, \\ \forall k\in \mathrm{K}_N \setminus \mathrm{K}_N^s \tag{159}$$

or, using the first condition of the *Theorem* , we get

$$\|\mathbf{x}(k)\| < \sqrt{\beta}, \qquad \forall k\in \mathrm{K}_N \setminus \mathrm{K}_N^s \tag{160}$$

so that the practical stability on the discrete time interval $\mathrm{K}_N \setminus \mathrm{K}_N^s$ is confirmed.

Using the analogous procedure, starting with eq.(157) for the discrete time interval K_N^s, we get

$$\|\mathbf{x}(k)\| < \sqrt{\gamma}, \qquad \forall k\in \mathrm{K}_N^s \tag{161}$$

which concludes the proof of the *Theorem*. ■

Bearing in mind the fact that all the theorems give only the sufficient conditions of practical and finite time stability, some better estimations of the systems behavior may be gained if potential time intervals of *practical unstability* are determined.

Then we can estimate from the "*right side*". The results, according to *Definitions 12-13*, are presented in the sequel.

Theorem 14. The system given by eq.(4) is *practically unstable w.r.t.* $\{k_0, k_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if there exists a real, positive number δ, $\delta \in\,]\,0, \alpha\,[$ and the discrete moment k, $k = k^*$: $\exists(k^* > k_0)\in \mathrm{K}_N$ and if the following condition is satisfied

$$\prod_{j=k_0}^{j=k_0+k^*-1}\lambda_{\min}(j) > \beta/\delta, \qquad k^*\in \mathrm{K}_N \tag{162}$$

where $\lambda_{\min}(j) = \min\lambda\left(A^T(j)A(j)\right)$, *Debeljković (1979.a)*.

Proof. Let $V\left(\mathbf{x}(k)\right) = \ln \mathbf{x}^T(k)\mathbf{x}(k)$.

Then

$$\Delta V\left(\mathbf{x}(k)\right) = \ln\frac{\mathbf{x}^T(k)A^T(k)A(k)\mathbf{x}(k)}{\mathbf{x}^T(k)\mathbf{x}(k)} \geq \\ \geq \ln\lambda_{\min}\left(A^T(k)A(k)\right) \tag{163}$$

The summation $\sum_{j=k_0}^{j=k_0+k-1}$ of the previous inequality for any k, $k\in \mathrm{K}_N$, gives

$$\ln\mathbf{x}^T(k_0+k)\mathbf{x}(k_0+k) \geq \\ \sum_{j=k_0}^{j=k_0+k-1}\ln\lambda_{\min}(j) + \ln\mathbf{x}^T(k_0)\mathbf{x}(k_0) \tag{164}$$

Taking into account that for some $\mathbf{x}_0$ the condition $\delta < \|\mathbf{x}_0\|^2 < \alpha$ is fullfilled and combining the previous equation with eq.(162)

$$\exists(k^* > k_0)\in \mathrm{K}_N \ni \\ \ln\mathbf{x}^T(k_0+k^*)\mathbf{x}(k_0+k^*) > \ln\beta \tag{165}$$

which concludes the proof of the *Theorem*. ■

Theorem 15. The system given by eq.(5) is *practically unstable w.r.t.* $\{k_0, k_N, \alpha, \beta, \|(\cdot)\|\}$, $\alpha < \beta$, if there exists a real, positive number δ, $\delta \in\,]\,0, \alpha\,[$ and the discrete moment k, $k = k^*$: $\exists(k^* > k_0)\in \mathrm{K}_N$ and if the following condition is fullfilled

$$\lambda_{\min}^{k^*} > \beta/\delta, \qquad k^*\in \mathrm{K}_N \tag{166}$$

where $\lambda_{\min} = \min\lambda(A^TA)$.

Proof. As in the previous *Theorem*, if

$$\lambda_{\min}(j) = \lambda_{\min}(A^TA) = \text{const.} ■$$

When the forced discrete time systems are considered in the sense of *practical unstability,* it is convinient to use the following result.

Lemma 2. Let us consider, again, the system given by eq(4).

If

$$\mathbf{q}(k,h) = \Phi(k,h)\mathbf{v}(h) \tag{167}$$

then

$$\|\mathbf{q}(k,h)\|^2 \geq \|\mathbf{v}(h)\|^2\cdot\prod_{j=h}^{j=k-1}\lambda_{\min}(j) \tag{168}$$

where $\lambda_{\min} = \min\lambda(A^TA)$, *Debeljković (1979.a)*.

Proof. The proof is analogous to the proof of *Lemma* 1 and is based on the very well known relation from the quadratic forms theory. ■

In a particular case, when $A(k) = A$, then

$$\|\mathbf{q}(k,h)\| \geq \|\mathbf{v}(h)\|\lambda_{\min}^{0.5(k-h)} \tag{169}$$

After this, the next results can be given.

Theorem 16. The system given by eq.(3) is *practically unstable w.r.t.* $\{k_0, k_N, \alpha, \beta, \|(\cdot)\|\}$, $\alpha < \beta$, if there exist real, positive numbers δ and ε_0, such that $\delta < \|x_0\|^2 < \alpha$ and $\varepsilon_0 < \|f(k)\| < \varepsilon$, $\forall k \in \mathrm{K}_N$ and the discrete moment k, $k = k^*$: $\exists!(k^* > k_0)\in \mathrm{K}_N$ for which the following condition is fullfilled

$$\left|\sqrt{\delta}\lambda_{\min}^{0.5k^*} - k^*\cdot\varepsilon \quad \lambda_{\min}^{0.5(k^*-1)}\right| > \sqrt{\beta}, \qquad k^*\in \mathrm{K}_N \tag{170}$$

Proof. The solution of eq.(3) is given with

$$\Phi(k,0)\mathbf{x}_0 = \mathbf{x}(k) - \sum_{j=0}^{j=k-1}\Phi(k-1,j)\cdot\mathbf{f}(j), \\ \forall k\in \mathrm{K}_N \tag{171}$$

Using the well known properties of the utilized norm, we get

$$\|\Phi(k,0)\mathbf{x}_0\| \leq \\ \leq \|\mathbf{x}(k) - \sum_{j=0}^{j=k-1}\Phi(k-1,j)\cdot\mathbf{f}(j)\|, \\ \forall k\in \mathrm{K}_N \tag{172}$$

or

$$\|\Phi(k,0)\mathbf{x}_0\| \le \|\mathbf{x}(k)\| + \sum_{j=0}^{j=k-1}\|\Phi(k-1,j)\cdot\mathbf{f}(j)\|, \forall k\in \mathrm{K}_N \tag{173}$$

If we proceed further

$$\|\mathbf{x}(k)\| \ge \|\Phi(k,0)\mathbf{x}_0\| - \sum_{j=0}^{j=k-1}\|\Phi(k-1,j)\cdot\mathbf{f}(j)\|, \quad \forall k\in \mathrm{K}_N \tag{174}$$

Using the same procedure, starting from eq.(170)

$$\|\mathbf{x}(k)\| \ge \sum_{j=0}^{j=k-1}\|\Phi(k-1,j)\cdot\mathbf{f}(j)\| - \|\Phi(k,0)\mathbf{x}_0\|, \quad \forall k\in \mathrm{K}_N \tag{175}$$

eqs.(173) and (175) can be combined in the following way

$$\|\mathbf{x}(k)\| \ge \Big|\|\Phi(k,0)\mathbf{x}_0\| - \sum_{j=0}^{j=k-1}\|\Phi(k-1,j)\cdot\mathbf{f}(j)\|\Big|, \quad \forall k\in \mathrm{K}_N \tag{176}$$

Using the basic result of *Lemma* 2, we get

$$\|\mathbf{x}(k)\| \ge \Big|\|\mathbf{x}_0\|\cdot\lambda_{\min}^{0.5k} - k\|\mathbf{f}(j)\|\lambda_{\min}^{0.5(k-1)}\Big|, \quad \forall k\in \mathrm{K}_N \tag{177}$$

Taking into account that for some $\mathbf{x}_0$ the condition $\delta < \|\mathbf{x}_0\|^2 < \alpha$ is fullfilled, for a $\mathbf{f}(k)$, $\varepsilon_0 < \|\mathbf{f}(k)\| < \varepsilon$, and for a discrete moment $k = k^* \in \mathrm{K}_N$. It follows that

$$\|\mathbf{x}(k^*)\| \ge \Big|\sqrt{\delta}\lambda_{\min}^{0.5k^*} - k^*\varepsilon_0\lambda_{\min}^{0.5(k-1)}\Big|, \quad k^*\in \mathrm{K}_N \tag{178}$$

Using this theorem condition we finally get

$$\|\mathbf{x}(k^*)\| > \sqrt{\beta}, \quad k^*\in \mathrm{K}_N \tag{179}$$

which was meant to be proved. ■

The recent results of the author, concerning the same problems are presented in the end, *Debeljković* (*1993*).

Theorem 17. The time invariant, autonomous system given by eq.(1) is *practically stable w.r.t.* $\{k_0, k_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if there exists a real, symmetric, positive definite matrix $M=M^T > 0$, which satisfies the following condition

$$\psi^T\big(\mathbf{x}(k)\big)Q\psi\big(\mathbf{x}(k)\big) < \mathbf{x}^T(k)M\mathbf{x}(k), \quad \forall \mathbf{x}(k)\in S_\beta \tag{180}$$

and if the next condition is fullfiled as well

$$\nu\eta^k < \beta/\alpha, \quad \forall k\in \mathrm{K}_N \tag{181}$$

where ν and η are given by

$$\nu = \frac{\Lambda(Q)}{\lambda_{\min}(Q)}, \quad \eta = \frac{\Lambda(M)}{\lambda_{\min}(Q)} \tag{182}$$

and the matrix $Q = Q^T > 0$, is also a real, symmetric, positive definite matrix.

Proof. Let $V\big(\mathbf{x}(k)\big) = \ln \mathbf{x}^T(k)Q\mathbf{x}(k)$.

Then

$$\Delta V\big(\mathbf{x}(k)\big) = \ln\frac{\psi^T\big(\mathbf{x}(k)\big)Q\psi\big(\mathbf{x}(k)\big)}{\mathbf{x}^T(k)Q\mathbf{x}(k)} \tag{183}$$

Should we use the well known nature of the positive definite quadratic forms

$$\lambda_{\min}(H)\|\mathbf{x}(k)\|^2 \le \|\mathbf{x}(k)\|_H^2 \le \Lambda(H)\|\mathbf{x}(k)\|^2 \tag{184}$$

it is easy to see that

$$\Delta V\big(\mathbf{x}(k)\big) \le \ln\frac{\Lambda(M)\|\mathbf{x}(k)\|^2}{\lambda_{\min}(Q)\cdot\|\mathbf{x}(k)\|^2} < \frac{\Lambda(M)}{\lambda_{\min}(Q)} \tag{185}$$

The summation $\sum_{j=k_0}^{j=k_0+k-1}$ of the previous inquality, makes

$$\ln\mathbf{x}^T(k_0+k)\,Q\,\mathbf{x}(k_0+k) - \ln\mathbf{x}^T(k_0)\,Q\,\mathbf{x}(k_0) \le \le \sum_{j=k_0}^{j=k_0+k-1}\ln\eta \le \ln\prod_{j=k_0}^{j=k_0+k-1}\eta \le \ln\eta^k \tag{186}$$

or

$$\begin{aligned}\ln\mathbf{x}^T(k_0+k)\;\;Q\;\;\mathbf{x}(k_0+k) &\le \ln\eta^k + \ln\Lambda(Q)\|\mathbf{x}_0\|^2\\ &\le \ln\eta^k + \ln\Lambda(Q)\alpha\\ &\le \ln\alpha\eta^k\Lambda(Q)\end{aligned} \tag{187}$$

If the expression on the left side of the previous equation is replaced by the corresponding term from eq.(184) and then eq.(182) is used

$$\|\mathbf{x}(k)\|^2 \le \alpha\nu\eta^k \tag{188}$$

in other words, if eq.(181) is used we get

$$\|\mathbf{x}(k_0+k)\|^2 = \|\mathbf{x}(k)\|^2 < \beta, \quad \forall k\in \mathrm{K}_N \tag{189}$$

which concludes the proof. ■

Theorem 18. The system given by eq.(7), is *practically stable w.r.t.* $\{k_0, k_N, \alpha, \beta, \|(\cdot)\|^2\}$, $\alpha < \beta$, if there exists a real, positive number θ such that

$$\theta = \sup_{\mathbf{x}\in S_\beta}\Lambda\big(A^T(\mathbf{x})A(\mathbf{x})\big) \tag{190}$$

and if the following condition is satisfied

$$\theta^k < \beta/\alpha, \quad \forall k\in \mathrm{K}_N \tag{191}$$

Proof. Let $V\big(\mathbf{x}(k)\big) = \ln\|\mathbf{x}(k)\|_I^2$. The rest of the proof is completely metodologically identical to the previous one and omitted here for the sake of brevity. ■

Conclusion

This layout of the results of numerous authors who have been working on the problems of nonlyapunov stability over the last fifty years deals primarily with the concept of finite time stability, practical stability and technical stability. The presented results are treating only a particular class of discrete time systems and are given only in the form of sufficient conditions.

Some of these results are extracts from author's doctoral dissertation.

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Stabilnost linearnih diskretnih sistema na konačnom vremenskom intervalu: retrospektiva rezultata

U radu je dat iscrpan, hronološki pregled rezultata koji se bave problematikom stabilnosti ove klase sistema na konačnom vremenskom intervalu. Navedene su brojne definicije, a kroz selektivno odabrane teoreme izloženi su najnoviji ranije publikovani i danas aktuelni rezultati, koji specificiraju uslove stabilnosti i nestabilnosti linearnih, diskretnih sistema koji svoje ponašanje ostvaruju ili u slobodnom ili u prinudnom radnom režimu. Velika većina rezultata data je u formi dovoljnih uslova ovog koncepta stabilnosti.

Ključne reči: **diskretni sistemi, stabilnost na konačnom vremenskom intervalu, praktična stabilnost, Bellman-Gronwallova lema.**

Stabilité des systèmes linéaires discrèts dans l'intervalle de temps fini: retrospective des résultats

L'etude détaillée et chronologique des résultats concernant le problème de la stabilité de cette classe de systèmes dans l'intervalle de temps fini est donnée. Les définitions nombreuses sont présentées aussi bien que les théorèmes choisis et les résultats auparavant publiés mais toujour actuels sur les conditions de la stabilité et de la non-stabilité des systèmes linéaires et discrets dans le régime du travail libre ou imposé. La plupart des résultats est donnée en forme des conditions suffisantes de cette conception de la stabilité.

Mots-clés: **systèmes discrets, stabilité dans l'intervalle de temps fini, stabilité pratique, lemma de Bellman-Gronwall.**